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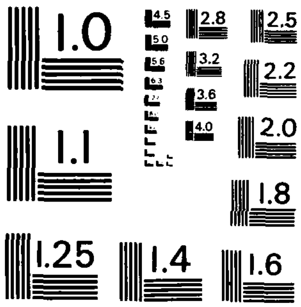

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ON THE CHARACTERIZATION OF SIMPLE CLOSED SURFACES IN THREE-DIMENSIONAL DIGITAL IMAGES

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ABSTRACT

This is a continuation of a series of papers on the digital geometry of three-dimensional digital images. In earlier reports, D. Morgenthaler and A. Rosenfeld gave symmetric definitions for simple surface points under the concepts of 6-connectivity and 26-connectivity, and they non-trivially characterized a simple closed surface (i.e., a subset of the image which separates its complement into an "inside" and an "outside") as a connected collection of "orientable" simple surface points. Later, the author and A. Rosenfeld established that the computationally costly assumption of orientability is unnecessary for 6-connectivity by proving that orientability, a local property, is implicitly guaranteed within the $(3\times3\times3)$-neighborhood definition of a 6-connected simple surface point. However, they also showed that no such guarantee exists for 26-connectivity. In this report, the author completes this investigation of simple closed surfaces by showing that orientability is ensured globally by 26-connectivity. Hence, a simple closed surface may be efficiently charactered as a connected collection of simple surface points regardless of the type of connectivity in consideration.

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1. Introduction

Geometrical and topological characterizations of subsets of digital pictures play an important role in computer image analysis and pattern recognition [1]. Topological concepts such as connectedness and simple closed curves are well-understood in two-dimensional arrays [2] and have proven to be useful tools for a wide variety of image analysis tasks such as object extraction, thinning, and skeletonization.

With the increased interest in computed tomography and the three-dimensional representation of microscopic cross-sections and time sequences of images, it has now become desirable to develop a consistent and efficient theory for the understanding of geometric and topological properties of subsets of three-dimensional digital arrays. Early work in this area was done by Gray [3] and Park [4], and other authors [5,6] have considered generalizations of specific two-dimensional results to higher dimensions. However, the current series of papers on three-dimensional digital geometry written at the Computer Vision Laboratory of the University of Maryland ([7],[5],[9],[10],[11],[12],[13]) appears to be the first systematic study designed to develop the desired theory. This report is the latest in this series, and it completes the characterization of simple closed surfaces.

Simple surface points and simple closed surfaces were introduced in [10] to establish the three-dimensional analog of the two-dimensional Jordan Curve Theorem. The goal was to define computationally efficient properties such that a subset of the 3D-lattice satisfying those properties separated the complement of that subset into an "inside" and an "outside." Indeed, this goal was to a large extent achieved in [10] where it was shown that a connected collection of "orientable" simple surface points provided the desired characterization. Unresolved, however, was the necessity of orientability, a property that unlike the other components of the characterization required the computationally costly examination of the (5×5×5) neighborhoods of the points under consideration. In [13], it was shown that orientability was actually guaranteed by the (3×3×3)-neighborhood restrctions imposed locally by simple surface points under 6-connectivity (one of the two types of connectedness to be considered). However, it was also shown in [13] that no such guarantee was imposed locally under 26-connectivity. In this paper, we succeed in showing that orientability is imposed globally by a 26-connected set of simple surface points. Hence, the desired characterization, which requires only the examination of (3×3×3)-neighborhoods, is etablished under both types of connectivity.

The approach to the characterization of surface properties in this paper and others in this series where surfaces are considered to be sets of voxels should be contrasted with that of Artzy, Frieder, and Herman [14] and Herman and Webster [15] in which surfaces are considered to be faces of voxels. The two approaches are complementary.

2. Connectivity and simple closed surfaces

Let Σ denote a 3D array of lattice points, which, without loss of generality, we may assume to be defined by integer valued triples of Cartesian coordinates (x,y,z). We consider two types of neighbors of a point $p = (x_p, y_p, z_p) \in \Sigma$:

(i) the neighbors (u,v,w) such that $|x_p-u|+|y_p-v|+|z_p-w| = 1$

(ii) the neighbors (u,v,w) such that $\max\{|x_p-u|, |y_p-v|, |z_p-w|\} = 1$

We refer to the neighbors of type (i) as 6-neighbors of p (the face neighbors) and to the neighbors of type (ii) as 26-neighbors of p (the face, edge, and corner neighbors). The 6-neighbors are said to be 6-adjacent to p, and the 26-neighbors are said to be 26-adjacent to p. The statement that α is a path from point p to point q in Σ means that there exists a positive integer n such that $\alpha=\{p_0, p_1, ..., p_n\} \subseteq \Sigma$ where $p_0=p$, $p_n=q$ and p_i is adjacent to p_{i-1} for $1 \le i \le n$. The terms 6-path and 26-path are utilized depending on the type of adjacency under consideration.

Let S denote a non-empty subset of Σ which, without loss of generality, we may assume does not meet the border of Σ. The points p and q of S are said to be connected in S provided there is a path from p to q which is contained in S. Connectivity is an equivalence relation, and the classes under this relation are called components. Again, the terms 6-connectivity, 26-connectivity, 6-components, and 26-components are utilized depending on the type of path under consideration.

