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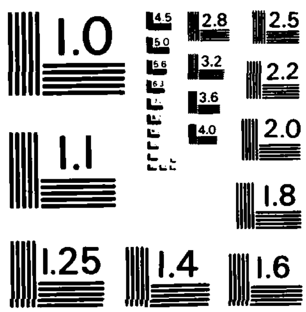

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A NEW CLASS OF STATISTICAL PERFORMANCE CRITERIA FOR STOCHASTIC LINEAR CONTROL SYSTEMS*

by

R.C. Hartwig and S.R. Liberty**

ABSTRACT

A set of performance statistics known as cumulants show the way to new design criteria for stochastic linear control systems. The new criteria allow the designer to affect the probability distribution of performance measures which do away with the need for stochastic simulation. Controllers arising in this new format are linear-dynamical and exhibit the classical separation property.

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**The authors are with the Department of Electrical Engineering, Texas Tech University, Lubbock, Texas 79409.

I. INTRODUCTION

Three commonly (but not consistently) used terms in the control literature are "performance measure", "performance index", and "performance criterion". The reader should note that by "performance measure" we mean a functional that assigns a nonnegative real number to each sample run of the control system. Thus, for stochastic systems performance measures are random variables.

We call statistical quantities associated with performance measures for the sake of optimization "performance indices". In this paper optimization consists of minimizing weighted sums of performance indices, and "performance criterion" is the term we use to refer to the objective of achieving such a minimum.

The research reported in this paper is the culmination of approximately a decade of effort originated by Sain [6], [7], [8] continued by Liberty [4] and Sain and Liberty [10], and completed by Liberty and Hartwig [5] and Hartwig [2]. In the sequel we present new analytical results in Linear-Quadratic-Gaussian (LQG) control theory which reflect a design philosophy that significantly departs from traditional lines of thought.

The objectives of the overall research effort were two fold: first, to solve LQG control problems with broader criteria than minimum average performance; and second, to develop broad statistical performance analysis techniques for LQG systems.

Original work directed at the first objective resulted in the solution of the open-loop minimum-variance problem [7], [9], while that directed at the second resulted in the development of various partial and complete statistical descriptions of performance [4], [6],

[8], [10] with the complete solution of the performance analysis problem appearing in [5].

These performance analysis results are utilized in section V to demonstrate the characteristics of an LQG control system designed according to criteria developed herein.

Although solution of the open loop minimum variance control problem provided new insight into the LQG theory, it had little practical impact on LQG design. This, of course, was simply due to its inherent lack of feedback structure.

With this history in mind and with insight gained from the work reported in [5] we now set out to develop a general LQG design philosophy and an accompanying design procedure.

The following objectives form the framework of our development:

i. We require that the performance criteria we decide upon yield linear control laws. This is desirable for implementation considerations and will guarantee that the final system will still lie in the LQG class thus assuring the applicability of the design performance analysis results of [5].

ii. We demand physically realizable controllers. (It may seem silly to even state this objective, but actually one must carefully guard against non-causality that can artificially arise in optimization.)

iii. We do not want to sacrifice any more computational tractability than necessary for the sake of achieving more general results.

iv. We want to select performance criteria that will allow the designer to affect the probability distribution of his performance measure(s) in a desirable way. (For the purpose of performance analysis there may be more than one performance measure for each control system.)

In section II we state several well known facts from the LQG theory. For the sake of conciseness we heavily reference the tutorial paper by Tse [11] and use his notation as much as possible. This section also contains the definition of quadratic performance measures to be utilized in the control system design procedure.

Section III contains the development of a complete statistical description of performance in terms of an "accessible state". In addition, design performance indices are defined in this section.

In section IV we state the new LQG performance criteria and solve a general problem within this class. Section V contains a numerical example of design according to the new criteria.

