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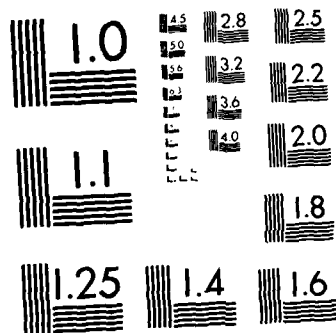
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MOMENTS OF THE MINIMUM OF A RANDOM WALK

AND COMPLETE CONVERGENCE

BY

MICHAEL HOGAN

TECHNICAL REPORT NO. 21

JANUARY 1983

DEPARTMENT OF STATISTICS
STANFORD UNIVERSITY
STANFORD, CALIFORNIA

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Abbreviated Title: Moments of Minimum of a Random Walk

Key Words: Random Walk, Renewal Theorem, Complete Convergence,
Strong Law of Large Numbers

Summary: Moments of the Minimum of a Random walk and Complete Convergence.

Let S_n be a random walk with positive drift. Let $S_{min} = \inf_{n>0} \{S_n\}$. New proofs are given of the following: For $p \geq 1$ $E|S_{min}|^p < \infty \iff$ $E(|S_1|^{p+1} 1_{(S_1<0)}) < \infty$; $\Sigma P\{S_n < 0\} < \infty \iff E(|S_1|^2 1_{(S_1<0)}) < \infty$, and some related results.

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1. Introduction

This paper gives new proofs of the equivalences that are stated as Theorem 1 in Section 3. Robbins and Hsu [6] first showed $c \Rightarrow e$ in 1948 with $p = 1$. They considered the problem in the context of a random walk generated by X_i with $E\,X_i = 0$, and showed that $E\,X_i^2 < \infty$ $\Rightarrow \sum_n P\{\frac{|S_n|}{n} > \varepsilon\} < \infty$. They called the finiteness of this sum complete convergence. It implies the strong law of large numbers by an application of the Borel-Cantelli lemma. Erdos [3] proved the reverse direction in 1949 and Baum and Katz [1] added the equivalence of (d) in 1965. Kiefer and Wolfowitz [5] estiblished the equivalence of (c) and (f) and the (c)<=>(g) is in Taylor [7]. Independent discovery of both of the results were credited by the respective authors to unpublished work of Darling, Erdos and Kakutani. These results are partially restated as Theorems 2 and 3 of section 4.

The new proofs provide an ε-free approach to these problems. The elementary Renewal Theorem, time reversal, and Wald's identities are the primary tools, and suffice for the case $p=2$. For larger p, the martingale conditional square function has to be used to replace Wald's identities to show the existence of moments in stopped random walks.

2. Notation and Conventions.

Fix the following notation and conventions. X_i is an i.i.d. sequence with $\mu = EX_i > 0$; $X^- = -X\,1_{\{X<0\}}$; $S_0 = 0$, $S_i = \sum_{j=1}^{i} X_j$ for $i > 0$; $S_{min} = \inf\{S_i; i > 0\}$; τ_+ is the first strict ascending ladder epoch, $\tau_+^{(j)}$ is the jth strict ascending ladder epoch: τ_- is the first weak descending ladder epoch, or $+\infty$ if none exists, $\tau_-^{(j)}$ is the jth weak descending ladder epoch, or $+\infty$ if none exists (see Feller [4], Sec. 12.1 for definition);

$$L(0) = \sum_{n=1}^{\infty} 1_{\{\inf_{j \geq n} S_j \leq 0\}}, \quad N(0) = \sum_{n=1}^{\infty} 1_{\{S_n \leq 0\}},$$

i.e. $L(0)$ is the last time the process is non-positive, and $N(0)$ is the number of times the process is non-positive; $\inf\{\ \} = \infty$; $\tau(a) = \inf\{n > 0: S_n \geq a\}$; $t(a) = \inf\{n > 0: S_n \leq a\}$; K and C will be positive constants, not necessarily the same from line to line; $E\{Y;A\} = E(Y1_A)$; $\omega = (X_1, X_2, \ldots)$, $\omega_a^+ = (X_{a+1}, X_{a+2}, \ldots)$. E^x denotes expectation of the random walk started from x; $E = E^0$.