Similarly, we can consider the components of the complement $\overline{S}$ of S. Exactly one of these components contains the border of Σ; this component is called the background of S. All other components of $\overline{S}$, if any, are called cavities in S. As is the custom in 2D (and 3D) digital geometry, opposite types of connectivity are assumed for S and $\overline{S}$ to avoid ambiguous situations. Finally, let p be a point of S. We let $N_{27}(p)$ denote the 27 points in the (3×3×3) neighborhood of p, and we let $N_{125}(p)$ denote the 125 points in the (5×5×5) neighborhood centered at p.

Surfaces

In [10], the above structure on the 3D-lattice was utilized to introduce the concept of a simple closed surface in providing a non-trivial 3D analog of the 2D Jordan Curve Theorem.

A point $p \in S$ is a simple surface point provided:

(i) $S \cap N_{27}(p)$ has exactly one component adjacent to p (in the S sense); denote this component A_p.

(ii) $\overline{S} \cap N_{27}(p)$ has exactly two components, C_1 and C_2, adjacent to p (in the $\overline{S}$ sense).

(iii) If $q \in S$ and q is adjacent to p (in the S sense) q is adjacent (in the $\overline{S}$ sense) to both C_1 and C_2.

As observed in [10], there are at most two components of $\overline{S} \cap N_{125}(p)$ adjacent (in the $\overline{S}$ sense) to a simple surface point p. Thus, suppose that p is a simple surface point of S and that each element of A_p is also a simple surface point of S (i.e., p is not near an "edge"). When $\bar{S} \cap N_{125}(p)$ has two components adjacent to p, (the surface at) p is said to be <u>orientable</u> and A_p is called a <u>disk</u>. When $\bar{S} \cap N_{125}(p)$ has only one component adjacent to p, (the surface at) p is said to be <u>non-orientable</u> and A_p is called a <u>cross-cap</u>.

<u>Theorem 0.1</u> [10] If S is a connected collection of orientable simple surface points, then S has exactly one cavity, and S is said to be a <u>simple closed surface</u>.

<u>Theorem 0.2</u> [13] There does not exist a 6-connected cross-cap. That is, if S is a 6-connected subset of Σ and p is a simple surface point in S such that each element of A_p is also a simple surface point of S, then $N_{125}(p) \cap \bar{S}$ has two components 26-adjacent to p and hence p is orientable.

Thus, the computationally costly assumption of orientability in Theorem 0.1 is unnecessary for 6-connectivity. However, the following example shows that the situation is more complex with respect to 26-connectivity.

<u>Example 0.1</u> [13] There exists a 26-connected cross-cap. The following set S (of "1's") is 26-connected, the central point p in the third plane is a simple surface point of S and each element of A_p is also a simple surface point of p, yet $N_{125}(p) \cap \bar{S}$ has only one component 6-adjacent to p and hence p is not orientable.

1st plane	2nd plane	3rd plane	4th plane	5th plane
0 0 0 0 0	0 0 0 1 0	0 0 0 0 0	0 0 0 1 0	0 0 0 0 0
0 0 0 1 0	0 0 1 0 1	0 0 0 1 1	0 0 1 0 1	0 0 0 1 0
0 0 1 0 0	0 1 0 0 0	0 0 1 0 0	0 1 0 0 0	0 0 1 0 0
0 0 0 1 0	0 0 1 0 1	0 0 0 1 1	0 0 1 0 1	0 0 0 1 0
0 0 0 0 0	0 0 0 1 0	0 0 0 0 0	0 0 0 1 0	0 0 0 0 0

3. 26-connected simple closed surfaces

To show that orientability is not necessary in the characterization of 26-connected simple closed surfaces, let us first outline the proof of Theorem 0.1 given in [10]. Suppose S is a 26-connected collection of [orientable], simple surface points.

(1) For each $p = (x_p, y_p, z_p) \in \overline{S}$, let $H_p = \{(x,y,z) \in \Sigma \mid x=x_p, y=y_p, \text{ and } z \geq z_p\}$, the vertical half-line emanating upward from p.

(2) Suppose $p \in \overline{S}$ and $\alpha = p_1, p_2, \ldots, p_n$ is a connected path in S along H_p such that p_0 and p_{n+1} (the points preceding and following p along H_p) are both in $\overline{S}$. Consider $M = \cup\{N_{27}(x) \mid x \in \alpha\}$. If p_0 and p_{n+1} are 26-connected by a path in $M \cap \overline{S}$, then we say that H_p <u>touches</u> S in α. If p_0 and p_{n+1} are not 26-connected by a path in $M \cap \overline{S}$, then we say that H_p <u>crosses</u> S in α.

(3) If $p \in \overline{S}$, and H_p crosses S an odd number of times, we say that p is <u>inside</u> S. If $p \in \overline{S}$, and H_p crosses S an even number of times, we say that p is <u>outside</u> S.

(4) If $p, q \in \overline{S}$, let $C_{p,q} = \{A^*_{p,q} \mid A^*_{p,q} \text{ is a component of } S \cap (H_p \cup H_q)\}$.

