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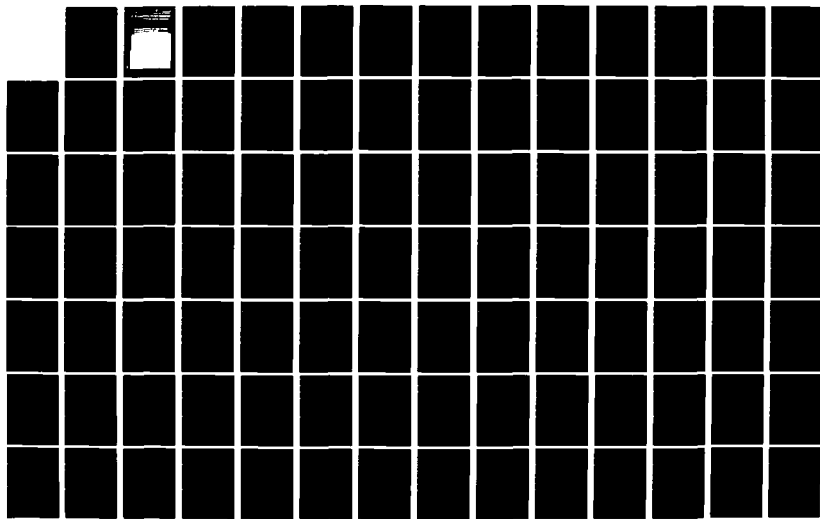
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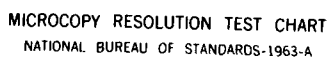
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**DECOMPOSITION OF LINEAR SYSTEMS
IN LINEAR SYSTEMS AND
MARKOVIAN DECISION PROCESSES**



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DECOMPOSITION OF TIME SCALES IN LINEAR SYSTEMS
AND MARKOVIAN DECISION PROCESSES

by

Randolph Gale Phillips

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DECOMPOSITION OF TIME SCALES IN LINEAR SYSTEMS
AND MARKOVIAN DECISION PROCESSES

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THESIS

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in the Graduate College of the
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Thesis Adviser: Professor P. V. Kokotovic

Urbana, Illinois

DECOMPOSITION OF TIME SCALES IN LINEAR SYSTEMS
AND MARKOVIAN DECISION PROCESSES

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University of Illinois at Urbana-Champaign, 1981

The presence of "slow" and "fast" dynamics in large scale systems has motivated the use of singular perturbations as a means of obtaining reduced order models for analysis and control law design. In this thesis we establish how systems having this "two-time-scale" property can use singular perturbation modeling to make this property explicit enabling various reduced order analysis and design techniques to be applied. For deterministic linear time-invariant systems, various techniques for obtaining reduced order models are unified through left and right eigenspace decompositions. A general two stage control design procedure for separate fast and slow subsystems is developed which can be applied to both continuous and discrete time models. Finally, Markov chain models of stochastic systems with "weak" and "strong" transition probabilities lead to a singularly perturbed model from which we obtain the concept of the reduced order "aggregate" chain. For controlled Markov chains the aggregate model is used to develop decentralized optimization algorithms for the discounted and average cost per stage problems.

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CHAPTER 1

INTRODUCTION

The analysis and control of large scale systems has always been a challenging problem to control engineers. The large dimension of the system model and the rich interactions make conventional simulation and design algorithms impractical if not impossible to use. The research direction in this area has centered on obtaining reduced order models from large scale models without sacrificing significant accuracy. These reduced order models are usually constructed in one of two ways. First, within a large scale model, several smaller subsystems are identified by their weak couplings to the remainder of the system. These reduced order subsystems are treated separately for simulation and design purposes with interactions between subsystems taken care of separately [1,2]. Second, the entire large scale model is approximated by one reduced order model where the dynamics of the reduced order model are determined by the "dominant" dynamics of the large scale model [3,4]. Over the years, many names have been given to various methods of order reduction. Of these methods aggregation and singular perturbations seem to be the most well known [5]. The analysis and design of singularly perturbed systems has been well documented [6,7,8]. The multiple-time-scale property of these systems has been used in developing reduced order models and control laws for high order "stiff" models. This thesis further contributes to the theory of multiple-time-scale systems and how they can be used as a powerful order reduction technique for both the analysis and design of large scale systems.

In Chapter 2 we consider the linear time-invariant system

$$\frac{dx(t)}{dt} = Ax(t) + Bu(t) \quad (1.1)$$

where $x(t) \in \mathbb{R}^q$, $u(t) \in \mathbb{R}^p$, and $\{a_{ij}\}, \{b_{ij}\} \in \mathbb{R} \quad \forall i, j$. Many seemingly unrelated iterative techniques [9-12] have been proposed for transforming (1.1) into

$$\begin{bmatrix} \dot{y} \\ \dot{w} \end{bmatrix} = \begin{bmatrix} A^* & 0 \\ 0 & D^* \end{bmatrix} \begin{bmatrix} y \\ w \end{bmatrix} + \begin{bmatrix} E^* \\ F^* \end{bmatrix} u \quad (1.2)$$

where $y \in \mathbb{R}^n$, $w \in \mathbb{R}^m$, and

$$\min_{i=1, \dots, n} |\lambda_i(A^*)| < \max_{j=1, \dots, m} |\lambda_j(D^*)| \quad (1.3)$$

(in (1.3), $\lambda_i(Z)$ is the i th eigenvalue of the matrix Z). When such a transformation is possible, (1.1) is said to have a "two-time-scale property." These spectrum separation techniques can then be used for reduced order modeling and design. Unfortunately, the convergence results of these iterative techniques are either heuristic or conservative [10,12]. In this chapter we unify and extend the results of time scale decomposition techniques to form a composite theory from which all previous methods can be derived as variations to the basic results.

In Chapter 3 we consider the time scale decomposition of singularly perturbed systems. For this problem (1.1) takes the form

$$\begin{bmatrix} \dot{x}_1 \\ \mu \dot{x}_2 \end{bmatrix} = \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} B_1 \\ B_2 \end{bmatrix} u \quad (1.4)$$

where $x_1 \in \mathbb{R}^n$, $x_2 \in \mathbb{R}^m$, and $0 < \mu \ll 1$. In this case, the singular perturbation parameter μ explicitly defines the presence of a two-time-scale phenomena. Using this parameter, two-time-scale asymptotic expansions [8,13] of the state vector can be used to obtain a reduced order solution to (1.4). In this chapter we show the relationship between the "spectrum separating" iterative techniques of Chapter 2 and matched asymptotic expansions used for singularly perturbed models. The results of these two chapters establish a unified theory of time scale decompositions in linear time-invariant systems. In the remainder of this thesis we consider a class of stochastic systems in which the results from both singular perturbations and aggregation techniques are applied.

Markov chains and Markovian decision processes have long been used in the analysis and design of stochastic systems [14,15]. Some of the application areas of Markov modeling included the following:

- i. Numerical solutions to stochastic control problems [16]
- ii. Inventory theory [17]
- iii. Queuing theory [18].

Markovian decision processes can be traced back to Bellman's development of Dynamic Programming [19,20] where many of the Markov chain control problems were formulated. Since this time, many design algorithms for a variety of controlled Markov chain problems have been developed. The theoretical richness of this area has kept it popular with researchers. However, the practical usefulness of Markov models and Markovian decision processes has been severely limited due to the extremely large dimension of most Markov chains. The computational burden of these problems has discouraged systems engineers from using Markov chains for modeling purposes. Recent applications in queueing theory [21] and the management of hydrodams [22] have exhibited

Markov chain models with a "weakly coupled" structure suitable for perturbational analysis. In recent years, authors [23-25] have used this structure to construct reduced order aggregate models for large Markov chains. Since the aggregate models were developed using an intuitive understanding of the process dynamics, the results were limited and a more complete theory needed in order for this concept to become useful for analysis and design. Therefore, in Chapter 4 we show how a weakly coupled Markov chain can be transformed into a singularly perturbed system model. Then, the decomposition techniques of Chapters 2 and 3 can readily be applied.

In Chapters 5 and 6 we consider the problem of optimally controlling Markov chains with respect to certain performance measurements. In general, these problems are computationally horrendous. However, by applying the results of Chapter 4, a near optimal policy can be found using a simplified decentralized optimization algorithm. In Chapter 5 we consider the discounted cost problem [15] and in Chapter 6 the average cost per stage problem [15].

Finally, in Chapter 7 we draw conclusions and point to a number of research directions to which this thesis leads.

CHAPTER 2

EIGENSTRUCTURE DECOMPOSITION OF TIME SCALES
IN LINEAR-TIME INVARIANT SYSTEMS2.1. Introduction

Many control theory concepts are valid for any system order, however, their actual use is limited to low order models. Large scale systems result not only in a formidable amount of computation, but also in ill-conditioned initial and two point boundary value problems. The interaction of fast and slow phenomena in high-order systems results in stiff numerical problems which require expensive integration routines. The singular perturbations approach to decomposing fast and slow phenomena involves using a time-scale separation technique. In this case a reduced order "steady state" and "boundary layer" solutions are obtained from a high order model. Control designs and simulations for the high order model are then carried out on the reduced order subsystems.

It is the purpose of this chapter to unify and extend the results of previous authors [6-12] and attempt to provide a sense of completeness to the theory of time-scale separation in linear systems. Given the linear time invariant homogeneous system

$$\begin{bmatrix} \dot{y} \\ \dot{z} \end{bmatrix} = \begin{bmatrix} A & B \\ C & D \end{bmatrix} \begin{bmatrix} y \\ z \end{bmatrix} \quad \begin{matrix} y(t_0) = y_0 \\ z(t_0) = z_0 \end{matrix} \quad (2.1)$$

$$y \in R^n, \quad z \in R^m$$

our purpose is to transform it into either

$$\begin{bmatrix} \dot{y} \\ \dot{n} \end{bmatrix} = \begin{bmatrix} A_\infty & B \\ 0 & D_\infty \end{bmatrix} \begin{bmatrix} y \\ n \end{bmatrix} \quad (2.2)$$

or

$$\begin{bmatrix} \dot{\xi} \\ \dot{z} \end{bmatrix} = \begin{bmatrix} A_{\infty}^* & 0 \\ C & D_{\infty}^* \end{bmatrix} \begin{bmatrix} \xi \\ z \end{bmatrix} \quad (2.3)$$

such that

$$\min |\sigma(D_{\infty})| > \max |\sigma(A_{\infty})| \quad (2.4)$$

or

$$\min |\sigma(D_{\infty}^*)| > \max |\sigma(A_{\infty}^*)|. \quad (2.5)$$

Any system (2.1) which has this property is said to satisfy the two-time-scale property for dimensions n and m .

In Section 2.2 earlier methods of time-scale decomposition used to transform (2.1) into (2.2) and (2.3) are briefly reviewed. Then using a modified form of dominant left and right eigenspace power iterations the equivalence of these past iterative schemes is established. This enables us to define unified conditions for convergence of algorithms as well as the convergence rates.

Section 2.3 completes the block diagonalizations of (2.2) and (2.3) and identifies "fast" and "slow" components of our original state vectors. The explicit invertibility of the transformation matrices is shown. This becomes very important in later chapters.

In Section 2.4 we consider the problem of properly ordering the state variables. A recently developed "grouping algorithm" [34] used for power networks is shown to be directly applicable.

Finally, in Section 2.5, an example is given of the eigenspace decompositions.

2.2. Left and Right Eigenspace Decomposition

In the first part of this section we briefly review two well known iterative methods for transforming (2.1) into (2.2) or (2.3). We then introduce the dominant left and right eigenspace power iterations. Using modified versions of these iterations, it is possible to show the equivalence of the earlier methods. Less conservative conditions for the convergence of algorithms as well as convergence rates are also obtained.

1) Quasi steady state method [6]

This method was motivated by singularly perturbed models. Assume that the states are ordered such that D^{-1} exists. This assumption is standard in studies of singularly perturbed systems. However, as will be presented later, this assumption can be justified.

If the eigenvalues of D are such that the real parts are large and negative, then the homogeneous solution of z converges to a steady-state rapidly. If this convergence is assumed to be instantaneous, then $\dot{z} = 0$ and this quasi steady state assumption yields

$$z_s = -D^{-1}Cy_s. \quad (2.6)$$

Next we try to remove this slow part of z by introducing

$$\eta_1 = z + D^{-1}Cx \quad (2.7)$$

which transforms (2.1) into

$$\begin{aligned}
 \begin{bmatrix} \dot{y} \\ \dot{\eta}_1 \end{bmatrix} &= \begin{bmatrix} (A-BD^{-1}C) & B \\ D^{-1}C(A-BD^{-1}C) & D+D^{-1}CB \end{bmatrix} \begin{bmatrix} y \\ \eta_1 \end{bmatrix} \\
 &= \begin{bmatrix} A_1 & B \\ C_1 & D_1 \end{bmatrix} \begin{bmatrix} y \\ \eta_1 \end{bmatrix}.
 \end{aligned} \tag{2.8}$$

Repeating steps (2.6) and (2.7) k times results in the following

$$\eta_k = \eta_{k-1} + D_{k-1}^{-1} C_{k-1} y \quad \eta_0 = z \tag{2.9}$$

$$\begin{bmatrix} \dot{y} \\ \dot{\eta}_k \end{bmatrix} = \begin{bmatrix} A_k & B \\ C_k & D_k \end{bmatrix} \begin{bmatrix} y \\ \eta_k \end{bmatrix} \tag{2.10}$$

where the subsystem matrices are defined as

$$A_k = A_{k-1} - BD_{k-1}^{-1} C_{k-1} \quad A_0 = A \tag{2.11}$$

$$C_k = D_{k-1}^{-1} C_{k-1} A_k \quad C_0 = C \tag{2.12}$$

$$D_k = D_{k-1} + D_{k-1}^{-1} C_{k-1} B \quad D_0 = D. \tag{2.13}$$

Experimental results, motivated by singular perturbations, have converged to the form (2.2) satisfying spectral property (2.4).

The dual to this procedure involves removing the fast parts of the y states. Such a procedure would transform (2.1) into (2.3) satisfying condition (2.5). [12] proposed this dual procedure which led to matrix recursions

$$A_k = A_{k-1} - B_{k-1} D_{k-1}^{-1} C \quad A_0 = A \tag{2.14}$$

$$B_k = A_k B_{k-1} D_{k-1}^{-1} \quad B_0 = B \tag{2.15}$$

$$D_k = D_{k-1} + CB_{k-1} D_{k-1}^{-1} \quad D_0 = D. \tag{2.16}$$

Again, while experimental results have proved successful, conditions for convergence as well as convergence rates are unavailable.

ii) Algebraic Riccati equation method [10]

In [10, 11, 26, 27], the transformation of the form

$$\eta = z + Py \quad (2.17)$$

is proposed in an attempt to transform (2.1) into (2.2). By applying (2.17) to (2.1), we obtain

$$\begin{bmatrix} \dot{y} \\ \dot{\eta} \end{bmatrix} = \begin{bmatrix} A-BP & B \\ C-DP+PA-PBP & D+PB \end{bmatrix} \begin{bmatrix} y \\ \eta \end{bmatrix}. \quad (2.18)$$

The problem is to find the solution P to the Riccati type equation

$$R(P) = C - DP + PA - PBP = 0 \quad (2.19)$$

such that $A-BP$ and $D+PB$ have the spectral properties (2.4). Such spectrum dependent solutions have been referred to as "dichotomic" [28]. We will throughout the rest of this thesis continue to refer to this solution by this label.

Earlier work by [10] and more recent work by [9] have resulted in the following iterative recursion formula for obtaining the dichotomic solution to (2.19)

$$P_{k+1} = P_k + (D + P_k B)^{-1} \cdot R(P_k) \quad (2.20)$$

$$P_0 = D^{-1}C$$

which gives the subsystem matrices in (2.2) at each k as

$$A_k = A - BP_{k-1} \quad A_0 = A \quad (2.21)$$

$$D_k = D + P_{k-1}B \quad D_0 = D \quad (2.22)$$

$$C_k = R(P_{k-1}) = C - DP_{k-1} + P_{k-1}A - P_{k-1}BP_{k-1} \quad C_0 = C. \quad (2.23)$$

Likewise, [11,35] have proposed the dual to the Riccati method via the transformation

$$\xi = y - \hat{P}z \quad (2.24)$$

which transforms (2.2) into

$$\begin{bmatrix} \dot{\xi} \\ \dot{z} \end{bmatrix} = \begin{bmatrix} A - \hat{P}C & A\hat{P} + B - \hat{P}C\hat{P} - \hat{P}D \\ C & D + C\hat{P} \end{bmatrix} \begin{bmatrix} \xi \\ z \end{bmatrix}. \quad (2.25)$$

The problem is again to obtain the dichotomic solution $\hat{P}$ to the Riccati type equation

$$S(\hat{P}) = A\hat{P} + B - \hat{P}C\hat{P} - \hat{P}D = 0 \quad (2.26)$$

such that $A - \hat{P}C$ and $D + C\hat{P}$ have spectral properties (1.5). From [11,36] the following iterative scheme was derived for obtaining the dichotomic solution to (2.26)

$$\hat{P}_{k+1} = \hat{P}_k + S(\hat{P}_k)(D + C\hat{P}_k)^{-1} \quad (2.27)$$

$$\hat{P}_0 = BD^{-1} \quad (2.28)$$

which leads to the matrix equations for (2.3) as

$$A_k = A - \hat{P}_{k-1}C \quad A_0 = A \quad (2.29)$$

$$D_k = D + C\hat{P}_{k-1} \quad D_0 = D \quad (2.30)$$

$$B_k = S(\hat{P}_{k-1}) = B_{k-1} - \hat{P}_{k-1}D + A\hat{P}_{k-1} - \hat{P}_{k-1}C\hat{P}_{k-1} \quad B_0 = B. \quad (2.31)$$

While some convergence results are available [9,10] for the matrix recursion (2.27), they are either conservative or limited to solving only (2.26). We now unify and extend these results by showing that these iterative techniques are equivalent to either dominant right or left eigenspace power iterations.

We now give a lemma which establishes a convergence criterion for (2.27) based on dominant left eigenspace power iterations. In the process, we shall show that (2.11)-(2.13) and (2.21)-(2.23) are equivalent at every iterate.

Lemma 2.1: Given (2.1), if the spectrum is concentrated in two groups of m such n eigenvalues such that

$$\max_{i=1,n} |\lambda_i| < \min_{j=1,m} |\lambda_j|. \quad (2.32)$$

Then under mild restrictions [29] on the initial iterate P_0 , (2.27) will converge to the dichotomic solution of (2.26) at a convergence rate of ϵ^k , where

$$\epsilon = \frac{\max_{i \in 1,n} |\lambda_i|}{\min_{j \in 1,m} |\lambda_j|}. \quad (2.33)$$

Proof: The well known power iteration method [30,31] for computing a m -dimensional basis for the dominant left eigenspace of (2.1) is of the form

$$\begin{bmatrix} M_k & N_k \end{bmatrix} = R_k \begin{bmatrix} M_{k-1} & N_{k-1} \end{bmatrix} \begin{bmatrix} A & B \\ C & D \end{bmatrix}. \quad (2.34)$$

R_k is a nonsingular $m \times m$ scaling matrix used, for example, to keep the rows of $\begin{bmatrix} M_k & N_k \end{bmatrix}$ strongly independent and the individual components within a practical range of computation [31]. Many methods have been proposed

for selecting the sequence of R_k 's and the interested reader is referred to [29] and [31]. The analytical convergence of (2.34), however, is independent of R_k . Thus, under condition (2.32) and mild conditions on $[M_0 \ N_0]$, it is known that (2.34) converges to the dominant m -dimensional left eigenspace of (2.1).

Expressing the common iterates as

$$M_k = M_{k-1}A + N_{k-1}C \quad M_0 = C \quad (2.35)$$

$$N_k^{-1} = (D + N_{k-1}^{-1}M_{k-1}B)^{-1}N_{k-1}^{-1} \quad N_0^{-1} = D^{-1}. \quad (2.36)$$

We can form the product

$$N_k^{-1}M_k = (D + N_{k-1}^{-1}M_{k-1}B)^{-1}(C + N_{k-1}^{-1}M_{k-1}A). \quad (2.37)$$

Letting $P_k = N_k^{-1}M_k$ gives (2.37) as

$$P_k = (D + P_{k-1}B)^{-1}(C + P_{k-1}A) \quad P_0 = D^{-1}C \quad (2.38)$$

which is equivalent to (2.27) $\forall k \geq 0$.

To show the solution is dichotomic, P is of the form

$$P = N^{-1}M. \quad (2.39)$$

Without loss of generality we can take $[M \ N] = [V_1 \ V_2]$ where $[V_1 \ V_2]$ are the m left eigenvectors corresponding to the m dominant eigenvalues, thus

$$P = V_2^{-1}V_1$$

and,

$$\begin{bmatrix} v_1 & v_2 \end{bmatrix} \begin{bmatrix} A & B \\ C & D \end{bmatrix} = \begin{bmatrix} J_1 & 0 \\ 0 & J_2 \end{bmatrix} \begin{bmatrix} v_1 & v_2 \end{bmatrix} \quad (2.40)$$

or,

$$v_1 A + v_2 C = J_1 v_1 \quad (2.41)$$

$$v_1 B + v_2 D = J_2 v_2. \quad (2.42)$$

However, from (1.16)

$$C + v_2^{-1} v_1 A = D v_2^{-1} v_1 + v_2^{-1} v_1 B v_2^{-1} v_1$$

which leaves both (2.41) and (2.42) in the form

$$v_2 (D + v_2^{-1} v_1 B) v_2^{-1} = J_2 \quad (2.43)$$

verifying the desired spectral decomposition we can now show the equivalence of (2.11)-(2.13) and (2.21)-(2.23).

Corollary 2.2: The matrix iterations (2.11)-(2.13) and (2.21)-(2.23) are equivalent at every k . Thus, the just provided convergence properties of (2.21)-(2.23) are propagated to (2.11)-(2.13).

Proof: Substitution of (2.20) into (2.21)-(2.23) results in

$$A - B P_{k-1} = A - B P_{k-2} - B (D + P_{k-2} B)^{-1} \cdot R(P_{k-2}) \quad (2.44)$$

$$D + P_{k-1} B = D + P_{k-2} B + (D + P_{k-2} B)^{-1} \cdot R(P_{k-2}) \cdot B \quad (2.45)$$

$$C_k = R(P_{k-1}) = (D + P_{k-2} B)^{-1} \cdot R(P_{k-2}) \cdot (A - B P_{k-1}). \quad (2.46)$$

Letting

$$\alpha_k = A - B P_{k-1} \quad (2.47)$$

$$\gamma_k = D + P_{k-1} B \quad (2.48)$$

$$\sigma_k = C_k \quad (2.49)$$

(2.47)-(2.49) become

$$\alpha_k = \alpha_{k-1} - B\gamma_{k-1}^{-1}\sigma_{k-1} \quad \alpha_0 = A \quad (2.50)$$

$$\gamma_k = \gamma_{k-1} + \gamma_{k-1}^{-1}\sigma_{k-1}B \quad \gamma_0 = D \quad (2.51)$$

$$\sigma_k = \gamma_{k-1}^{-1}\sigma_{k-1}\alpha_k \quad \sigma_0 = C \quad (2.52)$$

which are equivalent to (2.11)-(2.13) $\forall k \geq 0$.

We now cite a lemma [9] which is the dual to Lemma 2.1 and establishes the conditions of convergence of (2.27) based on dominant right eigenspace power iterations.

Lemma 2.3: If the spectrum of (2.1) satisfies (2.32), then under mild restrictions on $\hat{P}_0$ [9], (2.27) will converge to the dichotomic solution at a convergence rate of ε^k where ε is defined in (2.33).

Proof: The well known power iteration method [29,30] for computing an M dimensional basis for the dominant right eigenspace of (2.1) is of the form

$$\begin{bmatrix} M_k \\ N_k \end{bmatrix} = \begin{bmatrix} A & B \\ C & D \end{bmatrix} \begin{bmatrix} M_{k-1} \\ N_{k-1} \end{bmatrix} R_k \quad \begin{matrix} M_0 = B \\ N_0 = D \end{matrix} \quad (2.53)$$

where the scaling matrix R_k serves the same purpose as explained in Lemma 2.1. Thus, under mild restrictions on $\begin{bmatrix} M_0 \\ N_0 \end{bmatrix}$, it is well known that (2.53) converges to the dominant m -dimensional right eigenspace of (2.1).

Expressing (2.53) as

$$\begin{aligned} M_k &= AM_{k-1} + BN_{k-1} & M_0 &= B \\ N_k^{-1} &= N_{k-1}^{-1}(D + CM_{k-1}N_{k-1}^{-1})^{-1} & N_0^{-1} &= D^{-1} \end{aligned}$$

form the product

$$M_k N_k^{-1} = (B + A M_{k-1} N_{k-1}^{-1}) (D + C M_{k-1} N_{k-1}^{-1})^{-1}. \quad (2.54)$$

Letting

$$\hat{P}_k = M_k N_k^{-1},$$

$$\hat{P}_{k+1} = (B + A \hat{P}_k) (D + C \hat{P}_k)^{-1} \quad (2.55)$$

$$\hat{P}_0 = B D^{-1} \quad (2.56)$$

which is equivalent to (2.27) $\forall k \geq 0$. Proving $\hat{P}$ is dichotomic is carried out as in Lemma 2.1 and can be seen in [9, 36].

Finally, we can show the equivalence of (2.14)-(2.16) and (2.29)-(2.31).

Corollary 2.4: The matrix recursions (2.14)-(2.16) and (2.29)-(2.31) are equivalent at every k , thus, the convergence properties of (2.55) are propagated to (2.14)-(2.16).

Proof: Substitution of (2.27) into (2.29)-(2.31) gives

$$A - P_{k-1} C = A - P_{k-2} C - S(P_{k-2}) (D + C P_{k-2})^{-1} C \quad (2.57)$$

$$D + C P_{k-1} = D + C P_{k-2} + C S(P_{k-2}) (D + C P_{k-2})^{-1} \quad (2.58)$$

$$B_k = S(P_{k-1}) = (A - P_{k-1} C) S(P_{k-2}) (D + C P_{k-2})^{-1}. \quad (2.59)$$

Letting

$$\alpha_k = A - P_{k-1} C \quad (2.60)$$

$$\gamma_k = D + C P_{k-1} \quad (2.61)$$

$$\beta_k = B_k \quad (2.62)$$

gives

$$\alpha_k = \alpha_{k-1} - \beta_{k-1} \gamma_{k-1}^{-1} C \quad \alpha_0 = A \quad (2.63)$$

$$\gamma_k = \gamma_{k-1} + C \beta_{k-1} \gamma_{k-1}^{-1} \quad \gamma_0 = D \quad (2.64)$$

$$\beta_k = \alpha_k \beta_{k-1} \gamma_{k-1}^{-1} \quad \beta_0 = B \quad (2.65)$$

which are equivalent to (2.14)-(2.16).

Thus, the dominant left and right eigenspace iterations of (2.38) and (2.55) can be used to transform (2.1) into (2.2) and (2.3) respectively, requiring only that there exists a corresponding separation in the original spectrum. In the next section (2.2) and (2.3) are block diagonalized so as to isolate the fast and slow dynamics of (2.1).

2.3. Block Diagonalization and Identification of Fast and Slow State Vector Components

Once we have transformed (2.1) into (2.2) or (2.3) satisfying conditions (2.4) or (2.5) respectively, block diagonalization is always possible.

Consider form (2.2), and the transformation (2.17) used to obtain this form. The dichotomic solution matrix P is of the form

$$P = N^{-1}M$$

where the rows of $[M \ N]$ span the dominant left eigenspace of (2.1). Thus, the exact form of (2.2) is

$$\begin{bmatrix} \dot{y} \\ \dot{w} \end{bmatrix} = \begin{bmatrix} A-BP & B \\ 0 & D+PB \end{bmatrix} \begin{bmatrix} y \\ w \end{bmatrix}. \quad (2.66)$$

Now, let $x = y - Qw$. This leaves (2.55) in the form

$$\begin{bmatrix} \dot{x} \\ \dot{w} \end{bmatrix} = \begin{bmatrix} A - BP & (A - BP)Q - Q(D + PB) + B \\ 0 & D + PB \end{bmatrix} \begin{bmatrix} x \\ w \end{bmatrix}. \quad (2.67)$$

Thus, we seek Q to satisfy the Lyapunov type equation

$$(A - BP)Q - Q(D + PB) + B = 0 \quad (2.68)$$

Such a Q will always exist since

$$\sigma(A - BP) \cap \sigma(D + PB) = \emptyset \quad (2.69)$$

(2.68) may be solved algebraically [32] or iteratively [10]. One obvious iterative scheme is to apply the dominant right eigenspace iterations used for transforming (2.1) into (2.3). Since (2.66) satisfies (2.32) convergence is assured. Such an iteration would take the form

$$Q_{k+1} = (B + (A - BP)Q_k) \cdot (D + PB)^{-1}$$

$$Q_0 = 0$$

whichever method used, the resulting system is of the form

$$\begin{bmatrix} \dot{x} \\ \dot{w} \end{bmatrix} = \begin{bmatrix} A - BP & 0 \\ 0 & D + PB \end{bmatrix} \begin{bmatrix} x \\ w \end{bmatrix} \quad (2.70)$$

and the composite transformation is

$$\begin{bmatrix} y \\ z \end{bmatrix} = \begin{bmatrix} I & Q \\ -P & I - PQ \end{bmatrix} \begin{bmatrix} x \\ w \end{bmatrix} \quad (2.71)$$

which possesses the explicit inverse

$$\begin{bmatrix} x \\ w \end{bmatrix} = \begin{bmatrix} I - QP & -Q \\ P & I \end{bmatrix} \begin{bmatrix} y \\ z \end{bmatrix} \quad (2.72)$$

Thus, we have decomposed the y and z state vectors into their respective "fast" and "slow" components. Namely

$$y = x + Qw = y_{\text{slow}} + y_{\text{fast}} \quad (2.73)$$

$$z = -Px + (I - PQ)w = z_{\text{slow}} + z_{\text{fast}} \quad (2.74)$$

where

$$\begin{aligned} x(t) &= e^{(A - BP)t} x_0 & x_0 &= (I - QP)y_0 - Qz_0 \\ w(t) &= e^{(D + PB)t} w_0 & w_0 &= Py_0 + z_0 \end{aligned}$$

Such decompositions will become more important when we consider singularly perturbed systems in the next chapter. There, the fast and slow components take on the names of "boundary layer" and "steady state components".

