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SPLINE FITTING THROUGH PARALLEL PROCESSING(U) VIRGINIA
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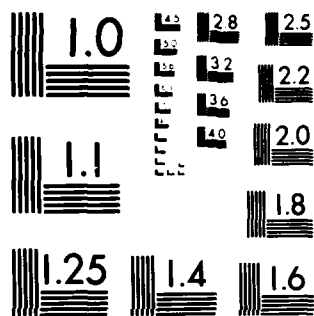
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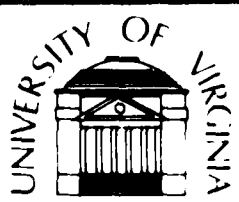
Submitted to:
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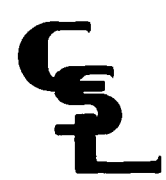
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COMMUNICATIONS SYSTEMS LABORATORY
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A Technical Report

SPLINE FITTING THROUGH PARALLEL PROCESSING

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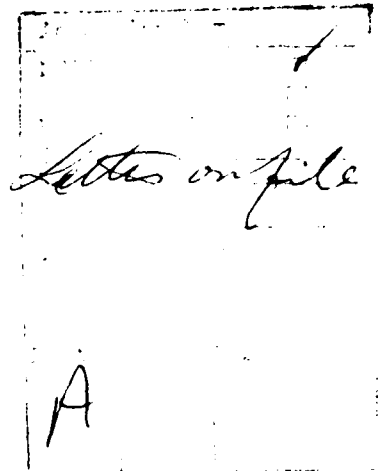
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ABSTRACT

A new method is presented for fitting polynomial splines to n equispaced data. Using Jain's [32] cyclic decomposition of banded Toeplitz matrices, we show that the operations can be performed by n -point Fast Fourier Transforms (FFT). Thus, the use of parallel processing FFT techniques provides a speed of $O(2\log_2 n)$, independently of the degree of the spline. Explicit solutions are derived for the cubic, quartic and quintic spline.



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Introduction

✓ In many applications there appears the need to represent a set of raw data by fitting to them a smooth function or set of functions. A great amount of theoretical work has been done within the framework of approximation theory in studying properties of curve fitting for various families of approximating functions.

One of the most attractive and well structured families of approximating functions are the splines. They have been extensively studied in the mathematical literature, for example in [1] - [7] and elsewhere.

— Splines have been very useful in statistics. References [7] - [18] represent the most significant spline application papers in the statistical literature.

— In the engineering literature, spline functions have been used as approximating tools, in the areas of Systems ([19] - [22]) and Pattern Recognition. ([23] - [30]).

In ~~the present~~ paper we concentrate on a fast, parallel computation technique for fitting a spline to n equispaced data points. Existing techniques can fit splines in $O(n)$ time, with recursive (serial) processing of the data, as in [16]. The new technique is based on the use of Fast Fourier Transform, and its parallel processing capability. As a result, our technique achieves the fitting of a spline in $O(\log_2 n)$ time. The n data points must be equispaced.

Cubic Spline Fit

Let $\{(x_i, y_i) \mid i = 0, 1, \dots, n\}$ be a set of data points with $a = x_0 < x_1 < \dots < x_n = b$. We would like to fit to the data a function $S(x)$ that has two derivatives. We require that $S(x_i) = y_i$, $i = 0, 1, \dots, n$ and that under the above requirement S minimizes the integral

$$I_2(S) = \int_a^b [S''(x)]^2 dx \quad (1)$$

under the conditions $S'(a) = S'(b) = 0$.

In the theory of spline functions it is shown [2] that the above constrained minimum is achieved when $S(x)$ is a set of piecewise cubic polynomials with continuous first and second derivatives at the points $\{x_i, i=1, \dots, n-1\}$. We will be concerned here only with uniformly spaced x_i 's, hence $x_i = x_0 + ih$, $i = 0, 1, \dots, n$. h = increment. The continuity requirement demands that:

$$S'(x_i^-) = S'(x_i^+), \quad S''(x_i^-) = S''(x_i^+) \quad (2)$$

for $i = 1, 2, \dots, n-1$.