II. SYSTEM MODELS AND PERFORMANCE MEASURES

Let R^n denote the n-fold Cartesian product of the real line, and let J denote the real line interval $[t_0, t_f]$. The linear system to be controlled is described by

$$\frac{dx(t)}{dt} = A(t)x(t) + B(t)u(t) + \xi(t), \tag{1}$$

and

$$z(t) = C(t)x(t) + \theta(t), \; t\varepsilon J \; , \tag{2}$$

where the state $x(t) \varepsilon R^n$, the control action $u(t) \varepsilon R^m$, and the observation process $z(t) \varepsilon R^r$. The initial condition for (1), $x(t_0)$, is assumed to be Gaussian with mean

$$x_0 = E\{x(t_0)\} \tag{3}$$

and covariance

$$\Sigma_0 = E\{[x(t_0)-x_0][x^T(t_0)-x_0^T]\} \tag{4}$$

where $(^T)$ denotes matrix transposition. The state noise process, $\xi(t)$ is zero-mean Gaussian-white with covariance kernel

$$E\{\xi(t)\xi^T(\tau)\} = \Xi(t)\delta(t-\tau), \; t,\tau\varepsilon J \tag{5}$$

where $\Xi(t)$ is symmetric and positive semi-definite on J. The measurement noise process, $\theta(t)$, is zero-mean Gaussian-white with covariance kernel

$$E\{\theta(t)\theta^T(\tau)\} = \Theta(t)\delta(t-\tau), \quad t,\tau\varepsilon J \tag{6}$$

where $\Theta(t)$ is symmetric and positive definite on J. For convenience we assume that $\xi(t)$, $\theta(t)$, and $x(t_0)$ are all uncorrelated. That is,

$$E\{\xi(t)\theta^T(\tau)\} = 0, \quad t,\tau\varepsilon J \tag{7}$$

$$E\{[x(t_0)-x_0]\xi^T(t)\} = 0, \quad t\varepsilon J, \tag{8}$$

and

$$E\{[x(t_0)-x_0]\theta^T(t)\} = 0, \quad t\varepsilon J . \tag{9}$$

The control action, $u(t)$, is assumed to be a causal function of the observation process. That is,

$$u(t) = \psi(t,z(\tau);\ \tau\varepsilon[t_0,\ t]), \tag{10}$$

where $\psi(t,\cdot)$ satisfies the Lipschitz condition in [11]. All matrix functions including the mapping $\psi(t,\cdot)$ are assumed to be smooth enough to guarantee mean square continuity of the x-process on J. Consequently x is a finite energy process on J.

We define three measures of system performance. The first, henceforth referred to as the design performance measure, is given by

$$J = x^T(t_f)Sx(t_f) + \int_{t_0}^{t_f} [x^T(t)Q(t)x(t) + u^T(t)R(t)u(t)]dt, \tag{11}$$

where the terminal penalty weighting, S, is symmetric and positive semi-definite as is the weighting $Q(t)$ on J. The weighting $R(t)$ is symmetric and positive definite on J. We also assume that $Q(t)$ and $R(t)$ are continuous on J. The performance measure, J, describes a-priori design specifications involving the relative importance of state regulation and control effort.

The remaining two measures of system performance will be referred to as design-analysis performance measures and are defined by

$$J_x = \int_{t_0}^{t_f} x^T(t)X(t)x(t)dt, \tag{12}$$

and

$$J_u = \int_{t_0}^{t_f} u^T(t)U(t)u(t)dt. \tag{13}$$

The weighting matrices $X(t)$ and $U(t)$ are assumed to be symmetric and positive semi-definite on J. Conceptually, J_x measures the state regulating quality of a given design, and J_u measures the corresponding control effort. The weightings $X(t)$ and $U(t)$ are chosen by the designer to select and/or weight particular components of x or u for analysis. Performance analysis as demonstrated in section V consists of obtaining probabilistic or statistical descriptions of J_x and J_u after a design is achieved.

Quantities of special interest in our development are the conditional mean and conditional covariance of $x(t)$ given by,

$$\hat{x}(t) = E\{x(t)|F_t\}, \quad t\varepsilon J, \tag{14}$$

and

$$K_{\hat{x}}(t,\tau) = E\{[x(t)-\hat{x}(t)][x^T(\tau)-\hat{x}^T(\tau)]|F_\sigma\}, \tag{15}$$

$$\sigma = t\vee\tau,\ t,\ \tau\varepsilon J$$

where F_t is the sigma algebra induced by the measurements $\{x(\tau), \tau \in [t_o, t]\}$. When $t = t_f$ we will write F without a subscript. Define the single argument covariance $\Sigma(t)$ as

$$\Sigma(t) \underline{\Delta}\, K_{\hat{x}}(t, t), \quad t\varepsilon J \,. \tag{16}$$