Now consider form (2.3) and the transformation (2.24) used to obtain this form. The dichotomic solution matrix $\hat{P}$ is of the form

$$\hat{P} = MN^{-1} \quad (2.75)$$

Where the columns of $\begin{bmatrix} M \\ N \end{bmatrix}$ span the dominant right eigenspace of (2.1). Thus, the exact form of (2.3) is

$$\begin{bmatrix} \dot{x} \\ \dot{z} \end{bmatrix} = \begin{bmatrix} A - \hat{P}C & 0 \\ C & D + C\hat{P} \end{bmatrix} \begin{bmatrix} x \\ z \end{bmatrix} \quad (2.76)$$

Now, let $w = z + \hat{Q}x$. This transforms (2.76) into the form

$$\begin{bmatrix} \dot{x} \\ \dot{w} \end{bmatrix} = \begin{bmatrix} A - \hat{P}C & 0 \\ \hat{Q}(A - \hat{P}C) - (D + C\hat{P})\hat{Q} + C & D + C\hat{P} \end{bmatrix} \begin{bmatrix} x \\ w \end{bmatrix} \quad (2.77)$$

Thus we seek $\hat{Q}$ to satisfy the Lyapunov type equation

$$\hat{Q}(A - \hat{P}C) - (D + \hat{C}\hat{P})\hat{Q} + C = 0 \quad (2.78)$$

such a $\hat{Q}$ will always exist since

$$\sigma(A - \hat{P}C) \cap \sigma(D + \hat{C}\hat{P}) = \emptyset \quad (2.79)$$

Again, (2.78) may be solved iteratively or algebraically. Applying the dominant left eigenspace iterations to (2.76) convergence is assured. This iteration takes the form

$$\hat{Q}_{k+1} = (D + \hat{C}\hat{P})^{-1} \cdot (C + \hat{Q}_k(A - \hat{P}C)) \quad \hat{Q}_0 = D^{-1}C \quad (2.80)$$

The resulting system is of the form

$$\begin{bmatrix} \dot{x} \\ \dot{w} \end{bmatrix} = \begin{bmatrix} A - \hat{P}C & 0 \\ 0 & D + \hat{C}\hat{P} \end{bmatrix} \begin{bmatrix} y \\ z \end{bmatrix} \quad (2.81)$$

and the composite transformation is

$$\begin{bmatrix} \dot{x} \\ w \end{bmatrix} = \begin{bmatrix} I & -\hat{P} \\ \hat{Q} & I - \hat{Q}\hat{P} \end{bmatrix} \begin{bmatrix} y \\ z \end{bmatrix} \quad (2.82)$$

with also possesses the explicit inverse

$$\begin{bmatrix} y \\ z \end{bmatrix} = \begin{bmatrix} I - \hat{P}\hat{Q} & \hat{P} \\ -\hat{Q} & I \end{bmatrix} \begin{bmatrix} x \\ w \end{bmatrix} \quad (2.83)$$

Thus, we have again decomposed the y and z state vectors into "fast" and "slow" components. Namely

$$y = (I - \hat{P}\hat{Q})x + \hat{P}w = \hat{y}_{\text{slow}} + \hat{y}_{\text{fast}} \quad (2.84)$$

$$z = -\hat{Q}x + w = \hat{z}_{\text{slow}} + \hat{z}_{\text{fast}} \quad (2.85)$$

where

$$\begin{aligned} x(t) &= e^{(A - \hat{P}\hat{C})t} x_0 & x_0 &= y_0 - \hat{P}z_0 \\ w(t) &= e^{(D + \hat{C}\hat{P})t} w_0 & w_0 &= \hat{Q}y_0 + (I - \hat{Q}\hat{P})z_0 \end{aligned}$$

The relationships between various fast and slow components of (2.1) will be an important topic of the next chapter.

2.4. Ordering of State Variables

In the results of the previous two sections it was always assumed that there existed an ordering of states such that D^{-1} existed as well as N^{-1} in (2.39) and (2.54). While [37] guarantees the existence of such an ordering, until recently no systematic algorithm was available to achieve this ordering of states. [34] has developed a "grouping" algorithm for the area decomposition of power networks. By applying this algorithm to the left or right dominant eigenspace of (2.1) the necessary ordering of states can be obtained.

Let F be the system matrix in (2.1) and let V correspond to the matrix of right eigenvectors. Thus, $F = V\Lambda V^{-1}$ where Λ is the eigenvalue matrix. If $\dot{x} = Fx$, and we let $x = Vy$, then

$$x(t) = Ve^{\Lambda t}y(0). \quad (2.86)$$

Partition V as $V = [V_f : V_s]$ where V_f and V_s are the right eigenvectors corresponding to the fast and slow spectrum respectively. Each row i of V_f "weighs" the contribution of the fast modes to state x_i . If V_f has m columns, we want the m rows of V_f that are the most linearly independent to correspond to our fast states. This can be done by performing a Gaussian

elimination with complete pivoting [34] on the rows of V_f . If we call V_f^2 the m most linearly independent rows of V_f and permute the states such that

$$V_f = \begin{bmatrix} V_f^1 \\ V_f^2 \end{bmatrix}. \quad (2.87)$$

Then since $FV_f = V_f \Lambda_f$, we obtain

$$D + C(V_f^1(V_f^2)^{-1}) = V_f^2 \Lambda_f (V_f^2)^{-1} \quad (2.88)$$

where Λ_f is the dominant eigenvalue matrix of dimension m . By this construction, $V_f^2 = N$ in (2.54) and the proof of dominant right eigenspace iterations follows.

Now partition V^{-1} as $\begin{bmatrix} W_f \\ W_s \end{bmatrix}$ where W_f and W_s are the left eigenvectors corresponding to the fast and slow spectrums of F respectively.

Again, since W_f has m rows, we want the m columns of W_f that are the most linearly independent to correspond to our fast states. This again can be done using Gaussian elimination. If we call W_f^2 the m most linearly independent columns of W_f and permute the states such that

$$W_f = [W_f^1 : W_f^2]. \quad (2.89)$$

Then since $W_f F = \Lambda_f W_f$, we obtain

$$D + (W_f^2)^{-1} W_f^1 B = (W_f^2)^{-1} \Lambda_f W_f^2. \quad (2.90)$$

Thus, N^{-1} will exist in (2.37) and the proof for the dominant left eigenspace iterations follows.

Thus, depending on whether you want to transform (2.1) into (2.2) or (2.3), the ordering of states can possibly be different using the above methods. However, in general, the state orderings resulting from these methods are not the only such orderings which possess the desired properties. In many cases a given ordering of states will satisfy the conditions necessary for the application of both left and right eigenspace iteration. The application of the "grouping" algorithm merely assures us that there exists orderings of states which satisfy the conditions assumed in the lemmas. In many cases, such as singularly perturbed models, the proper ordering of states can be done by inspection.

2.5. Example - Decomposition of States into Fast and Slow Components

In [39], the 8th order model of an isolated mixed power system is given as

$$\dot{x} = \begin{bmatrix} -.2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 4.75 & -5 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & .16667 & -.16667 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 2 & -2 & 0 & 0 & 0 & 0 \\ 0 & -.08 & -.07467 & -.112 & -3.9944 & 10 & -.92778 & -9.1 \\ 0 & 0 & 0 & 0 & .2 & -.5 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & -1.39 & -.278 \\ 0 & .01 & .0093 & .014 & -.06319 & 0 & .11597 & -.112361 \end{bmatrix} x \quad (2.91)$$

using a permutation of $x = Py$

$$P = (e_1, e_3, e_6, e_8, e_5, e_4, e_7, e_2)$$

gives

$$\dot{y} = \begin{bmatrix} -.2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & -.167 & 0 & 0 & 0 & 0 & 0 & .167 \\ 0 & 0 & -.5 & 0 & .2 & 0 & 0 & 0 \\ 0 & .009 & 0 & -.112 & -.063 & .014 & .116 & .01 \\ 0 & -.075 & 10.00 & -9.101 & -3.994 & -.112 & -.927 & -.08 \\ 0 & 2.00 & 0 & 0 & 0 & -2.00 & 0 & 0 \\ 0 & 0 & 0 & -.277 & 1.319 & 0 & -1.386 & 0 \\ 4.75 & 0 & 0 & 0 & 0 & 0 & 0 & -5.00 \end{bmatrix} y$$

(2.92)

The eigenvalues of (2.92) are

$$\begin{array}{ll} -1.3884147 & + \quad 0.0000000J \\ -0.1291288 & + \quad 0.2124795J \\ -0.1291288 & - \quad 0.2124795J \\ -4.3489879 & + \quad 0.0000000J \\ -2.0000000 & + \quad 0.0000000J \\ -0.1666700 & + \quad 0.0000000J \\ -5.0000000 & + \quad 0.0000000J \\ -0.2000000 & + \quad 0.0000000J \end{array} \quad (2.93)$$

For $n = m = 4$, we obtain an ε of .1792. Using the dominant left eigenspace iterations we obtain

$$A - P_0 B = \begin{bmatrix} -0.20000 & 0.00000 & 0.00000 & 0.00000 \\ 0.15834 & -0.16667 & 0.00000 & 0.00000 \\ -0.00312 & -0.00766 & -0.08981 & -0.36571 \\ 0.00877 & 0.02153 & 0.09635 & -0.22145 \end{bmatrix} \quad (2.94)$$

which has eigenvalues

$$\begin{aligned}
 & -0.15563 + 0.17579J \\
 & -0.15563 - 0.17579J \\
 & -0.16667 + 0.00000J \\
 & -0.20000 + 0.00000J
 \end{aligned} \tag{2.95}$$

$$D + BP_o = \begin{bmatrix} -4.52014 & -0.08640 & -0.71572 & -0.05533 \\ 0.00000 & -2.00000 & 0.00000 & -0.16667 \\ 0.80733 & 0.02712 & -1.16426 & 0.02543 \\ 0.00000 & 0.00000 & 0.00000 & -5.00000 \end{bmatrix} \tag{2.96}$$

which has eigenvalues

$$\begin{aligned}
 & -4.33808 + 0.00000J \\
 & -1.34632 + 0.00000J \\
 & -2.00000 + 0.00000J \\
 & -5.00000 + 0.00000J
 \end{aligned} \tag{2.97}$$

using the dominant right eigenspace iterations we get

$$A - C\hat{P}_o = \begin{bmatrix} -0.20000 & 0.00000 & 0.00000 & 0.00000 \\ 0.15834 & -0.16667 & 0.00000 & 0.00000 \\ -0.00312 & -0.00766 & -0.08981 & -0.36571 \\ 0.00877 & 0.02153 & 0.09635 & -0.22145 \end{bmatrix} \tag{2.98}$$

which has eigenvalues

$$\begin{aligned}
 & -0.15563 + 0.17579J \\
 & -0.15563 - 0.17579J \\
 & -0.16667 + 0.00000J \\
 & -0.20000 + 0.00000J
 \end{aligned} \tag{2.99}$$

$$D + \hat{P}_o C = \begin{bmatrix} -4.31691 & -0.03023 & 0.04757 & -0.05415 \\ 0.00000 & -2.00000 & 0.00000 & -0.06667 \\ 1.32208 & 0.00179 & -1.36749 & 0.00051 \\ 0.00000 & 0.00000 & 0.00000 & -5.00000 \end{bmatrix} \tag{2.100}$$

Which has eigenvalues

$$\begin{aligned}
 &-4.33808 + 0.00000J \\
 &-1.34632 + 0.00000J \\
 &-2.00000 + 0.00000J \\
 &-5.00000 + 0.00000J
 \end{aligned} \tag{2.101}$$

To show how this accuracy may be improved, after two iterations we obtain

$$A - P_1 B = \begin{bmatrix} -0.20000 & 0.00000 & 0.00000 & 0.00000 \\ 0.16492 & -0.16667 & 0.00000 & 0.00000 \\ -0.00233 & -0.00741 & -0.08420 & -0.36234 \\ 0.00979 & 0.02836 & 0.13210 & -0.17633 \end{bmatrix} \tag{2.102}$$

with eigenvalues

$$\begin{aligned}
 &-0.13027 + 0.21388J \\
 &-0.13027 - 0.21388J \\
 &-0.16667 + 0.00000J \\
 &-0.20000 + 0.00000J
 \end{aligned} \tag{2.103}$$

$$D + B P_1 = \begin{bmatrix} -4.52468 & -0.08664 & -0.71767 & -0.05571 \\ 0.00000 & -2.00000 & 0.00000 & -0.18172 \\ 0.76779 & 0.02154 & -1.21045 & 0.01334 \\ 0.00000 & 0.00000 & 0.00000 & -5.00000 \end{bmatrix} \tag{2.104}$$

with eigenvalues

$$\begin{aligned}
 &-4.34912 + 0.00000J \\
 &-1.38601 + 0.00000J \\
 &-2.00000 + 0.00000J \\
 &-5.00000 + 0.00000J
 \end{aligned} \tag{2.105}$$

$$A - \hat{C} P_1 = \begin{bmatrix} -0.20000 & 0.00000 & 0.00000 & 0.00000 \\ 0.16379 & -0.16667 & 0.00000 & 0.00000 \\ -0.00287 & -0.00609 & 0.00444 & -0.45814 \\ 0.00821 & 0.02283 & 0.13946 & -0.26497 \end{bmatrix} \tag{2.106}$$

with eigenvalues

$$\begin{array}{rcl}
 -0.13027 & + & 0.21388J \\
 -0.13027 & - & 0.21388J \\
 -0.16667 & + & 0.00000J \\
 -0.20000 & + & 0.00000J
 \end{array} \quad (2.107)$$

$$D + \hat{P}_1 C = \begin{bmatrix} -4.37192 & -0.03420 & -0.05145 & -0.05565 \\ 0.00000 & -2.00000 & 0.00000 & -0.06896 \\ 1.32327 & 0.00202 & -1.36321 & 0.00048 \\ 0.00000 & 0.00000 & 0.00000 & -5.00000 \end{bmatrix} \quad (2.108)$$

with eigenvalues

$$\begin{array}{rcl}
 -4.34912 & + & 0.00000J \\
 -1.38601 & + & 0.00000J \\
 -2.00000 & + & 0.00000J \\
 -5.00000 & + & 0.00000J
 \end{array} \quad (2.109)$$

We now give graph of selected states along with their fast and slow components using (1.85), (1.86), (1.96) and (1.97) for both the left and right eigenspace decompositions. The plots will be based on the P_0 and $\hat{P}_0$ iterates. On the graphs of the individual components, the following legend will be in effect

ACTUAL STATE _____
 SLOW COMPONENT -----
 FAST COMPONENT

On the graphs of the actual state versus the approximated state

ACTUAL STATE _____
 APPROXIMATED STATE

The plots appear on the next several pages. The system is perturbed

with an initial state vector of

$$\mathbf{x}_0^T = (1, 2, 3, -2, 1, -1, 4, 2) . \quad (2.110)$$

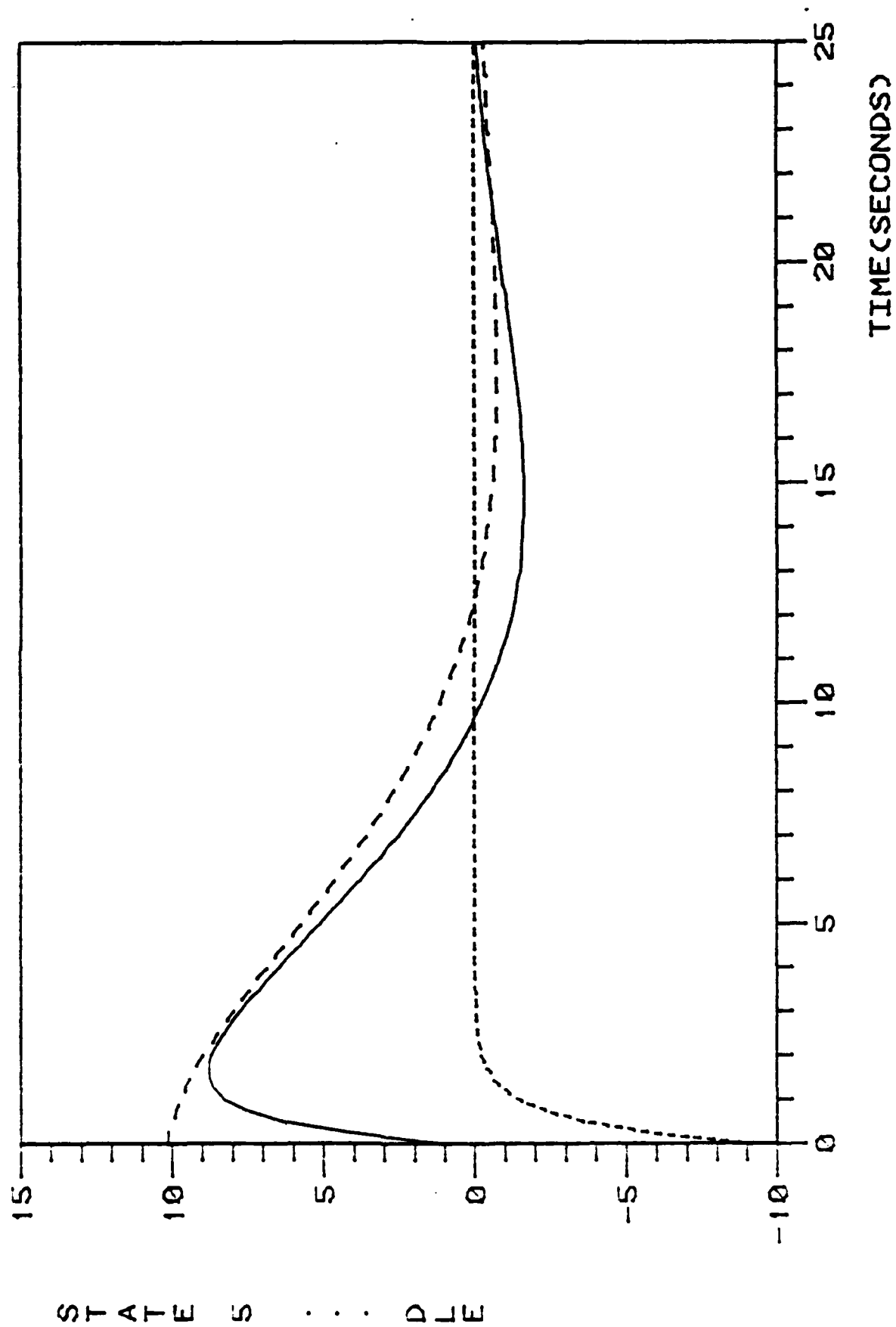


Figure 1.1. State 5 and components using left eigenspace iterations.

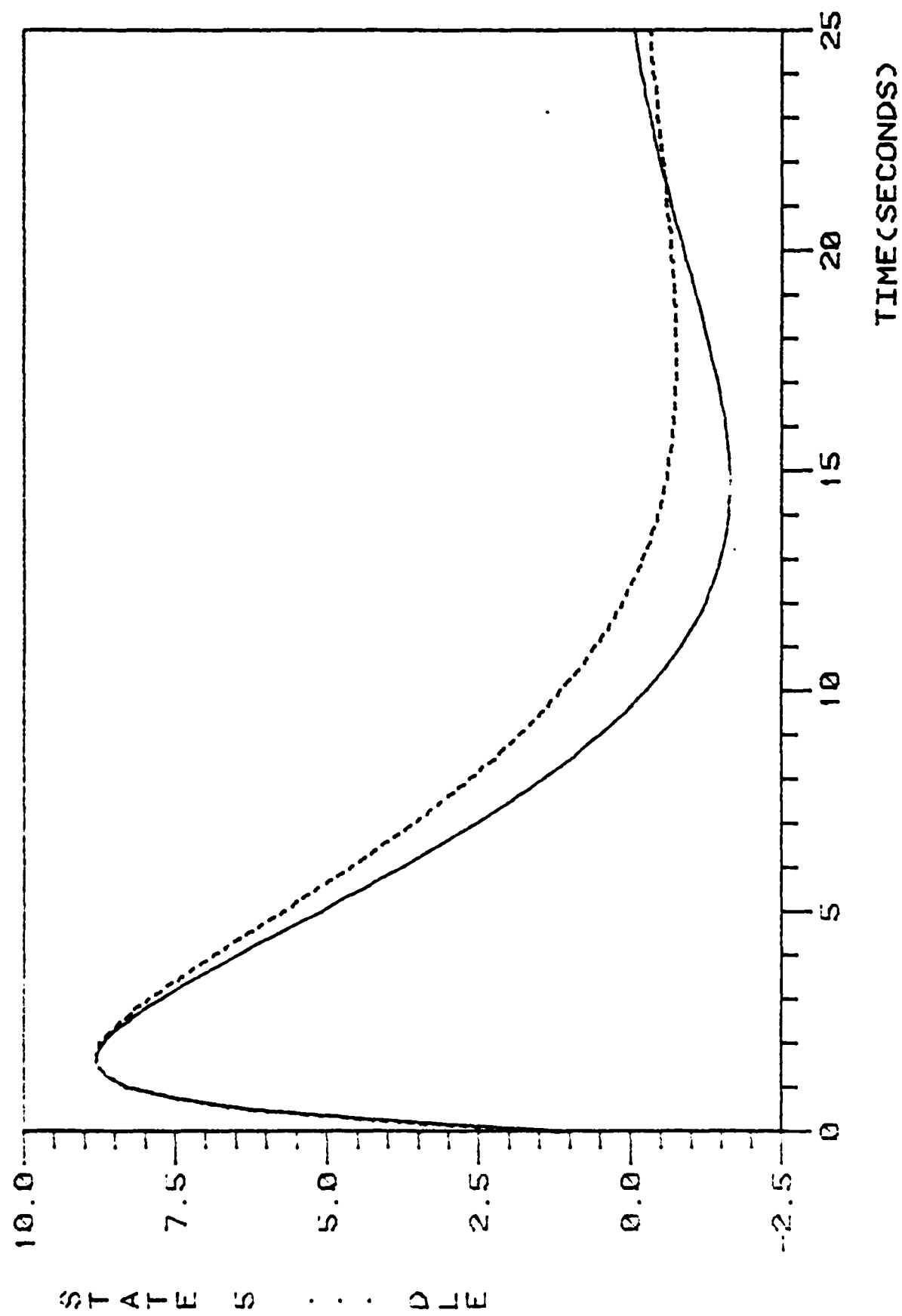


Figure 1.2. State 5 and added components using left eigenspace iterations.

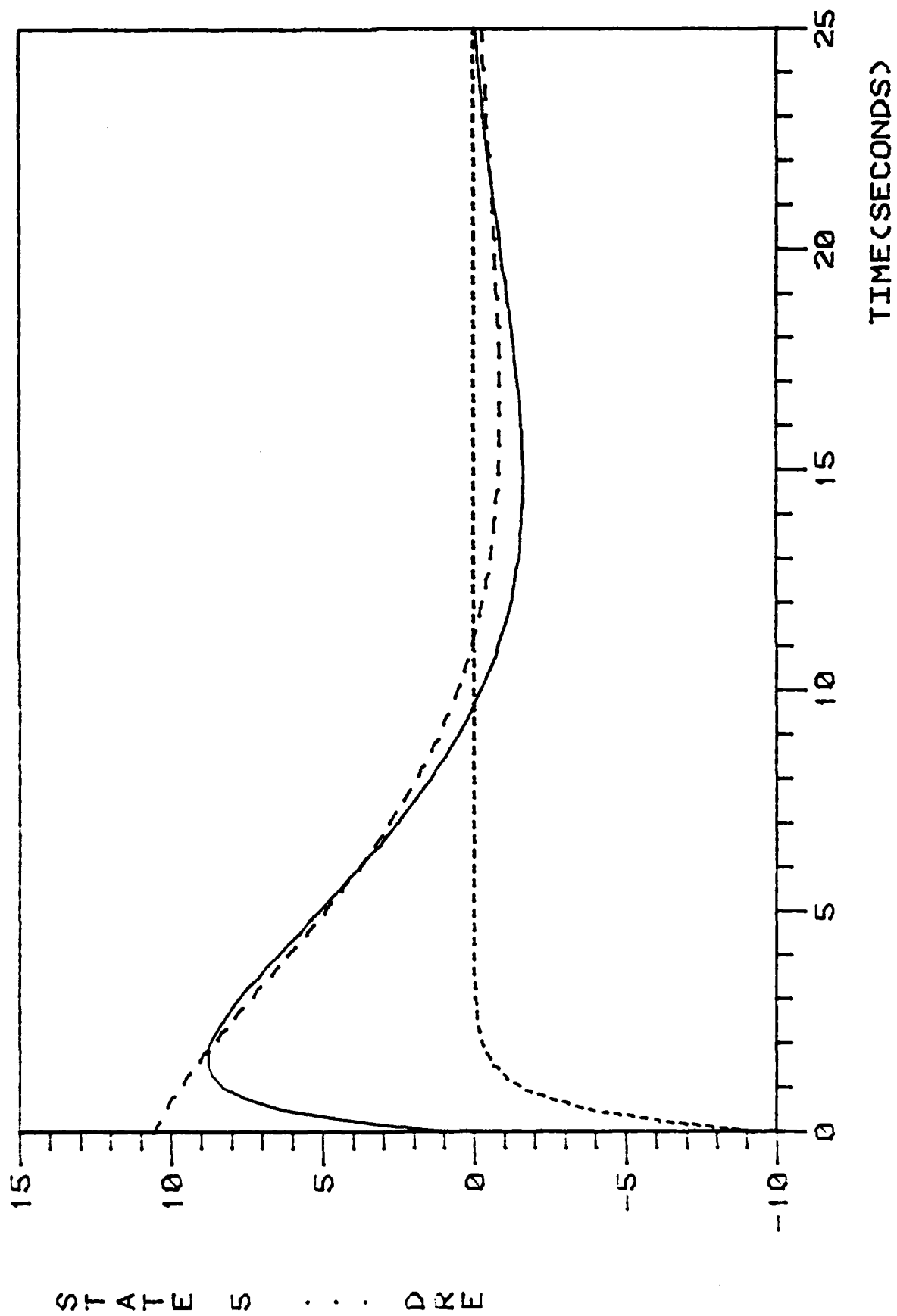


Figure 1.3. State 5 and components using right eigenspace iterations.