We denote M_i the second derivative of $S(x)$ at x_i . Then

$$S''(x) = M_{i-1}(x_i - x)h^{-1} + M_i(x - x_{i-1})h^{-1}, \quad x_{i-1} \leq x \leq x_i \quad (3)$$

Integrating (3) twice and using the conditions $\{S(x_i) = y_i\}$ we obtain:

$$\begin{aligned} S(x) = & M_{i-1}(x_i - x)^3(6h)^{-1} + M_i(x - x_{i-1})^3(6h)^{-1} + \\ & + (h^{-1}y_i - hM_i6^{-1})(x - x_{i-1}) + (h^{-1}y_{i-1} - hM_{i-1}6^{-1})(x_i - x), \\ & x_{i-1} \leq x \leq x_i \end{aligned} \quad (4)$$

Differentiating (4) once and using the continuity of the first derivative for $i=1, \dots, n-1$, we find:

$$M_{i-1} + 4M_i + M_{i+1} = 6h^{-2}(y_{i-1} - 2y_i + y_{i+1}) \quad i=1, 2, \dots, n-1 \quad (5)$$

We assume M_0 and M_n are known. Then the set of unknowns is

$M_1, M_2, \dots, M_{n-1}$, and we have a set of $n-1$ equations from (5) with an equal number of unknowns. In matrix notation the set of equations is:

$$TM = 6h^{-2}H \quad (6)$$

where

$$M = \begin{bmatrix} M_1 \\ M_2 \\ \vdots \\ M_{n-1} \end{bmatrix}, \quad T = \begin{bmatrix} 4 & 1 & & & & & \\ 1 & 4 & 1 & & & & 0 \\ & 1 & 4 & 1 & & & \\ & & & \ddots & & & \\ & & & & \ddots & & \\ & 0 & & & & \ddots & \\ & & & & & 1 & 4 & 1 \\ & & & & & & 1 & 4 \end{bmatrix} \quad (7)$$

$$H = \begin{bmatrix} y_0 = 2y_1 + y_2 - h^2 M_0 6^{-1} \\ y_1 = 2y_2 + y_3 \\ y_2 = 2y_3 + y_4 \\ \vdots \\ y_{n-3} = 2y_{n-2} + y_{n-1} \\ y_{n-2} = 2y_{n-1} + y_n - h^2 M_n 6^{-1} \end{bmatrix} = \begin{bmatrix} h_1 \\ h_2 \\ \vdots \\ h_{n-1} \end{bmatrix}$$

and T is an $(n-1) \times (n-1)$ matrix, H is an $(n-1)$ vector. Clearly the basic problem here is the efficient inversion of T and the multiplication $T^{-1}H$. We define the following $(n-1) \times (n-1)$ matrix:

$$T_p = \begin{bmatrix} 1 & p & & & & \\ p & 1 & p & 0 & & \\ & p & 1 & p & & \\ & & \ddots & \ddots & \ddots & \\ & & & \ddots & \ddots & \\ & & & & 1 & p \\ & & & & p & 1 \end{bmatrix}$$

It is a known result from [31] that the eigenvalues q_k and eigenvectors V_k of T_p are:

$$q_k = 1 + 2p \cos(k\pi n^{-1}) \quad (8)$$

$$V_k = [\sin(k\pi n^{-1}), \sin(2k\pi n^{-1}), \dots, \sin((n-1)k\pi n^{-1})] \sqrt{2/n} \\ k = 1, \dots, n-1 \quad (9)$$

If we divide T by 4, we have the special case of (7) with $p = 1/4$. Using the eigenvector-eigenvalue expansion of T , we have:

$$T = 4 \sum_{k=1}^{n-1} [1 + 0.5 \cos(k\pi n^{-1})] V_k^t V_k \quad (10)$$

The inverse of T has the same eigenvectors and inverse eigenvalues.

Therefore,

$$T^{-1} = 1/4 \sum_{k=1}^{n-1} [1 + 0.5 \cos(k\pi n^{-1})]^{-1} V_k^t V_k \quad (11)$$

The equation for the coefficient vector M is:

$$M = 1.5h^{-2} \sum_{k=1}^{n-1} s_k V_k^t V_k H \quad (12)$$

where

$$s_k = [1 + 0.5 \cos(k\pi n^{-1})]^{-1} \quad (13)$$

The matrix $V_k^t V_k$ has elements:

$$\{2n^{-1} \sin(km\pi n^{-1}), \sin(kq\pi n^{-1}), m, q=1, \dots, n-1\} \quad (14)$$