<u>Proposition 1</u>. If $p, q \in \overline{S}$, p is 6-adjacent to q, and $A^*_{p,q} \in C_{p,q}$, then $M \cap \overline{S}$ has two components which are 6-adjacent to each element of $A^*_{p,q}$, where $M = \cup\{N_{27}(x) \mid x \in A^*_{p,q}\}$.

Proposition 2. If $p,q \in \overline{S}$, p is 6-adjacent to q, and $A^*_{p,q} \in C_{p,q}$, then H_p crosses S an odd number of times in $A^*_{p,q}$ iff H_q crosses S an odd number of times in $A^*_{p,q}$.

Proposition 3. If p is 6-connected to q in $\overline{S}$, then either both are inside of S or both are outside of S.

Proposition 4. The inside and outside of S are both non-empty and S has exactly one cavity.

A detailed examination of the above proof shows that the use of orientability is restricted to establishing Proposition 1, where it is assumed that for each $x \in A^*_{p,q}$, $N_{125}(x) \cap \overline{S}$ has two components which are each 6-adjacent to x. Thus, our goal is to provide a proof of Proposition 1 without such an assumption. As in [13], to establish properties of simple surface points, in which symmetry between the two types of connectivity fails, requires considerable combinatorial detail and explicit notation.

Notation. If $M \subseteq \Sigma$, let $\overline{M} = M \cap \overline{S}$. For each $p = (x_p, y_p, z_p)$, let $p(i,j,k) = (x_p+i, y_p+j, z_p+k)$. In addition, let p+ denote p(0,0,1) and p- denote p(0,0,-1).

For k an integer, let:

(1) $N_p^k = \{p(i,j,k) \mid -1 \le i \le 1, -1 \le j \le 1\}$,

(2) $N_p^{k=m,n} = \bigcup_{k=m,n} N_p^k$, and

(3) $N_p = N_p^{k=-1,1}$.

For example:

$N_p = N_{27}(p)$

N_p^{-2} = the 3×3 plane centered on $p(0,0,-2)$

$\bar{N}_p^1 = N_p^1 \cap \bar{S}$

$H_p = \{p(0,0,k) \mid k \geq 0\}$

Finally, if p,q $\bar{S}$, p is 6-adjacent to q, and $A^*_{p,q}$ is a component of $(H_p \cup H_q) \cap S$, then there exist integers n_0 and m such that $A^*_{p,q} = \bigcup_{n_0,m} (A^*_{p,q} \cap \{p(0,0,n), q(0,0,n)\})$.

(1) For $n_0 \leq n \leq m$, let $A^*_{p,q}(n) = A^*_{p,q} \cap \{p(0,0,n), q(0,0,n)\} \neq \emptyset$.

(2) For $n_0 \leq n \leq m$, let $B^*_{p,q}(n) = \{p(0,0,n), q(0,0,n)\}$.

(3) $B^*_{p,q} = \bigcup_{n=n_0,m} B^*_{p,q}(n)$.

Our goal is now to establish that if $\{p,q\} \subseteq \bar{S}$, p is 6-adjacent to q, and $A^*_{p,q}$ is a component of $(H_p \cup H_q) \cap S$, then $\bar{M}$, where $M = \bigcup\{N_x \mid x \in B^*_{p,q}\}$, has two components which are each 6-adjacent to every element of $A^*_{p,q}$.

Lemma 1. If p is a simple surface point of S and $x \in \bar{N}_p$, then x is 6-connected to p in $\bar{N}_p$.

Proof: Suppose not. Then W.L.G., either (1) $x \in N_p^0$, (2) $x = p(1,1,1)$ and $p(1,1,0) \in S$, or (3) $x = p(0,1,1)$ and $p(1,1,1) \in S$.

If (1), then W.L.G. let $x = p(1,1,0)$. Now, $\{p(1,0,0), p(0,1,0)\} \subseteq S$ and either (i) $\{p(1,1,1), p(1,1,-1)\} \cap S \neq \emptyset$ or (ii) $\{p(1,1,1), p(1,1,-1)\} \subseteq \bar{S}$. If (i), then W.L.G. let $p(1,1,1) \in S$. But now p cannot be 6-adjacent to two components of $\bar{N}_{p(1,1,1)}$. # If (ii), then $p(1,0,0)$ cannot be 6-adjacent to two components of $\bar{N}_p$ which are 6-adjacent to p. #

If (2), then p+ cannot be 6-adjacent to two components of $\overline{N}_{p(1,1,0)}$. Hence $p+\in\overline{S}$, and therefore $\{p(0,1,1), p(1,0,1)\}\subseteq S$. Now, $\overline{N}_{p(0,1,1)}$ has exactly two components, C_1 and C_2, 6-adjacent to p(0,1,1). Since p(1,1,1) is 6-adjacent to p(0,1,1), p(1,1,1) must be in one of C_1 and C_2, say C_1. Now, since p(1,0,0) cannot be 6-connected to p(0,1,1) in $\overline{N}_{p(0,1,1)}$, it can be in neither C_1 or C_2. Thus $p(0,0,1)\in C_2$ or else p(1,0,1) could not be 6-adjacent to both C_1 and C_2. But p is 6-adjacent to both C_1 and C_2 in $\overline{N}_{p(0,1,1)}$. Therefore, either (i) $p(0,1,0)\in C_1$, or (ii) $p(-1,0,0)\in C_1$. If (i), then $p(1,2,0)\in C_2$ or else p(1,1,0) could not be 6-adjacent to C_2. Thus, $\{p(0,2,0), p(1,2,1)\}\subseteq S$ or else C_1 would be 6-adjacent to C_2. However, now there can be no 6-path in $\overline{N}_{p(0,1,1)}$ from $p(1,2,0)\in C_2$ to p(0,1,1). # If (ii), then $p(-1,0,1)\in S$. Hence, since these must be a 6-path in $\overline{N}_{p(0,1,1)}$ from p(-1,0,0) to p(0,1,1), it follows that $p(-1,1,0)\in C_1$. But $p(0,1,0)\in\overline{S}$ since it cannot be 6-adjacent to two components of $\overline{N}_{p(-1,0,1)}$. Thus, $p(0,1,0)\in C_1$ and we again have case (i). #