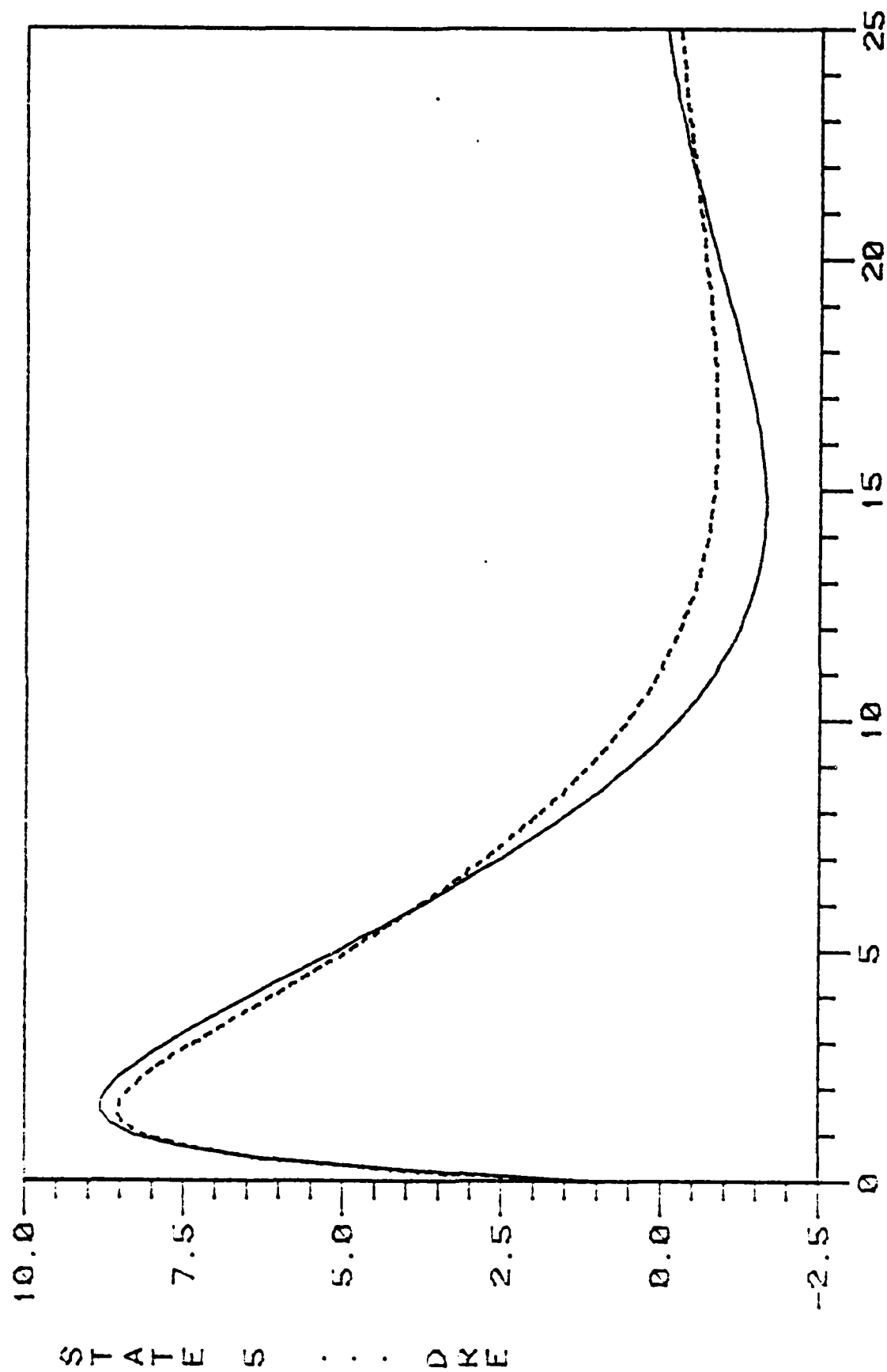


Figure 1.4. State 5 and added components using right eigenspace iterations.

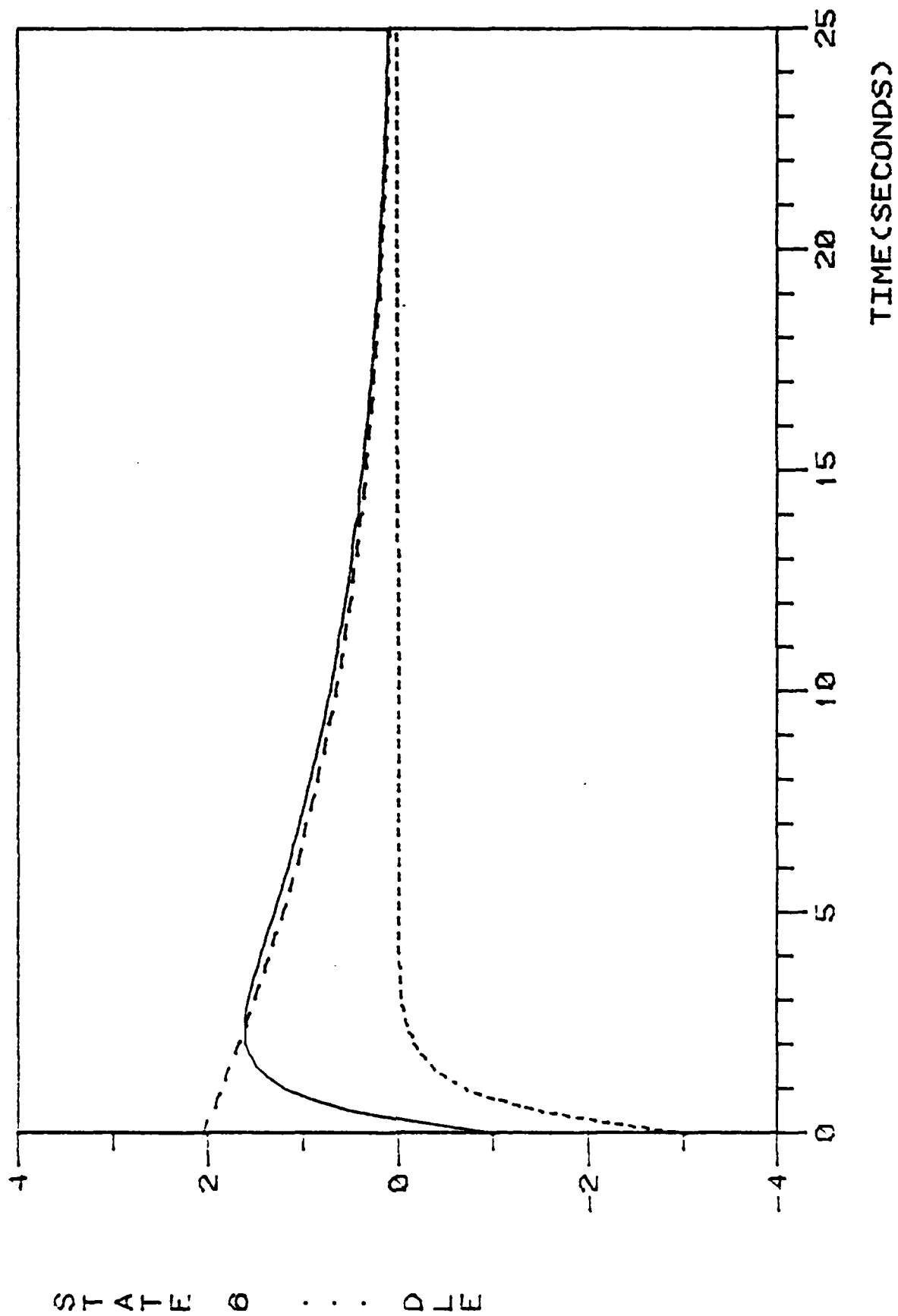


Figure 1.5. State 6 and components using left eigenspace iterations.

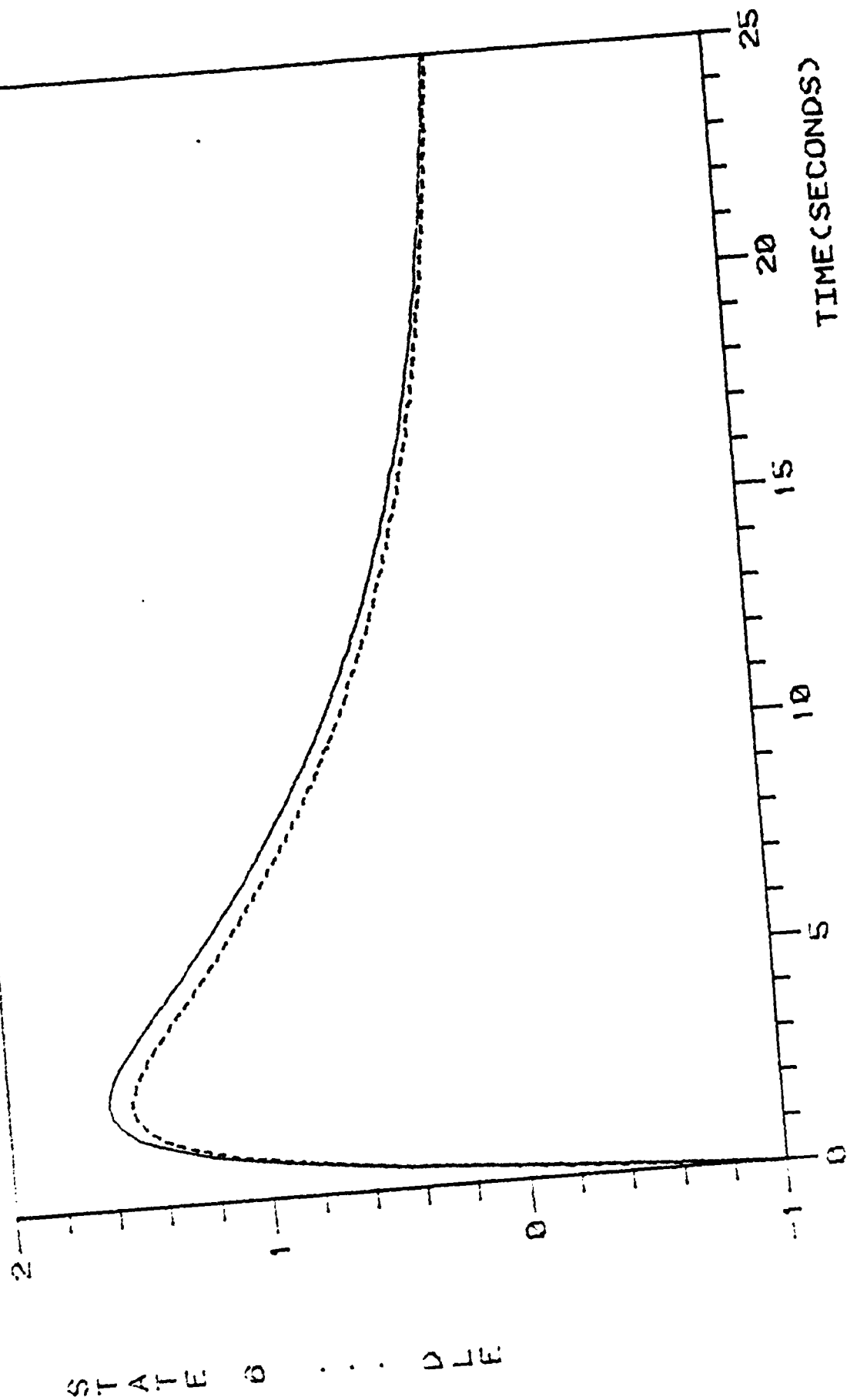


Figure 1.6. State 6 and added components using left eigenspace iterations.

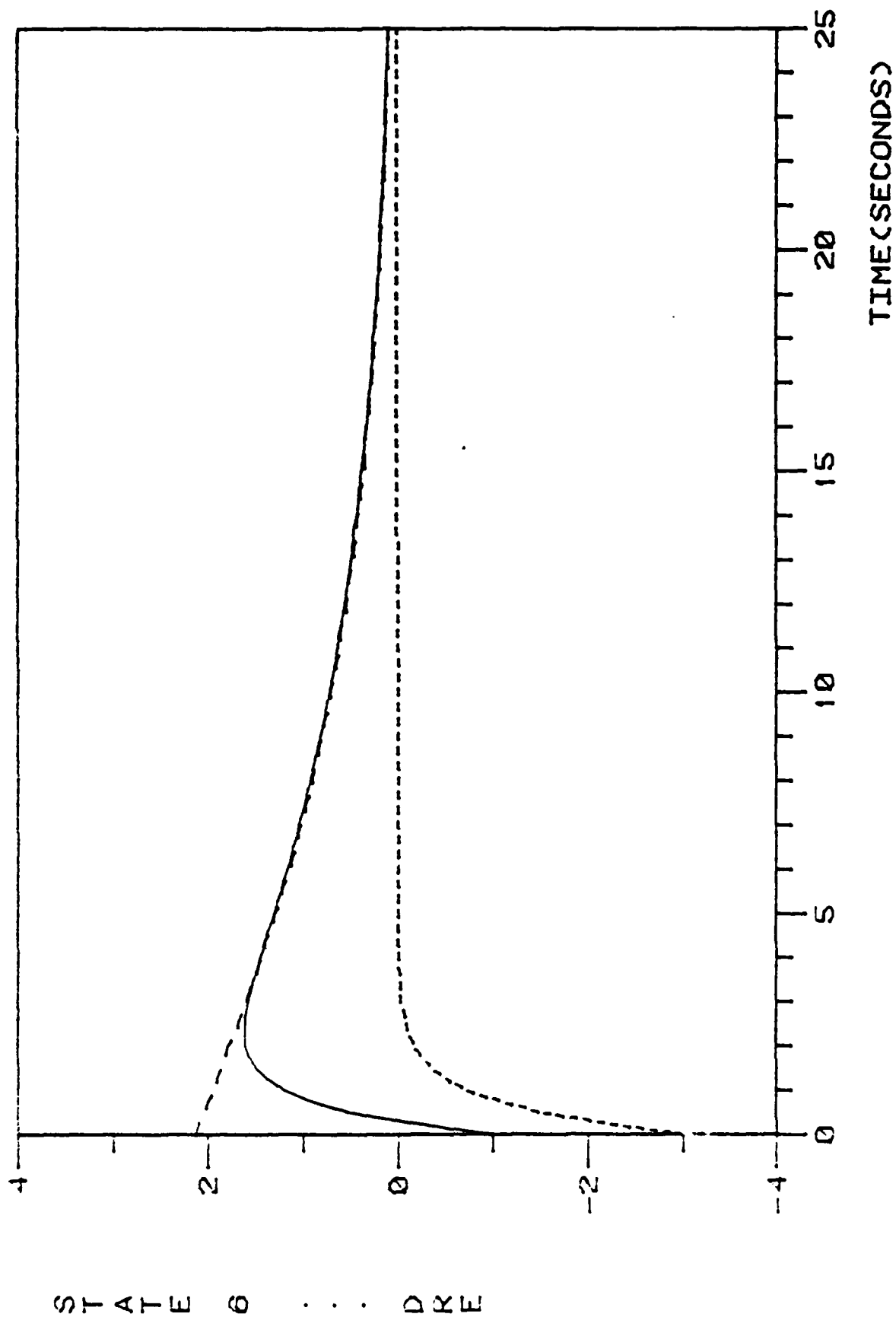


Figure 1.7. State 6 and components using right eigenspace iterations.

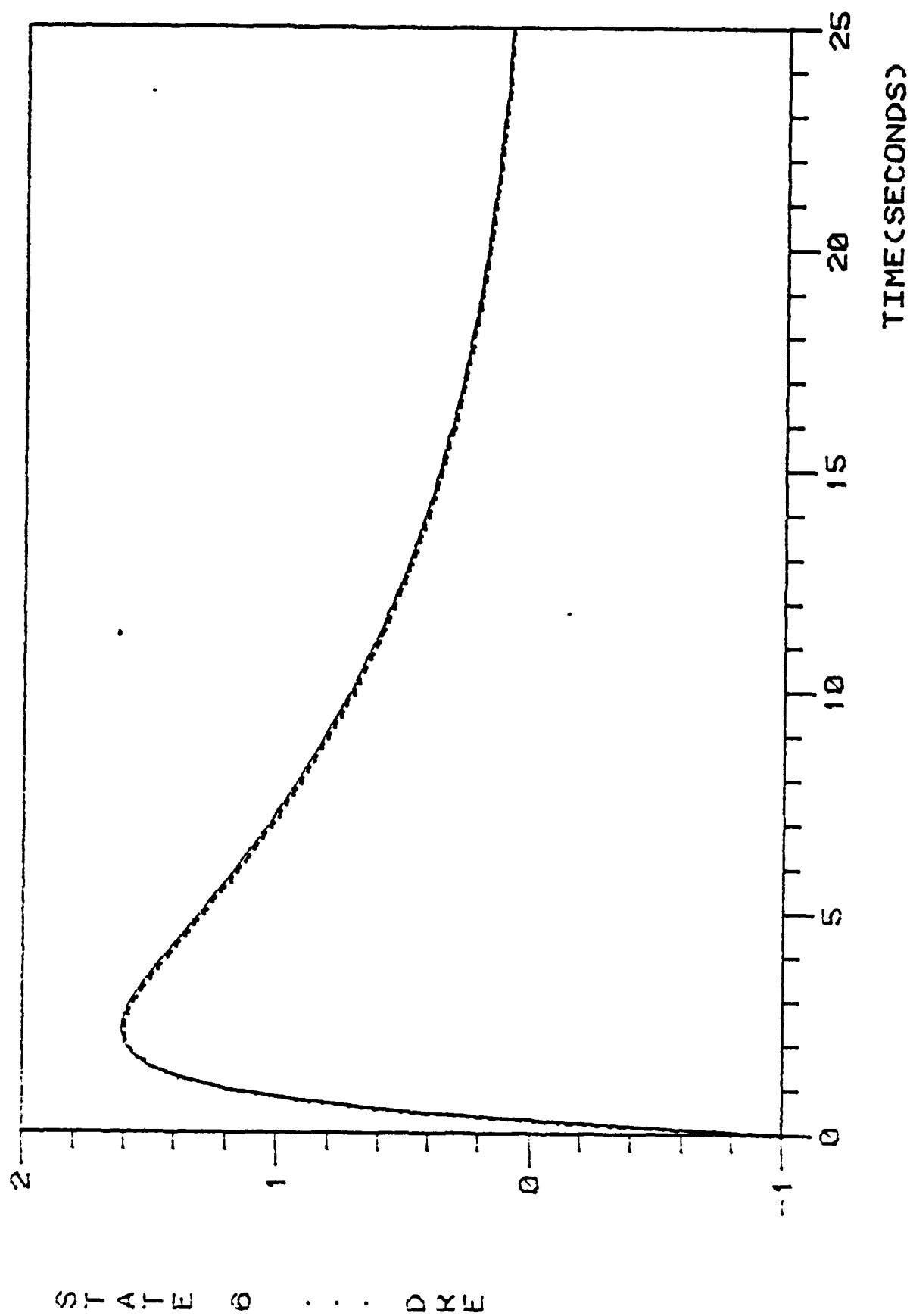


Figure 1.8. State 6 and added components using right eigenspace iterations.

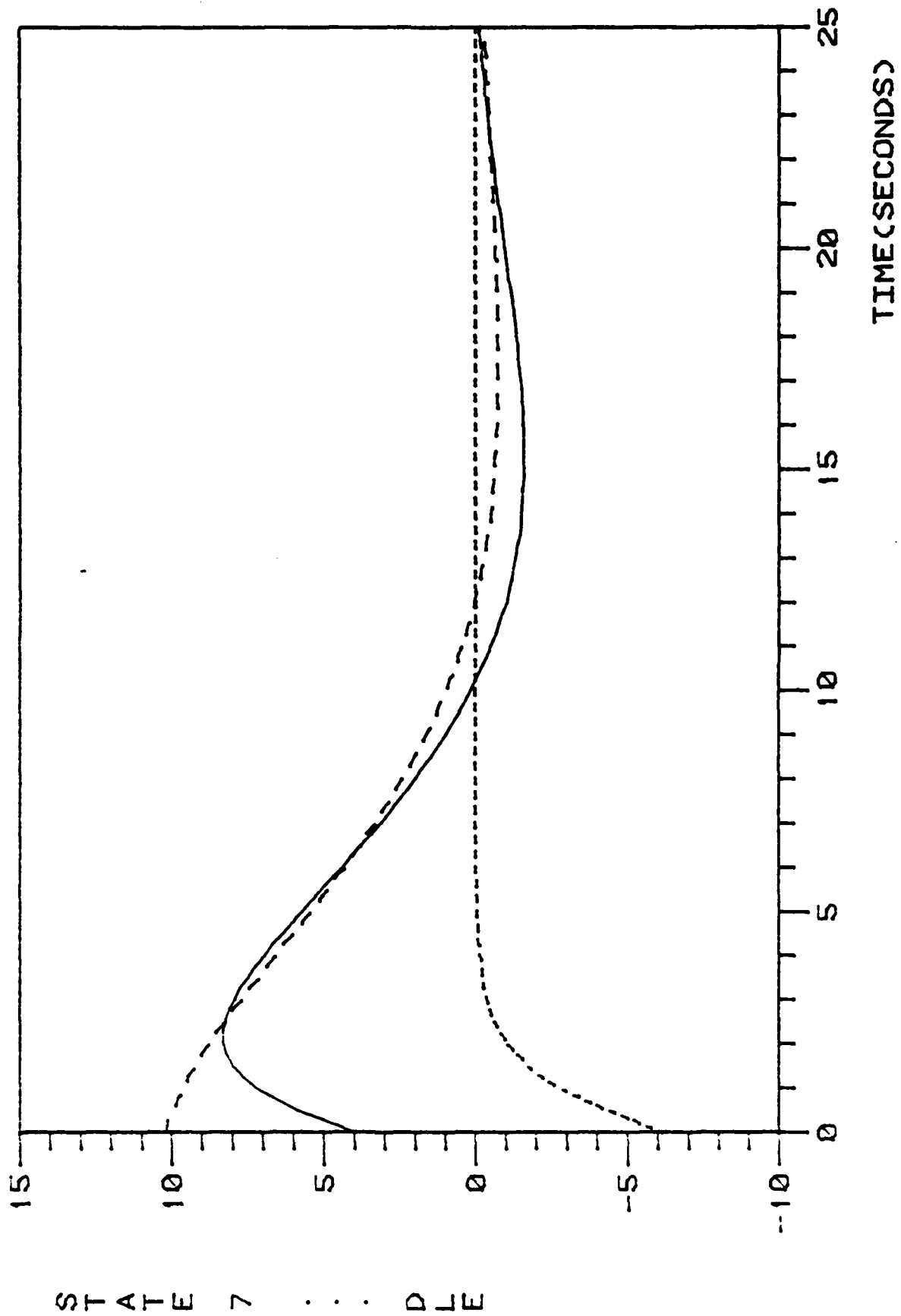


Figure 1.9. State 7 and components using left eigenspace iterations.

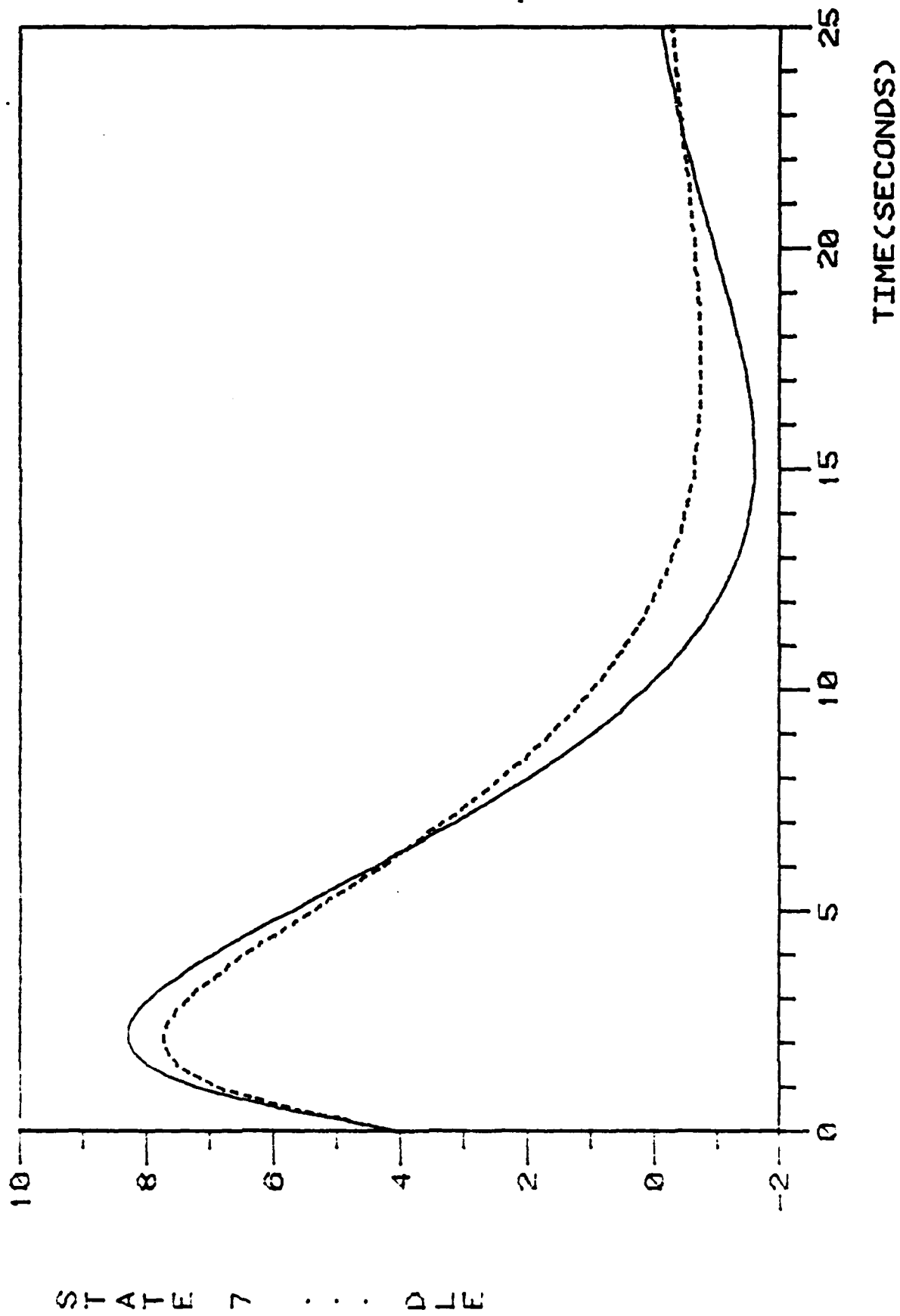


Figure 1.10. State 7 and added components using left eigenspace iterations.

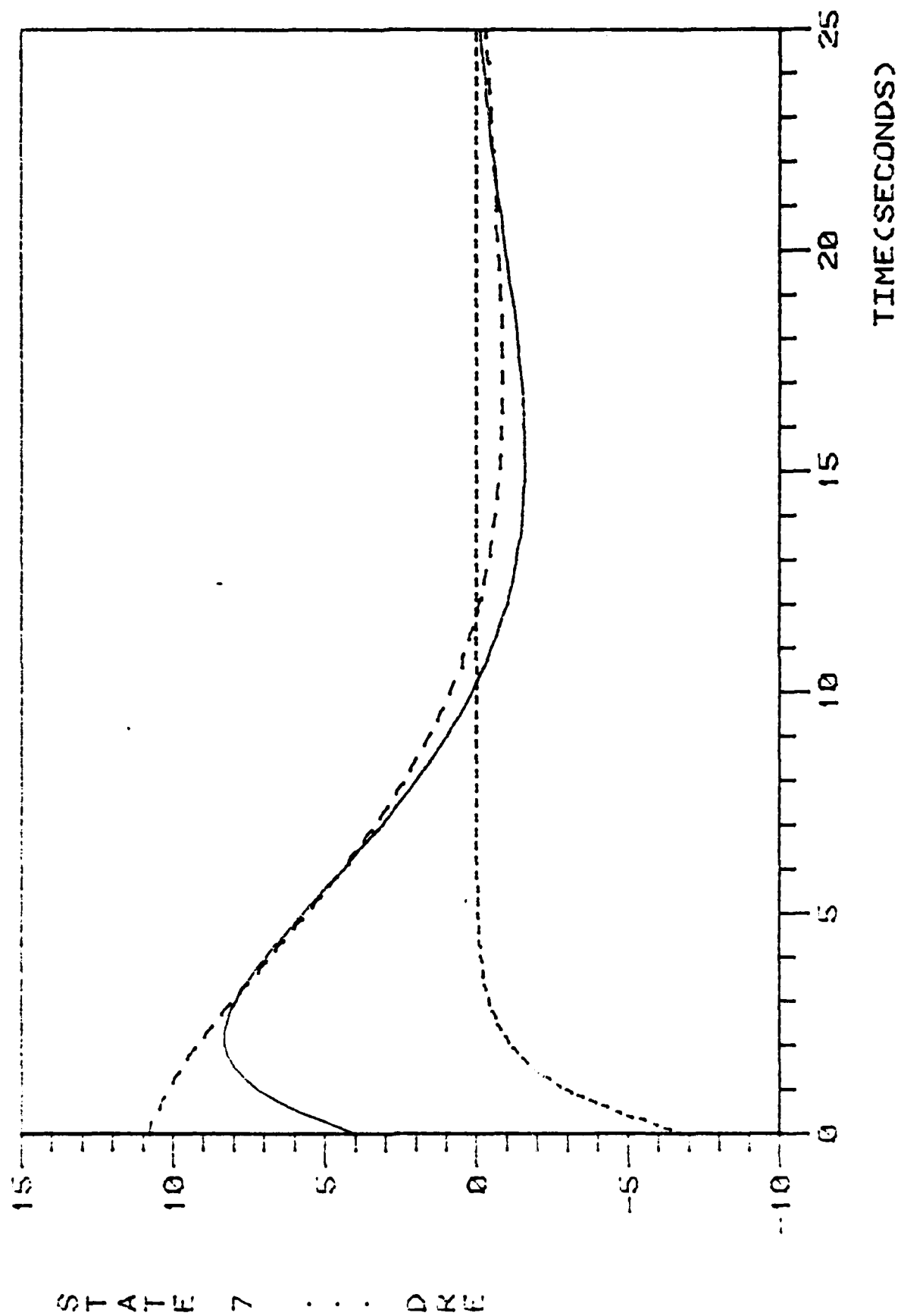


Figure 1.11. State 7 and components using right eigenspace iterations.