If we define $F = (f_1 \dots f_{n-1})$,

$$f_k = n^{-1/2} \sum_{q=1}^{n-1} h_q \sin(kq\pi n^{-1}), k=1, \dots, n-1 \quad (15)$$

then the m th component of the vector M is:

$$M_m = 3h^{-2} n^{-1/2} \sum_{k=1}^{n-1} s_k f_k \sin(km\pi n^{-1}) \quad (16)$$

Computations (15) and (16) are slightly modified versions of Finite Fourier Transforms, and each of them can be completed in $O(\log_2 n)$ time, if Fast Fourier Transform methods are used. In other words, F is produced from H by a Fast Fourier Transform, and then M is produced by the vector $(s_1 f_1, \dots, s_{n-1} f_{n-1})$ by another Fast Fourier Transform. Hence the computation of M requires computation time of the order $O(\log_2 n)$.

The reason for obtaining such a simple solution is the fortunate event of the eigenvalues and eigenfunctions of T being expressed in a closed form that relates them to the Fast Fourier Transform. For higher order splines our solution requires to develop methods based on the approximation of Toeplitz matrices by circulant ones, to be presented in the next section. We will utilize a method of partitioning and cyclical decomposition of banded Toeplitz matrices, which is due to A.K. Jain [32].

Circulant and Toeplitz Matrices

Part of the exposition in the present section follows Gray [33].

A circulant matrix C is one having the form:

$$C = \begin{bmatrix} c_0 & c_1 & c_2 & \cdot & \cdot & \cdot & \cdot & c_{n-1} \\ c_{n-1} & c_0 & c_1 & c_2 & \cdot & \cdot & \cdot & c_{n-2} \\ \cdot & c_{n-1} & & & & & & \cdot \\ \cdot & & & & & & & \cdot \\ \cdot & & & & & & c_2 & \\ \cdot & & & & & & c_1 & \\ c_1 & \cdot & \cdot & \cdot & \cdot & c_{n-1} & & c_0 \end{bmatrix} \quad (17)$$

The eigenvalues q_m and eigenvectors V_m of C are the solutions of

$$CV = qV, \quad V = [v_0, \dots, v_{n-1}]^t \quad (18)$$

or equivalently of the n difference equations

$$\sum_{k=0}^{m-1} c_{n-m+k} v_k + \sum_{k=m}^{n-1} c_{k-m} v_k = q v_m, \quad m=0, 1, \dots, n-1 \quad (19)$$

It is easily verified for any $m=0, 1, \dots, n-1$, that $v_k = \exp\{-2\pi i m k n^{-1}\}$ is a solution to (19), resulting in the eigenvalues

$$q_m = \sum_{k=0}^{n-1} c_k \exp\{-2\pi i m k n^{-1}\} \quad (20)$$

$$(i = \sqrt{-1})$$

and the corresponding eigenvectors

$$v_m = n^{-1/2} [1, \exp(-2\pi i m n^{-1}), \dots, \exp(-2\pi i m (n-1) n^{-1})]^t \quad (21)$$

We can now write

$$C = n^{-1} \sum_{m=0}^{n-1} q_m v_m v_m^{*t} \quad (22)$$

$$C^{-1} = n^{-1} \sum_{m=0}^{n-1} q_m^{-1} v_m v_m^{*t} \quad (23)$$

We observe that all circulant matrices have the same set of eigenvectors. Also, the inverse of a circulant is also a circulant. The multiplication of C^{-1} to any vector $H = (h_0 \dots h_{n-1})^t$ can be done by Fast Fourier Transform techniques as follows:

$$\text{Let } Z = (z_0 z_1 \dots z_{n-1})^t,$$

$$Z = C^{-1} H \quad (24)$$

$$\text{Let } f_k = n^{-1/2} \sum_{s=0}^{n-1} h_s \exp(-2\pi i s k n^{-1}) \quad (25)$$

$$k = 0, \dots, n-1$$

be the Finite Fourier Transform coefficients of H . Then, the components of Z ,

$$z_m = n^{-1/2} \sum_{k=1}^{n-1} q_k^{-1} f_k \exp(2\pi i m k n^{-1}) \quad (26)$$

$$m = 0, \dots, n-1$$

are Finite Fourier Transform coefficients. Hence, Z is computable from (26) in $O(\log_2 n)$ computation time. A Toeplitz matrix T_n of order (n, m, s) , with $s < n$, $m < n$ is defined as an $n \times n$ matrix with entries $t(k, j)$ such that

$$t(k, j) = \begin{cases} t(k-j) & \text{for } -s \leq k-j \leq m \\ 0 & \text{otherwise} \end{cases} \quad (27)$$

and $t(m) \neq 0$, $t(-s) \neq 0$.