3. Statement and Proof of Theorem:

Theorem 1: For $p \geq 2$ the following are equivalent:

a). $E(\tau_+^{\ p}) < \infty$

b). $E(\tau_-^{\ p-1}; \tau_- < \infty) < \infty$

c). $E((X^-)^p) < \infty$

d). $E(L(0)^{p-1}) < \infty$

e). $E(N(0)^{p-1}) < \infty$

f). $E(|S_{min}|^{p-1}) < \infty$

g). $E(|S_{\tau_-}|^{p-1}; \tau_- < \infty) < \infty$.

Four lemmas will be given first, then the proof proceeds as follows:

a <=> b; a <=> c; d => b; a and b => d; d => e; e => g; g -> f; f => e; f => c.

Lemma 1: If $E(\tau_+^{\ p}) < \infty$ then $E(t(-x)^{p-1}; t(-x) < \infty) < K \quad \forall \ x > 0$.

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<u>Proof:</u> By time-reversal one has, $\forall\, K > 0$,

$$P\{t(-K) = n\} = P\{S_1 > -K,\ldots,S_{n-1} > -K,\ S_n < -K\}$$

$$= P\{S_n - S_{n-1} > -K,\ldots,S_n - S_1 > -K,\ S_n < -K\}$$

$$\leq P\{S_1 < 0,\ldots,S_n < 0\}$$

$$= P\{\tau_+ > n\}.$$

Multiply the first and last lines by n^{p-1} and summing implies the result, with $K = C_p\, E(\tau_+^{\,p})$.

<u>Lemma 2:</u>   If $E(\tau_+^{\,p}) < \infty$ then $E(\tau(y)^p) \leq K(y + 1)^p$ $\forall\, y > 0$.

<u>Proof:</u> First notice that $E(\tau_+^{\,p}) < \infty \Rightarrow E(\tau(y)^p) < \infty$, $\forall\, y > 0$. For if $P\{X < 0\} > 0$ one conditions on the random walk at time 1 to show $E(t(\varepsilon)^p) < \infty$, $\exists\, \varepsilon > 0$, from which $E(\tau(y)^p) < \infty$ $\forall y$ follows as below, if $P\{X < 0\} = 0$ the one-sided hitting problem is the same as a two-sided problem, for which Stein's Lemma (cf Feller [5], Sec. 18.2) says $\tau(y)$ has moments of all orders. To proceed with the proof, observe that for $K > 0$ an integer

$$\tau(Ky) \leq \tau(y) + \tau(y)(\omega_{\tau(y)}^+) + \ldots + \tau(y)(\omega_{\tau((K-1)y)}^+) \quad \text{and the}$$

$\tau(y)(\omega_{\tau(jy)}^+)$ are i.i.d. Hence by Minkowski's Inequality.

$$E(\tau(Ky)^p) \leq K^p\, E(\tau(y)^p)$$

and so

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$$E(\tau(y)^p) \leq E \; \tau([y] + 1)^p$$

$$\leq E(\tau(1)^p)([y] + 1)^p$$

$$\leq E(\tau(1)^p)(y + 1)^p$$

<u>Lemma 3</u>: For $x > 0$ let

$$R_{-x} = \begin{array}{ll} -S_{t(-x)} -x , & t(-x) < \infty \\ 0 , & t(-x) = \infty. \end{array}$$

Then $\forall \; p \geq 1 \; E((X^-)^p) < \infty \Rightarrow E(R_{-x}^{p-1}; t(-x) < \infty) < K$, where K is independent of x.