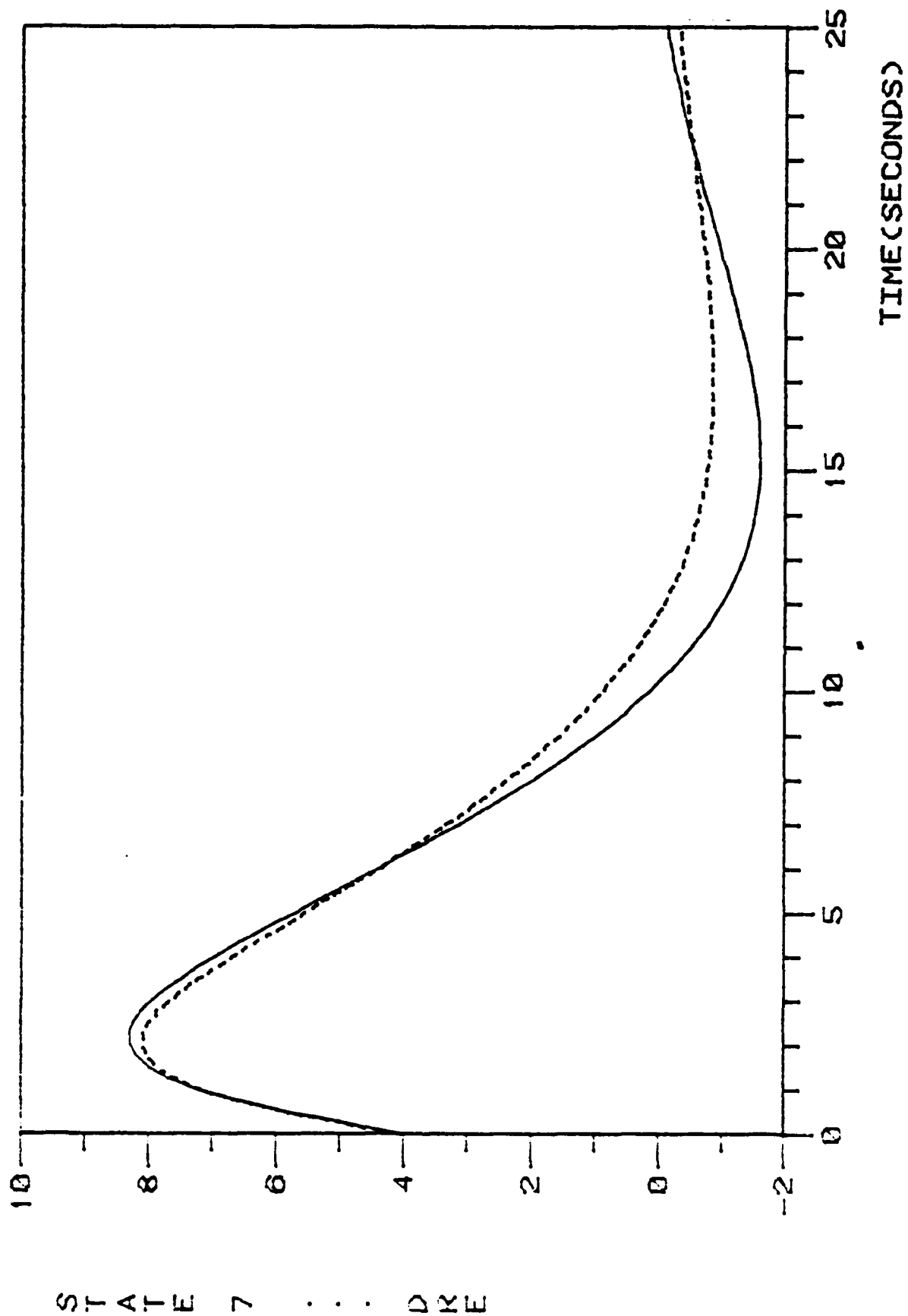


Figure 1.12. State 7 and added components using right eigenspace iterations.

CHAPTER 3

ASYMPTOTIC SERIES DECOMPOSITION OF TIME-SCALES
IN LINEAR TIME-INVARIANT SYSTEMS3.1. Introduction

When small parameters are present in differential equations defining initial or boundary value problems, one of the popular methods used [30] is to obtain an asymptotic power series expansion of the solution. Such techniques have been well documented and can produce approximate solutions to problems where otherwise explicit analytic solutions are impossible or exact numerical solutions are computationally not practical. Such systems are of the form

$$\dot{x} = f(x, t, \epsilon) \quad x(0) = x_0 \quad (3.1)$$

and a solution is sought of the form

$$x(t) = x^0(t) + \epsilon x^1(t) + \dots \quad (3.2)$$

When such an expansion converges uniformly in t as $\epsilon \rightarrow 0$ we have a regular perturbation problem [8]. If there is a region of nonuniformity, usually at one of the boundaries, we have a singular perturbation problem. In most cases, the dynamics of the solution vector within this region of nonuniform convergence involve fast transients or the so called "boundary layer phenomena." Thus, such singularly perturbed systems [8] are said to possess an inherent two-time-scale property characterized by a steady state or "outer solution" which is defined by the regions of uniform convergence of (3.2), and the boundary layer or "inner solution" where a stretched time variable is usually introduced in order to achieve asymptotic convergence on the total time interval. In the linear case such systems take the form

$$\begin{aligned} \dot{y} &= Ay + Bz & y(0) &= y_0 \\ \mu \dot{z} &= Cy + Dz & z(0) &= z_0 \end{aligned} \tag{3.3}$$

Much work has been done to exploit the multiple time scale property of (3.3) when designing regulators, pole placement, reduced order modeling, etc. [3,7,40]. As a result, for a system which is known to have fast and slow phenomena, the systems engineer is motivated to permute the state in order to attain the above structure and take advantage of these decomposition techniques. It is the purpose of this chapter to use multiple time scale asymptotic expansions to obtain a "steady state" and "boundary layer" decomposition in (3.3) and show how this decomposition compares to the eigenspace decompositions of Chapter 2.

In Section 3.2 we obtain power series representations of our dichotomic transformation matrices P and $\hat{P}$.

In Section 3.3 we derive important relationships between various fundamental sets of solutions to (3.3) and any system satisfying (2.32). These fundamental sets are based on our reduced order subsystem matrices and the dichotomic transformation matrices P , $\hat{P}$, Q , and $\hat{Q}$.

In Section 3.4 we use Vasil'eva's method of matched asymptotic expansions to obtain the "boundary layer" and "steady state" components of the solution vectors $y(t)$ and $z(t)$. It is shown that this decomposition is equivalent to the eigenspace decompositions of Chapter 2 by using one of the fundamental solution sets established in Section 3.3.

Section 3.5 discusses some computational simplifications to the dominant left and right eigenspace iterations based on system (3.3). The

simplifications involve eliminating the necessity to take an inverse at every iteration.

Finally, in Section 3.6 we discuss an important application of these techniques, namely, reduced order control law design.

3.2. Series Solutions to Riccati Iterations

For proper spectral decomposition and dimensions m and n in (2.1), it was shown in Section 2.2 of Chapter 2, that the following matrix recursion equation for finding the dominant left eigenspace of (2.1)

$$P_{k+1} = -P_k - (D + P_k B)^{-1} \cdot (DP_k - C + P_k B P_k - P_k A) \quad (3.4)$$

$$P_0 = D^{-1}C$$

will converge to the dichotomic solution of

$$R(P) = DP - C + PBP - PA = 0. \quad (3.5)$$

When (2.1) is in the form of a singularly perturbed model, (3.4) and (3.5) become

$$P_{k+1} = P_k - (D + \epsilon P_k B)^{-1} \cdot (DP_k - C + \epsilon P_k B P_k - \epsilon P_k A) \quad (3.6)$$

and

$$R(P) = DP - C + \epsilon PBP - \epsilon PA = 0 \quad (3.7)$$

respectively. Approximate solutions to (3.7) have been used in [6,7,26] to construct near optimal control laws for singularly perturbed systems.

In this section we construct an asymptotic approximation (to N terms) of the matrix function $P(\epsilon)$ as $\epsilon \rightarrow 0$ with respect to the asymptotic sequence

$\{\mu^k\}$. Thus, we seek a solution to (3.7) of the form

$$P(\mu) = P^0 + \mu P^1 + \dots \quad (3.8)$$

Substituting (3.8) into (3.7) and matching separately like powers of μ to zero we obtain

$$DP^0 - C = 0 \quad (3.9)$$

$$DP^1 + P^0 BP^0 - P^0 A = 0 \quad (3.10)$$

$$\begin{aligned} &\vdots \\ DP^N + \sum_{j=0}^{N-1} P^{N-1-j} BP^j - P^{N-1} A &= 0 \end{aligned} \quad (3.11)$$

Hence, each term in the series (3.8) is uniquely defined as follows:

$$P^0 = D^{-1}C \quad (3.12)$$

$$\begin{aligned} P^1 &= -D^{-1}P^0BP^0 + D^{-1}P^0A \\ &\vdots \\ P^N &= -D^{-1} \sum_{j=0}^{N-1} P^{N-1-j}BP^j + D^{-1}P^{N-1}A \end{aligned} \quad (3.13)$$

the asymptotic correctness of this series is obvious and we thus write (3.8) in the form

$$P(\mu) = \sum_{j=0}^N \mu^j P^j + O(\mu^{N+1}) \quad (3.14)$$

or using standard notation

$$P(\mu) \sim \sum_{j=0}^{\infty} \mu^j P^j \quad (3.15)$$

One question we might ask is how do the iterations of (3.6) relate to the individual terms in the series (3.14). After considerable algebra, it is possible to show that

$$\sum_{i=0}^N P^i - P_N = O(\mu^{N+1}) \quad (3.16)$$

for $N = 0, 1, \dots$

This is to be expected, since (3.6) converges to the dichotomic solution of (3.7) which, due to the uniqueness of (3.14) with respect to the asymptotic sequence $\{\mu^k\}$, must have this asymptotic expansion.

One area of research in the use of asymptotic expansions that has received little attention is nonlinear difference equations. If we interpret (3.6) with $P_0 = D^{-1}C$ as an initial value problem for a matrix system of nonlinear difference equations, we can construct an asymptotic series solution to (3.6) of the form

$$P_k = P_k^0 + \mu P_k^1 + \dots \quad (3.17)$$

Note that such an expansion converges as $\mu \rightarrow 0$ uniformly in k and thus defines a regular perturbation problem [8].

Expressing (3.6) in a more convenient form

$$(D + \mu P_k B) P_{k+1} = C + \mu P_k A. \quad (3.18)$$

We now substitute in the series (3.17) and obtain

$$\begin{aligned} & [(D + \mu(P_k^0 + \mu P_k^1 + \dots)B)(P_{k+1}^0 + \mu P_{k+1}^1 + \dots)] \\ & = C + \mu(P_k^0 + \mu P_k^1 + \dots)A. \end{aligned}$$

Equating like powers of μ we obtain the so called "equations of the variations" [13].

$$DP_{k+1}^0 = C \quad (3.19)$$

$$DP_{k+1}^1 + P_k^0 BP_{k+1}^0 = P_k^0 A \quad (3.20)$$

$$DP_{k+1}^N + \sum_{j=0}^{N-1} P_k^{N-1-j} BP_{k+1}^j = P_k^{N-1} A. \quad (3.21)$$

Since D^{-1} exists, we can solve (3.19) as

$$P_{k+1}^0 = D^{-1}C \quad (3.22)$$

and since our matching condition implies

$$P_0^0 = D^{-1}C \quad (3.23)$$

$$P_0^N = 0, \quad N > 0$$

The solution to (3.22) is thus,

$$P_k^0 = D^{-1}C \quad \forall k \geq 0$$

Likewise, the solution to (3.20) is found from

$$P_{k+1}^1 = -D^{-1}P_k^0 BP_{k+1}^0 + D^{-1}P_k^0 A.$$

However, since $P_{k+1}^0 = P_k^0 \quad \forall k \geq 0$

$$P_{k+1}^1 = -D^{-1}P_k^0 BP_k^0 + D^{-1}P_k^0 A = \text{constant matrix}.$$

Thus,

$$P_k^1 = 0 \quad k = 0$$

$$= -D^{-1}P_k^0 BP_k^0 + D^{-1}P_k^0 A \quad \forall k \geq 1. \quad (3.24)$$

Continuing in this manner, the equilibrium solution for the N^{th} variation is attained after N iterations and is of the form.

$$P_k^N = -D^{-1} \sum_{j=0}^{N-1} P_k^{N-1-j} B P_k^j + D^{-1} P_k^{N-1} A. \quad (3.25)$$

which is equivalent to (3.13) $\forall k \geq N$. While it is clear that (3.17) asymptotically solves (3.6) for any finite time interval, general stability and asymptotic correctness results for discrete time perturbation problems remains an open research area.

In a completely analogous manner, the series solution for the equilibrium of the dominant right eigenspace iterations

$$\hat{P}_{k+1} = \hat{P}_k + (\mu B + \mu A \hat{P}_k - \hat{P}_k D - \hat{P}_k C \hat{P}_k) \cdot (D + C \hat{P}_k)^{-1} \quad (3.26)$$

$$\hat{P}_0 = \mu B D^{-1}$$

has the form

$$\hat{P} = \hat{P}^0 + \mu \hat{P}^1 + \dots \quad (3.27)$$

where

$$\hat{P}^0 = 0 \quad (3.28)$$

$$\hat{P}^1 = BD^{-1} \quad (3.29)$$

$$\vdots$$

$$\hat{P}^N = \left(- \sum_{j=1}^{N-1} P^{N-j} C P^j \right) \cdot D^{-1} + A P^{N-1} D^{-1}. \quad (3.30)$$

Thus,

$$\hat{P}(\mu) = \sum_{j=0}^N \mu^j \hat{P}^j + O(\mu^{N+1}). \quad (3.31)$$

The series solutions (3.14) and (3.31) will be of importance in Section 3.4 when we obtain a two-time-scale asymptotic series solution to (3.3).

3.3. Fundamental Sets of Solutions

In this section we develop some basic properties relating the dichotomic dominant left and right eigenspace transformations of Chapter 2. The need for these properties will become apparent in the next section.

One of the basic properties of linear homogeneous systems of differential equations of the form

$$\dot{x} = Fx \quad (3.32)$$

is that a fundamental matrix [41] is of the form

$$K(t) = e^{Ft}. \quad (3.33)$$

For a given initial value problem $x(t_0) = x_0$, the solution to (3.32) for $t \geq t_0$ is uniquely given by

$$x(t) = X(t)X(t_0)^{-1}x_0 \quad (3.34)$$

or when using (3.33)

$$x(t) = e^{F(t-t_0)} x_0.$$

Another property of the fundamental matrix (3.33) is that given any nonsingular matrix M ,

$$Y(t) = e^{Ft} \cdot M \quad (3.35)$$

is also a fundamental matrix.

For system (2.1) satisfying condition (2.32) we have established transformation matrices P , Q , $\hat{P}$, and $\hat{Q}$ such that

$$\begin{bmatrix} A & B \\ C & D \end{bmatrix} = \begin{bmatrix} I & Q \\ -P & I-PQ \end{bmatrix} \begin{bmatrix} A-BP & 0 \\ 0 & D+PB \end{bmatrix} \begin{bmatrix} I-QP & -Q \\ P & I \end{bmatrix} \quad (3.36)$$

$$\begin{bmatrix} A & B \\ C & D \end{bmatrix} = \begin{bmatrix} I-\hat{P}\hat{Q} & \hat{P} \\ -\hat{Q} & I \end{bmatrix} \begin{bmatrix} A-\hat{P}C & 0 \\ 0 & D+\hat{C}\hat{P} \end{bmatrix} \begin{bmatrix} I & -\hat{P} \\ \hat{Q} & I-\hat{Q}\hat{P} \end{bmatrix}. \quad (3.37)$$

Thus,

$$e^{\begin{bmatrix} A & B \\ C & D \end{bmatrix} t} = \begin{bmatrix} I & Q \\ -P & I-PQ \end{bmatrix} \begin{bmatrix} e^{(A-BP)t} & 0 \\ 0 & e^{(D+PB)t} \end{bmatrix} \begin{bmatrix} I-QP & -Q \\ P & I \end{bmatrix} \quad (3.38)$$

is a fundamental matrix for (2.1).

Thus, by (3.35), so is

$$e^{-\begin{bmatrix} A & B \\ C & D \end{bmatrix}t} \begin{bmatrix} I & Q \\ -P & I-PQ \end{bmatrix} = \begin{bmatrix} I & Q \\ -P & I-PQ \end{bmatrix} e^{(A-BP)t} \begin{bmatrix} 0 \\ e^{(D+PB)t} \end{bmatrix} \quad (3.39)$$

or, the columns of

$$X(t) = \begin{bmatrix} e^{(A-BP)t} & Qe^{(D+PB)t} \\ -Pe^{(A-BP)t} & (I-PQ)e^{(D+PB)t} \end{bmatrix} \quad (3.40)$$

form a fundamental set for the system (2.1). Likewise, using a similar argument for (3.37), the columns of

$$X(t) = \begin{bmatrix} (I-\hat{P}\hat{Q})e^{(A-\hat{P}C)t} & \hat{P}e^{(D+\hat{C}\hat{P})t} \\ -\hat{Q}e^{(A-\hat{P}C)t} & e^{(D+\hat{C}\hat{P})t} \end{bmatrix} \quad (3.41)$$

Now, by the dichotomic property of the transformations, there exist nonsingular matrices T_1 , T_2 , T_3 and T_4 such that

$$T_1^{-1}(A-BP)T_1 = \Lambda_1 \quad (3.42)$$

$$T_2^{-1}(D+PB)T_2 = \Lambda_2 \quad (3.43)$$

$$T_3^{-1}(A-\hat{P}C)T_3 = \Lambda_1 \quad (3.44)$$

$$T_4^{-1}(D+\hat{C}\hat{P})T_4 = \Lambda_2 \quad (3.45)$$

where Λ_2 is the dominant eigenvalue matrix and Λ_1 is the eigenvalue matrix consisting of the rest of the spectrum of (2.1).

Using (3.42), (3.43), (3.44), and (3.45) in (3.36) and (3.37)

gives

$$\begin{bmatrix} A & B \\ C & D \end{bmatrix} = \begin{bmatrix} (I - \hat{P}\hat{Q})T_1 & \hat{P}T_2 \\ -\hat{Q}T_1 & T_2 \end{bmatrix} \begin{bmatrix} \Lambda_1 & 0 \\ 0 & \Lambda_2 \end{bmatrix} \begin{bmatrix} T_1^{-1} & -T_1^{-1}\hat{P} \\ T_2^{-1}\hat{Q} & T_2^{-1}(I - \hat{Q}\hat{P}) \end{bmatrix} \quad (3.46)$$

$$\begin{bmatrix} A & B \\ C & D \end{bmatrix} = \begin{bmatrix} T_3 & QT_4 \\ -PT_3 & (I - PQ)T_4 \end{bmatrix} \begin{bmatrix} \Lambda_1 & 0 \\ 0 & \Lambda_2 \end{bmatrix} \begin{bmatrix} T_3^{-1}(I - QP) & -T_3^{-1}Q \\ T_4^{-1}P & T_4^{-1} \end{bmatrix} \quad (3.47)$$

which, by definition identifies

$$\begin{bmatrix} (I - \hat{P}\hat{Q})T_1 & \hat{P}T_2 \\ -\hat{Q}T_1 & T_2 \end{bmatrix}, \begin{bmatrix} T_3 & QT_4 \\ -PT_3 & (I - PQ)T_4 \end{bmatrix} \quad (3.48)$$

as eigenvector matrices for (2.1).

While the magnitudes of eigenvectors are not unique, their directions are. Thus,

$$\begin{bmatrix} \hat{P}T_2 \\ T_2 \end{bmatrix} e^{\Lambda_2 t}, \begin{bmatrix} T_3 \\ -PT_3 \end{bmatrix} e^{\Lambda_1 t} \quad (3.49)$$

also serve as a fundamental set for (2.1), or in matrix form

$$\begin{bmatrix} T_3 e^{\Lambda_1 t} & \hat{P}T_2 e^{\Lambda_2 t} \\ -PT_3 e^{\Lambda_1 t} & T_2 e^{\Lambda_2 t} \end{bmatrix} \quad (3.50)$$

However, postmultiplying (3.50) by the nonsingular matrix

$$\begin{bmatrix} T_3^{-1} & 0 \\ 0 & T_2^{-1} \end{bmatrix} \quad (3.51)$$

gives

$$Y(t) = \begin{bmatrix} e^{(A-BP)t} & \hat{P}e^{(D+\hat{C}\hat{P})t} \\ -Pe^{(A-BP)t} & e^{(D+\hat{C}\hat{P})t} \end{bmatrix} \quad (3.52)$$

Also, by a similar argument on (3.46) and (3.47),

$$\hat{Y}(t) = \begin{bmatrix} (I-\hat{P}\hat{Q})e^{(A-\hat{P}\hat{C})t} & \hat{Q}e^{(D+\hat{P}\hat{B})t} \\ -\hat{Q}e^{(A-\hat{P}\hat{C})t} & (I-\hat{P}\hat{Q})e^{(D+\hat{P}\hat{B})t} \end{bmatrix} \quad (3.52)$$

also qualifies as a fundamental matrix for (2.1).

Thus, in this section we have established the existence of four fundamental matrices for (2.1) based on the dichotomic transformation matrices P , Q , $\hat{P}$ and $\hat{Q}$. This flexibility will prove valuable in the next section concerning asymptotic expansions of our singularly perturbed model (3.3).

3.4. Solution by Asymptotic Expansion Using the Method of Vasil'eva

Using the results established in the first two sections of this chapter, we will attempt to solve (3.3) using asymptotic expansion techniques.

In [13], the method of matched asymptotic expansions was proposed as a method of obtaining asymptotic solutions to the general nonlinear singularly perturbed initial value problem

$$\begin{aligned} \frac{dy}{dt} &= f(z, y, t) & y(0) &= y_0 \\ \mu \frac{dz}{dt} &= F(z, y, t) & z(0) &= z_0. \end{aligned} \quad (3.54)$$

To use this method, it is assumed that the root $z = \varpi(y, t)$ of the equation

$$F(y, z, t) = 0 \quad (3.55)$$

is stable in the first approximation or specifically, the real parts of the roots of the characteristic equation

$$\text{DET} \left(\frac{\partial F}{\partial z} \bigg|_{z=\varpi(y,t)} - \lambda I \right) = 0 \quad (3.56)$$

be negative in D , where D is a closed bounded domain in the variables $t_0 \leq t \leq t_1$, $|z| < K_1$, $|y| < K_2$, and $0 \leq \mu < \mu_0$. Under this assumption, the method can be applied to (3.3) as clearly carried out in [13].

In our case (3.54) reduces to

$$\begin{aligned} \frac{dy}{dt} &= Ay + Bz & y(0) &= y_0 \\ \mu \frac{dz}{dt} &= Cy + Dz & z(0) &= z_0 \end{aligned} \quad (3.57)$$

and the assumption becomes

$$\operatorname{Re}(\lambda_i) < 0 \quad \forall \lambda_i \in \sigma(D) \quad i=1, \dots, m$$

with this assumption satisfied, we can proceed with the asymptotic solution method proposed by Vasil'eva.

A solution to (3.57) is sought of the form

$$y = \bar{y} + \pi_y \quad (3.58)$$

$$z = \bar{z} + \pi_z \quad (3.59)$$

where

$$y = y_0(t) + \mu y_1(t) + \dots \quad (3.60)$$

denotes a formal power series in μ whose coefficients depend on t ,

and

$$\pi_y = \pi_0 y(\tau) + \mu \pi_1 y(\tau) + \dots \quad (3.61)$$

Denotes a formal power series in μ whose coefficients depend on $\tau = t/\mu$.

Substitution of (3.60), (3.61) and the analogous expansions for z into (3.57) yields

$$\mu \frac{d\bar{y}}{dt} + \frac{d}{d\tau} \pi_y = \mu A(\bar{y} + \pi_y) + \mu B(\bar{z} + \pi_z) \quad (3.62)$$

$$\mu \frac{d\bar{z}}{dt} + \frac{d}{d\tau} \pi_z = C(\bar{y} + \pi_y) + D(\bar{z} + \pi_z).$$

Equating the coefficients of equal powers of μ , those depending on t and those depending on τ being treated separately, we arrive at the following equations for the variations.

Zeroth order,

$$C\bar{y}_0(t) + D\bar{z}_0(t) = 0 \quad (3.63)$$

$$\frac{d}{d\tau} \pi_0 z(\tau) = C\pi_0 y(\tau) + D\pi_0 z(\tau) \quad (3.64)$$

$$\frac{d\bar{y}_0(t)}{dt} = A\bar{y}_0(t) + B\bar{z}_0(t) \quad (3.65)$$

$$\frac{d}{d\tau} \pi_0 y(\tau) = 0. \quad (3.66)$$

First order,

$$\frac{d\bar{z}_1(t)}{dt} = C\bar{y}_1(t) + D\bar{z}_1(t) \quad (3.67)$$

$$\frac{d\pi_1 z(\tau)}{d\tau} = C\pi_1 y(\tau) + D\pi_1 z(\tau) \quad (3.68)$$

$$\frac{d\bar{y}_1(t)}{dt} = A\bar{y}_1(t) + B\bar{z}_1(t) \quad (3.69)$$

$$\frac{d\pi_1 y(\tau)}{d\tau} = A\pi_0 y(\tau) + B\pi_0 z(\tau) \quad (3.70)$$

k^{th} order

$$\frac{d\bar{z}_{k-1}(t)}{dt} = C\bar{y}_k(t) + D\bar{z}_k(t) \quad (3.71)$$

$$\frac{d\pi_k z(\tau)}{d\tau} = C\pi_k y(\tau) + D\pi_k z(\tau) \quad (3.72)$$

$$\frac{d\bar{y}_k(t)}{dt} = A\bar{y}_k(t) + B\bar{z}_k(t) \quad (3.73)$$

$$\frac{d\pi_k y(\tau)}{d\tau} = A\pi_{k-1} y(\tau) + B\pi_{k-1} z(\tau). \quad (3.74)$$

Since we are considering the initial value problem, the matching conditions become

$$\pi_0 z(0) + \bar{z}_0(0) = z^0 \quad (3.75)$$

$$\pi_0 y(0) + \bar{y}_0(0) = y^0 \quad (3.76)$$

and for $k \geq 1$

$$\pi_k z(0) + \bar{z}_k(0) = 0 \quad (3.77)$$

$$\pi_k y(0) + \bar{y}_k(0) = 0 \quad (3.78)$$

and, due to our stability assumption,

$$\pi_k y(\infty) = \pi_k z(\infty) = 0 \quad k \geq 0. \quad (3.79)$$

Solutions of this type are referred to as "inner" and "outer", "fast" and "slow", or "steady state" and "boundary layer" depending on the author.

Our purpose here is to show that the series

$$\begin{aligned} y &= \bar{y} + \pi_y \\ z &= \bar{z} + \pi_z \end{aligned} \quad (3.80)$$

are equivalent to the solution of (3.3) obtained using the fundamental matrix (3.52). In other words,

$$\begin{bmatrix} \bar{y} \\ \bar{z} \end{bmatrix} = \begin{bmatrix} I \\ -P \end{bmatrix} e^{(A-BP)t} \bar{y}_0(0) \quad (3.81)$$

$$\begin{bmatrix} \pi_y \\ \pi_z \end{bmatrix} = \begin{bmatrix} \hat{P} \\ I \end{bmatrix} e^{(D+\hat{C}\hat{P})t} \pi_z(0). \quad (3.82)$$

The approach here is to show that y_{slow} and z_{slow} of (2.73) and (2.74) found using dominant left eigenspace iterations are equivalent to $\bar{y}$ and $\bar{z}$ respectively in (3.80). We then show that $\hat{y}_{\text{fast}}$ and $\hat{z}_{\text{fast}}$ of (2.84) and (2.85) found using the dominant right eigenspace iterations are equivalent to π_y and π_z respectively, in (3.80). This is the purpose of using fundamental matrix (3.52) since it expresses the solution in terms of these components. Other fundamental matrices involve the Lyapunov solutions Q and $\hat{Q}$ that possess complex series expansions which we want to avoid.

First, we seek an asymptotic solution to y_{slow} of the form

$$x(t) = x_0(t) + \mu x_1(t) + \dots \quad (3.83)$$

and a solution of z_{slow} in the form

$$-Px(t) = -P_0 x_0(t) - \mu(P_1 x_0 + P_0 x_1) \dots$$

where $x(t)$ is the transformation variable of (2.70). When (3.3) is used as the system model

$$\begin{bmatrix} \dot{x} \\ \dot{z} \end{bmatrix} = \begin{bmatrix} A-P(\mu)B & 0 \\ 0 & \frac{D}{\mu} + BP(\mu) \end{bmatrix} \begin{bmatrix} x \\ z \end{bmatrix} \quad (3.84)$$

Substituting in the formal power series (3.83) and (3.7) into (3.84) and equating like powers of u , we obtain the following equations of the variations

$$\dot{x}_0 = (A - BP_0)x_0 \quad (3.85)$$

$$\dot{x}_1 = (A - BP_0)x_1 - BP_1x_0 \quad (3.86)$$

⋮

$$\dot{x}_k = (A - BP_0)x_k - B \sum_{j=0}^{k-1} P_{k-j}x_j \quad (3.87)$$

We will now show that the differential equations necessary to solve for $\bar{y}_k$, $k \geq 0$ are equivalent to (3.87) $\forall k$. The equivalence of $\bar{z}_k$ and $(-Px)_k$ is a byproduct of the derivation for the equivalence of $\bar{y}_k$ and x_k .