If (3), then $\{p(0,0,1), p(1,0,0)\}\subseteq\overline{S}$. However, now p cannot be 6-adjacent to two components of $\overline{N}_{p(1,1,1)}$. #

<u>Lemma 2</u>. If $p,q\in S$ and $q\in A_p$, then $\overline{N}_p\cup\overline{N}_q$ has two components each of which is 6-adjacent to both p and q. Equivalently, the two components of $\overline{N}_p$ are not merged by a 6-path in $\overline{N}_q$.

Proof: Suppose not. Let C_1 and C_2 denote the two components of $\overline{N}_p$, and let C_1' and C_2' denote the two components of $\overline{N}_q$. Now, let $M=N_q \cap N_p$. There must exist a 6-path α contained in $\overline{N}_q/M$ such that α is 6-adjacent to $y_1 \in C_1 \cap M$ and to $y_2 \in C_2 \cap M$. Furthermore, q is 6-adjacent to both $C_1 \cap M$ and $C_2 \cap M$. Due to symmetry, we can assume W.L.G. that (1) q = p+, (2) q = p(1,1,0), or (3) q = p(1,1,1).

If (1), it follows that $C_2' \cap M$ must also be 6-adjacent to q. Suppose not. Then $p(0,0,2) \in C_2'$ since q must be 6-adjacent to C_2' in N_q. Thus, W.L.G., either (i) $p(0,1,1) \in C_1$ and $p(1,0,1) \in C_2$ or (ii) $p(0,1,1) \in C_1$ and $p(0,-1,1) \in C_2$. If (i), then $p(1,1,1) \in S$. But then p(1,1,1) cannot be adjacent to C_2' in N_q without merging C_1' and C_2'. # If (ii), then one of p(1,0,1) and p(-1,0,1) must be in C_2' since there must exist a 6-path in $\overline{N}_q$ from p(0,0,2) to p. # Therefore, q is 6-adjacent to each of $C_1 \cap M$, $C_2 \cap M$ and $C_2' \cap M$. W.L.G., let {p(0,1,1), p(0,-1,1), p(1,0,1)} contain y_1, y_2 and an element of $C_2' \cap M$. Note that $\{p(1,1,1), p(1,-1,1)\} \subseteq S$ and that some $x \in \{p(-1,1,1), p(-1,0,1), p(-1,1,1)\}$ must also be in S. Now, α cannot connect p(0,1,1) and p(1,0,1) or else p(1,1,1) could not be 6-adjacent to two components of $\overline{N}_q$. Similarly, α cannot connect p(0,-1,1) and p(1,0,1) or else p(1,-1,1) could not be 6-adjacent to two components of $\overline{N}_q$. Thus α must connect p(0,1,1) and p(0,-1,1). But then x cannot be 6-connected to p(1,0,1) in $\overline{N}_q$. (# from which (1) follows).

If (2), then W.L.G., either (i) $p(0,1,0)\in C_1$ and $p(1,0,0)\in C_2$, (ii) $p(0,1,0)\in C_1$ and $p(1,1,1)\in C_2$, or (iii) $p(1,1,-1)\in C_1$, $p(1,1,1)\in C_2$, and $\{p(0,1,0), p(1,0,0)\}\subseteq S$. If (i), then p is 6-adjacent to only one component of $\overline{N}_q$. # If (ii), then $p(0,1,1)\in S$. Hence, $p(1,0,0)\in\overline{S}$ since it can be 6-adjacent to only one component of $\overline{N}_{p(0,1,1)}$. However, if $p(1,0,0)\in C_2$ then we have case (i) again. But if $p(1,0,0)\in C_1$, q can be adjacent to only one component of $\overline{N}_p$. # If (iii), then each point of M not in N_p^o must be in $\overline{S}$ and $\overline{M}$ has only two components. (# from which (2) follows).

If (3), then W.L.G., either (i) $p(0,1,0)\in C_1$ and $p(1,0,1)\in C_2$ or (ii) $p(0,1,1)\in C_1$ and $p(1,1,0)\in C_2$. In either case, p can be adjacent to only one component of $\overline{N}_q$. # This completes the proof.

Lemma 3. If $p\in S$, $q\in\overline{S}$, and q is 6-adjacent to p, then $\overline{N}_p\cup\overline{N}_q$ has two components each of which is 6-adjacent to p.