$$T_n = \begin{bmatrix} t(0) & t(-1) & . & . & t(-s) & & & 0 \\ . & t(0) & & & . & & & \\ . & & . & & . & & & \\ . & & & . & . & & & \\ t(m) & & & t(0) & . & . & t(-s) & \\ . & & & . & & . & . & \\ 0 & & & t(m) & . & . & t(0) & \end{bmatrix} \quad (28)$$

With the exception of the upper right and lower left corners T_n looks like a circulant matrix; i.e., each row is the row above it shifted to the right one place. If we fill in the upper right and lower left corners by the appropriate entries, we can make T_n exactly a circulant. Define the circulant matrix C in this way

$$c_k = \begin{cases} t(-k), & k = 0, \dots, s \\ t(n-k), & k = n-m, \dots, n-1 \\ 0 & \text{otherwise} \end{cases} \quad (29)$$

C , defined as above, is a prime candidate for approximating T_n . The rationale for such an approximation is the desire to approximate any operation $T_n^{-1}Y$, Y = vector, by the operation $C_n^{-1}Y$, which can be performed in $O(\log_2 n)$ computation time using the Fast Fourier Transform.

Let D be the difference

$$D = C - T_n \quad (30)$$

D has the form

$$D = \begin{bmatrix} 0 & \bar{Q} \\ P & 0 \end{bmatrix} \text{ where } Q = \begin{bmatrix} t(m) & t(m-1) & \dots & t(1) \\ & t(m) & & t(m-1) \\ & 0 & \cdot & \\ & & & t(m) \end{bmatrix}$$

$$P = \begin{bmatrix} t(-s) & & & 0 \\ t(1-s) & \cdot & & \\ \cdot & & \cdot & \\ t(-1) & \cdot & \cdot & t(1-s) & t(-s) \end{bmatrix}$$

Now suppose that we want to solve the equation $T_n X = Y$, where X, Y are n -vectors. Substitute $T_n = C - D$. Then we have

$$CX = DX + Y \text{ or } X = C^{-1}DX + C^{-1}Y \quad (31)$$

We partition X and $C^{-1}Y$ as follows:

$$X = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}, \quad C^{-1}Y = \begin{bmatrix} w_1 \\ w_2 \\ w_3 \end{bmatrix} \quad (32)$$

(10)

where X_1, W_1 have dimension s , W_3, X_3 have dimension m .

We also partition C^{-1}

$$C^{-1} = \begin{bmatrix} A_{11} & A_{12} & A_{13} \\ A_{21} & A_{22} & A_{23} \\ A_{31} & A_{32} & A_{33} \end{bmatrix} \quad (33)$$

The dimensions of A_{ij} will become obvious from the next equations.
Equation (31) becomes

$$X_1 = A_{11}QX_3 + A_{13}PX_1 + W_1 \quad (34)$$

$$X_2 = A_{21}QX_3 + A_{23}PX_1 + W_2 \quad (35)$$

$$X_3 = A_{31}QX_3 + A_{33}PX_1 + W_3 \quad (36)$$

Now, we can solve (34) and (36) for (X_1, X_3) and then substitute them in (35) to find X_2 . Solution:

$$\begin{bmatrix} X_1 \\ X_3 \end{bmatrix} = \left[I - \begin{bmatrix} A_{13}P & A_{11}Q \\ A_{33}P & A_{31}Q \end{bmatrix} \right]^{-1} \begin{bmatrix} W_1 \\ W_3 \end{bmatrix} \quad (37)$$

The method of solution of equations (34) - (36) through partitioning and cyclical decomposition of the banded Toeplitz matrices, is due to A.K. Jain [32] and is a very efficient method for solving a system of linear equations when the matrix T_n is of the banded Toeplitz form.

Using parallel processing techniques through Fast Fourier Transform architecture, the solution of equations (34) - (37) is achieved in $(s+m)^3 + O(2\log_2 n)$ time.