It is well known that the conditional covariance kernel is nonrandom [11], and that $\hat{x}$ evolves according to

$$\frac{d}{dt}\hat{x}(t) = A(t)\hat{x}(t) + B(t)u(t) + W(t)\nu(t), \quad t\varepsilon J \tag{17}$$

where

$$\hat{x}(t_o) = x_o, \tag{18}$$

$$W(t) = \Sigma(t)C^T(t)\Theta^{-1}(t), \quad t\varepsilon J, \tag{19}$$

and $\Sigma(t)$ satisfies

$$\begin{aligned}\frac{d}{dt}\Sigma(t) &= A(t)\Sigma(t) + \Sigma(t)A^T(t) + \Xi(t) \\ &\quad - \Sigma(t)C^T(t)\Theta^{-1}(t)C(t)\Sigma(t), \quad t\varepsilon J,\end{aligned} \tag{20}$$

with

$$\Sigma(t_o) = \Sigma_o. \tag{21}$$

The control action, $u(t)$, is as in (10) and the "innovations process", $\nu(t)$, defined by

$$\nu(t) \underline{\Delta}\, z(t) - C(t)\hat{x}(t), \quad t\varepsilon J, \tag{22}$$

is zero-mean Gaussian-white with covariance kernel

$$E\{\nu(t)\nu^T(\tau)\} = \Theta(t)\delta(t-\tau),\ t,\tau\epsilon J \tag{23}$$

The process, $\hat{x}$, will be an "accessible state" in our formulation and the $\hat{x}$-process model in (17) alone plays the key dynamical role in the determination of u(t). However, both (17) and (1) are utilized in post-design performance analysis.

III. PERFORMANCE STATISTICS AND PERFORMANCE INDICES

The process generated by (1) is a nonzero-mean Gauss-Markov process which we expand in an orthonormal series

$$x(t) \sim \sum_{i=1}^{\infty} x_i \phi_i(t), \quad t \epsilon J \tag{24}$$

where the x_i are conditionally Gaussian scalar random variables given by

$$x_i = x^T(t_f) S \phi_i(t_f) + \int_{t_o}^{t_f} x^T(t) Q(t) \phi_i(t) dt \tag{25}$$

with conditional means

$$m_i = E\{x^T(t_f) | F\} S \phi_i(t_f)$$

$$+ \int_{t_o}^{t_f} E\{x^T(t) | F_t\} Q(t) \phi_i(t) dt, \tag{26}$$

In addition, the x_i are assumed to be conditionally uncorrelated, that is,

$$E\{[x_i - m_i][x_j - m_j] | F\} = \lambda_i \delta_{ij} \quad \forall i,j. \tag{27}$$

The nonrandom vector valued functions, $\phi_i(t)$, are chosen to satisfy the orthonormality condition,

$$\phi_i^T(t_f) S \phi_j(t_f) + \int_{t_o}^{t_f} \phi_i^T(t) Q(t) \phi_j(t) dt = \delta_{ij} \quad \forall i,j. \tag{28}$$

Under the assumptions we have made on the x-process, J is finite with probability one; see Doob [1]. It follows from Parseval's Theorem that

$$J = \sum_{i=1}^{\infty} x_i^2 + \int_{t_0}^{t_f} u^T(t)R(t)u(t)dt, \tag{29}$$

where convergence is with probability one; see Kolmogorov and Fomin [3].

Since the x_i are conditionally Gaussian and conditionally uncorrelated, they are conditionally independent as are their squares. The conditional characteristic function of each x_i^2 term in (29) is of the noncentral chi-square type given by,

$$C_{x_i^2|F}(j\omega) = (1 - 2j\omega\lambda_i)^{-1/2} \exp[j\omega m_i^2(1 - 2j\omega\lambda_i)^{-1}]. \tag{30}$$

The conditional characteristic function of J follows as,

$$C_{J|F}(j\omega) = [\prod_{i=1}^{\infty}(1 - 2j\omega\lambda_i)^{-1/2}] \cdot \exp [j\omega \int_{t_0}^{t_f} u^T(t)R(t)u(t)dt + \sum_{i=1}^{\infty} j\omega m_i^2(1 - 2j\omega\lambda_i)^{-1}]. \tag{31}$$