<u>Proof</u>: This is essentially the same as Theorem 2.4 in Woodroofe [8].

$$P\{R_{-x} > y\} = \sum_{n=1}^{\infty} P\{t(-x) \geq n, S_n < -x - y\}.$$

$$\leq \sum_{n=1}^{\infty} P\{S_{n-1} \geq -x, S_n < -x - y\}$$

$$= \sum_{n=1}^{\infty} \int_{-x}^{\infty} F(-x - y - s) \; F^{*(n-1)}(ds)$$

$$= \int_{-x}^{\infty} F(-x - y - s)U \; (ds)$$

$$\leq C \sum_{k>-x} F(-x - y - k)$$

$$\leq C \sum_{k>0} F(-y - k)$$

$$\leq C \int_{-\infty}^{-y} F(z)dz.$$

where F^{*j} is the j-fold convolution of F with itself, and U is the renewal measure: $U(x) - U(y) = \sum\limits_{m=0} [F^{*m}(x) - F^{*m}(y)]$. Multiplying the first and last statements by y^{p-2} and integrating gives the stated result.

Lemma 4: $E(|S_{\tau-}|^p; \tau- < \infty) < \infty \Rightarrow E((X^-)^{p+1}) < \infty \; \forall \; p > 0.$

Proof: The statement is invariant under change of scale, so if X is lattice one may assume that the span of X is less than 1. It may also be assumed that X is not bounded below, for otherwise the statement of the lemma is trivial. In this case, with

$$R_x = S_{\tau(x)} - x,$$

since the asymptotic distribution of R_x has positive mass on $[0,1)$ (see Woodroofe [8], Sec. 2.2),

$$0 < r = \inf_{x \geq 0} P\{R_x < 1\}.$$

By time-reversal, for $n \geq 0$

$P\{S_{\tau-} \in (-n - 1, -n), \tau- < \infty\}$

$= \sum\limits_{j=1}^{\infty} P\{S_1 > 0, \ldots, S_{j-1} > 0, S_j \in (-n - 1, -n)\}$

$= \sum\limits_{j=1}^{\infty} P\{S_j > S_1, \ldots, S_j > S_{j-1}, S_j \in (-n - 1, -n)\}.$

$\geq \sum\limits_{j=1}^{\infty} P\{-n - 1 > S_1, \ldots, -n - 1 > S_{j-1}, S_j \in (-n - 1, -n)\}$

$= \sum\limits_{j=1}^{\infty} P\{\tau_{-n-1} = j, R_{-n-1} < 1\}$

$= P\{R_{-n-1} < 1\}.$

And

$$P\{R_{-n-1} < 1\} \geq P\{R_{-n-1} < 1, X_1 \leq -n - 1\}$$

$$= \int_{n+1}^{\infty} P\{R_{x-n-1} < 1\} \, P\{X_1^- \in dx\}.$$

$$\geq r \, P\{X_1^- > n + 1\}.$$

Thus

$$P\{S_{\tau_-} \in (-n - 1, -n)\} \geq r \, P\{X_1^- \geq n + 1\}.$$

Multiplying by n^p and summing gives the statement of the lemma.

Proof of the Theorem.

<u>a => b</u> $E(\tau_+^p) < \infty$ => $E(\tau_-^{p-1}; \tau_- < \infty) < \infty$. By a standard time reversal argument (See Feller [4], Sec. 12.2),

$$P\{\tau_+ > n\} = \sum_{j=1}^{\infty} P\{\tau_-^{(j)} = n\}.$$

In particular

$$P\{\tau_+ > n\} \geq P\{\tau_- = n\}.$$

Multiplying by n^{p-1} and summing gives the result.

<u>b => a</u> $E(\tau_-^{p-1}; \tau_- < \infty) < \infty$ => $E(\tau_+^p) < \infty$. Note that conditioned on $\tau_-^{(j)} < \infty$ $\tau_-^{(j)} = \sum_{i=1}^{j} Y_i$, where the Y_i are i.i.d. with

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$P\{Y_i < y\} = P\{\tau_- < y \mid \tau_- < \infty\}$. Thus $\sum\limits_{m=1}^{\infty} n^{p-1} P\{\tau_-^{(j)} = n\} =$

$E(|\tau_-^{(j)}|^{p-1}; \tau_-^{(j)} < \infty) \le j^{p-1} E(\tau_-^{p-1} \mid \tau_- < \infty) P\{\tau_- < \infty\}^j$