Proof: Suppose not. W.L.G., let q=p+, let C_1 and C_2 denote the two components of $\overline{N}_p$ with $p+\in C_1$, and let α denote a 6-path in $\overline{N}_q^2$ from $y_1\in N_p^1\cap C_1$ to $y_2\in N_p^1\cap C_2$. Then $y_2\in\{p(-1,-1,1),p(-1,1,1), p(1,-1,1),p(1,1,1)\}$. W.L.G., let $y_2=p(1,1,1)$ and $\{p(0,1,1), p(1,0,1)\}\subseteq S$.

[There is no 6-path in $\overline{N}_p^1\cup\overline{N}_p^2$ from p+ to y_2.] Suppose there is such a path β, where W.L.G. β is minimal. Then $p(1,1,2)\in\beta$ and one of $p(1,0,2)$ or $p(0,1,2)$, say $p(1,0,2)$, must be in β between p+ and y_2. Observe that $N_{p(1,0,1)}\cap\beta$ cannot be 6-adjacent

to p+ or else p(0,1,1) could not be 6-adjacent to two components of $\overline{N}_{p(1,0,1)}$. Hence $p(0,0,2)\in S$. Now, since β must be 6-adjacent to p+, $p(1,-1,2)\in\beta$, $p(0,-1,2)\in\beta$ and $p(0,-1,1)\in S$. Furthermore, $p(1,-1,1)\in\overline{S}$ since it can be adjacent to only one component of $\overline{N}_{p(0,0,2)}$. However, now β must be 6-adjacent to p+ via p(-1,0,1) which is the only remaining possibility. But then p+ is 6-adjacent to $\beta\cap N_{p(0,-1,1)}$ and p(1,0,1) cannot be 6-adjacent to two components of $\overline{N}_{p(-1,0,1)}$. # Hence, p+,y_1, and y_2 must belong to three distinct components of $\overline{N}^1_p$. Thus (i) $y_1\in\{p(1,-1,1),\ p(-1,1,1)\}$ or (ii) y_1=p(-1,-1,1). If (i), W.L.G. let y_1=p(1,-1,1). Then $p(0,-1,1)\in S$. Also, since C_1 and C_2 must be 6-adjacent to p, $p(1,1,0)\in C_2$ and $p(1,-1,0)\in C_2$. However, now p(1,0,0) must be in S and $p+\in\overline{N}_{p(1,0,0)}$. But p+ cannot be 6-connected to p(1,0,0) in $\overline{N}_{p(1,0,0)}$.#
If (ii), then $\{p(-1,0,1),\ p(0,-1,1)\}\subseteq S$. Furthermore, since p(1,-1,1) cannot be 6-adjacent to two components of $\overline{N}_p$, $p(1,-1,1)\in\overline{S}$. Now, since each of y_1,y_2, and p(1,-1,1) must be 6-connected to p in $\overline{N}_p$, $\{p(1,1,0),\ p(1,-1,0),\ p(-1,-1,0)\}\subseteq\overline{S}$. Thus, one of x=p(1,0,0) or x=p(0,-1,0) must be in S or C_1 would be 6-adjacent to C_2 in N_p. However, in either case, we would have $p+\in\overline{N}_x$ but p+ cannot be 6-connected to x in $\overline{N}_x$.# The proof is complete.

<u>Lemma 4</u>. If p and q=p(1,0,1) are in S, $p+\in\overline{S}$, and $y\in\{p(0,-1,2),\ p(0,0,2),\ p(0,1,2)\}\cap\overline{S}$, then y is 6-connected to p via a path in $\overline{N}_p\cup\overline{N}^2_p$.

Proof: Suppose not. Then $y \neq p(0,0,2)$, hence W.L.G. let $y=p(0,1,2)$. Note $\{p(0,0,2), p(0,1,1)\} \subseteq S$ or else y would be 6-adjacent to $\bar{N}_p$ and thus 6-connected to p in $\bar{N}_p \cup \bar{N}_p^2$. Now, since y must be 6-connected to q in $\bar{N}_q$, $p(1,1,2) \in \bar{S}$. Again, $p(1,1,1)$ must then be in S. However, now $p(1,1,1)$ cannot be 6-adjacent to two components of $\bar{N}_{p(0,0,2)}$. #

Lemma 5. Suppose p and $q=p(1,0,-1)$ are in S and $p(0,0,-1) \in \bar{S}$. If $y \in \{p(-1,1,0), p(-1,0,0), p(-1,-1,0)\}$ and y is 6-connected to $\bar{N}_p^{-2}$ in $\bar{N}_p^{i=-2,0}$, then there is a 6-path α in $\bar{N}_p^0 \cup \bar{N}_p^{-1}$ from y to $\bar{N}_q$.