From (3.63) and (3.65)

$$\begin{aligned} \bar{z}_0 &= -D^{-1}C\bar{y}_0 \\ &= -P_0\bar{y}_0 \\ \frac{d\bar{y}_0}{dt} &= (A - BP_0)\bar{y}_0 \end{aligned} \quad (3.89)$$

Now, from (3.67) and (3.69)

$$\begin{aligned} \bar{z}_1 &= -P_0\bar{y}_1 + D^{-1} \frac{d\bar{z}_0}{dt} \\ &= -P_0\bar{y}_1 - D^{-2}C \frac{d\bar{y}_0}{dt} \\ &= -P_0\bar{y}_1 - D^{-2}C(A - BD^{-1}C)\bar{y}_0 \\ &= -P_0\bar{y}_1 - P_1\bar{y}_0 \end{aligned} \quad (3.90)$$

$$\frac{d\bar{y}_1}{dt} = (A - BP_0)\bar{y}_1 - BP_1\bar{y}_0 \quad (3.91)$$

and thus, for (3.71) and (3.73)

$$\begin{aligned} \bar{z}_k &= -P_0 y_k + D^{-1} \frac{dz_{k-1}}{dt} \\ \frac{dz_{k-1}}{dt} &= -P_0 \frac{d\bar{y}_{k-1}}{dt} - P_1 \frac{d\bar{y}_{k-2}}{dt} - \dots - P_{k-1} \frac{d\bar{y}_0}{dt} \end{aligned} \quad (3.92)$$

$$\begin{aligned} \bar{z}_k &= -P_0 \bar{y}_k \\ &\quad - D^{-1} P_0 A + D^{-1} P_0 B P_0 \\ &\quad - \dots - \\ &\quad - D^{-1} P_{k-1} A + D^{-1} \sum_{j=0}^{k-1} P_j B P_{k-1-j} \\ &= -P_0 \bar{y}_k - P_1 \bar{y}_{k-1} - \dots - P_k \bar{y}_0 \\ &= - \sum_{j=0}^k P_{k-j} \bar{y}_j \end{aligned} \quad (3.93)$$

$$\frac{d\bar{y}_k}{dt} = (A - BP_0)\bar{y}_k - B \sum_{j=0}^{k-1} P_{k-j} \bar{y}_j \quad (3.94)$$

which is equivalent to (3.87) $\forall k$. Plus, it is obvious from (3.93) that

$$\bar{z}_k = - (Px)_k, \quad \forall k.$$

We now consider the fast components.

Using the dominant right eigenspace iterations, the singularly perturbed model (3.3) is transformed into

$$\begin{bmatrix} \dot{\hat{x}} \\ \dot{\hat{w}} \end{bmatrix} = \begin{bmatrix} A - \hat{P}(\mu)C & 0 \\ 0 & \frac{D + C\hat{P}(\mu)}{\mu} \end{bmatrix} \begin{bmatrix} \hat{x} \\ \hat{w} \end{bmatrix} \quad (3.95)$$

where

$$\begin{aligned} \hat{x}_0 &= y_0 - \hat{P}z_0 \\ \hat{w}_0 &= \hat{Q}y_0 + (I - \hat{Q}\hat{P})z_0. \end{aligned}$$

$\hat{P}$ is obtained from (2.26) and $\hat{Q}$ is then obtained from (2.78).

The dichotomic nature of our fast and slow components led us to identify fast and slow components of y and z as

$$\begin{aligned} y_{\text{slow}} &= (I - \hat{P}\hat{Q})\hat{x} & y_{\text{fast}} &= \hat{P}\hat{w} \\ z_{\text{slow}} &= -\hat{Q}\hat{x} & z_{\text{fast}} &= \hat{w} \end{aligned}$$

The differential equation for the fast state vector is

$$\dot{\hat{w}} = \frac{D + C\hat{P}}{\mu} \hat{w}. \quad (3.96)$$

Let $\tau = t/\mu$, then (3.96) becomes

$$\frac{d\hat{w}}{d\tau} = (D + C\hat{P})\hat{w}(\tau). \quad (3.97)$$

We now seek an asymptotic solution to z_{fast} of the form

$$w(\tau) = w_0(\tau) + \mu w_1(\tau) + \dots \quad (3.98)$$

and a solution to y_{fast} of the form

$$\hat{P}w(\tau) = \hat{P}_0 w_0(\tau) + \mu(\hat{P}_0 w_1(\tau) + \hat{P}_1 w_0(\tau)) + \dots$$

Substituting the formal power series (3.98) and (3.27) into (3.97) and equating like powers of u , we obtain the following equations of the variations

$$\frac{d\hat{w}_0(\tau)}{d\tau} = D\hat{w}_0(\tau) \quad (3.99)$$

$$\frac{d\hat{w}_1(\tau)}{d\tau} = D\hat{w}_1 + CP_1 w_0 \quad (3.100)$$

⋮

$$\frac{d\hat{w}_k(\tau)}{d\tau} = D\hat{w}_k + C \sum_{j=0}^{k-1} \hat{P}_{k-j} \hat{w}_j. \quad (3.101)$$

We will now show that the differential equations necessary to solve for $\pi_k z(\tau)$, $k \geq 0$ are equivalent to (3.101) $\forall k$. The equivalence of $\pi_k y(\tau)$ and $(\hat{P}w)_k$ is a byproduct of this derivation. From (3.66) and (3.79)

$$\pi_0 y = 0. \quad (3.102)$$

Thus,
$$\frac{d\pi_0 z}{d\tau} = D\pi_0 z. \quad (3.103)$$

From (3.70)

$$\frac{d\pi_1 y(\tau)}{d\tau} = A\pi_0 y + B\pi_0 z. \quad (3.104)$$

Thus,

$$\pi_1 y(\tau) = \pi_1 y(0) + \int_0^\tau [A\pi_0 y(\tau) + B\pi_0 z(\tau)] d\tau. \quad (3.105)$$

To establish $\pi_1 y(0)$, from (3.79)

$$0 = \pi_1 y(0) + \int_0^\infty [A\pi_0 y(\sigma) + B\pi_0 z(\sigma)] d\sigma. \quad (3.106)$$

Thus,

$$\pi_1 y(\tau) = - \int_\tau^\infty [A\pi_0 y(\sigma) + B\pi_0 z(\sigma)] d\sigma \quad (3.107)$$

Since

$$\pi_0 y(\sigma) = 0$$

$$\pi_1 y(\tau) = - \int_\tau^\infty B\pi_0 z(\sigma) d\sigma \quad (3.108)$$

$$= - \int_\tau^\infty BD^{-1} \frac{d\pi_0 z(\sigma)}{d\sigma} d\sigma$$

$$= - BD^{-1} [\pi_0 z(\infty) - \pi_0 z(\tau)]$$

$$= BD^{-1} \pi_0 z(\tau) = \dot{P}_{10} z(\tau) \quad (3.109)$$

and as a result

$$\frac{d\pi_1 z(\tau)}{d\tau} = D\pi_1 z(\tau) + C\dot{P}_{10} z(\tau). \quad (3.110)$$

Finally, for the k^{th} variation

$$\pi_k y(\tau) = - \int_\tau^\infty [A\pi_{k-1} y(\sigma) + B\pi_{k-1} z(\sigma)] d\sigma \quad (3.111)$$

$$= - BD^{-1} \int_\tau^\infty \frac{d\pi_{k-1} z(\sigma)}{d\sigma} d\sigma$$

$$= - \int_\tau^\infty (A - BD^{-1}D) \dot{P}_{10} \pi_{k-2} z(\sigma) d\sigma$$

$$- \int_{\tau}^{\infty} (A - BD^{-1}C) \hat{P}_2 \pi_{k-3} z(\sigma) d\sigma - \dots - \int_{\tau}^{\infty} (A - BD^{-1}C) \hat{P}_{k-1} \pi_0 z(\sigma) d\sigma \quad (3.112)$$

using

$$\pi_{k-1}(\tau) = D^{-1} \frac{d\pi_{k-1} z(\tau)}{d\tau} - D^{-1} C \sum_{j=0}^{k-2} \hat{P}_{k-1-j} \pi_j z(\tau) \quad (3.113)$$

$$\begin{aligned} \pi_k y(\tau) &= - \hat{P}_1 \int_{\tau}^{\infty} \frac{d\pi_{k-1} z(\sigma)}{d\sigma} d\sigma - \hat{P}_2 \int_{\tau}^{\infty} \frac{d\pi_{k-2} z(\sigma)}{d\sigma} d\sigma - \\ &\quad \dots - \hat{P}_k \int_{\tau}^{\infty} \frac{d\pi_0 z(\sigma)}{d\sigma} d\sigma \\ &= \hat{P}_1 \pi_{k-1} z(\tau) + \hat{P}_2 \pi_{k-2} z(\tau) + \dots + \hat{P}_k \pi_0 z(\tau) \\ &= \sum_{j=0}^{k-1} \hat{P}_{k-j} \pi_j z(\tau). \end{aligned} \quad (3.114)$$

$$\text{Thus,} \quad \frac{d\pi_k z(\tau)}{d\tau} = D\pi_k z(\tau) + C \sum_{j=0}^{k-1} \hat{P}_{k-j} \pi_j z(\tau) \quad (3.116)$$

Which is equivalent to (3.101), $\forall k \geq 0$. Also, from (3.115),

$$(\pi_y)_k = (\hat{P}\pi)_k \quad \forall k \geq 0. \quad (3.117)$$

Thus, we have shown that $\bar{y}_0$ satisfies

$$\frac{dy_0}{dt} = (A - BP)\bar{y}_0 \quad (3.118)$$

and that

$$\frac{d\pi_z(\tau)}{d\tau} = (D + CP)\pi_z(\tau) \quad (3.119)$$

likewise

$$\begin{aligned}\pi_y &= \hat{P}\pi_z(\tau) \\ \bar{z}_0 &= -P\bar{y}_0(t).\end{aligned}\tag{3.120}$$

Thus

$$\begin{aligned}y(t) &= \bar{y}_0(t) + \pi_y(\tau) \\ &= e^{(A-BP)t}\bar{y}_0(0) + \hat{P}e^{(D+C\hat{P})\tau}\pi_z(0)\end{aligned}\tag{3.121}$$

$$\begin{aligned}z(t) &= \bar{z}_0(t) + \pi_z(\tau) \\ &= -Pe^{(A-BP)t}\bar{y}_0(0) + e^{(D+C\hat{P})\tau}\pi_z(0)\end{aligned}\tag{3.122}$$

$$\begin{bmatrix} y(t) \\ z(t) \end{bmatrix} = \begin{bmatrix} e^{(A-BP)t} & \hat{P}e^{(D+C\hat{P})\tau} \\ -Pe^{(A-BP)t} & e^{(D+C\hat{P})\tau} \end{bmatrix} \begin{bmatrix} \bar{y}_0(0) \\ \pi_z(0) \end{bmatrix}\tag{3.122}$$

The matching conditions (3.75)-(3.80) thus reduce to

$$\begin{aligned}y_0 &= \bar{y}(0) + \pi_y(0) \\ z_0 &= \bar{z}(0) + \pi_z(0).\end{aligned}\tag{3.124}$$

However, from (3.120)

$$y_0 = \bar{y}(0) + \hat{P}\pi_z(0)\tag{3.125}$$

$$z_0 = -P\bar{y}(0) + \pi_z(0)\tag{3.126}$$

or

$$\begin{bmatrix} y_0 \\ z_0 \end{bmatrix} = \begin{bmatrix} I & \hat{P} \\ -P & I \end{bmatrix} \begin{bmatrix} \bar{y}(0) \\ \pi_z(0) \end{bmatrix}.\tag{3.127}$$

Thus,

$$\begin{bmatrix} \bar{y}(0) \\ \pi_z(0) \end{bmatrix} = \begin{bmatrix} I & \hat{P} \\ -P & I \end{bmatrix}^{-1} \begin{bmatrix} y_0 \\ z_0 \end{bmatrix}\tag{3.128}$$

which, together with (3.123), uniquely defines the asymptotic series solution to (3.3) in terms of fundamental matrix (3.52).

In conclusion, given the initial value problem (3.3) that satisfies (2.32), the presence of the singular perturbation parameter μ suggests seeking a series solution in two time scales. Vasil'eva's method of matched asymptotic expansions is used for decomposing the solution vector of (3.3) into fast and slow series components that are individually easier to solve than the original higher order system. The attractive feature of series solutions is that no state transformation is necessary. The series terms for the actual states are computed as they are needed. In the eigenspace iterations, this is not true. Transformed fast and slow states are found, solved, and the actual states attained through an inverse transformation. From a computational point of view this method has to be preferred since only two reduced order systems of differential equations are solved in attaining the high order solution. In the series method, two reduced order systems of differential equations are solved for every term in the series. Thus, it is practical only when the number of terms in the series required is small. In this section we have shown that the separation of time scales attained in asymptotic series solution is equivalent to the dominant eigenspace decompositions of Chapter 2, in that the convergence of both methods is dependent on the existence of a dichotomic solution to the Riccati equations (2.19) and (2.26). Thus, for a given state, the "fast" and "slow" components of that state obtained using either decomposition algorithm will be the same.

3.5. Simplified Iterative Schemes

One of the computational drawbacks of the dominant left and right eigenspace iterations is the computation of the inverses

$$(D + P_k B)^{-1} \quad (3.129)$$

$$(D - C\hat{P}_k)^{-1} \quad (3.130)$$

at every iteration.

Looking at the left eigenspace case, the iterative matrix recursion is

$$P_{k+1} = (D + P_k B)^{-1} (C + P_k A) \quad (3.131)$$

which can be expressed as

$$DP_{k+1} = -P_k B P_{k+1} + C + P_k A. \quad (3.132)$$

If this is approximated by

$$\begin{aligned} D\tilde{P}_{k+1} &= -\tilde{P}_k B \tilde{P}_k + C + \tilde{P}_k A \\ \tilde{P}_{k+1} &= D^{-1} (C + \tilde{P}_k A - \tilde{P}_k B \tilde{P}_k). \end{aligned} \quad (3.133)$$

Then we will have eliminated the need for the inverse in (3.131) at every iteration.

Likewise, looking at the right eigenspace case, the iterative matrix recursion is

$$\hat{P}_{k+1} = (B + A\hat{P}_k)(D + C\hat{P}_k)^{-1} \quad (3.139)$$

which can be expressed as

$$\hat{P}_{k+1} D = -\hat{P}_{k+1} C \hat{P}_k + B + A\hat{P}_k. \quad (3.135)$$

If this is approximated by

$$\begin{aligned}\hat{P}_{k+1}D &= -\hat{P}_k C \hat{P}_k + B + A \hat{P}_k \\ \hat{P}_{k+1} &= (-\hat{P}_k C \hat{P}_k + B + A \hat{P}_k) D^{-1}\end{aligned}\tag{3.136}$$

we will have again eliminated the need for taking the inverse in (3.134) at every iteration. [10] has used a contraction mapping argument to develop conditions under which (3.133) will converge to the dichotomic Riccati solution. This methodology can easily be extended to (3.136). When the conditions of [10] are satisfied, (3.133) and (3.136) are computationally superior to (3.131) and (3.134). Unfortunately, the conditions of [4] are somewhat conservative and are not satisfied by many systems which we know can be decomposed using (3.131) and (3.134).

3.6. Partial and Full Pole Placement

There are many applications using the techniques developed in Chapters 2 and 3. They include robust designs, reduced order regulator designs, and reduced order modeling to only mention a few. In this section we will show how the time scale decomposition techniques can be used to implement partial or full pole placement design.

We will now be considering the completely state controllable system

$$\begin{bmatrix} \dot{y} \\ \dot{z} \end{bmatrix} = \begin{bmatrix} A & B \\ C & D \end{bmatrix} \begin{bmatrix} y \\ z \end{bmatrix} + \begin{bmatrix} G \\ H \end{bmatrix} u \quad u \in R^p.\tag{3.137}$$

If the open loop eigenvalues satisfy (2.32), then we can apply transformation (2.17) which transforms (3.137) into

$$\begin{bmatrix} \dot{y} \\ \dot{\eta} \end{bmatrix} = \begin{bmatrix} A-BP & B \\ 0 & D+PB \end{bmatrix} \begin{bmatrix} y \\ \eta \end{bmatrix} + \begin{bmatrix} G \\ H+PG \end{bmatrix} u \quad (3.138)$$

where P is obtained using either (3.131) or (3.133). The transformation involved here can be written as

$$\begin{bmatrix} y \\ \eta \end{bmatrix} = \begin{bmatrix} I & 0 \\ P & I \end{bmatrix} \begin{bmatrix} y \\ z \end{bmatrix} \quad (3.139)$$

which possesses the explicit inverse

$$\begin{bmatrix} y \\ z \end{bmatrix} = \begin{bmatrix} I & 0 \\ -P & I \end{bmatrix} \begin{bmatrix} y \\ \eta \end{bmatrix}. \quad (3.140)$$

Observe now that the pair $(D+PB, H+PG)$ spans only the "fast" controllable subspace. Let $D^* = D + \hat{P}B$ and $H^* = H + \hat{P}G$ and design a feedback gain F such that

$$(D^* + H^*F)$$

has m desired poles.

The control is of the form

$$u = F\eta \quad (3.141)$$

$$= F(Py + z)$$

$$= [FP : F] \begin{bmatrix} y \\ z \end{bmatrix} \quad (3.142)$$

and the resulting closed-loop system has n eigenvalues according to

$$\sigma(A-BP) \quad (3.143)$$

and m eigenvalues according to

$$\sigma(D^* + H^*F). \quad (3.144)$$

Now apply transformation (2.24) to (3.137). This gives

$$\begin{bmatrix} \dot{\xi} \\ \dot{z} \end{bmatrix} = \begin{bmatrix} A-\hat{P}C & 0 \\ C & D+C\hat{P} \end{bmatrix} \begin{bmatrix} \xi \\ z \end{bmatrix} + \begin{bmatrix} G-\hat{P}H \\ H \end{bmatrix} u$$

where $\hat{P}$ is obtained using either (3.134) or (3.136). The transformation involved here can be written as

$$\begin{bmatrix} \xi \\ z \end{bmatrix} = \begin{bmatrix} I & -\hat{P} \\ 0 & I \end{bmatrix} \begin{bmatrix} y \\ z \end{bmatrix} \quad (3.145)$$

which has the explicit inverse

$$\begin{bmatrix} y \\ z \end{bmatrix} = \begin{bmatrix} I & \hat{P} \\ 0 & I \end{bmatrix} \begin{bmatrix} \xi \\ z \end{bmatrix} \quad (3.146)$$

Observe now that the pair $(A-\hat{P}C, G-\hat{P}H)$ spans only the "slow" controllable subspace. Let $A^* = A-\hat{P}C$ and $G^* = G-\hat{P}H$ and design a feedback F such that

$$(A^* + G^*F)$$

has n desired poles.

The control is of the form

$$u = F\xi \quad (3.147)$$

$$= F(y - \hat{P}z)$$

$$= [F \quad -F\hat{P}] \begin{bmatrix} y \\ z \end{bmatrix} \quad (3.148)$$

and the resulting closed loop system has n eigenvalues of

$$\sigma(A^* + G^*F)$$

and m eigenvalues of

$$\tau(D + C\hat{P}).$$

In general, both slow and fast modes may be designed for. In this case a general two-state design procedure may be implemented. Assume we have block diagonalized (3.137) as done in Section 2.3.

$$\begin{bmatrix} \dot{x} \\ \dot{w} \end{bmatrix} = \begin{bmatrix} A^* & 0 \\ 0 & D^* \end{bmatrix} \begin{bmatrix} x \\ w \end{bmatrix} + \begin{bmatrix} G^* \\ H^* \end{bmatrix} u. \quad (3.149)$$

We arbitrarily chose to design for the slow subsystem first. Thus, we chose an F_s such that

$$\sigma(A^* + G^*F_s)$$

has n desired "slow" eigenvalues. Letting

$$u = u_s + u_f$$

where now

$$u_s = F_s x,$$

the partially closed loop system looks like

$$\begin{bmatrix} \dot{x} \\ \dot{w} \end{bmatrix} = \begin{bmatrix} A^* + G^*F_s & 0 \\ H^*F_s & D^* \end{bmatrix} \begin{bmatrix} x \\ w \end{bmatrix} + \begin{bmatrix} G^* \\ H^* \end{bmatrix} u_f. \quad (3.150)$$

Now, let

$$v = w + Sx$$

which transforms (3.150) into

$$\begin{bmatrix} \dot{x} \\ \dot{v} \end{bmatrix} = \begin{bmatrix} A^* + G^*F_s & 0 \\ H^*F_s + S(A^* + G^*F_s) - D^*S & D^* \end{bmatrix} \begin{bmatrix} x \\ v \end{bmatrix} + \begin{bmatrix} G^* \\ H^* + SG^* \end{bmatrix} u_f. \quad (3.151)$$

We pick S such that

$$H^*F_s + S(A^* + G^*F_s) - D^*S = 0. \quad (3.152)$$

This Lyapunov type equation has a unique solution if

$$\sigma(D^*) \cap \sigma(A^* + G^*F_s) = \emptyset \quad (3.153)$$

and thus can be solved algebraically [32]. If

$$\min |\sigma(D^*)| > \max |\sigma(A^* + G^*F_s)|. \quad (3.154)$$

Then the iterative scheme (2.80) may be used to solve (3.152). With this S , the pair $(D^*, H^* + SG^*)$ now spans only the "fast" controllable subspace, and we can design a feedback gain F_f such that

$$\sigma(D^* + (H^* + SG^*)F_f)$$

has m desired eigenvalues.

Thus, the composite control is

$$\begin{aligned} u &= F_s x + F_f v \\ &= G_s x + F_f (w + Sx) \\ &= (F_s + F_f S)x + F_f w \\ &= [(F_s + F_f S) : F_f] \begin{bmatrix} x \\ w \end{bmatrix} \end{aligned} \quad (3.155)$$

and using either transformation (2.72) or (2.83), (3.155) can be expressed in terms of our original state variables. This control places n eigenvalues of

$$\sigma(A^* + G^*F_s) \quad (3.156)$$

and m eigenvalues of

$$\sigma(D^* + (H^* + SG^*)F_f). \quad (3.157)$$

This technique has been applied to singularly perturbed systems [42] where it is shown to be a generalization to results obtained in [7,26,40]. This technique is also applicable to discrete time models as shown in [43]. In this case, the dominant eigenvalues are part of the "slow" spectrum.

CHAPTER 4

SINGULAR PERTURBATION MODELING OF MARKOV CHAINS

4.1. Introduction

The previous two chapters have unified and extended many results on the control and analysis of deterministic two-time-scale systems. For stochastic systems modeled as Markov chains we can extend the theory of time scale decomposition to these probabilistic models where now "slow" and "fast" eigenmodes correspond to "weak" and "strong" transition probabilities.

Markov chain models are well known in the analysis and control of stochastic systems [14,15,21,44]. Until recently, however, this analysis was limited to simple systems due to the tremendous dimensionality requirements of most Markov models. Recent applications, such as management of hydrodams [22,25] and queueing network models of computer systems [21,45,46], have accentuated the need for reduced order approximations of large scale Markov chains. In this regard particularly promising is a perturbational decomposition-aggregation method of Pervozvanski, Smirnov and Gaitsgori [23,24,47,48], and Delebecque and Quadrat [25,49]. The method assumes that the groups of strongly interacting states are known and treats the weak interactions between these groups as perturbations. The result is a short-term decomposition. Over a longer period the weak interactions become significant, while each group of strongly coupled states can be replaced by an aggregate state. A long-term aggregate model is thus obtained. It is the purpose of this chapter to show how such weakly coupled Markov processes can be modeled as a singularly perturbed system. This enables us to apply the decomposition techniques of chapters 2 and 3.

In Section 4.2 we introduce the two-time-scale Markov chain. Slow and fast components of the chain are identified and used to construct the singularly perturbed model.

In Section 4.3, a grouping algorithm [34] is used to show how the states of an arbitrary Markov chain can be ordered so as to exhibit the two-time-scale property.

Finally, in Section 4.4, a two-time-scale asymptotic expansion of the steady state probability distribution is constructed using the singularly perturbed model. An example is then given to show how the series solutions can be used to efficiently calculate the invariant probability measure of a large queueing network.

4.2. Singular Perturbation Modeling

In this section we introduce the two-time-scale Markov chain and show how it can be put into standard singularly perturbed form.

Consider the n -state Markov chain

$$\frac{dp(\tau)}{d\tau} = p(\tau)(A + \epsilon B) \quad (4.1)$$

where $p(\tau)$ is the n -dimensional row vector of probabilities $p_i(\tau)$ to be in state i at time τ . Hence,

$$\sum_{i=1}^n p_i(\tau) = 1 \quad \tau \geq 0. \quad (4.2)$$

A and B are both n -dimensional Markov generators and A has the form

$$A = \begin{bmatrix} A_1 & & & 0 \\ & A_2 & & \\ & & \ddots & \\ 0 & & & A_N \end{bmatrix} \quad (4.3)$$

where A_j , $j=1, \dots, N$ are n_j dimensional Markov generators.

Thus, in (4.1) N groups of strongly interacting states have been identified. Group j consists of n_j states and $\sum_{j=1}^N n_j = n$. The weak interactions between states in different groups are modeled as multiples of a small positive scalar ϵ . We assume throughout the thesis that for $0 < \epsilon \leq \epsilon^*$ the process (4.1) has a single ergodic class with unique stationary probability distribution $\bar{p}$ defined by

$$0 = \bar{p}(A + \epsilon B) \quad (4.4)$$

Furthermore, let each of the N generators A_j define a Markov process with a single ergodic class. This implies that each A_j has one zero eigenvalue.

The corresponding right eigenvector t_j is the n_j -dimensional column made of ones. The left eigenvector v_j is the n_j -dimensional row of stationary probabilities for the states in the j -th group when $\epsilon=0$ in (4.4). The matrix form of $A_j t_j = 0$, $v_j A_j = 0$, and $v_j t_j = 1$, $j=1, \dots, N$, is

$$AT = 0, \quad VA = 0, \quad VT = I_N \quad (4.5)$$

$$T = \begin{bmatrix} t_1 & 0 & 0 & \dots & 0 \\ 0 & t_2 & 0 & \dots & 0 \\ \dots & \dots & \dots & \dots & \dots \\ 0 & 0 & 0 & \dots & t_N \end{bmatrix}, \quad V = \begin{bmatrix} v_1 & 0 & 0 & \dots & 0 \\ 0 & v_2 & 0 & \dots & 0 \\ \dots & \dots & \dots & \dots & \dots \\ 0 & 0 & \dots & 0 & v_N \end{bmatrix} \quad (4.6)$$

where I_N is the $N \times N$ identity, the j -th row of T is made of n_j -dimensional columns and the j -th column of V is made of n_j -dimensional rows, that is T is $n \times N$ and V is $N \times n$. The influence of weak interactions ϵB in (4.1) will become significant after a long period of time τ . Hence τ -scale is called "fast time." To see the influence of ϵ "sooner" we introduce the "slow time" $t = \epsilon \tau$. If, for example, τ is in weeks and $\epsilon = \frac{1}{52}$, then t is in years. In t -scale model (4.1) becomes

$$\dot{p}(t) = p(t) \left(\frac{A}{\epsilon} + B \right) \quad (4.7)$$

where the dot denotes $\frac{d}{dt}$. Initially $p(t)$ will rapidly approach the null-space of A^T as if the N groups were separated from each other. After that, pB is no longer negligible with respect to $p \frac{A}{\epsilon}$. This behavior is a characteristic of singularly perturbed systems [3,13,35]. As in [35] our goal is to transform (4.7) into a standard singularly perturbed form which makes the slow and fast parts of $p(t)$ more explicit. For N slow variables we take the elements π_j of the row

$$\pi = pT \quad (4.8)$$

because π_j is the probability of the process (4.7) to be in group j . Since the transitions between the groups are slow, π_j will change slowly. After the fast transient is over, probability p_i is approximated by $\pi_j v_j^i$, where v_j^i is the stationary probability for the process to be in state i , once it is in group j . Thus the difference $Y_i = p_i - \pi_j v_j^i$ is the fast part of p_i . Of n such differences, $n-N$ are independent and are defined by

$$YW = p - \pi V, \quad WT = 0. \quad (4.9)$$

W is an $(n-N) \times n$ block-diagonal matrix of the form

$$W = \begin{bmatrix} w_1 & & & & & \\ & w_2 & & & & \\ & & w_3 & & & \\ & & & \ddots & & \\ & 0 & & & w_N & \\ & & & & & 0 \end{bmatrix} \quad (4.10)$$

where each w_j is an $(\pi_j - 1) \times \pi_j$ matrix of the form

$$W_j = \begin{bmatrix} 1 & 0 & \dots & 0 & 0 & -1 \\ 0 & 1 & \dots & 0 & 0 & -1 \\ \vdots & \vdots & & \vdots & \vdots & \vdots \\ 0 & 0 & & 1 & 0 & -1 \\ 0 & 0 & \dots & 0 & 1 & -1 \end{bmatrix} \quad (4.11)$$

This defines the transformation

$$p = [\pi, Y] \begin{bmatrix} V \\ W \end{bmatrix} \quad (4.12)$$

to perform the inverse transformation, we define the $n \times (n-N)$ block diagonal matrix S such that

$$WS = I_{n-N}, \quad VS = 0. \quad (4.13)$$

This gives the explicit inverse of (4.12) as

$$[\pi, Y] = p[T, S] \quad (4.14)$$

of the form

$$S = \begin{bmatrix} S_1 & & 0 \\ & S_2 & \\ 0 & & \ddots \\ & & & S_N \end{bmatrix} \quad (4.15)$$

and each S_j is an $n_j \times (n_j - 1)$ matrix of the form

$$S_j = \begin{bmatrix} 1-v_j^1 & -v_j^2 & \cdots & -v_j^{n_j-1} \\ -v_j^1 & 1-v_j^2 & & -v_j^{n_j-1} \\ \vdots & \vdots & & \vdots \\ & & & 1-v_j^{n_j-1} \\ -v_j^1 & -v_j^2 & & -v_j^{n_j-1} \end{bmatrix} \quad (4.16)$$

using (4.14) and (4.12), (4.7) can now be transformed into a standard singular perturbation form

$$\begin{aligned} \dot{\bar{y}} &= \bar{W}VBT + YWBT \\ \epsilon \dot{\bar{y}} &= \epsilon \bar{W}VBS + YW(A + \epsilon B)S \end{aligned} \quad (4.17)$$

whose properties are well known [3,12]. The crucial stability condition on $W(A + \epsilon B)S$ is satisfied by the fact that the j -th block of the block diagonal matrix WAS is Λ_j deflated for the zero eigenvalue. Thus, from the basic assumptions on (4.1), $\text{Re } \lambda_i(WAS) < 0$, $i = 1, \dots, n-N$. Assuming that $\epsilon \dot{\bar{y}} \sim 0$ and substituting

$$Y = -\epsilon \bar{W}VBS(WAS)^{-1} \quad (4.18)$$

from (4.17) an ϵ -corrected slow model is obtained

$$\dot{\tilde{\eta}} = \tilde{\eta}(\text{VBT} - \epsilon \text{VBS}(\text{WAS})^{-1} \text{WBT}) \quad (4.19)$$

which for $\epsilon = 0$ reduces to the aggregate model obtained in [23]. Note that the inverse in the correction term is "decentralized," and consists of N inverses of smaller diagonal blocks. From (4.18) we see that the slow part of Y is only $O(\epsilon)$. If we express (4.17) in the fast time scale $\tau = t/\epsilon$, or, equivalently, if we apply transformation (4.12) to (4.1), we would obtain

$$\frac{d\tilde{\eta}}{d\tau} = O(\epsilon), \quad \frac{dY}{d\tau} = Y\text{WAS} + O(\epsilon). \quad (4.20)$$

Thus, in the fast time scale as $\epsilon \rightarrow 0$, the slow variable $\tilde{\eta}$ tends to a constant and the system matrix for fast variable Y is WAS. In this manner two-time scale asymptotic expansions for $\tilde{\eta}$, Y will be constructed in section 4.4. Note that in the fast time τ , to $O(\epsilon)$, the interactions between groups are neglected. In the slow time t this is not true. This points out the necessity of posing the problem in the slow time scale if we are to use perturbation methods in the analysis and control of Markov chains on the infinite horizon.