The second conditional characteristic function, $\tau_{J|F}(j\omega)$, is defined as the natural logarithm of $C_{J|F}(j\omega)$, that is,

$$\tau_{J|F}(j\omega) = \ln[C_{J|F}(j\omega)]. \tag{32}$$

The MacLaurin series representation of $\tau_{J|F}(j\omega)$ is given by

$$\tau_{J|F}(j\omega) = \sum_{i=1}^{\infty} \kappa_{i|F} \frac{(j\omega)^i}{i!} \tag{33}$$

where the coefficients $\kappa_{i|F}$ are called conditional cumulants. Utilizing (30), it can be easily shown that the first conditional cumulant of J is given by

$$\kappa_{1|F} = \sum_{i=1}^{\infty}\lambda_i + \sum_{i=1}^{\infty} m_i^2 + \int_{t_o}^{t_f} u^T(t)R(t)u(t)dt, \tag{34}$$

while the remaining conditional cumulants are of the form

$$\kappa_{k|F} = (k-1)!\, 2^{k-1} \sum_{i=1}^{\infty} \lambda_i^k + k!2^{k-1} \sum_{i=1}^{\infty} m_i^2 \lambda_i^{k-1} \quad, k > 1. \tag{35}$$

It is easily shown that

$$\kappa_{1|F} = E\{J|F\}, \tag{36}$$

$$\kappa_{2|F} = E\{(J - E\{J|F\})^2|F\}, \tag{37}$$

and that in general the conditional noncentral moments, $\mu_{k|F}$, are related to the conditional cumulants by

$$\mu_{k+1|F} = \sum_{j=0}^{k} \binom{k}{j} \mu_{k-j|F}\kappa_{j+1|F}. \tag{38}$$

The noncentral moments, μ_k, are related to the conditional noncentral moments by

$$\mu_k = E\{\mu_{k|F}\} . \tag{39}$$

The relationship between the cumulants, κ_k, and the conditional cumulants can be determined by first noting that the noncentral moments and the cumulants are related in a manner identical to that for the conditional statistics, that is,

$$\mu_{k+1} = \sum_{j=0}^{k} \binom{k}{j} \mu_{k-j}\kappa_{j+1}. \tag{40}$$

It then follows that

$$\kappa_1 = E\{\kappa_{1|F}\} \tag{41}$$

$$\kappa_2 = E\{\kappa_{2|F}\} + Var\{\kappa_{1|F}\} \tag{42}$$

and in general

$$\kappa_k = E\{\kappa_{k|F}\} + \{\text{statistics of lower order conditional cumulants}\} \tag{43}$$

The next step is to express (34) and (35) in terms of the non-random conditional covariance of the x-process. This cannot be done explicitly but requires the definition of the "iterated kernels"

$$K_{\hat{x}}^{(1)}(t,\tau) \triangleq K_{\hat{x}}(t,\tau) \tag{44}$$

and

$$K_{\hat{x}}^{(k)}(t,\tau) \triangleq K_{\hat{x}}(t, t_f)S\, K_{\hat{x}}^{(k-1)}(t_f,\tau) + \int_{t_0}^{t_f} K_{\hat{x}}(t,\sigma)Q(\sigma)K_{\hat{x}}^{(k-1)}(\sigma,\tau)d\sigma, \quad k>1. \tag{45}$$

It can be inductively shown that

$$K_{\hat{x}}^{(k)}(t,\tau) = \sum_{i=1}^{\infty} \lambda_i^k \phi(t)\phi_i^T(\tau) \tag{46}$$

The expression, $\sum_{i=1}^{\infty} \lambda_i^k$, in (34) and (35) can be written as

$$\sum_{i=1}^{\infty} \lambda_i^k = Tr[S\, K_{\hat{x}}^{(k)}(t_f, t_f) + \int_{t_0}^{t_f} Q(t)K_{\hat{x}}^{(k)}(t, t)dt] \tag{47}$$

where Tr[·] denotes the trace of the enclosed matrix. The expression, $\sum_{i=1}^{\infty} m_i^2 \lambda_i^{k-1}$, in (35) follows as

$$\sum_{i=1}^{\infty} m_i^2 \lambda_i^{k-1} = \hat{x}^T(t_f) S\, K_{\hat{x}}^{(k-1)}(t_f, t_f) S \hat{x}(t_f)$$