Quartic and Quintic Spline Fit

In the present section we will use the method of approximating a Toeplitz matrix by a circulant to solve efficiently the spline fit problem for higher degree splines. Explicit solutions will be given the quartic and quintic spline, which are the simplest and hence more useful higher degree splines. To avoid proliferation of notation, we will use M_j to denote the k th derivative at the knot x_j of a $k+1$ degree spline. Hence, $k=2, 3, 4$ correspond to the cubic, quartic and quintic spline. The number of knots used will be $n+k$ for the $k+1$ degree spline, so that k of them serve as boundary points with parameters assumed known, leaving exactly n unknown parameters. They form the vector $M = [M_1 \dots M_n]^t$. Hence the matrix to be inverted will be always $n \times n$. The vector M of k th derivatives together with the continuity of the first k derivatives are sufficient to define the spline completely.

We consider now the quartic spline. Let

$$I_4 = \{x_j = jh, a = -h, b = (n+1)h, j = -1, 0, 1, \dots, n, n+1\} \quad (38)$$

be a set of $n+3$ equispaced knots on $[a, b]$. A quartic spline $S(x)$ defined on $[a, b]$ is a piecewise fourth degree polynomial between knots (x_j, x_{j+1}) that fits a set of data $\{S(x_j) = y_j, j \in I_4\}$ and has continuous the first three derivatives at the knots, i.e.

$$S^{(k)}(x_j^-) = S^{(k)}(x_j^+), \quad k = 0, 1, 2, 3, j \in I_4 \quad (39)$$

$$\text{Let } M_j = S^{(3)}(x_j^-) = S^{(3)}(x_j^+), \quad j \in I_4 \quad (40)$$

We have $(n+3)$ parameters $\{M_j, j \in I_4\}$. The quartic spline requires specification of the following 3 boundary numbers:

$$S'(a), S''(a), S'(b),$$

It is shown in [4] that the 3 boundary conditions specify uniquely the "boundary parameters" M_{-1}, M_0, M_{n+1} . Therefore, we will consider them known. It is further shown in [4] that M_j satisfy the following equations:

$$M_{k-2} + 11M_{k-1} + 11M_k + M_{k+1} = 24h^{-3}(-y_{k-2} + 3y_{k-1} - 3y_k + y_{k+1})$$

for $k = 1, 2, \dots, n$ (41)

We have now n equations and an equal number of unknown parameters, constituting the vector $M = [M_1 \dots M_n]^t$.

We define then constants

$$f_k = \begin{bmatrix} -h^3 M_0 \cdot 11/24 - h^3 M_{-1}/24 + (y_2 - 3y_1 + 3y_0 - y_{-1}), & k=1 \\ -h^3 M_0/24 + (y_3 - 3y_2 + 3y_1 - y_0), & k=2 \\ \vdots & \\ (y_{j+1} - 3y_j + 3y_{j-1} - y_{j-2}), & 3 \leq k = j \leq n-1 \\ \vdots & \\ h^3 M_{n+1}/24 + (y_{n+1} - 3y_n + 3y_{n-1} - y_{n-2}), & k=n \end{bmatrix} \quad (42)$$

Let $F = (f_1 \dots f_n)^t$

We also define the $n \times n$ Toeplitz matrix

(13)

$$T = \begin{bmatrix} b & 1 & & & 0 \\ & b & b & & 1 \\ & 1 & b & & \\ & & 1 & & \\ 0 & & & b & b & 1 \\ & & & 1 & b & b \end{bmatrix}, \quad b = 11 \quad (43)$$

Then, the vector M from the solution of equations (43) is:

$$M = 24h^{-3}T^{-1}F \quad (44)$$

At this point, we are ready to apply directly the procedure of Section II because T is banded Toeplitz. If we identify T with T_n from eq. (30), we have: $s = 1$, $m = 2$, $t(0) = t(1) = b$, $t(2) = 1$, $t(-1) = 1$. The matrices P , Q are:

$$Q = \begin{bmatrix} 1 & b \\ 0 & 1 \end{bmatrix}, \quad P = 1 \quad (\text{scalar}) \quad (45)$$

Also,

$$Y = 24h^{-3}F, \quad x_1 = \text{scalar}$$

x_3 = two dimensional column vector. Here the unknown vector M is identified with X . Using equations (34) - (37), the solution for $M=X$ is immediate. The time required for the operations is $3^3 + O(2 \log_2 n)$.