Let us now consider the discrete time model

$$p(k+1) = p(k)P = p(k)\left(\frac{A}{\epsilon} + I + B\right) \quad (4.21)$$

where P is the probability transition matrix and A and B are generators. As in (4.7), the strong interactions appear as multiples of $\frac{1}{\epsilon}$, that is, (4.21) is expressed in the slow time scale. The transformation (4.12) results in

$$\begin{aligned} \tilde{\eta}(k+1) &= \tilde{\eta}(k)(I + \text{VBT}) + Y(k)\text{WBT} \\ Y(k+1) &= \tilde{\eta}(k)\text{VBS} + Y(k)W\left(\frac{A}{\epsilon} + I + B\right)S. \end{aligned} \quad (4.22)$$

Properties of this type of discrete time models are discussed in [43,50,51].

Note that the same model is obtained from (4.17) if $\hat{\eta}, \dot{\gamma}$ are replaced

$$\eta(k+1) - \eta(k), \quad \gamma(k+1) - \gamma(k),$$

that is for the step size 1. The slow model analogous to (4.19) is obtained by neglecting $\epsilon[\gamma(k+1) - \gamma(k)]$.

Throughout the remainder of this thesis singularly perturbed models (4.17) and (4.22) will be used extensively. It is important to realize that by just finding the steady state probabilities $v_j, j=1, \dots, N$, we can construct the transformations (4.12) and (4.14). Thus, these singularly perturbed models are obtained with little computational burden once the structure of (4.1) is identified. The problem of permuting the states of an arbitrary generator to exhibit this form is the topic of the next section.

4.3. Ordering of States

The preceding section assumes, as do the earlier references [23-25, 47-49], that the N groups of strongly interacting states are known and the generator of the process (4.1) is of the form $G = A + \epsilon B$, where A is block-diagonal and ϵB is small. This situation, convenient for an asymptotic analysis, is seldom met in reality. The ability to identify groups of strongly coupled states given an arbitrary generator is an important modeling task. In this section we address this task. Our main tool is a state "grouping" algorithm developed in [34] for power system matrices which we apply here to Markov generators.

Consider the generator

$$G = \begin{bmatrix} -.05 & .05 & & & & & & & & \\ .05 & -.46 & .05 & & .36 & & & & & \\ & .05 & -.46 & .05 & & .36 & & & & \\ & & .05 & -.41 & & & .36 & & & \\ & .2 & & & -.25 & .05 & & & & \\ & & .2 & & .05 & -.66 & .05 & .36 & & \\ & & & .2 & & .05 & -.61 & & .36 & \\ & & & & .2 & & & -.25 & .05 & \\ & & & & & .2 & .05 & -.61 & .36 & \\ & & & & & & .2 & & & -.2 \end{bmatrix}$$

(4.23)

This generator describes the transitions between the states in a queueing network of a computer system [21,46] consisting of a filing device D, a secondary memory M, and a processor C. Assuming that there are three users, the states $x_1, \dots, x_{10}$ are defined in Table 4.1 whose entries are the numbers of jobs in D, C, and M queues.

	x_1	x_2	x_3	x_4	x_5	x_6	x_7	x_8	x_9	x_{10}
D	3	2	1	0	2	1	0	1	0	0
C	0	1	2	3	0	1	2	0	1	0
M	0	0	0	0	1	1	1	2	2	3

Table 4.1. States of Queueing Network Model

The main difficulty in determining whether a state interacts weakly with a group of states is that its interactions with each state in the group

can be small, but the sum of these interactions can be significant enough to be considered strong. In other words, in practice ϵ is not infinitesimal and, if $\epsilon = 0.1$ is considered as weak, 6ϵ is strong. Thus, already for (4.23), and certainly for more complex forms of G , a systematic procedure is required to determine the strongly interacting states. Such a procedure is Avramovic's grouping algorithm.

The grouping algorithm is based on the following property of a process with as yet unknown groups of strongly interacting states: If there are N such groups, then matrix G will have $N-1$ small eigenvalues which are clustered near its zero eigenvalue. Let the columns of an $n \times N$ matrix $\tilde{M}$ be the right eigenvector of G for the N smallest eigenvalues, including $\lambda = 0$. Each row of $\tilde{M}$ corresponds to one of the n states. We observe that T in (4.6) is the limiting form of $\tilde{M}$ when interactions are neglected and the states are grouped. Note that states in the same group have identical rows in T while states not in the same group have rows in T that are perpendicular to one another. By continuity we expect that the corresponding rows in $\tilde{M}$ should be "nearly identical" and hence close to being linearly dependent. Instead of investigating "nearly identical" rows of $\tilde{M}$, Avramovic's algorithm does the opposite: it starts by determining N rows of $\tilde{M}$ which are as linearly independent as possible. In the algorithm, these rows are found by a simple Gaussian elimination with full pivoting. The corresponding N states are called the reference states around which the remaining $n-N$ states should be grouped. When the N reference rows of $\tilde{M}$ are found, a permutation π is performed so that these rows appear as the first N rows. Thus the $N \times N$ matrix M_1 in

$$\tilde{M} = M = \begin{bmatrix} M_1 \\ M_2 \end{bmatrix} \quad (4.24)$$

is nonsingular and a new basis of the same eigenspace is

$$M M_1^{-1} = \begin{bmatrix} I_N \\ M_2 M_1^{-1} \end{bmatrix} = \begin{bmatrix} I_N \\ L \end{bmatrix}. \quad (4.25)$$

In [28] important properties of matrix L are deduced from the fact that it is the "dichotomic" solution of the Riccati equation (2.26). A property to be used here is that the sum of entries in each row of L is 1. Thus, if $\tilde{M}$ has "nearly identical" rows, each row of L will have an entry close to 1, and all other entries close to 0. The criterion for grouping is simple. A row of L belongs to the group defined by that reference row which has entry 1 in the same column in which the examined row of L has its largest entry.

We now proceed to apply this algorithm to determine four groups of strongly interacting states in (4.23). The four smallest eigenvalues of G are 0, -0.025, -0.065, -0.107. The eigenvector matrix $\tilde{M}$ and the matrix (4.25) are as follows:

$$\tilde{M} = \begin{bmatrix} I_4 \\ L \end{bmatrix} = \begin{matrix} 1. & \begin{bmatrix} .32 & .84 & -.81 & .42 \end{bmatrix} \\ 2. & \begin{bmatrix} .32 & .41 & .25 & -.49 \end{bmatrix} \\ 3. & \begin{bmatrix} .32 & .06 & .28 & .26 \end{bmatrix} \\ 4. & \begin{bmatrix} .32 & -.12 & -.04 & .04 \end{bmatrix} \\ 5. & \begin{bmatrix} .32 & .38 & .35 & -.57 \end{bmatrix} \\ 6. & \begin{bmatrix} .32 & .03 & .28 & .32 \end{bmatrix} \\ 7. & \begin{bmatrix} .32 & -.14 & -.07 & -.01 \end{bmatrix} \\ 8. & \begin{bmatrix} .32 & -.01 & .27 & .43 \end{bmatrix} \\ 9. & \begin{bmatrix} .32 & -.17 & -.13 & -.07 \end{bmatrix} \\ 10. & \begin{bmatrix} .32 & -.19 & -.19 & -.16 \end{bmatrix} \end{matrix} \begin{matrix} 1. & \begin{bmatrix} 1 & 0 & 0 & 0 \end{bmatrix} \\ 5. & \begin{bmatrix} 0 & 1 & 0 & 0 \end{bmatrix} \\ 8. & \begin{bmatrix} 0 & 0 & 1 & 0 \end{bmatrix} \\ 10. & \begin{bmatrix} 0 & 0 & 0 & 1 \end{bmatrix} \\ 2. & \begin{bmatrix} .08 & .92 & 0 & 0 \end{bmatrix} \\ 3. & \begin{bmatrix} 0 & .17 & .83 & 0 \end{bmatrix} \\ 4. & \begin{bmatrix} 0 & 0 & .34 & .66 \end{bmatrix} \\ 6. & \begin{bmatrix} 0 & .11 & .89 & 0 \end{bmatrix} \\ 7. & \begin{bmatrix} 0 & 0 & .26 & .74 \end{bmatrix} \\ 9. & \begin{bmatrix} 0 & 0 & .14 & .86 \end{bmatrix} \end{matrix} \quad (4.26)$$

Note that the rows are labeled with the index of the state. An excellent grouping is achieved, because each row of L has one distinctly large entry. Therefore the groups are $\{4,7,9,10\}$, $\{3,6,8\}$, $\{2,5\}$, $\{1\}$. The permutation of the generator (4.23) to this ordering of the states is

$$\begin{bmatrix} \begin{array}{cc|cc} -.41 & .36 & & .05 \\ .2 & -.61 & .36 & .05 \\ .2 & -.61 & .36 & .05 \\ .2 & -.2 & & \end{array} \\ \hline \begin{array}{cc|cc} .05 & & -.46 & .36 & .05 \\ .05 & & .2 & -.66 & .36 & .05 \\ .05 & & .2 & -.25 & & \end{array} \\ \hline \begin{array}{cc|cc} & & .05 & & -.46 & .36 & .05 \\ & & .05 & & .2 & -.25 & \\ & & & & .05 & & -.05 \end{array} \end{bmatrix} \quad (4.27)$$

where the weak coupling is apparent. The physical interpretation of the grouping is now clear. The aggregate $\pi_j(t)$ is the probability that at time t there are $j-1$ jobs in the D-queue. This is intuitively clear since the mean service time of a filing device D is typically much slower than that of secondary memory M or processor C. The $Y(t)$ variables describe fast fluctuations between the C and M while the D-queue is in a given state. The accuracy of the approximation using the aggregate matrix

$$VBT = \begin{bmatrix} -.025 & .025 & 0 & 0 \\ .05 & -.073 & .023 & 0 \\ 0 & .05 & -.068 & .018 \\ 0 & 0 & .05 & -.05 \end{bmatrix} \quad (4.28)$$

can be judged from the fact that its eigenvalues 0, $-.027$, $-.071$, $-.118$ are close (less than 10% error) to the corresponding eigenvalues of G . With a corrected model (4.19) they are within 2%.

4.4. Two-Time-Scale Asymptotic Expansion

One of the applications of the singularly perturbed model (4.17) is the ability to obtain a two-time-scale asymptotic expansion of the solution vector $p(t)$. Then "slow" and "fast" components can be solved independently rather than solving the high order system of "stiff" differential equations. It is the purpose of this section to construct such an expansion. As time tends to infinity, the asymptotic series of differential equations will reduced to algebraic equations which can be used to solve for $\bar{p}$ in (4.4) in a computationally attractive manner. The ability to compute $\bar{p}$ efficiently for large chains has many applications [46].

We seek a solution to (4.17) of the form [8,13]

$$\eta = \hat{\eta}(t) + L_{\eta}(\tau) \quad (4.29)$$

$$Y = \hat{Y}(t) + L_Y(\tau) \quad (4.30)$$

where each term is a power series in ϵ with coefficients depending either on t , for the slow ("outer") series, or on $\tau = \frac{t}{\epsilon}$, for the fast ("inner") series.

$$\hat{\eta}(t) = \hat{\eta}_0(t) + \epsilon \hat{\eta}_1(t) + \dots \quad (4.31)$$

$$L_{\eta}(\tau) = L_{\eta}^0(\tau) + \epsilon L_{\eta}^1(\tau) + \dots \quad (4.32)$$

$$\hat{Y}(t) = \hat{Y}_0(t) + \epsilon \hat{Y}_1(t) + \dots \quad (4.33)$$

$$L_Y(\tau) = L_Y^0(\tau) + \epsilon L_Y^1(\tau) + \dots \quad (4.34)$$

Substituting (4.29) through (4.34) into (4.17) and equating the terms with like powers in ϵ , separately for t and τ series, we obtain, for zeroth order terms

$$\hat{\eta}_0(t) = \hat{\eta}_0(t) \text{VBT} \quad \hat{\eta}_0(0) = \eta(0) \quad (4.35)$$

$$L_{\eta}^0(\tau) = 0 \quad (4.36)$$

$$\hat{Y}_0(t) = 0 \quad (4.37)$$

$$\frac{dL_Y^0(\tau)}{d\tau} = L_Y^0(\tau) \text{ WAS } L_Y^0(0) = Y(0) \quad (4.38)$$

We see that within $O(\epsilon)$ the fast part of $\eta(t)$ and the slow part of $Y(t)$ are zero for all t . Furthermore, due to the asymptotic stability of (4.38) the fast term $L_Y^0(\tau) \rightarrow 0$ as $\tau = \frac{t}{\epsilon} \rightarrow \infty$. For small ϵ this "boundary layer term" is negligible for all $t > \hat{t}$, where $\hat{t}$ is of order $-\epsilon \ln \epsilon$. Thus, for $t > \hat{t}$ $p(t)$ is approximated within $O(\epsilon)$, by $\hat{\eta}_0(t) \text{V}$.

For first order terms we obtain

$$\hat{\eta}_1(t) = \hat{\eta}_1(t) VBT + \hat{\gamma}_1(t) WBT \quad (4.39)$$

$$\frac{dL_{\eta}^1(\tau)}{d\tau} = L_Y^0(\tau) WBT \quad (4.40)$$

$$\hat{\gamma}_1(t) = \hat{\eta}_0(t) VBS (WAS)^{-1} \quad (4.41)$$

$$\frac{dL_Y^1(\tau)}{d\tau} = L_Y^1(\tau) WAS + L_Y^0(\tau) WBS \quad (4.42)$$

The matching conditions become $L_{\eta}^1(0) = -\hat{\eta}_1(0)$ and $L_Y^1(0) = -\hat{\gamma}_1(0)$. By direct integration of (4.40) we obtain

$$L_{\eta}^1(\tau) = L_{\eta}^0(0) + \int_0^{\tau} L_Y^0(\sigma) WBT d\sigma. \quad (4.43)$$

Since $L_{\eta}^1(\infty) = 0$,

$$L_{\eta}^1(\tau) = -\int_{\tau}^{\infty} L_Y^0(\sigma) WBT d\sigma \quad (4.44)$$

using (4.38), this becomes

$$\begin{aligned} L_{\eta}^1(\tau) &= -\int_{\tau}^{\infty} \frac{dL_Y^0(\sigma)}{d\sigma} (WAS)^{-1} WBT d\sigma = -[L_Y^0(\infty) - L_Y^0(\tau)] (WAS)^{-1} WBT \\ &= L_Y^0(\tau) (WAS)^{-1} WBT \end{aligned} \quad (4.45)$$

thus at each stage only separate fast and slow differential equations need to be solved. An important property is that the fast equations are "decentralized" groups of states due to the fact that WAS is block-diagonal. From (4.35) through (4.45) we have

$$\eta(t) = \hat{\eta}_0(t) + \epsilon(\hat{\eta}_1(t) + L_{\eta}^1(\tau)) + O(\epsilon^2) \quad (4.46)$$

$$Y(t) = L_Y^0(\tau) + \epsilon(\hat{Y}_1(t) + L_Y^1(\tau)) + O(\epsilon^2)$$

higher order terms are obtained similarly. The computational advantages of the eigenspace iterations over the series solutions have been discussed in Chapter 3. However, for an intuitive understanding of how the solution vector decomposes, the series solutions are more attractive. As $t \rightarrow \infty$, the L-terms vanish and the equilibrium or steady state distribution $\bar{p}$ is given as

$$\bar{p} = (\bar{\eta}_0 + \epsilon \bar{\eta}_1 + \dots)V + (\epsilon \bar{Y}_1 + \epsilon^2 \bar{Y}_2 + \dots)W \quad (4.47)$$

where $\bar{\eta}_i = \hat{\eta}_i(t)$ and $\bar{Y}_i = \hat{Y}_i(t)$ as $t \rightarrow \infty$. The terms in (4.47) are easily computed from the following sequence of algebraic equations.

$$\bar{\eta}_0 \text{VBT} = 0 \quad \sum_{i=1}^N \bar{\eta}_0^i = 1 \quad (4.48)$$

$$\bar{Y}_1 = -\bar{\eta}_0 \text{VBS}(\text{WAS})^{-1} \quad (4.49)$$

$$\bar{\eta}_1 \text{VBT} + \bar{Y}_1 \text{WBT} = 0 \quad \sum_{i=0}^N \bar{\eta}_1^i = 0 \quad (4.50)$$

$$\bar{Y}_k = -(\bar{Y}_{k-1} \text{WBS} + \bar{\eta}_{k-1} \text{VBS})(\text{WAS})^{-1} \quad (4.51)$$

$$\bar{\eta}_k \text{VBT} + \bar{Y}_k \text{WBT} = 0 \quad \sum_{i=1}^N \bar{\eta}_k^i = 0 \quad (4.52)$$

Note that VBT is an $N \times N$ matrix and $W_j A_j S_j$ is $(\bar{\eta}_{j-1}) \times (\bar{\eta}_{j-1})$ for $j = 1, \dots, N$. Thus we need only solve N and $\bar{\eta}_i - 1, i = 1, \dots, N$ dimensional systems of linear equations to obtain a good approximation to $\bar{p}$. If we were to solve (4.4) directly not only can the dimension of $A + \epsilon B$ be large, but also the presence of ϵ could result in an ill-conditioned problem. Note that (4.48)-(4.52) are independent of ϵ .

Queueing network models of computer systems are often in the form of a large Markov generator [21,45,46]. Performance analysis measures are usually functions of $\bar{p}$. By using (4.48)-(4.52) these performance measures can be efficiently computed on the subsystem and aggregate level. We illustrate this concept on the model used in [46]. In this paper various iterative techniques for computing $\bar{p}$ are compared from a computational point of view. Direct methods of solution are not considered due to the large storage requirements. However, it is acknowledged that the results obtained using direct methods are more accurate and require substantially less total time than iterative techniques. The singular perturbation method we have proposed is a hybrid of direct and iterative methods. Each series term of $\bar{p}$ is solved directly on a significantly reduced order basis. However, to improve the approximation successive series terms must be solved (which can be viewed as an iterative process with convergence rate ϵ^k). We will now compute $\bar{p}$ using (4.48)-(4.52) for the model in [46]. This queueing model represents the architecture of a time-shared multiprogrammed paged virtual memory computer system. Assuming three jobs in the system, the Markov generator is of 20th order. Using the asymptotic series solutions (4.47)-(4.52), $\bar{p}_0 = \bar{\pi}_0 V$ and $\bar{p}_1 = \bar{\pi}_0 V + \epsilon(\bar{\pi}_1 V + \bar{Y}_1 W)$ were computed and compared to the actual steady state distribution $\bar{p}$ as follows:

$\bar{p} =$	0.0002493	$\bar{p}_0 =$	0.0002483	$\bar{p}_1 =$	0.0002493
	0.0004452		0.0004455		0.0004452
	0.0007962		0.0007992		0.0007961
	0.0014284		0.0014338		0.0014282
	0.0002509		0.0002502		0.0002508
	0.0004483		0.0004489		0.0004483
	0.0008027		0.0008052		0.0008028
	0.0002530		0.0002502		0.0002530
	0.0004527		0.0004489		0.0004527
	0.0002530		0.0002502		0.0002530
	0.0082679		0.0082697		0.0082677
	0.0080455		0.0080753		0.0080451
	0.0078604		0.0078856		0.0078601
	0.0082979		0.0082697		0.0082984
	0.0080917		0.0080753		0.0080921
	0.0082914		0.0082697		0.0082919
	0.1049480		0.1049442		0.1049481
	0.0363037		0.0362582		0.0363041
	0.1048605		0.1049442		0.1048591
	0.6996535		0.6996279		0.6996541

Note the accuracy using just one or two terms in the series.

In the remaining chapters the concepts developed in this chapter are applied to controlled Markov chain problems. Efficient two-time-scale design algorithms result in a near optimal control policy.

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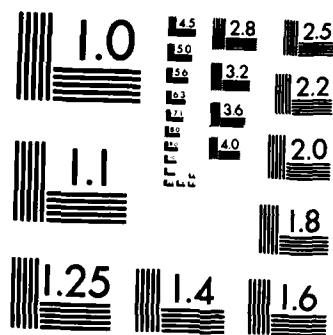
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CHAPTER 5

THE DISCOUNTED COST PROBLEM FOR CONTROLLED MARKOV CHAINS

5.1. Introduction

In this chapter we consider the controlled Markov chain

$$\dot{p}(t) = p(t) \left(\frac{A(u)}{\epsilon} + B(u) \right). \quad (5.1)$$

When the process is in a given state x , the control policy $u(x)$ determines the transition rates of the chain.

We assume that $a_{ij}(u(i))$ and $b_{ij}(u(i))$ are continuous on the compact set U_i for all $i, j = 1, \dots, n$. Thus, a policy $u \in \mathbb{R}^n$ is of the form

$$u = \begin{bmatrix} u(1) \\ \vdots \\ u(n) \end{bmatrix}$$

and can take any value in the Cartesian product $U_1 \times U_2 \times \dots \times U_n$. A control policy u is usually chosen to meet some performance specification. In this chapter we seek a control policy that minimizes the infinite horizon discounted cost [14,15]

$$J = E_x \int_0^\infty e^{\alpha\sigma} f(x_\sigma, u(x_\sigma)) d\sigma \quad (5.2)$$

where α is the discount rate and $f(i, u(i))$ is the instantaneous cost of being in state i using control $u(i)$. It is assumed that $f(i, u(i))$ is continuous on U_i , for $i = 1, \dots, n$.

In general, there are no closed form solutions to this problem. Thus, iterative techniques must be used to obtain the optimal policy. These methods may be classified as either policy iterations [14] or value iterations [15,44]. Policy iterations usually require more computations per iterate,

however, they often converge faster than value iterations [15]. The large dimension of the generator in (5.1) can make policy iterations impractical. Also, the order of magnitude difference between $\frac{A(u)}{\epsilon}$ and $B(u)$ can result in ill-conditioned nonlinear programming problems during the minimization phase of these algorithms. In this chapter, we apply the results of Chapter 4 to obtain a near optimal policy with less computational problems.

In Section 5.2 we motivate our approach to the problem by decomposing a finite time discounted cost for a fixed policy into aggregate and fast components. This enables us to identify an "aggregate" discounted cost problem that we will use to approximate the high order problem (5.2).

In Section 5.3 we consider the solution to the problem (5.1) and (5.2) as $\epsilon \rightarrow 0$. We then establish in what sense the resulting policy is near optimal.

Finally, in Section 5.4, a decentralized algorithm is presented for obtaining a near optimal control in a computationally attractive manner.

5.2. Decomposition of the Cost for a Fixed Policy

When the policy $u(x)$ is fixed and time is finite, (5.2) reduces to

$$J(x_0, t) = E \int_0^t e^{\alpha\sigma} f(x_\sigma) d\sigma. \quad (5.3)$$

It is well known [14] that J is the solution of

$$\dot{J} + \alpha J = \left(\frac{A}{\epsilon} + B\right)J + f \quad (5.4)$$

where

$$f = [f(1), f(2), \dots, f(n)]^T \quad (5.5)$$

and $J(t)$ is an n -dimensional column vector whose i th entry is the cost incurred starting in state i at time t .

As in (4.7), (4.12), (4.14) we transform (5.5) using

$$J_\eta = VJ, \quad J_Y = WJ \quad (5.6)$$

and obtain

$$\dot{J}_\eta + \alpha J_\eta = VBTJ_\eta + VBSJ_Y + Vf \quad (5.7)$$

$$\epsilon \dot{J}_Y + \epsilon \alpha J_Y = \epsilon WBTJ_\eta + W(A + \epsilon B)SJ_Y + \epsilon Wf. \quad (5.8)$$

Since this system is in standard singular perturbation form, we can apply Vasileva's two-time scale expansion procedure [13],

$$J_\eta(t) = \bar{J}_\eta(t) + \mathcal{L}_\eta(\tau) \quad (5.9)$$

$$J_Y(t) = \bar{J}_Y(t) + \mathcal{L}_Y(\tau) \quad (5.10)$$

where each term is a power series in ϵ with coefficients depending either on t , for the slow ("outer") series, or on $\tau = \frac{t}{\epsilon}$, for the fast ("inner") series,

$$\bar{J}_\eta(t) = \bar{J}_\eta^0(t) + \epsilon \bar{J}_\eta^1(t) + \dots \quad (5.11)$$

$$\mathcal{L}_\eta(\tau) = \mathcal{L}_\eta^0(\tau) + \epsilon \mathcal{L}_\eta^1(\tau) + \dots \quad (5.12)$$

$$\bar{J}_Y(t) = \bar{J}_Y^0(t) + \epsilon \bar{J}_Y^1(t) + \dots \quad (5.13)$$

$$\mathcal{L}_Y(\tau) = \mathcal{L}_Y^0(\tau) + \epsilon \mathcal{L}_Y^1(\tau) + \dots \quad (5.14)$$

Substituting (5.9) through (5.14) into (5.7), (5.8) and equating the terms with like powers in ϵ , separately for t and τ series, we obtain, for zeroth order terms

$$\frac{d\bar{J}_\eta^0(t)}{dt} + \alpha \bar{J}_\eta^0(t) = VBT\bar{J}_\eta^0(t) + Vf, \quad \bar{J}_\eta^0(0) = J_\eta(x_0, 0) \quad (5.15)$$

$$f_{\eta}^0(\tau) = 0 \quad (5.16)$$

$$\bar{J}_{\gamma}^0(t) = 0 \quad (5.17)$$

$$\frac{d f_{\gamma}^0(\tau)}{d\tau} = \text{WAS} f_{\gamma}^0(\tau), \quad f_{\gamma}^0(0) = J_{\gamma}(x_0, 0). \quad (5.18)$$

We see that within $O(\epsilon)$ the fast part of J_{η} and the slow part of J_{γ} are zero for all t . Furthermore, due to the asymptotic stability of (5.18) the fast term $f_{\gamma}^0(\tau) \rightarrow 0$ as $\tau = \frac{t}{\epsilon} \rightarrow \infty$. For small ϵ this "boundary layer term" is negligible for all $t > \hat{t}$, where $\hat{t}$ is of order $-\epsilon \ln \epsilon$. Thus, for $t > \hat{t}$ cost J is approximated within $O(\epsilon)$ by the "aggregate" cost $\bar{J}_{\eta}^0(t)$ defined by (5.15).