Proof: Suppose not. Note that if $y=p(-1,0,0)$, then one of $p(-1,1,0)$ or $p(-1,-1,0)$ must be in $\bar{S}$, or else $p(-1,0,-1) \in \bar{S}$ and y is 6-connected to $\bar{N}_q$ via $p(-1,0,-1)$. Hence, W.L.G., let $y=p(-1,1,0)$ which implies that $p(0,1,0) \in S$. Now, either (1) $p(0,1,-1) \in \bar{S}$ or (2) $p(0,1,-1) \in S$. If (1), then $p(-1,1,-1) \in S$ or else y would be 6-connected to $\bar{N}_q$ via $p(0,1,-1)$. Hence, since y is 6-connected to $\bar{N}_p^{-2}$ in $\bar{N}_p^{i=-2,0}$, $p(-1,0,0) \in \bar{S}$ and again $p(-1,0,-1) \in S$. Now, $p(-1,-1,0)$ must be in $\bar{S}$, and $p(-1,-1,-1)$ must also be in $\bar{S}$. Thus, $\{p(0,-1,0), p(0,-1,-1)\} \subseteq S$. However, $p(0,0,-1) \in \bar{N}_{p(0,-1,0)}$ but since $q \in S$, $p(0,0,-1)$ cannot be 6-connected to $p(0,-1,0)$ in $\bar{N}_{p(0,-1,0)}$. # If (2), then $p(0,0,-1) \in \bar{N}_{p(0,1,0)}$. Hence, since $\{p,q\} \subseteq S$, $p(-1,0,-1) \in \bar{S}$ and one of $p(-1,1,-1)$ and $p(-1,0,0)$ must also be in S. But now we have a 6-path in $\bar{N}_p^0 \cup \bar{N}_p^{-1}$ from y to $\bar{N}_q$. # This completes the proof.

Lemma 6. If $\{p,q\}\subseteq\overline{S}$, p is 6-adjacent to q, and $A^*_{p,q}$ is a component of $(H_p \cup H_q)\cap S$, then $M\cap\overline{S}$, where $M=\cup\{N_x \mid x\in B^*_{p,q}\}$, has two components, C_1 and C_2, which are each 6-adjacent to every element of $A^*_{p,q}$.

Proof: W.L.G. $q=p(0,0,1)$ or $q=p(1,0,0)$. If $q=p(0,0,1)$, then the proof follows immediately by induction on Lemma 2. Thus, assume $q=p(1,0,0)$. Hence, there exist integers n_o and m such that $A^*_{p,q}=\bigcup_{i=n_o,m} A^*_{p,q}(i)$, where for $n_o\le i\le m$, $A^*_{p,q}(i)\neq\emptyset$. To simplify notation, for each $n_o\le i\le m$, let $A_i=\bigcup_{n_o\le j\le i} A^*_{p,q}(j)$, $B_i=\bigcup_{n_o\le j\le i} B^*_{p,q}(j)$, and $M_i=\cup\{N_x \mid x\in B_i\}$. Note that $B^*_{p,q}(i+1)\subseteq M_i$ for each $n_o\le i<m$. From Lemma 2 and Lemma 3, it follows immediately that $\overline{M}_{n_o}$ has two components which are each 6-adjacent to each element of A_{n_o}. We now proceed by induction. Assume $n_o\le n<m$ and $\overline{M}_n$ has two components, C_1 and C_2, which are each 6-adjacent to every element of A_n. [To show: $\overline{M}_{n+1}$ has two components which are each 6-adjacent to every element of A_{n+1}.] Suppose not, then there must exist a 6-path α in $\cup\{\overline{N}^1_x \mid x\in B^*_{p,q}(n+1)\}$ from $y_1\in C_1\cap(\cup\{\overline{N}^0_x \mid x\in B^*_{p,q}(n+1)\})$ to $y_2\in C_2\cap(\cup\{\overline{N}^0_x \mid x\in B^*_{p,q}(n+1)\})$. Due to the geometric symmetries involved, we need only consider the following four cases:

(1) $A^*_{p,q}(n) = \{p(0,0,n)\}$, $A^*_{p,q}(n+1) = (p(0,0,n+1))$,

(2) $A^*_{p,q}(n) = \{p(1,0,n)\}$, $A^*_{p,q}(n+1) = \{p(0,0,n+1)\}$,

(3) $A^*_{p,q}(n) = \{p(1,0,n)\}$, $A^*_{p,q}(n+1) = \{p(0,0,n+1), p(1,0,n+1)\}$,

(4) $A^*_{p,q}(n) = \{p(0,0,n), p(1,0,n)\}$.

(1). $(A^*_{p,q}(n)=\{p(0,0,n)\},\ A^*_{p,q}(n+1)=\{p(0,0,n+1)\})$. It follows immediately from Lemma 2 that $\overline{M}$, where $M=M_n \cup N^1_{p(0,0,n+1)}$, has two components, C'_1 and C'_2, each of which is 6-adjacent to every element of A_{n+1}. Furthermore, $C_1 \subseteq C'_1$ and $C_2 \subseteq C'_2$. Hence, we can assume that α is a 6-path, contained in the two rightmost columns of $\overline{N}^1_{p(1,0,n+1)}$, which is 6-adjacent to $y_1 \in C'_1$ and $y_2 \in C'_2$ where $\{y_1, y_2\}$ is contained in the union of the rightmost column of $N^1_{p(0,0,n+1)}$ and the rightmost column of $N^0_p(1,1,n+1)$. Note that one of y_1 and y_2 (say y_1) cannot be 6-connected to $\overline{N}_{p(0,0,n+1)}$ via a path in $\overline{N}_{p(1,0,n+1)} \cap \overline{N}_{p(2,0,n+1)}$, or else the two components of $\overline{N}_{p(0,0,n+1)}$ would be merged by the rightmost plane of $N_{p(1,0,n+1)}$ in contradiction (by symmetry) to Lemma 3. However, each of y_1 and y_2 must be 6-connected to $p(0,0,n+1)$ by a path in $\overline{M}$. Thus, W.L.G. let $y_1=p(2,1,n+1)$ which implies $\{p(2,1,n+1),\ p(2,1,n+2),\ p(2,1,n)\} \subseteq \overline{S}$ and $\{p(1,1,n+1),\ p(2,0,n+1),\ p(1,1,n),\ p(2,0,n)\} \subseteq S$.