Finally, we will derive the solution to the quintic spline fit problem. Let

$$I_5 = \{x_j = jh, j = -1, 0, 1, \dots, n, n+1, n+2, a = -h, b = (n+2)h\} \quad (46)$$

be a set of $(n+4)$ equispaced knots in $[a, b]$. A quintic spline $S(x)$ defined on $[a, b]$ is a function that has continuous the three first derivatives on $[a, b]$, fits a set of data $y_j = S(y_j)$, $j \in I_5$, and

minimizes the integral

$$\int_a^b [S^{(3)}(x)]^2 dx \quad (47)$$

under 4 specified boundary conditions:

$$S'(a), S''(a), S'(b), S''(b).$$

(The quintic spline corresponds to a differential operator $L=D^3$). In spline theory [4] it is shown that the solution to the above constrained minimization problem is a piecewise fifth order polynomial with the first four derivatives continuous at the joints. Hence,

$$S(x_j) = y_j, S^{(k)}(x_j^-) = S^{(k)}(x_j^+), k = 0, 1, 2, 3, 4, j \in I_5 \quad (48)$$

$$\text{Let } M_j = S^{(4)}(x_j^-) = S^{(4)}(x_j^+), j \in I_5 \quad (49)$$

The 4 boundary conditions specify uniquely the 4 "boundary parameters" $M_{-1}, M_0, M_{n+1}, M_{n+1}'$, as shown in [4]. Therefore, we will consider them known. It is shown in [4] that the following set of n equations is satisfied by the M_j 's:

$$\begin{aligned} M_{k-2} + 26M_{k-1} + 66M_k + 26M_{k+1} + M_{k+2} = \\ 120h^{-4} [y_{k-2} - 4y_{k-1} + 6y_k - 4y_{k+1} + y_{k+2}] \end{aligned}$$

(50)

Now we have n equations and n unknown parameters, the components of the vector $M = [M_1 M_2 \dots M_n]$. Let $F = [f_1 f_2 \dots f_n]$, where:

(15)

$$f_k = \left[\begin{array}{l} (y_{-1} - 4y_0 + 6y_1 - 4y_2 + y_3) - 26 \cdot 120^{-1}h^4M_0 - \\ - 120^{-1}h^4M_{-1} \text{ for } k=1 \\ (y_0 - 4y_1 + 6y_2 - 4y_3 + y_4) - 120^{-1}h^4M_0 \text{ for } k=2 \\ \vdots \\ (y_{k-2} - 4y_{k-1} + 6y_k - 4y_{k+1} + y_{k+2}), \text{ for } 3 \leq k \leq n-2 \\ (y_{n-3} - 4y_{n-2} + 6y_{n-1} - 4y_n + y_{n+1}) - 120^{-1}h^4M_{n+1}, \\ \text{for } k = n-1 \\ (y_{n-2} - 4y_{n-1} + 6y_n - 4y_{n+1} + y_{n+2}) - 26 \cdot 120^{-1}h^4M_{n+1} - \\ - h^4 120^{-1}M_{n+2}, \text{ for } k=n \end{array} \right] \quad (51)$$

Let also

$$T = \left[\begin{array}{cccccccc} 66 & 26 & 1 & . & . & . & . & . \\ 26 & 66 & 26 & . & . & . & . & 0 \\ 1 & 26 & . & . & . & . & . & . \\ . & . & . & . & . & . & . & . \\ . & 1 & . & . & . & . & . & . \\ . & . & . & . & . & . & . & 1 \\ 0 & . & . & . & . & . & 66 & 26 \\ . & . & . & . & . & . & . & . \\ . & . & 1 & 26 & 66 & . & . & . \end{array} \right] \quad (52)$$

Then the set of equations (52) becomes:

$$TM = 120^{-4}F \quad (53)$$

Identifying the parameters of the present problem to the previous results of Section II, we have: $s=m=2$

(16)

$$P = \begin{bmatrix} 1 & 0 \\ 26 & 1 \end{bmatrix} \quad Q = \begin{bmatrix} 1 & 26 \\ 0 & 1 \end{bmatrix} \quad (54)$$

$$X_1 = \begin{bmatrix} M_1 \\ M_2 \end{bmatrix} \quad X_3 = \begin{bmatrix} M_{n-1} \\ M_n \end{bmatrix}$$

$$Y = 120h^{-4}F \quad (55)$$

Using (39) we can compute X_1, X_3 with 4^3 operations. Combining with (37) we compute X_2 . The total computational time required is: $4^3 + O(2\log_2 n)$.