For first order terms we obtain

$$\frac{d \bar{J}_{\eta}^1(t)}{dt} + \alpha \bar{J}_{\eta}^1(t) = \text{VBT} \bar{J}_{\eta}^1(t) + \text{VBS} \bar{J}_{\gamma}^1(t) \quad (5.19)$$

$$\frac{d f_{\eta}^1(\tau)}{d\tau} = \text{VBS} f_{\eta}^0(\tau), \quad f_{\eta}^1(0) = -\bar{J}_{\eta}^1(0) \quad (5.20)$$

$$\bar{J}_{\gamma}^1(t) = -(\text{WAS})^{-1} (\text{WBT} \bar{J}_{\eta}^0(t) + \text{Wf}) \quad (5.21)$$

$$\frac{d f_{\gamma}^1(\tau)}{d\tau} + \alpha f_{\gamma}^1(\tau) = \text{WAS} f_{\gamma}^1(\tau) + \text{WBS} f_{\gamma}^0(\tau), \quad f_{\gamma}^1(0) = -\bar{J}_{\gamma}^1(0). \quad (5.22)$$

Observe that at $t=0$, $\tau=0$ the first order terms in each series sum to zero. Also observe that $\tau \rightarrow \infty$ all f terms tend to zero. Hence, by direct integration (5.20) yields an algebraic expression for $f_{\eta}^1(\tau)$ in terms of $f_{\gamma}^0(\tau)$,

$$f_{\eta}^1(\tau) = \text{VBS}(\text{WAS})^{-1} f_{\gamma}^0(\tau) \quad (5.23)$$

that is, at each stage, only separate fast and slow equations need to be solved. An important property is that the fast equations are "decentralized" groups of states due to the fact that WAS is block-diagonal.

From (5.9), (5.10), (5.16), and (5.17) we have

$$J_{\eta}(t) = \bar{J}_{\eta}^0(t) + \varepsilon(\bar{J}_{\eta}^1(t) + \ell_{\eta}^1(\tau)) + O(\varepsilon^2) \quad (5.24)$$

$$J_{\gamma}(t) = \ell_{\gamma}^0(\tau) + \varepsilon(\bar{J}_{\gamma}^1(t) + \ell_{\gamma}^1(\tau)) + O(\varepsilon^2). \quad (5.25)$$

Higher order terms can be determined in an analogous manner, or using techniques [12,13] which are computationally more efficient. Recalling (4.5), (4.12), (4.14), and (5.6) we consolidate (5.24) and (5.25) into

$$J = TJ_{\eta} + SJ_{\gamma} \quad (5.26)$$

which represents the two-time expansion of J for all t . For t large, $t \rightarrow \infty$, the ℓ -terms vanish and the equilibrium (infinite horizon) cost expansion is

$$J = TJ_{\eta}^0 + \varepsilon(TJ_{\eta}^1 + SJ_{\gamma}^1) + \varepsilon^2(TJ_{\eta}^2 + SJ_{\gamma}^2) + O(\varepsilon^3) \quad (5.27)$$

where $J_{\eta}^0 = \bar{J}_{\eta}^0$, $J_{\eta}^1 = \bar{J}_{\eta}^1$, $J_{\gamma}^1 = \bar{J}_{\gamma}^1$, etc. are uniquely defined as the equilibrium solutions of (5.15), (5.19), (5.21), etc. We see that aggregate cost TJ_{η}^0 is an $O(\varepsilon)$ approximation of the full cost J . This fact serves as motivation for the two-time-scale algorithm developed in Section 5.4.

5.3. Near Optimal Control

The infinite horizon discounted cost problem (5.1) and (5.2) is well defined and can, in principle, be solved using any one of a number of methods [15,44,53]. In all these methods, the computational burden can be prohibitive for large chains. However, in singularly perturbed models to reduce computations and improve convergence of design algorithms [7], a reduced or unperturbed ($\varepsilon=0$) problem is solved. The resulting policy

obtained is termed "near optimal." In this section we define the "reduced problem" and in what sense the resulting policy is near optimal.

For the purpose of intuitively understanding the near optimality results, let us first consider a special case of the more general problem outlined in Section 5.1. Assume that the control parameter u is a scalar and that u can take on any value in a segment U_0 . The Hamilton Jacobi equation for this problem is

$$0 = \min_{u \in U_0} \left\{ \left(\frac{A(u)}{\epsilon} + B^\alpha(u) \right) J + f(u) \right\} = \min_{u \in U_0} \{ \varphi(u, \epsilon) \} \quad (5.28)$$

where

$$f(u) = [f(1, u(1)), \dots, f(n, u(n))] \quad (5.29)$$

and $B^\alpha(u) = B(u) - \alpha I$. We now approximate the full optimal control problem (5.28), (5.2) for $\epsilon > 0$ by a simpler problem defined at $\epsilon = 0$. In (5.20) we cannot let $\epsilon = 0$, but if we substitute (5.27) into (5.28),

$$0 = \min_{u \in U} \{ A(u) S J_Y^1 + B^\alpha(u) T J_\eta^1 + f(u) + \epsilon [A(u) S J_Y^2 + B^\alpha(u) T J_Y^1 + B^\alpha(u) S J_Y^1] + O(\epsilon^2) \} \quad (5.30)$$

we obtain at $\epsilon = 0$ the "reduced" optimality condition

$$0 = \min_{u \in U} \{ A(u) S J_Y^1 + B^\alpha(u) T J_\eta^0 + f(u) \}. \quad (5.31)$$

We denote a control minimizing (5.31) by u_η . To avoid technicalities we assume that the derivatives A_u , B_u^α , and f_u of $A(u)$, $B^\alpha(u)$, and $f(u)$ are continuous and that the unique minima for each row in (5.31) are reached in the interior of U_0 , that is u_η satisfies

$$A_u(u_\eta) S J_Y^1 + B_u^\alpha(u_\eta) T J_\eta^0 + f_u(u_\eta) = 0. \quad (5.32)$$

Note that with u_n given, the substitution into (5.31) yields

$$A(u_n)SJ_Y^1 + B^\alpha(u)TJ_n^0 + f(u_n) = 0 \quad (5.33)$$

which, when multiplied by $V(u_n)$ and W , respectively, results in

$$V(u_n)B^\alpha(u_n)TJ_n^0 + V(u_n)f(u_n) = 0 \quad (5.34)$$

$$WA(u_n)SJ_Y^1 + WB^\alpha(u_n)TJ_n^0 + WF(u_n) = 0. \quad (5.35)$$

Conditions (5.33), (5.34), and (5.35) uniquely determine u_n , TJ_n^0 , and SJ_Y^1 . To see in what sense u_n is near-optimal, we represent the optimal control u^* for (5.28) in the form

$$u^* = u^0 + \epsilon u^1 + O(\epsilon^2). \quad (5.36)$$

Then we substitute

$$A(u^*) = A(u^0) + \epsilon A_u(u^0)u^1 + O(\epsilon^2) \quad (5.37)$$

and similar expressions for $B^\alpha(u^*)$ and $f(u^*)$ into (5.33), that is into

$$0 = A(u^*)SJ_Y^{1*} + B^\alpha(u^*)TJ_n^{0*} + f(u^*) + \epsilon [A(u^*)SJ_Y^{2*} + B^\alpha(u^*)(TJ_n^{1*} + SJ_Y^{1*})] + O(\epsilon^2) \quad (5.38)$$

and obtain

$$0 = A(u^0)SJ_Y^{1*} + B^\alpha(u^0)TJ_n^{0*} + f(u^0) + \epsilon \{ [A_u(u^0)SJ_Y^{1*} + B_u^\alpha(u^0)TJ_n^{0*} + f_u(u^0)]u^1 + A(u^0)SJ_Y^{2*} + B_u^\alpha(u^0)(TJ_n^{1*} + SJ_Y^{1*}) \} + O(\epsilon^2) = L(\epsilon). \quad (5.39)$$

We see from (5.31), (5.33), and (5.39) at $\epsilon = 0$ that $u^0 = u$ and, hence, J_Y^{1*} and J_n^{0*} are determined by $u^0 = u$. Next we note that the ϵ -term in (5.39) is zero for all $\epsilon \geq 0$ because u^* is optimal for all $\epsilon \geq 0$. This term involves the unknown first order optimal control term u^1 . However, by (5.32) the expression multiplying u^1 is zero. Thus the cost terms SJ_Y^{2*} and TJ_n^{1*} are uniquely

determined by u^0 . This establishes that when u^0 is found, the series (5.27) for the optimal cost is matched up to $O(\epsilon^2)$. Thus u_η is near optimal in the sense that

$$J(u^*) - J(u_\eta) = O(\epsilon^2). \quad (5.40)$$

This property that each term in the optimal control series matches two terms in the optimal cost series is found in other control problems [58].

For the more general problem posed in Section 5.1, it can be shown that when u_η is unique, (5.40) holds without the differentiability requirements on $A(u)$, $B^\alpha(u)$, and $f(u)$ needed in (5.37). This has been shown in [25] based on the theory of extremum in quasi-differentiable functions [60]. The basic result of [60] used in [25] is that

$$\left. \frac{\partial \mu(\epsilon)}{\partial \epsilon^+} \right|_{\epsilon=0} = \left. \frac{\partial \varphi(u^0, \epsilon)}{\partial \epsilon^+} \right|_{\epsilon=0} = A(u^0) S J_Y^2 + B^\alpha(u^0) (T J_\eta^1 + S J_Y^1) \quad (5.41)$$

where $\mu(\epsilon)$ and $\varphi(u, \epsilon)$ are defined in (5.39) and (5.28), respectively. (5.41) shows that the $O(\epsilon)$ coefficient of the expansion in (5.39) is dependent only on u^0 . Thus, J_Y^2 , J_η^1 , and J_Y^1 are determined uniquely by u^0 . For the special case assumed in the above deviation we showed this explicitly by observing that in (5.39) the only nonzero part in the coefficient of ϵ was given by (5.41).

When the reduced policy u_η is not unique, then (5.41) may not hold and we can only guarantee

$$J(u_\eta) - J^* = O(\epsilon), \quad (5.42)$$

although in practice we could expect much greater accuracy. The importance of the reduced problem will now be used to develop a decentralized optimization algorithm for the infinite horizon discounted cost problem.

5.4. Decentralized Optimization

In this section a decentralized algorithm is presented for obtaining a near-optimal control policy to the singularly perturbed controlled Markov chain problem of Section 5.1. Since the algorithm is based on value iterations, it is more convenient to work with the discrete-time version of (5.1), namely

$$p(k+1) = p(k) \left(\frac{A(u)}{\epsilon} + B(u) + I \right) \quad (5.42)$$

with discounted cost

$$J(x_0) = \min_{u \in U} E_x \sum_{k=0}^{\infty} \rho^{k+1} f(x_k, u(x_k)) \quad (5.43)$$

where $0 < \rho < 1$. The optimality condition becomes

$$J^* = \min_{u \in U} \rho \left\{ \frac{A(u)}{\epsilon} + B(u) + I \right\} J^* + f(u). \quad (5.44)$$

J^* is the n -dimensional optimal cost vector where J_i^* is the cost if the process starts in state i , for $i=1, \dots, n$.

For any given policy cost J can be found by solving the set of n linear equations

$$J = \rho \left[\left(\frac{A}{\epsilon} + B + I \right) J + f \right] \quad (5.45)$$

which we can rewrite as

$$\begin{bmatrix} J_n \\ J_Y \end{bmatrix} = \rho \begin{bmatrix} I+VBT & VBS \\ WBT & W(\frac{A}{\epsilon} + B + I)S \end{bmatrix} \begin{bmatrix} J_n \\ J_Y \end{bmatrix} + \rho \begin{bmatrix} V\bar{f} \\ W\bar{f} \end{bmatrix} \quad (5.46)$$

where $J_n = VJ$, $J_Y = WJ$ as in (5.6). Substitution of $J_n = J_n^0 + O(\epsilon)$ and $J_Y = J_Y^0 + O(\epsilon)$

into (5.46) gives $J_Y^0 = 0$ and

$$J_\eta^0 = \rho(I + VBT)J_\eta^0 + \rho Vf \quad (5.47)$$

and hence,

$$J = TJ_\eta^0 + O(\epsilon) \quad (5.48)$$

which agrees with (5.27). The form of the aggregate cost equation (5.47) suggests that J_η^0 can be considered as the optimal cost J_η^* for an aggregate problem whose optimality condition is

$$J_\eta^* = \min_{u \in U} \rho[(I + V(u)B(u)T)J_\eta^* + V(u)f(u)]. \quad (5.49)$$

An optimization algorithm known as the Jacobi iterations [15,44], when applied to (5.44) and (5.49) is, respectively

$$J^{k+1} = \min_u \rho\left\{\left(\frac{A(u)}{\epsilon} + B(u) + I\right)J^k + f(u)\right\} \quad (5.50)$$

$$J_\eta^{k+1} = \min_u \rho\{(I + V(u)B(u)T)J_\eta^k + V(u)f(u)\}. \quad (5.51)$$

In the aggregate problem (5.51) the dimension is reduced from n to N , but a difficulty is that it is not obvious how the control obtained in (5.51) depends on the original states. To avoid this difficulty, we rewrite (5.51) in the form

$$J_\eta^{k+1} = \min_u \rho\{V(u)[(I + B(u))TJ_\eta^k + f(u)]\}, \quad (5.52)$$

and interpret the term in the brackets as the cost $g^k(u)$ of an average cost-per-stage problem. It is an n -column vector which can be partitioned into N subvectors $g_j^k(u^j)$ corresponding to strongly interacting groups of states,

described by decoupled fast chains $A_j(u^j)$ where u^j denotes controls for states in j -th group. The solutions for the average-cost-per-stage problem for each fast chain exist under the ergodicity assumption on $A_j(u^j)$. They can be found using algorithms such as [54,55]. Then (5.52) is rewritten in a decentralized form

$$J_{\eta_j}^{k+1} = \min_{u^j} \rho [v_j(u^j) g_j^k(u^j)] \quad (5.53)$$

for each group $j=1, \dots, N$. Therefore, if at step k a "coordinator" obtains the results of (5.53) calculated locally in each group, its role is to consolidate the result in the form of J_{η}^{k+1} . This information is then used to form the new fast cost $g^{k+1}(u)$ according to

$$g^{k+1}(u) = [I + B(u)] T J_{\eta}^{k+1} + f(u). \quad (5.54)$$

Graphically, this algorithm has the decentralized structure in Figure 5.1.

The aggregate Markov process assumes that the "fast chains" have reached their steady state probabilities V_j . Each aggregate iteration (5.53) is in fact an infinite horizon problem for the "fast chains." These infinite horizon minimizations are in the form of N separate average cost per stage problems with respect to the costs $g^k(u^j)$, $j=1, \dots, N$. These costs contain not only the instantaneous subsystem cost $f(u^j)$, but also the cost contributions due to coupling to other subsystems. It is the latter that enables the fast problems to be solved independently. Other iterative algorithms, such as Gauss-Seidel [15] can be decentralized in a similar fashion.

We can now show that the limiting ($k \rightarrow \infty$) policy u_{η} in (5.53) satisfies (5.31) with $\rho = \frac{1}{1+\alpha}$, thus establishing the near optimality of u_{η} .

Lemma 5.1: The policy obtained in (5.53) as $k \rightarrow \infty$ satisfies the reduced optimality equation (5.31). Thus, if the policy u_η is unique, then $J(u_\eta) - J^* = O(\epsilon^2)$.

Proof: From [55], we can express the limit in (5.53) as N average cost per stage optimality equations of the form

$$J_{\eta_j} = \min_{u^j} \rho(A_j(u^j)c_j + g(u^j)) \quad (5.55)$$

$$= \min_{u^j} \rho(A_j(u^j)c_j + B_{j1}(u^j)T_{1\eta_1} + \dots + B_{jN}(u^j)T_{N\eta_N} + T_{j\eta_j} + f(u^j)) \quad (5.56)$$

or in vector form

$$TJ_\eta = \min_u \rho(A(u)c + B(u)TJ_\eta + TJ_\eta + f(u)) \quad (5.57)$$

where $c \in R^n$ are dual variables as defined in [14,55]. When $u = u_\eta$, (5.57) becomes

$$TJ_\eta = \rho(A(u_\eta)c + B(u_\eta)TJ_\eta + TJ_\eta + f(u_\eta)). \quad (5.58)$$

Premultiplying by W we obtain

$$0 = WA(u_\eta)c + WB(u_\eta)TJ_\eta + Wf(u_\eta) \quad (5.59)$$

which, from (5.35) uniquely defines c as SJ_Y^1 . Thus, letting $\rho = \frac{1}{1+\alpha}$, $\alpha > 0$, (5.57) takes the form

$$\alpha TJ_\eta^0 = \min_u \{A(u)SJ_Y^1 + B(u)TJ_\eta^0 + f(u)\} \quad (5.60)$$

which is equivalent to (5.31), thus from Section 5.3, the $O(\epsilon^2)$ optimality of the cost using policy u_η is established. In concluding this chapter, several useful properties of this algorithm should be cited.

1. No system of linear equations of any order need ever be solved.

2. Storing "aggregate" iterations of the form BT rather than just B reduces the number of computations and memory requirements.
3. The algorithm is independent of the small parameter ϵ . Thus, in general the algorithmic stiffness of the high order problem has been removed.
4. Fast chains perform their local minimizations in parallel. Fast costs are updated on the slow time scale by receiving only $J_n^{k+1} \in \mathbb{R}^N$ from the aggregate coordinator.

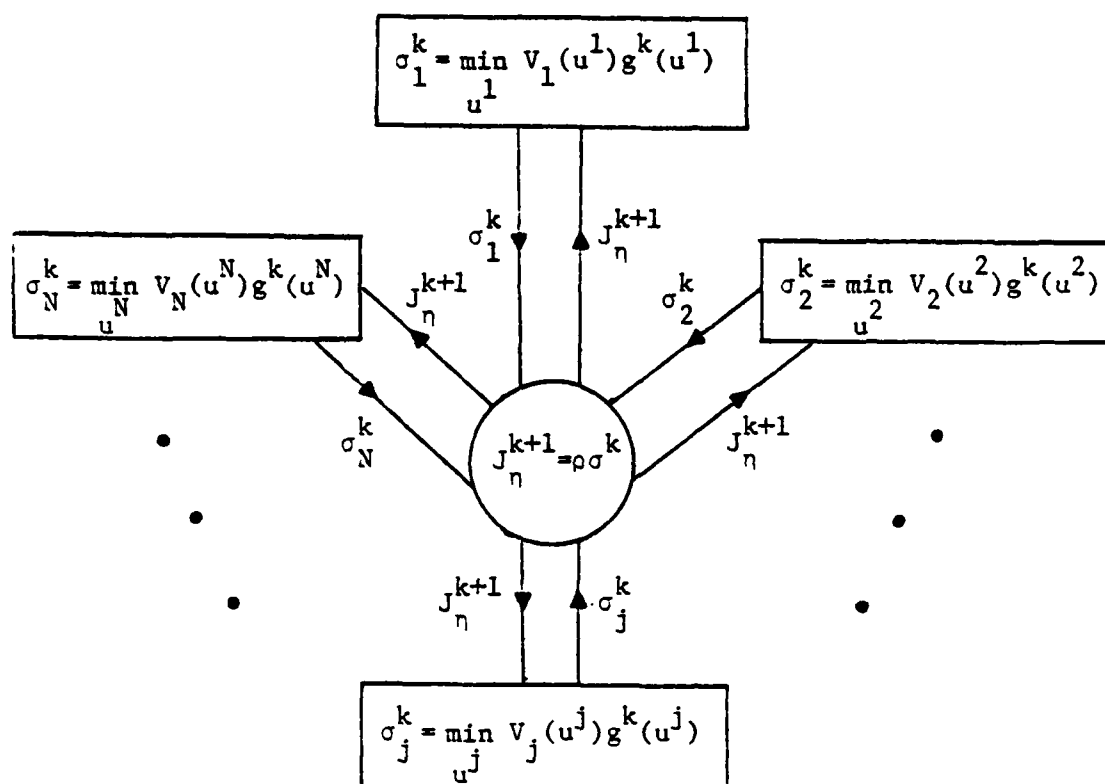


Figure 5.1. Two-time-scale algorithm for discounted cost problem.

CHAPTER 6

THE AVERAGE COST PER STAGE PROBLEM FOR CONTROLLED MARKOV CHAINS

6.1. Introduction

In many controlled Markov chain problems the discounted cost is not a physically meaningful measure of system performance. In such cases the average cost per stage [55,44,15] is commonly used. In discrete time, this cost function takes the form

$$J = \min_{u \in U} \lim_{T \rightarrow \infty} \frac{1}{1+T} E_x \sum_{k=0}^T f(x_k, u(x_k)) \quad (6.1)$$

where U and $f(i, u(i))$ are defined as in the discounted case.

In this chapter we again use the two-time-scale property of (5.42) to develop a computationally efficient algorithm for obtaining a near optimal control policy u for the problem defined by (5.42) and (6.1).

In Section 6.2 the optimality conditions for the average cost per stage problem are preserved and the cost for a fixed policy decomposed into "fast" and "aggregate" components.

In Section 6.3, we consider the solution to the problem (6.1) as $\varepsilon \rightarrow 0$ in (5.42). This results in a reduced problem for which a near-optimal policy is obtained.

In Section 6.4 a decentralized algorithm is developed for obtaining a near optimal policy in a computationally attractive manner.

Finally, in Section 6.5, an example is given illustrating the decentralized algorithm.

6.2. Decomposition of the Cost for a Fixed Policy

In this section we assume that the control policy u is fixed and we wish to obtain the corresponding cost (6.1). By using the two-time-scale property of (5.42) we will attempt to decompose the cost J into "fast" and "aggregate" components. This will lead us to the concept of the dual variables $c \in \mathbb{R}^n$, and their decomposition into "fast" and "aggregate" components. These results will be of importance in the next two sections.

Under the assumptions of Chapter 4 and Section 5.1, (6.1) can be written in the convenient form

$$J = \min_{u \in U} \bar{p}(u)f(u) \quad (6.2)$$

where $f(u)$ is defined in (5.29) and $\bar{p}(u)$ is the n -dimensional row vector of steady state probabilities defined by

$$\bar{p}(u) = \bar{p}(u) \left(\frac{A(u)}{\epsilon} + B(u) + I \right)$$

or

$$0 = \bar{p}(u) \left(\frac{A(u)}{\epsilon} + B(u) \right) \quad (6.3)$$

and

$$\sum_{i=1}^n \bar{p}_i(u) = 1. \quad (6.4)$$

Note that unlike in Chapter 5, here J is independent of initial state and is thus a scalar. Hence the optimal cost per stage is the same for all states.

Assume that there are only a small number of policies from which to choose. In this case, it may be simpler to evaluate (6.2) for each policy and pick the minimum. Unfortunately, (6.3) usually represents a large set of ill-conditioned linear equations. However, as we have shown in Chapter 4, $\bar{p}(u)$ can be written in the form

$$\bar{p}(u) = \bar{n}(u)V(u) + \bar{y}(u)W. \quad (6.5)$$

Then, for a given policy, (6.2) takes the form

$$\begin{aligned} J &= \bar{\eta} V f + \bar{\gamma} W f \\ &= \bar{\eta} f_{\eta} + \bar{\gamma} f_{\gamma} \end{aligned} \quad (6.6)$$

where $\bar{\eta}$ and $\bar{\gamma}$ can be computed efficiently using the asymptotic series solution developed in Chapter 4.

As an alternative to this approach, consider the optimality equation for the average cost per stage problem

$$J^* \underline{1} = \min_{u \in U} \left\{ \frac{A(u)}{\epsilon} + B(u) c^* + f(u) \right\} \quad (6.7)$$

where J^* is the optimal average cost per stage and $c^* \in \mathbb{R}^n$ the optimal dual variables as defined in [55]. Given a policy we could solve [14]

$$J \underline{1} = \left(\frac{A(u)}{\epsilon} + B(u) \right) c + f(u) \quad (6.8)$$

for J and c and pick the policy with minimum J as optimal if the number of policies is small. Again, to avoid solving this usually large set of possibly ill-conditioned linear equations (6.8) we premultiply (6.8) by $\begin{bmatrix} V \\ - \\ W \end{bmatrix}$. Define

$$c_{\eta} = Vc, \quad c_{\gamma} = Wc \quad (6.9)$$

and obtain

$$\begin{bmatrix} J \underline{1} \\ 0 \end{bmatrix} = \begin{bmatrix} VBT & VBS \\ \epsilon WBT & W(A + \epsilon B)S \end{bmatrix} \begin{bmatrix} c_{\eta} \\ c_{\gamma} \end{bmatrix} + \begin{bmatrix} Vf \\ Wf \end{bmatrix}. \quad (6.10)$$

Expanding J , c_{η} , and c_{γ} in asymptotic power series of the form

$$J = J_0 + \epsilon J_1 + \dots \quad (6.11)$$

$$C_n = C_n^0 + \epsilon C_n^1 + \dots \quad (6.12)$$

$$C_Y = C_Y^0 + \epsilon C_Y^1 + \dots \quad (6.13)$$

we obtain $C_Y^0 = 0$, and the remaining terms defined by

$$J_0 \underline{1} = VBTC_n^0 + Vf \quad (6.14)$$

$$C_Y^1 = -(WAS)^{-1}(WBTC_n^0 + Wf) \quad (6.15)$$

$$J_1 \underline{1} = VBTC_n^1 + VBSC_Y^1 \quad (6.16)$$

$$C_Y^2 = -(WAS)^{-1}(WBTC_n^1 + WBSC_Y^1) \quad (6.17)$$

$$\vdots$$

$$J_k \underline{1} = VBTC_n^k + VBSC_Y^k \quad (6.18)$$

$$C_Y^{k+1} = -(WAS)^{-1}(WBTC_n^k + WBSC_Y^k). \quad (6.19)$$

(6.14)-(6.19) can be solved on a reduced order basis. Note that (6.14), (6.16)... (6.18) have more unknowns than linear independent equations. One way around this problem as recommended in [14] is to set one of the $C_{n_i}^k$'s equal to zero for all k . Once this is done the remaining $C_{n_i}^k$'s and J_k can be calculated uniquely.

Once C_n^k is found, C_Y^{k+1} is uniquely determined. This property of non-uniqueness of C_n^k is to be expected since if C^* is the optimal dual variable in (6.7), so is $C^* + \delta \underline{1}$, $\forall \delta \in \mathbb{R}$. Thus, if we follow the above procedure in computing J_k , C_n^k , and C_Y^k , we will obtain the unique average cost per stage given by (6.11) and the dual variables C given by

$$C = TC_n + SC_Y \quad (6.20)$$

will be unique to within an additive vector of the form $\delta \underline{1}$, $\delta \in R$. Using either (6.6) or (6.10) we can efficiently compute the average cost per stage for a fixed policy.

Before concluding this section, it is interesting to note that (6.14) is of the form of an "aggregate" average cost per stage equation with transition matrix $I + VBT$ and instantaneous cost Vf . This idea will be of significance in Section 6.4.

6.3. Near Optimal Control

The problem defined by (6.1) and (5.42) can, in theory, be solved by a number of different methods [14,54,55]. However, analogous to Section 5.3 for the discounted problem, we wish to take advantage of the singularly perturbed model of (5.42) to obtain a reduced or unperturbed ($\epsilon=0$) problem. In this section we define the "reduced" average cost per stage problem and establish in what sense the resulting policy is near optimal. The results of this section are obtained in much the same way as the near optimality results of Section 5.3 for the discounted cost problem. Therefore, to avoid redundancy, many references will be made to that section.