(i) $[p(1,0,n+1) \in C'_2]$ Since $y_1 \in C'_1$ must be 6-connected to $p(0,0,n)$ in $\overline{M}$, let k denote the greatest integer less than n such that one of $p(1,1,k)$ and $p(2,0,k)$ is in $\overline{S}$. Then observe that $p(1,0,k) \in S$ or else either $p(1,1,k+1)$ is not 6-adjacent to two components of $\overline{N}_{p(2,0,k+1)}$ or $p(2,0,k+1)$ is not 6-adjacent to two components of $\overline{N}_{p(1,1,k+1)}$. Also, note that if $k<i \le n$, $p(1,0,i) \in \overline{S}$. To see that this is true, suppose for some $k<i<n$, $p(1,0,i) \in S$. Let j be the greatest such i, then $p(1,0,j+1) \in \overline{S}$ and $\{p(1,1,j+1),\ p(2,0,j+1),\ p(1,0,j),\ p(2,0,j)\} \subseteq S$. But now $p(2,0,j)$ cannot be 6-adjacent to two components of $\overline{N}_{p(1,1,j+1)}$. # Hence, it follows

that $p(1,0,k+1)\in\overline{S}$ and $p(1,0,k+1)$ is 6-connected to $p(1,0,n+1)$ in $\overline{M}$. Now, $p(1,0,k+1)$ and $p(2,1,k+1)$ are in opposite components of $\overline{N}_{p(1,0,k)}$ or else one of $p(1,1,k+1)$ and $p(2,0,k+1)$ could not be 6-adjacent to two components of $\overline{N}_{p(1,0,k)}$. Thus, since $p(1,0,k)\in A_n$ and $N_{p(1,0,k)}\subseteq M$, $p(1,0,k+1)$ and $p(2,1,k+1)$ are in opposite components of $\overline{M}$. Hence, since $p(1,0,n+1)$ is 6-connected to $p(1,0,k+1)$ in $\overline{M}_n$ and y_1 is 6-connected to $p(2,1,k+1)$ in $\overline{M}$, we have $p(1,0,n+1)\in C_2'$.

(ii) (Suppose y_2 is also in the rightmost column of $N_p^0(1,0,n+1)$.) Thus $y_2=p(2,-1,0)$. Then since two components of $\overline{N}_{p(0,0,n+1)}$ cannot be merged by the rightmost plane of $N_{p(1,0,n+1)}$, it follows that the rightmost column of $N^1_{p(1,0,n+1)}\subseteq\overline{S}$. Hence $p(1,0,n+2)\in S$ or else $p(0,2,n+1)$ could not be 6-adjacent to two components of $\overline{N}_{p(1,1,n+1)}$. Furthermore, $p(1,-1,n+1)\in S$ or else $p(1,0,n+2)$ could not be 6-adjacent to two components of $p(2,0,n+1)$. Now, since y_2 is 6-connected to $p(0,0,n)$ in $\overline{M}$, it again follows as in (i) that y_2 and $p(1,0,n+1)$ must be in opposite components of $\overline{M}$. But $\{y_1,p(1,0,n+1)\}\subseteq C_2'$. #

(iii) (Suppose y_2 is in the rightmost column of $N^1_{p(0,0,n+1)}$.) Note that $p(1,0,n+2)\in S$, or else $\{p(1,1,n+2), p(2,0,n+2)\}\subseteq S$ and α could not connect y_1 to y_2. Thus $p(0,1,n+1)\in\overline{S}$ since it cannot be 6-adjacent to two components of $\overline{N}_{p(1,0,n+2)}$. Furthermore, $p(0,1,n+1)\in C_1'$ since $p(0,1,n+1)$ and $p(1,0,n+1)$

must be in different components of $\bar{N}_{p(0,0,n+1)}$ or else $p(1,1,n+1)$ could not be 6-adjacent to two components of $N_{p(0,0,n+1)}$. Finally, $p(0,1,n+2)\epsilon\bar{S}$ since otherwise $p(0,1,n+1)$ could not be 6-connected to $p(1,0,n+2)$ in $\bar{N}_{p(1,0,n+2)}$. Hence, $p(0,1,n+2)\epsilon C_1'$ and $y_2 \neq p(1,1,n+2)$. Thus, $y_2 = p(1,-1,n+2)$. Now, $p(1,-1,n+1)\epsilon\bar{S}$, or else $p(1,0,n+1)$ could not be 6-connected to $p(0,0,n+1)$ in $\bar{N}_{p(0,0,n+1)}$. But since $p(0,1,n+2)\epsilon C_1'$, $\{p(2,0,n+2), p(2,-1,n+2)\}\subseteq\alpha$. However, it then follows that $p(1,0,n+2)$ is 6-adjacent to only one component of $\bar{N}_{p(2,0,n+1)}$. (# from which (1) follows.)