So

$$\sum_{m=1}^{\infty} n^{p-1} P\{\tau_+ > n\} \le \sum_{j=1}^{\infty} j^{p-1} P\{\tau_- < \infty\}^{j-1} E(\tau_-^{p-1}; \tau_- < \infty)$$

$$\le K \, E(\tau_-^{p-1}; \tau_- < \infty).$$

$\underline{a \Rightarrow c}$ $\quad E(\tau_+^p) < \infty \Rightarrow E((X^-)^p) < \infty.$

By the Elementary Renewal Theorem (Chung [9], Thm. 5.5.2), $\exists\, c, K > 0$ such

that $E(\tau(x)) > cx \; \forall \; x > K$. So $E(\tau(x)^p) \ge (E(\tau(x)))^p \ge c^p x^p \; \forall \; x > K$

Conditioning on the first step of the random walk gives

$$\infty > E(\tau_+^p)$$

$$\ge \int_0^{\infty} E(\tau(x))^p \, P\{X^- \epsilon \, dx\}.$$

$$\ge \int_K^{\infty} c^p x^p \, P\{X^- \epsilon \, dx\}.$$

The last line implies $E((X^-)^p) < \infty.$

$\underline{c \Rightarrow a}$ $\quad E((X^-)^p) < \infty \Rightarrow E(\tau_+^p) < \infty.$ It suffices to assume $X_i \le c_x$ for some

$c > 0$; for, X_i can be truncated above to give $\tilde{X}_i$ with $E\,\tilde{X}_i > 0$,

and τ_+ for the random walk generated by the $\tilde{X}_i$ is larger than that for

the X_i random walk, so if the claim can be proven for the $\tilde{X}_i$ process

it follows for the X_i process.

In this case it may be assumed that the X_i have at least 2 moments.

Wald's identity for the 2nd moment gives

$$E(S_{\tau_+} - \mu\,\tau_+)^2 \;=\; (\text{Var } X_i)\; E(\tau_+) < \infty.$$

But $S_{\tau_+} < c$ so $E\,S_{\tau_+}^2 < \infty \Rightarrow E\,\tau_+^2 < \infty$. Let $\hat{q} = \sup\{p \geq 2:$

$E((X^-)^q) < \infty \Rightarrow E(\tau_+^{\,q}) < \infty \; \forall \; 2 \leq q \leq p\}$. Suppose $\hat{q} < \infty$. Let $\hat{q} \leq q < 2\hat{q}$.

Then $E(\tau_+^{\,q/2}) < \infty$. Therefore, by Burkholder and Gundy [2], Theorem 5.3.

$$E \mid S_{\tau_+} - \mu\,\tau_+ \mid^q \; < \; \infty$$

from which $E\,\tau_+^q < \infty$ follows as above. This is a contradiction.

$\underline{d \Rightarrow b}$ $\;\; E(L(0)^{p-1}) < \infty \Rightarrow E(\tau_-^{\,p-1};\; \tau_- < \infty) < \infty.$

Proof: $L(0) \geq \tau_- 1_{\{\tau_- < \infty\}}.$

$\underline{a \text{ and } b \Rightarrow d}$ $\;\; E(\tau_+^p) < \infty$, and $E(\tau_-^{\,p-1};\; \tau_- < \infty) < \infty \Rightarrow E(L(0)^{p-1}) < \infty.$

The idea of the proof is to express $L(0)$ as a sum of successive trips above and below the origin, until the random walk stays permanently above 0. Finding the random walk above 0 one must know the $p\text{-}1^{st}$ moment of the expected time to get back below 0 must be bounded no matter where the process is, provided it ever does. This is the content of Lemma 1. Having hit below 0 one must know that the $p\text{-}1^{st}$ moment of the expected time to reach 0 is finite. According to Lemma 2 this quantity is bounded by $K \int_0^\infty (|y| + 1)^{p-1}\, F(dy)$, where F denotes the hitting place of the non-positive axis.

Lemma 3 provides a uniform bound on the $p-1^{st}$ moments of these distributions F. The proof is then finished by observing that, because of the positive drift, this cycling behavior can only be repeated a few times.