Generally, to fit a $k+1$ degree polynomial spline to $n+k$ equispaced knots with data $\{y_j\}$, we have $n+k-1$ intervals and $n+k-2$ interior points. For each interval, specification of the spline requires $k+2$ constants to be solved for. Hence the number of unknown parameters is $(k+2)(n+k-1)$. The continuity of $S^{(j)}(x)$, $j = 0, 1, \dots, k$ at the interior knots (joints) provides $(k+1) \cdot (n+k-2)$ equations. By fitting the data points $\{y_j\}$ we get $(n+k)$ equations. Let s be the number of specified boundary conditions. In order to have equal number of unknown parameters and equations, we must have:

$$(k+2)(n+k-1) = (k+1)(n+k-2) + (n+k) + 2$$

hence

$$s=k.$$

The boundary conditions can be specified either by a number of derivatives at the end points a, b or by a number of boundary values of the moments M_j . For the cubic spline, $k=2$, and we picked M_0, M_{n+1} as the known boundary values. For the quartic

spline, we have $k=3$. The moments (M_{-1}, M_0, M_{n+1}) were assumed known. For the quintic, we have $k=4$ and we used the moments $(M_{-1}, M_0, M_{n+1}, M_{n+2})$ as known boundary conditions. An alternate set of boundary conditions that has been frequently used is the specification of a total of k derivatives of $S(x)$ at a and b . There is a one to one dependence between the two different boundary conditions [4]. In the present paper, the boundary moments were chosen as more convenient.

The k th derivative of a $(k+1)$ degree spline is a piecewise linear function, expressed as:

$$S^{(k)}(x) = M_{j-1}h^{-1}(x_j - x) + M_jh^{-1}(x - x_{j-1}) \text{ for } x_{j-1} \leq x \leq x_j \quad (56)$$

where

$$M_j = S^{(k)}(x_j)$$

Up to this point we have shown that the parameters $\{M_j\}$, which are the third derivatives at $\{x_j\}$ for the quartic spline and fourth derivatives at x_j for the quintic spline, can be computed from the data y_j in parallel processing time of $2\log_2 n$ operations.

Complete specification of the other parameters of the spline will be now undertaken.

For the quartic spline, we have:

$$S^{(3)}(x) = M_{j-1}h^{-1}(x_j - x) + M_jh^{-1}(x - x_{j-1}) \quad (57)$$

for $x_{j-1} \leq x \leq x_j$

Integrating three times, we have:

$$S(x) = -(24h)^{-1} M_{j-1} (x_j - x)^4 + (24h)^{-1} M_j (x - x_{j-1})^4 + \\ + 2^{-1} h B_j^1 (x - x_j)^2 + h^2 B_j^2 (x - x_j) + h^3 B_j^3, \quad x_{j-1} \leq x \leq x_j \quad (58)$$

where $\{B_j^1, B_j^2, B_j^3\}$ are constants that need to be determined from $\{M_j\}$, $\{y_j\}$, and the continuity requirement at the joints. We take the consecutive derivatives of $S(x)$, write the corresponding expressions of $S^{(k)}(x)$ for the intervals $[x_{j-1}, x_j]$ and $[x_j, x_{j+1}]$, then apply the continuity requirement for $S^{(k)}(x)$ at $x = x_j$, for $k = 1, 2$. The following equations result:

$$S^{(2)}(x_j^-) = S^{(2)}(x_j^+); \quad B_{j+1}^1 = B_j^1 + M_j, \quad j = 0, 1, \dots, n \quad (59)$$

$$S^{(1)}(x_j^-) = S^{(1)}(x_j^+); \quad B_{j+1}^2 = B_j^2 + B_{j+1}^1, \quad j = 0, 1, \dots, n \quad (60)$$

The equations $\{S(x_j^-) = y_j\}$ provide the following relationships:

$$y_j = S(x_j^-); \quad B_j^3 = h^{-3} y_j - 24^{-1} M_j, \quad j = 0, 1, \dots, n, n+1 \quad (61)$$

$$S(a^+) = y_{-1}; \quad 2^{-1} B_0^1 - B_0^2 = -B_0^3 + h^{-3} y_{-1} + 24^{-1} M_{-1} \quad (62)$$

We still need to determine the two initial values, (B_0^1, B_0^2) . For this purpose, we use one additional equation:

$$S(x_0^+) = y_0; \quad h^{-3} y_0 = -24^{-1} M_0 + 2^{-1} B_1^1 - B_1^2 + B_1^3 \quad (63)$$

Substituting (B_1^1, B_1^2) from (59), (60) for $j=0$, in terms of B_0^1, B_0^2 , we get:

$$2^{-1} B_0^1 + B_0^2 = B_1^3 - M_0 \cdot 13/24 - h^{-3} y_0 \quad (64)$$

From (62) and (64) we can solve for (B_0^1, B_0^2) . Then B_j^1, B_j^2, B_j^3 are computable from M_j .