(2). $(A^*_{p,q}(n)=\{p(1,0,n)\}, A^*_{p,q}(n+1)=\{p(0,0,n+1)\})$. Consider $N_{p(1,0,n)}\subseteq M_n$. Note $B_1\subseteq C_1$ and $B_2\subseteq C_2$ where B_1 and B_2 are the two components of $\bar{N}_{p(1,0,n)}$. From Lemma 5, if $y\epsilon\bar{S}$, y is in the leftmost column of $N^0_p(0,0,n+1)$, and y is 6-connected to $p(1,0,n)$ in M_n, then y is 6-connected to $p(1,0,n)$ in $\bar{N}^{i=-1,0}_{p(0,0,n+1)} \cup \bar{N}_{p(1,0,n)}$. Furthermore, from Lemma 2, it follows that B_1 and B_2 cannot be merged in $\bar{N}_{p(0,0,n+1)}$. Thus, $\bar{M}$, where $M=M_n\cup \bar{N}^{1}_{p(0,0,n+1)}$, has two components, C_1' and C_2', each of which is 6-adjacent to every element of A_{n+1}. Furthermore, $C_1\subseteq C_1'$ and $C_2\subseteq C_2'$. Hence, as in (1), we can assume that α is a 6-path in the two rightmost columns of $\bar{N}^{1}_{p(1,0,n+1)}$ which is 6-adjacent to $y_1\epsilon C_1'$ and $y_2\epsilon C_2'$ where $\{y_1,y_2\}\subseteq$ the union of the rightmost column of $N^{1}_{p(0,0,n+1)}$ and the rightmost column of $N^{0}_{p(1,0,n+1)}$.

However, by geometric symmetry to Lemma 4, if either of y_1 or y_2 is in the righthand column of $N^1_{p(0,0,n+1)}$, then it is 6-connected to $p(1,0,n)$ via a path in $\overline{N}_{p(1,0,n)} \cup \overline{N}^2_{p(1,0,n)}$. Thus, $\overline{N}^2_{p(1,0,n)}$ merges two components of $\overline{N}_{p(1,0,n)}$ in contradiction to Lemma 3. #

(3). $(A^*_{p,q}(n)=\{p(1,0,n)\}$, $A^*_{p,q}(n+1)=\{p(0,0,n+1), p(1,0,n+1)\})$. As in (2) above, $\overline{M}$, where $M=M_n \cup \overline{N}^1_{p(0,0,n+1)}$, has two components, C_1' and C_2', each of which is 6-adjacent to every element of A_{n+1}. Hence, again we can assume α is contained in the two rightmost columns of $\overline{N}^1_{p(1,0,n+1)}$. However, we now have two components of $\overline{N}_{p(1,0,n)}$ merged by a 6-path in $\overline{N}_{p(1,0,n+1)}$, which violates Lemma 2. #

(4). $(A^*_{p,q}(n)=\{(p(0,0,n), p(1,0,n)\})$. W.L.G., assume $p(1,0,n+1) \in S$. From either Lemma 2 or Lemma 3, we have that $\overline{M}$, where $M=M_n \cup \overline{N}^1_{p(0,0,n+1)}$, has two components, C_1' and C_2', each of which is 6-adjacent to every element of A_{n+1}. Hence, α is contained in the two rightmost columns of $\overline{N}^1_{p(1,0,n+1)}$. Again, we then arrive at a contradiction to Lemma 2 as in (3) by consideration of $N_{p(1,0,n+1)} \cup N_{p(1,0,n)}$. This completes the proof.

4. Conclusion

Theorem 1. If S is a connected collection of simple surface points then S has exactly one cavity, and S is said to be a simple closed surface.

Hence, we now have the above characterization of simple closed surfaces which holds for both 6-connectivity and 26-connectivity. Furthermore, this characterization is of minimal computational cost in that only the smallest three-dimensional neighborhoods ($3 \times 3 \times 3$) of the respective points need to be examined. This completes the study of [10] and [13].

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This is a continuation of a series of papers on the digital geometry of three-dimensional digital images. In earlier reports, D. Morgenthaler and A. Rosenfeld gave symmetric definitions for simple surface points under the concepts of 6-connectivity and 26-connectivity, and they non-trivially characterized a simple closed surface (i.e., a subset of the image which separates its complement into an "inside" and an "outside") as a connected collection of "orientable" simple surface points. Later, the author and A. Rosenfeld

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SECURITY CLASSIFICATION OF THIS PAGE (When Data Entered) established that the computationally costly assumption of orientability is unnecessary for 6-connectivity by proving that orientability, a local property, is implicitly guaranteed within the (3×3×3)-neighborhood definition of a 6-connected simple surface point. However, they also showed that no such guarantee exists for 26-connectivity. In this report, the author completes this investigation of simple closed surfaces by showing that orientability is ensured globally by 26-connectivity. Hence, a simple closed surface may be efficiently charactered as a connected collection of simple surface points regardless of the type of connectivity in consideration.

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