Proof: Set $p = (1-q) = P\{\tau_- = \infty\}$. Define

$$T_1 = \inf\{K \geq 1: S_K > 0 \text{ and } \exists \, m < K \text{ with } S_m \leq 0 ,$$

and for $n > 1$

$$T_n = \inf\{K > T_1 + \ldots + T_{n-1} = S_K > 0 \text{ and } \exists \, T_1 + \ldots + T_{n-1} < m < K \text{ with}$$

$$S_m \leq 0\} - (T_1 + \ldots + T_{n-1}), \quad T_{n-1} < \infty$$

$$= \infty, \quad T_{n-1} = \infty$$

$$[3.1] \qquad |L(0)|^{p-1} \leq \sum_{m=1}^{\infty} |T_1 + \ldots + T_n|^{p-1} 1_{\{T_n < \infty, T_{n+1} = \infty\}}.$$

$$E(|T_1 + \ldots + T_n|^{p-1}; T_{n<\infty}, T_{n+1=\infty})$$

$$\leq E(|T_1 + \ldots + T_n|^{p-1}; T_n < \infty)$$

$$\leq n^{p-1} E(T_1^{p-1} + \ldots + T_n^{p-1}; T_n < \infty)$$

$E(T_i^{p-1}; T_n < \infty)$ is estimated seperately when $i=n$, and $i<n$. First the case $i<n$.

$$E(T_i^{p-1}, T_n < \infty) = E(E(T_i^{p-1}; T_n < \infty | \mathfrak{F}_{T_1 + \ldots + T_{n-1}}))$$

$$= E(T_i^{p-1}; T_{n-1} < \infty \, P\{T_n < \infty | \mathfrak{F}_{T_1 + \ldots + T_{n-1}}\}; T_{n-1} < \infty)$$

$$\leq q \, E(T_i^{p-1}; T_{n-1} < \infty) \qquad (*)$$

and

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$$E(T_n^{p-1}; T_n < \infty) = E(E(T_n^{p-1}; T_n < \infty \mid \mathcal{F}_{T_1 + \ldots + T_{n-1}}))$$

$$= E(E^{S_{T_1 + \ldots + T_{n-1}}}(T_1^{p-1}; T_1 < \infty); T_{n-1} < \infty).$$

Consider for $x > 0$

$$E^x\{T_1^{p-1}; T_1 < \infty\} = E^x\{(\tau_- + \tau(-S_{\tau_-})(\omega_{\tau_-}^+))^{p-1}; \tau_- < \infty\}$$

$$\leq 2^{p-1}(E^x(\tau_-^{p-1}; \tau_- < \infty) + E^x(E^{S_{\tau_-}}(\tau(0)^{p-1}; \tau_- < \infty)).$$

The first term is $\leq K$ by Lemma 1. For the 2nd, using lemmas 2 and 3 it follows that

$$E^x(E^{S_{\tau_-}}(\tau(0)^{p-1}; \tau_- < \infty))$$

$$= \int_{-\infty}^0 E^y(\tau(0)^{p-1}) \, P^x\{S_{\tau_-} \in dy, \tau_- < \infty\}$$

$$\leq K' \int_{-\infty}^0 |y + 1|^{p-1} \, P^x\{S_{\tau_-} \in dy, \tau_- < \infty\}.$$

$$\leq K''.$$

Thus

$$E(T_n^{p-1}; T_n < \infty) \leq K \, P\{T_{n-1} < \infty\}$$

$$\leq K \, q^{n-1}. \quad (**).$$

Set $a_n = E(T_1^{p-1} + \ldots + T_n^{p-1}; T_n < \infty).$

Summing $(*)$ from 1 to $n-1$ and adding $(**)$ gives

$$a_n \leq q \, a_{n-1} + K \, q^{n-1},$$

Therefore, a_n is geometrically decreasing, and $\Sigma a_n < \infty$. A look at 3.1 shows that $\Sigma a_n < \infty \Rightarrow E\,L(0)^{P-1} < \infty$.

$\underline{d \Rightarrow e}$ $E(L(0)^{P-1}) < \infty \Rightarrow E(N(0)^{P-1}) < \infty$

Proof: $L(0) \geq N(0)$.