For the quintic spline, we have:

$$S^{(4)}(x) = M_{j-1} h^{-1}(x_j - x) + M_j h^{-1}(x - x_{j-1})$$

for $x_{j-1} \leq x \leq x_j$ (65)

Integrating four times, we find:

$$S(x) = (120h)^{-1} M_{j-1} (x_j - x)^5 + (120h)^{-1} M_j (x - x_{j-1})^5 +$$

$$+ 6^{-1} h B_j^1 (x - x_j)^3 + 2^{-1} h^2 B_j^2 (x - x_j)^2 + h^3 B_j^3 (x - x_j)$$

$$+ h^4 B_j^4$$

for $x_{j-1} \leq x \leq x_j$ (66)

The constants now are $B_j^1, B_j^2, B_j^3, B_j^4$. We take the consecutive derivatives $S^{(k)}(x)$ for the intervals $[x_{j-1}, x_j]$ and $[x_j, x_{j+1}]$, then apply the continuity requirement for $S^{(k)}(x)$ at $x = x_j$, $k = 1, 2, 3$. We also apply the condition $S(x_j^-) = y_j$. The resulting equations are:

$$S(x_j^-) = y_j; \quad B_j^4 = h^{-4} y_j - 120^{-1} M_j \quad j = 0, 1, \dots, n+2 \quad (67)$$

$$S^{(3)}(x_j^-) = S^{(3)}(x_j^+); \quad B_{j+1}^1 = B_j^1 + M_j \quad (68)$$

$$S^{(2)}(x_j^-) = S^{(2)}(x_j^+); \quad B_{j+1}^2 = B_{j+1}^1 + B_j^2 \quad (69)$$

$$S^{(1)}(x_j^-) = S^{(1)}(x_j^+); \quad B_{j+1}^3 = B_j^3 + B_{j+1}^2 - 2^{-1} B_{j+1}^1 + 12^{-1} M_j \quad (70)$$

for $j = 0, 1, \dots, n+1$

We need now to solve for the initial values (B_0^1, B_0^2, B_0^3) , which, together with the equations (67) - (70), will give the complete solution. For this purpose, we will formulate a set of 3 equations, in which the only unknown parameters are (B_0^1, B_0^2, B_0^3) .

Note that B_j^4 are immediately found from (67). From (66) we have:

$$h^{-4} \cdot S(x_{j-1}^+) = 120^{-1} M_{j-1} - 6^{-1} B_j^1 + 2^{-1} B_j^2 - B_j^3 + B_j^4 \quad (71)$$

for $j = 0, 1, \dots, n+2$

We apply (71) for $j = 0, 1, 2$. The result is:

$$y_{-1} h^{-4} = 120^{-1} M_{-1} - 6^{-1} B_0^1 + 2^{-1} B_0^2 - B_0^3 + B_0^4 \quad (72)$$

$$y_0 h^{-4} = 120^{-1} M_0 - 6^{-1} B_1^1 + 2^{-1} B_1^2 - B_1^3 + B_1^4 \quad (73)$$

$$y_1 h^{-4} = 120^{-1} M_1 - 6^{-1} B_2^1 + 2^{-1} B_2^2 - B_2^3 + B_2^4 \quad (74)$$

Using (68) - (70), we express $(B_1^1, B_1^2, B_1^3, B_2^1, B_2^2, B_2^3)$ in terms of (B_0^1, B_0^2, B_0^3) and substitute them in (72) - (74). Thus we finally have 3 equations with the unknown parameters (B_0^1, B_0^2, B_0^3) . After the determination of the initial values B_0^k , equations (68) - (69) provide all the parameters B_j^k .

Conclusions

We have presented an efficient algorithm for fitting a polynomial spline to a set of n equidistant data. The algorithm exploits the parallel nature of Fast Fourier Transform and the form of the difference equations relating the moments $\{M_j\}$ to the data $\{y_j\}$. Explicit formulas were obtained for the cubic, quartic and quintic spline, and the method is readily extensible to higher order and more general splines. The method involves only linear operations, and is much simpler than previous ones. The time complexity of the method is $O(2 \log_2 n)$, independently of the order of the spline.

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