Again, for intuitive understanding consider first the special case where u is a scalar and takes on values in a segment U_0 . The optimality equation for this problem can be written in the form

$$0 = \min_{u \in U_0} \left\{ \left(\frac{A(u)}{\epsilon} + B(u) \right) C + f(u) - J \right\} = \min_{u \in U_0} \{ \varphi(u, \epsilon) \} \quad (6.21)$$

where $f(u)$ is defined in (5.29). For any policy, C and J have the form (6.11)

and (6.20). Substituting these forms into (6.21) we obtain

$$0 = \min_{u \in U_0} \{A(u)SC_Y^1 + B(u)TC_\eta^0 + f(u) - J^0 \underline{1} + \varepsilon(A(u)SC_Y^2 + B(u)(TC_\eta^1 + JC_Y^1) - J^1 \underline{1}) + O(\varepsilon^2)\}. \quad (6.22)$$

Thus, the optimality condition for the reduced problem becomes

$$0 = \min_{u \in U_0} \{A(u)SC_Y^1 + B(u)TC_\eta^0 + f(u) - J^0 \underline{1}\}. \quad (6.23)$$

Again denote the control minimizing (6.23) as u_η . Assume that the derivatives A_u , B_u , and f_u are continuous and that the unique minima for each row in (6.23) are reached in the interior of U_0 , thus

$$A_u(u_\eta)SC_Y^1 + B_u(u_\eta)TC_\eta^0 + f_u(u_\eta) = 0 \quad (6.24)$$

and with u_η given, we obtain

$$J^0 \underline{1} = A(u_\eta)SC_Y^1 + B(u_\eta)TC_\eta^0 + f(u_\eta). \quad (6.25)$$

Premultiplying (6.25) by $V(u_\eta)$ and W results in

$$J^0 \underline{1} = V(u_\eta)B(u_\eta)TC_\eta^0 + V(u_\eta)f(u_\eta) \quad (6.26)$$

$$0 = WA(u_\eta)SC_Y^1 + WB(u_\eta)TC_\eta^0 + Wf(u_\eta) \quad (6.27)$$

which are in the form of (6.14) and (6.15) and hence can be solved uniquely for J^0 , SC_Y^1 , and TC_η^0 . To see in what sense u_η is near optimal, let the optimal control of (6.21) u^* , have the series form (5.36). Then substitute (5.37) and analogous expressions for $B(u^*)$ and $f(u^*)$ into (6.22) and obtain

$$0 = A(u^0)SC_Y^1 + B(u^0)TC_\eta^0 + f(u^0) - J^0 \underline{1} + \epsilon([A_u(u^0)SC_Y^1 + B_u(u^0)TC_\eta^0 + f_u(u^0)]u^1 + A(u^0)SC_Y^2 + B(u^0)(TC_\eta^1 + SC_Y^1) - J^1 \underline{1}) + O(\epsilon^2) = \mu(\epsilon). \quad (6.28)$$

From (6.25) and (6.23) at $\epsilon=0$, $u^0 = u_\eta$ and thus J^{0*} , C_η^{0*} , and C_Y^{1*} are determined by u_η . Next, note that due matching, the ϵ -term in (6.28) is zero $\forall \epsilon \geq 0$ since u^* is optimal $\forall \epsilon \geq 0$. This term involves the unknown first order optimal control term u^1 . However, the expression multiplying u^1 is, by (6.24), identically zero. Thus, J^{1*} , SC_Y^{2*} , and TC_η^{1*} are obtained from

$$J^1 \underline{1} = A(u^0)SC_Y^2 + B(u^0)(TC_\eta^1 + SC_Y^1) \quad (6.29)$$

and hence are uniquely defined by $u^0 = u_\eta$. Therefore, when u^0 is found, the series for the optimal cost (6.11) and optimal dual variables (6.12) and (6.13) are matched up to $O(\epsilon^2)$. Thus u_η is near optimal in the sense that

$$J(u_\eta) - J^* = O(\epsilon^2) \quad (6.30)$$

$$C(u_\eta) - C^* = O(\epsilon^2).$$

For the more general problem posed in Section 6.1, it can be shown [60] that when u_η is unique (6.30) and (6.31) hold without the differentiability requirements on $A(u)$, $B(u)$, and $f(u)$. This discussion is given in Section 5.3 and will not be repeated here.

6.4. Decentralized Optimization

In Section 6.2, the cost for a fixed policy was decomposed into "aggregate" and "fast" components. This enabled reduced order computations for finding the cost. Thus, for a small number of policies, the cost for each

policy computed and the optimal chosen. When the number of policies is large or infinite, an exhaustive search for the optimal policy is impractical and one of many iterative policy improvement algorithms must be used [14,54,55]. In this section we present a decentralized algorithm for computing a near optimal policy to the average cost per stage problem posed in Section 6.1. The structure of the algorithm is similar to that of Section 5.4 for discounted problem and hence offers the same computational advantages.

For every policy $u \in U$, $\bar{p}(u)$ can be written in terms of the "aggregate" and "fast" states as in (6.5). Substituting (6.5) into (6.2) we obtain

$$J = \min_{u \in U} \{ \bar{\eta}(u)V(u)f(u) + \bar{\gamma}(u)Wf(u) \}. \quad (6.32)$$

However, $\bar{\gamma}(u)$ is uniquely defined by

$$\begin{aligned} \bar{\gamma}(u) &= -\epsilon \bar{\eta}(u)V(u)B(u)S(u)(W(A(u) + \epsilon B))S(u))^{-1} \\ &= -\epsilon F(u). \end{aligned} \quad (6.33)$$

Thus, (6.32) takes the form

$$J = \min_{u \in U} \{ \bar{\eta}(u)\hat{f}(u) \} \quad (6.34)$$

where

$$\hat{f}(u) = (V(u) - \epsilon F(u)W)f(u).$$

Thus, the control problem has been reformulated in terms of the aggregate states $\eta \in R^N$. For a given policy we may compute J from (6.34) or by solving the following set of linear equations for J and the aggregate dual variables $C_\eta \in R^N$.

$$J \underline{1} = (V(u) - \epsilon F(u)W)B(u)TC_{\eta} + \hat{f}(u) \quad (6.35)$$

where $(V(u) - \epsilon F(u)W)B(u)T$ is the perturbed aggregate generator. Then the optimality equation for the aggregate problem (6.34) becomes

$$J \underline{1} = \min_{u \in U^*} \{ (V(u) - \epsilon F(u)W)B(u)TC_{\eta} + \hat{f}(u) \}. \quad (6.36)$$

Assume now that for an arbitrary policy u^k we have obtained the corresponding aggregate dual variables C_{η}^k . In most iterative algorithms [14,59,55], to improve the policy u^k , the following minimization is required

$$h(C_{\eta}^k) = \min_{u \in U} \{ (V(u) - \epsilon F(u)W)B(u)TC_{\eta}^k + \hat{f}(u) \}. \quad (6.37)$$

In the limit as $k \rightarrow \infty$, $h(C_{\eta}^k) \rightarrow J \underline{1}$. The elements of the aggregate generator and instantaneous cost are complex nonlinear functions of the controls for many of the original states and the minimization in (6.37) is difficult to carry out if not impossible. Thus, we have saved little by solving the low order aggregate problem (6.36) as opposed to (6.7) unless a further simplification is made. By letting $\epsilon \rightarrow 0$ in (6.37) we obtain

$$h(C_{\eta}^k) = \min_{u \in U} \{ V(u)B(u)TC_{\eta}^k + V(u)f(u) \}. \quad (6.38)$$

As expected, this optimization problem is equivalent to one we obtain if we were interested in minimizing only the 0th order aggregate cost term in (6.14). In other words, in (6.38) we are seeking a near optimal policy that satisfies the reduced optimality equation

$$J = \min_{u \in U} \{ V(u)B(u)TC_{\eta} + V(u)f(u) \}. \quad (6.39)$$

The analogy with the discounted problem of Section 5.4 should now be clear. We proceed to interpret the minimization operation in (6.38). The need to

know $V(u)$ as a function of u and the inability to disaggregate the policies obtained in (6.38) back to the original states makes the minimization of (6.38) impractical in its present form. This difficulty can be avoided by rewriting (6.38) in the form

$$h(C_n^k) = \min_{u \in U} \{V(u) [B(u)TC_n^k + f(u)]\}. \quad (6.40)$$

As in the discounted problem, partition u and $f(u)$ according to the fast subsystems

$$u = \begin{bmatrix} u^1 \\ \vdots \\ u^N \end{bmatrix} \quad f(u) = \begin{bmatrix} f(u^1) \\ \vdots \\ f(u^N) \end{bmatrix}. \quad (6.41)$$

To more clearly see the form of the vector $B(u)TC_n^k + f(u)$ we note that for a three class system it takes the form

$$\begin{bmatrix} B_{11}(u^1)T_1C_{n_1}^k + B_{12}(u^1)T_2C_{n_2}^k + B_{13}(u^1)T_3C_{n_3}^k + f(u^1) \\ B_{21}(u^2)T_1C_{n_1}^k + B_{22}(u^2)T_2C_{n_2}^k + B_{23}(u^2)T_3C_{n_3}^k + f(u^2) \\ B_{31}(u^3)T_1C_{n_1}^k + B_{32}(u^3)T_2C_{n_2}^k + B_{33}(u^3)T_3C_{n_3}^k + f(u^3) \end{bmatrix}. \quad (6.42)$$

It is crucial that each row i of (6.42) is a function of only one control $u(i)$, $i=1, \dots, n$. We can interpret each row of (6.42) as a cost $g^k(u(i))$ depending on a single control for a given C_n^k . If we partition this cost according to the fast subchain dimensions we obtain

$$g^k(u) = \begin{bmatrix} g^k(u^1) \\ g^k(u^2) \\ \vdots \\ g^k(u^N) \end{bmatrix}. \quad (6.43)$$

This, (6.40) takes the form

$$h(C_\eta^k) = \min_{u \in U} \{V(u)g^k(u)\}. \quad (6.44)$$

However, since $V(u)$ is block diagonal, (6.44) can be written as

$$h_1(C_\eta^k) = \min_{u^1 \in U^1} \{V_1(u^1)g^k(u^1)\} \quad (6.45)$$

$$\vdots$$

$$h_N(C_\eta^k) = \min_{u^N \in U^N} \{V_N(u^N)g^k(u^N)\}. \quad (6.46)$$

This formulation points out the possibility of decentralization, because (6.45)-(6.46) are average cost per stage problems for the fast chains $A_i(u^i)$ with respect to the instantaneous cost $g_i^k(u^i)$, $i = 1, \dots, N$ which can improve the controls at a "local" subsystem level using "local" costs $g^k(u^i)$. The $g^k(u^i)$'s are updated on a slower time scale at the aggregate level which assigns the new C_η^{k+1} .

Once the vector $h(C_\eta^k) \in R^N$ is computed, different algorithms [14,54,55] use different means to update the dual variables C_η^{k+1} . A recently developed algorithm for this problem is due to Varaiya [55]. We now use this algorithm to illustrate the hierarchical structure of the decentralized optimizations (6.45)-(6.46).

The first thing we must establish is that the problem we are attempting to solve is a well defined average cost per stage problem satisfying all the conditions necessary for Varaiya's algorithm to converge to the optimal policy. The problem we are trying to solve is summarized in (6.39). First note that $I + V(u)B(u)T$ is a well defined transition matrix whose elements depend continuously on $u \in U$. Second, under the ergodicity assumptions on (5.42), $I + V(u)B(u)T$ contains a single ergodic class $\forall u \in U$. And finally, $V(u)f(u)$ is a well defined vector of instantaneous costs whose elements depend continuously on the vector u . Thus, the problem satisfies the conditions in [55] necessary for the algorithm to converge to the optimal policy. We now review Varaiya's algorithm for the high order problem (6.7).

Given an arbitrary C^0 , define $h(C^0)$ as the vector of values resulting from the n pointwise minimizations

$$h(C^0) = \min_{u \in U} \left\{ \frac{A(u)}{\epsilon} + B(u)C^0 + f(u) \right\}. \quad (6.47)$$

The algorithm then proceeds as follows: Let

$$\bar{h}(C^0) = \max_i h_i(C^0) \quad (6.48)$$

$$\underline{h}(C^0) = \min_i h_i(C^0) \quad (6.49)$$

and find $C^1 = C^0 + \Delta t f(C^0)$ according to a discretization of the differential equation [56]

$$\frac{dc}{dt} = h(c) - \frac{1}{n} [h(c)^T \underline{1}] \underline{1} = f(c). \quad (6.50)$$

Then C^1 is used to obtain $h(C^1)$ as in (6.47) and the cycle continues.

Convergence is monotonic and can be "measured" since $\bar{h}(C^k) - \underline{h}(C^k) \rightarrow 0$ as $k \rightarrow \infty$.

We now apply this algorithm to solve (6.79). For a given initial C_n^0 , define $h(C_n^0) \in \mathbb{R}^N$ as

$$h(C_n^0) = \min_{u \in U} \{V(u)g^0(u)\} \quad (6.51)$$

where

$$g^0(u) = B(u)TC_n^0 + f(u). \quad (6.52)$$

In component form (6.51) becomes

$$\begin{aligned} h_1(C_n^0) &= \min_{u^1 \in U^1} \{V_1(u^1)g^0(u^1)\} \\ &\vdots \end{aligned} \quad (6.53)$$

$$h_N(C_n^0) = \min_{u^N \in U^N} \{V_N(u^N)g^0(u^N)\}. \quad (6.54)$$

Each "fast" chain average cost per stage minimization in (6.53)-(6.54) can be solved in parallel using, for example, Varaiya's algorithm. The results of the minimizations gives

$$\begin{aligned} h_1(C_n^0) \underline{1} &= \min_{u^1 \in U^1} \{A_1(u^1)C^1 + g^0(u^1)\} \\ &\vdots \\ h_N(C_n^0) \underline{1} &= \min_{u^N \in U^N} \{A_N(u^N)C^N + g^0(u^N)\} \end{aligned} \quad (6.55)$$

where $C^i \in \mathbb{R}^{n_i}$, $i=1, \dots, N$ are the dual variables during the fast optimizations. Note that $h_i(C_n^0)$ is the optimal average cost per stage for fast chain i under instantaneous cost $g^0(u^i)$, for $i=1, \dots, N$.

Let

$$\bar{h}(C_n^0) = \max_i h(C_n^0) \quad (6.56)$$

$$\underline{h}(C_n^0) = \min_i h(C_n^0) \quad (6.57)$$

and find $C_\eta^1 = C_\eta^0 + \Delta t_\eta f(C_\eta^0)$ according to a discretization of the differential equation

$$\frac{dc_\eta}{dt} = h(C_\eta) - \frac{1}{N} [h(C_\eta) \underline{1}] \underline{1} = f(C_\eta). \quad (6.58)$$

Use C_η^1 to update the fast cost

$$g^1(u) = B(u)TC_\eta^1 + f(u). \quad (6.59)$$

Intuitively, the C_η^i 's "distribute" information about the "motion" of the total system relative to each fast subsystem. The aggregate interactions, $B(u)T$ in (6.59), are "weighed" accordingly and the updated cost $g^1(u^i)$, $i=1, \dots, N$ is obtained. It is this component of the cost $g^1(u)$ in (6.59) that allows the optimizations of (6.55) to be carried out independently. Next, (6.51) is solved for $h(C^1)$ given $g^1(u)$ and the cycle continues. Convergence occurs when

$$\bar{h}(C_\eta^k) - \underline{h}(C_\eta^k) < \delta$$

where δ is some design tolerance. Thus, the aggregate serves as a "coordinator" passing and receiving the necessary information between subsystems allowing the subsystems to compute "local" controls that are "globally" near optimal. Graphically, the aggregate coordination scheme is given in Figure 6.1.

We now show that the limiting policy ($k \rightarrow \infty$) in (6.55) satisfies optimality equation (6.23). Thus, the near optimality results of Section 6.3 hold.

Lemma 6.1. Let u_η be the near optimal policy obtained using the two-time scale algorithm. Then if u_η is unique, $J(u_\eta)$ satisfies (6.30).

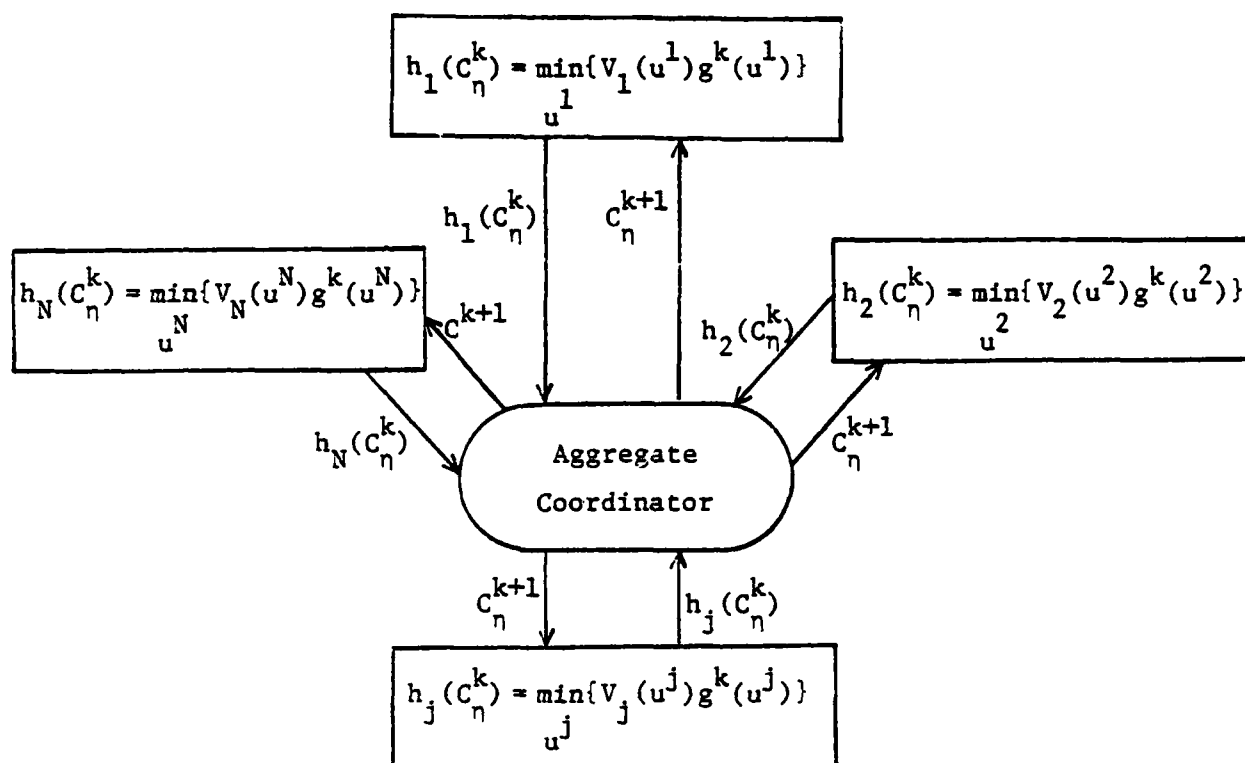


Figure 6.1. Two-time-scale average cost per stage algorithm

Proof: When the algorithm converges, (6.55) becomes

$$\begin{aligned} J(u_n) \underline{1} &= \min_{u^1 \in U^1} \{A_1(u^1)C^1 + g(u^1)\} \\ &\vdots \\ J(u_n) \underline{1} &= \min_{u^N \in U^N} \{A_N(u^N)C^N + g(u^N)\} \end{aligned} \quad (6.60)$$

or in vector form

$$J(u_n) \underline{1} = \min_{u \in U} \{A(u)C_F + g(u)\} \quad (6.61)$$

where

$$C_F = \begin{bmatrix} C^1 \\ C^2 \\ \vdots \\ C^N \end{bmatrix} \in R^n \quad (6.62)$$

is the vector of dual variables resulting from the fast subproblems. We now show how C_F relates to the expansion terms of Section 6.1.

From (6.59), (6.61) becomes

$$J(u_n) \underline{1} = \min_{u \in U} \{A(u)C_F + B(u)TC_n + f(u)\} \quad (6.63)$$

and at $u = u_n$,

$$J(u_n) \underline{1} = A(u_n)C_F + B(u_n)TC_n + f(u_n). \quad (6.64)$$

Premultiplying by $V(u_n)$ and W we obtain

$$J(u_n) \underline{1} = V(u_n)B(u_n)TC_n + V(u_n)f(u_n) \quad (6.65)$$

$$0 = WA(u_n)C_F + WB(u_n)TC_n + Wf(u_n) \quad (6.66)$$

which, from Section 6.2, uniquely defines C_F as SC_Y^1 and C_η as C_η^0 . Thus, (6.63) becomes

$$J(u_\eta) \stackrel{1}{=} \min_{u \in U} \{A(u)SC_Y^1 + B(u)TC_\eta^0 + f(u)\} \quad (6.67)$$

which is equivalent to (6.23). Thus, if u_η is unique, then $J(u_\eta) - J^* = O(\epsilon^2)$.

The attractive computational aspects of the algorithm are the same as the algorithm for the discounted problem and are listed at the end of Section 5.4. We close this chapter and this thesis in the next section with an example.

6.5. Example - Minimizing the Average Cost Per Stage

In this section we give an example of applying the algorithm presented in Section 6.4. The controlled Markov chain we will be considering has the following state transition matrix:

.45	.45	0	.05	.05	0	0	0	0
.27	.36	.27	.03	.04	.03	0	0	0
0	.72	.18	0	.08	.02	0	0	0
.5 μ	.5 μ	0	.45-.5 μ	.45-.5 μ	0	.05	.05	0
.3 μ	.4 μ	.3 μ	.27-.3 μ	.36-.4 μ	.27-.3 μ	.03	.04	.03
0	.8 μ	.2 μ	0	.72-.8 μ	.18-.2 μ	0	.08	.02
0	0	0	.5 u	.5 u	0	.5(1- u)	.5(1- u)	0
0	0	0	.3 u	.4 u	.3 u	.3(1- u)	.4(1- u)	.3(1- u)
0	0	0	0	.8 u	.2 u	0	.8(1- u)	.2(1- u)

(6.68)

where the states are defined by

	x_1	x_2	x_3	x_4	x_5	x_6	x_7	x_8	x_9
G	2	2	2	1	1	1	0	0	0
D	0	1	2	0	1	2	0	1	2

(6.69)

The control problem can be visualized as one of maintenance scheduling. From (6.69) the state of the process is defined by the variables G and D. G corresponds to the number of power generating units available while D corresponds to the demand in terms of generating units needed. Markov models of this type are common in optimal resource scheduling problems [22,59].

The units have a common failure rate over which we have no control. On the other hand, we can control the rate of repair. By increasing the repair rate, the probability of repairing the generator during the next unit time interval, μ , improves.

The higher probability of repair is obtained only by increasing the system cost due to assigning labor, new parts, etc. Therefore, the control variable will be the scalar quantity μ taking values in the closed interval $.02 \leq \mu \leq .2$. When the process arrives in one of the nine states identified in (6.69), the amount of maintenance scheduled is proportional to μ . Hence, the control problem is to find the policy $\mu(G,D)$ that minimizes the average cost per stage (6.1) with instantaneous cost

$$f(G,D,\mu(G,D)) = [(D-G)^+]^2 + K(\mu(G,D))^2 \quad (6.70)$$

where $(b)^+ = \max(0,b)$. This cost is composed of two terms: $[(D-G)^+]^2$ penalizes for not meeting the demand while $K(\mu(G,D))^2$ penalizes maintenance costs. Thus, the problem is well defined and possesses a nontrivial solution.

Note that (6.68) is of the form $\frac{A}{\epsilon} + B + I$ where

$$A = \left[\begin{array}{ccc|ccc|ccc} -.09 & .09 & 0 & & & & & & \\ .054 & -.108 & .054 & & & & & & \\ 0 & .144 & -.144 & & & & & & \\ \hline & & & -.08+1\mu & .09-.1\mu & 0 & & & \\ & & & .054-.06\mu & -.108+.12\mu & .054-.06\mu & & & \\ & & & 0 & .144-.16\mu & .144+.16\mu & & & \\ \hline & & & & & & -.1+.1\mu & .1-.1\mu & 0 \\ & & & & & & .06-.06\mu & -.12+.12\mu & .06-.06\mu \\ & & & & & & 0 & .16-.16\mu & -.16+.16\mu \end{array} \right] \quad (6.71)$$

and

$$B = \left[\begin{array}{ccc|ccc|ccc} -.1 & 0 & 0 & .05 & .05 & 0 & & & \\ 0 & -.1 & 0 & .03 & .04 & .03 & & & 0 \\ 0 & 0 & -.1 & 0 & .08 & .02 & & & \\ \hline .5\mu & .5\mu & 0 & -\mu-.1 & 0 & 0 & .05 & .05 & 0 \\ .3\mu & .4\mu & .3\mu & 0 & -\mu-.1 & 0 & .03 & .04 & .03 \\ 0 & .8\mu & .2\mu & 0 & 0 & -\mu-.1 & 0 & .08 & .02 \\ \hline & & & .5\mu & .5\mu & 0 & -\mu & 0 & 0 \\ & 0 & 0 & .3\mu & .4\mu & .3\mu & 0 & -\mu & 0 \\ & & & 0 & .8\mu & .2\mu & 0 & 0 & -\mu \end{array} \right] \quad (6.72)$$

where $\epsilon = .2$ in (6.68). First the problem was solved using $K = 30$ in (6.70).

Using the algorithm of Section 6.4, the following near optimal policy u_n was obtained

$$u_{\eta} = \begin{bmatrix} .02 \\ .02 \\ .02 \\ .111 \\ .111 \\ .09 \\ .162 \\ .142 \\ .086 \end{bmatrix} . \quad (6.73)$$

For the purpose of comparison, the optimal policy u^* was computed for several values of ϵ including $\epsilon = .2$. These results are given in Table 6.1.

Table 6.1. Optimal policies and cost for $K = 30$

ϵ	.5	.2	.15	.1	.05	.01
$u^*(G,D)$	.02	.02	.02	.02	.02	.02
	.02	.02	.02	.02	.02	.02
	.02	.02	.02	.02	.02	.02
	.092	.104	.106	.108	.110	.111
	.107	.111	.111	.111	.111	.111
	.178	.111	.107	.102	.097	.092
	.088	.126	.134	.143	.152	.16
	.138	.144	.144	.144	.144	.143
	.2	.149	.136	.121	.104	.090
J_{ϵ}^*	.65776	.671	.67145	.67095	.66913	.66648
$J(u_{\eta})$	.69722	.6811	.67767	.67399	.66999	.66654

Table 6.1 clearly illustrates the convergence of u^* to u_{η} as $\epsilon \rightarrow 0$. Also in Table 6.1, note that $J(u_{\eta}) - J_{\epsilon}^* < \epsilon^2$ for all values of ϵ . To see the effects

of changing the cost coefficients in (6.70), let $K = 20$. The near optimal policy u_n is then

$$u_n = \begin{bmatrix} .02 \\ .02 \\ .02 \\ .140 \\ .140 \\ .108 \\ .200 \\ .190 \\ .102 \end{bmatrix} \quad (6.74)$$

and the optimal policies for various values of ϵ given in Table 6.2.

Table 5.2. Optimal policies for $K = 20$

ϵ	.5	.2	.15	.1	.05	.01
$\mu^*(G,D)$	.02	.02	.02	.02	.02	.02
	.02	.02	.02	.02	.02	.02
	.02	.02	.02	.02	.02	.02
	.114	.129	.132	.135	.138	.140
	.135	.139	.140	.140	.141	.141
	.165	.140	.133	.126	.118	.110
	.116	.167	.178	.191	.200	.200
	.185	.192	.193	.193	.192	.191
	.200	.200	.180	.158	.122	.108

These results confirm our intuition in that lower values of K result in more maintenance scheduled (higher values of $\mu(G,D)$). From a computational point of view, the advantages of the two-time-scale algorithm over the standard method were significant. For this example, these advantages can be summarized

as follows:

- i) Memory requirements - The ability to aggregate interactions in the form of BT combined with having to solve only subsystem problems reduced on line storage requirements by about a 3 to 1 ratio.
- ii) Decentralized computations - The computations are performed in a decentralized manner allowing distributed computing when needed.
- iii) Ill-conditioning - As ϵ becomes smaller, the orders of magnitude difference between $\frac{A(u)}{\epsilon}$ and $B(u)$ causes the high order problem to converge very slowly. The two-time-scale solution is independent of ϵ and converges rapidly.
- iv) Computation time - Due to both the reduced dimension of the subsystem problems and the improved conditioning of the two-time-scale algorithm, CPU time was significantly reduced. For $\epsilon = .2$ the reduction in CPU time was about 3 to 1, for $\epsilon = .1$ about 6 to 1, and for $\epsilon = .05$ about 12 to 1.

In conclusion, this simple 9th order example has clearly demonstrated the usefulness of the two-time-scale algorithm. For more complex controlled Markov chain problems [21,59,62], the advantages of this algorithm will be of even greater significance.

CHAPTER 7

CONCLUSIONS

The need to analyze and control large scale systems is a research area that will always remain rich in potential. More demanding performance leads to more complex models necessitating the research and development of improved analysis and design techniques. This thesis has helped to unify one area of such research and open up another potentially promising new area.

In Chapters 2 and 3 we considered deterministic linear time-invariant systems and the existence of a "two-time-scale" property. It was shown that the reduced order modeling obtained through singular perturbations was directly related to dominant left and right eigenspace power iterations. This led to a unified design methodology for reduced order modeling and control of two-time-scale systems from which many previous design methods were shown to have been special cases.

In the remainder of this thesis, we considered stochastic systems which can be modeled as large finite state Markov processes. The "weak" and "strong" transition probabilities characteristic of many Markov chain models was interpreted as a two-time-scale property through singular perturbation modeling. This led to the concept of a reduced order "aggregate" Markov chain. This enabled reduced order asymptotic series solutions to be obtained for the steady state probability distribution, a problem frequently encountered in queueing theory. The aggregate model is then used to obtain near optimal policies for controlled Markov chain problems. The resulting optimization algorithms are decentralized in the sense that fast subsystems compute their controls "locally" with the aggregate coordinating necessary

information between subsystems on a slow time scale. This avoids much of the computational burden associated with many controlled Markov chain problems. The performance functions considered were the discounted cost and the average cost per stage.

The potential areas for further research and applications are numerous. Performance analysis of computer systems and numerical solutions to complex stochastic control problems seem to be the most immediate application areas. Research areas include incorporating dominant transient states [25] into the two-time-scale model (5.1), decentralized control over a finite time horizon, and Markov modeling techniques that result in the form (5.1). Many new results are now starting to appear in these areas [25,57,61,62]. The theoretical richness of this research area combined with numerous potential application fields should result in many more significant contributions to Markov modeling and Markovian decision processes in the near future.

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