$\underline{e \Rightarrow g}$ $E(N(0)^{P-1}) < \infty \Rightarrow E(|S_{\tau_-}|^{P-1};\ \tau_- < \infty) < \infty$

Proof: The amount of time spent getting back above ϕ after having hit below ϕ for the first time is $\tau(0)(\omega_{\tau_-}^+)\,1_{\{\tau_- < \infty\}}$. So

$$(1 + N(0))^{P-1} \geq \tau(0)(\omega_{\tau_-}^+)\,1_{\{\tau_- < \infty\}}.$$

and $\infty > E((1 + N(0))^{P-1}) \geq E(\tau^{P-1}(0)(\omega_{\tau_-}^+);\ \tau_- < \infty)$

$$= E(E(\tau^{P-1}(0)(\omega_{\tau_-}^+);\ \tau_- < \infty \mid F_{\tau_-}))$$
$$= E(E^{S_{\tau_-}}(\tau(0)^{P-1};\ \tau_- < \infty) \Rightarrow E(|S_{\tau_-}|^{P-1};\ \tau_- < \infty) < \infty$$

as in the last part of a and b $\Rightarrow$ d.

$\underline{g \Rightarrow f}$ $E(|S_{\tau_-}|^{P-1};\ \tau_- < \infty) < \infty \Rightarrow E(|S_{min}|^{P-1} < \infty)$

Proof: S_{min} can be written as Z_n, where Z_1 is a random walk with $P\{Z_1 < y\} = P\{S_{\tau_-} < y \mid \tau_- < \infty\}$, $P\{M = n\} = P\{\tau_- < \infty\}^n\,P\{\tau_- = \infty\}$, $n=0$, 1, ..., and M is independent of the Z_1. This can be seen intuitively by considering the decreasing ladder process, or a quick proof can be based on a comparison of the characteristic functions given in Feller [4], Chapt. 18. $E(|S_{\tau_-}|^{P-1};\ \tau_- < \infty) < \infty \Rightarrow E(|Z_1|^{P-1}) < \infty$, so

$$E(|S_{min}|^{P-1}) = \sum_n E(|Z_n|^{P-1})\,P\{M = n\}$$
$$\leq E(|Z_1|^{P-1}) \sum_n n^{P-1}\,P\{M = n\}$$
$$< \infty\ .$$

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$\underline{f \Rightarrow g} \quad E(|S_{min}|^{p-1}) < \infty \Rightarrow E\{|S_{\tau_-}|^{p-1}; \tau_- < \infty\} < \infty$

Proof: $\quad |S_{\tau_-}|1_{\{\tau_- < \infty\}} \leq |S_{min}|$

$\underline{f \Rightarrow e} \quad E(|S_{min}|^{p-1}) < \infty \Rightarrow E(N(0)^{p-1}) < \infty.$

Proof: Since $E(|S_{min}|^{p-1}) < \infty$, then $E\{|S_{\tau_-}|^{p-1}; \tau_- < \infty\} < \infty$. From lemma 4 $E((X^-)^{p+1}) < \infty$ so the result follows from $c \Rightarrow d \Rightarrow e$.

$\underline{f \Rightarrow c} \quad E(|S_{min}|^{p-1}) < \infty \Rightarrow E((X^-)^p) < \infty$

Proof: Follows from $f \Rightarrow g$ and Lemma 4.

4. Remarks and Applications.

Let Y_i be a i.i.d. sequence of random variables with $EY_i = 0$.
Let $S_n = Y_1 + \ldots + Y_n$

$$L(\varepsilon) = \sum_{n=1}^{\infty} 1_{\{\sup_{j \geq n} \frac{|S_j|}{j} > \varepsilon\}},$$

$$N(\varepsilon) = \sum_{n=1}^{\infty} 1_{\{|\frac{S_n}{n}| > \varepsilon\}}.$$

Theorem 2: For $p \geq 2$

(1) $E((N(\varepsilon)^{p-1}) < \infty \iff E(|Y|^p) < \infty$

(2) $E(L(\varepsilon)^{p-1}) < \infty \iff E(|Y|^p) < \infty$

(3) $\sum_{n=1}^{\infty} P\{\sup_{j \geq n} \frac{|S_j|}{j} > \varepsilon\} \cdot n^{p-2} < \infty \iff E|Y|^p < \infty$

(4) $\sum_{n=1}^{\infty} P\{|\frac{S_n}{n}| > \varepsilon\} \iff E|Y|^2 < \infty.$

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Remarks: The "only if" part of (4), for $p=1$ is due to Robbins and Hsu [6], (4) with $p=1$ is due to Erdos [3], (3) was first proved by Baum and Katz and can be found in [1].

Proof: (1) and (2) follow from the equivalence of c,d, and e by considering the random walks $S_n \pm n\varepsilon$. (3) is the same as (2) plus the observation that $P\{L(\varepsilon) > n\} = P\{\sup_{j \geq n} \frac{|S_j|}{j} > \varepsilon\}$, and (4) follows similarly from (1).

Let X_i be i.i.d. random variables with $E\,X_i = \mu\,\varepsilon\,(0,\infty)$, $S_n = X_1 + \ldots + X_n$.

Theorem 3: For $p \geq 1$ the following are equivalent:

(1) $E((X^-)^{p+1}) < \infty$

(2) $E(|S_{\tau_-}|^p; \tau_- < \infty) < \infty$

(3) $E(|S_{min}|^p) < \infty$.

Remark: The equivalence of (2) and (3) for $p=1$ is credited by Taylor [7], to unpublished work of Darling, Erdos and Kakutani, and Taylor adds a proof of the equivalence of (1). Kiefer and Wolfowitz [5] also credit the equivalence of (1) and (3) to unpublished results of Darling, Erdos and Kalutani; and they give their own proof. The moments of the minimum are of interest because the minimum has the distribution of the stationary distribution of a type of queueing process. See [4] p. 198.

Proof: This is the equivalence above, however, the tortuous path via the implications of Theorem 1 can be replaced by lemma 4.

I would like to thank Professor Siegmund for help received on this problem. In particular he showed me the time-reversal proof of b => a.

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4. TITLE *(and Subtitle)* Moments of the Minimum of a Random Walk and Complete Convergence		5. TYPE OF REPORT & PERIOD COVERED TECHNICAL REPORT
		6. PERFORMING ORG. REPORT NUMBER
7. AUTHOR(s) Michael Hogan		8. CONTRACT OR GRANT NUMBER(s) N00014-77-C-0306
9. PERFORMING ORGANIZATION NAME AND ADDRESS Department of Statistics Stanford University Stanford, California		10. PROGRAM ELEMENT, PROJECT, TASK AREA & WORK UNIT NUMBERS NR-042-373
11. CONTROLLING OFFICE NAME AND ADDRESS Statistics & Probability Program Code (411 (SP)) Office of Naval Research Arlington, Virginia 22217		12. REPORT DATE January 1983
		13. NUMBER OF PAGES 14
14. MONITORING AGENCY NAME & ADDRESS*(If different from Controlling Office)*		15. SECURITY CLASS. *(of this report)* UNCLASSIFIED
		15a. DECLASSIFICATION/DOWNGRADING SCHEDULE

16. DISTRIBUTION STATEMENT *(of this Report)*

Approved for Public Release: Distribution Unlimited

17. DISTRIBUTION STATEMENT *(of the abstract entered in Block 20, if different from Report)*

18. SUPPLEMENTARY NOTES

19. KEY WORDS *(Continue on reverse side if necessary and identify by block number)*

Random Walk, Renewal Theorem, Complete Convergence, Strong Law of Large Numbers

20. ABSTRACT *(Continue on reverse side if necessary and identify by block number)*

Let S_n be a random walk with positive drift. Let $S_{min} = \inf_{n>0} \{S_n\}$. New proofs are given of the following: For $p \geq 1$ $E|S_{min}|^p < \infty \iff E(|S_1|^{p+1} 1_{(S_1<0)}) < \infty$; $\Sigma P\{S_n < 0\} < \infty \iff E(|S_1|^2 1_{(S_1<0)}) < \infty$, and some related results.

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