$$+ \hat{x}^T(t_f) S \int_{t_0}^{t_f} K_{\hat{x}}^{(k-1)}(t_f, t) Q(t) \hat{x}(t) dt$$

$$+ \int_{t_0}^{t_f} \hat{x}^T(t) Q(t) K_{\hat{x}}^{(k-1)}(t, t_f) dt S \hat{x}(t_f)$$

$$+ \int_{t_0}^{t_f} \int_{t_0}^{t_f} \hat{x}^T(t) Q(t) K_{\hat{x}}^{(k-1)}(t,\tau) Q(\tau) \hat{x}(\tau) d\tau dt, \; k > 1, \tag{48}$$

For the case, $k = 1$, it is obvious that

$$\sum_{i=1}^{\infty} m_i^2 = \hat{x}^T(t_f) S \hat{x}(t_f) + \int_{t_0}^{t_f} \hat{x}^T(t) Q(t) \hat{x}(t) dt. \tag{49}$$

Recalling the objectives listed in section I we now examine the statistical expressions in (43), (47), (48) and (49).

i. Note that (48) and (49) are quadratic in $\hat{x}$.

ii. Note that (47) does not depend on u.

iii. Note from (43) that the value of every statistic of J is affected by the expected value of the corresponding conditional cumulant.

Thus we choose the expected value of the conditional cumulants of J as performance indices. Optimization performed over these objects will yield linear control laws and will provide the design engineer with a mechanism for affecting the probability distribution of J in a broad statistical sense.

IV. NEW PERFORMANCE CRITERIA FOR LQG CONTROL

We now select a class of performance criteria. In particular, we choose to minimize the expected value of a weighted sum of conditional cumulants of J. This minimization is to be done over the class of admissible control laws described in [11]. Thus, our criterion is

$$\min_{u} E\{\sum_{i=1}^{N} \alpha_i \kappa_i | F\}, \quad u \text{ admissible, } N \text{ finite,}$$

where the α_i are chosen by the designer to reflect the relative importance of affecting a particular statistic of J.

This criterion will preserve linearity while adding statistical breadth to the LQG design procedure. It should be noted that the classical minimum-mean LQG control problem is a special case of the new class of problems.

The objectives that we must yet meet are those of physical realizability and computational tractability. To achieve these we define a class of dynamical variables all driven by $\hat{x}$. The resulting dynamical systems become part of the feedback control structure. These variables are given by

$$\eta_k(t) \triangleq \int_{t_0}^{t} K^{(k)}_{\hat{x}}(t,\tau)Q(\tau)\hat{x}(\tau)d\tau, \quad t \in \mathcal{J}. \tag{50}$$

This variable arises naturally in each conditional cumulant expression by making the observation that the last term in (48) contains a symmetric (in argument) integrand. Thus

$$\int_{t_0}^{t_f}\int_{t_0}^{t_f} \hat{x}^T(t)Q(t)K^{(k-1)}_{\hat{x}}(t,\tau)Q(\tau)\hat{x}(\tau)d\tau dt = 2\int_{t_0}^{t_f}\int_{t_0}^{t} \hat{x}^T(t)Q(t)K^{(k-1)}_{\hat{x}}(t,\tau)Q(\tau)\hat{x}(\tau)d\tau dt$$

$$= 2\int_{t_0}^{t_f} \hat{x}^T(t)Q(t)\eta_{k-1}(t)dt. \tag{51}$$

It follows immediately that

$$\sum_{i=1}^{\infty} m_i^2 \lambda_i^{k-1} = \hat{x}^T(t_f) S k_{\hat{x}}^{(k-1)}(t_f,t_f) S \hat{x}(t_f) + 2\, \hat{x}^T(t_f) S \eta_{k-1}(t_f)$$

$$+ 2 \int_{t_o}^{t_f} \hat{x}^T(t) Q(t) \eta_{k-1}(t) dt. \tag{52}$$

At this point we can best illustrate the results by considering a single problem within the new class. Consider the problem of minimizing $E\{\kappa_{1|F} + \alpha\kappa_{2|F}\}$. Here we wish to decrease the variance of J at the expense of an increase in the mean of J. Disregarding the uncontrollable terms, this criterion may be explicitly written as

$$\min_u E\{\hat{x}^T(t_f) S \hat{x}(t_f) + \int_{t_o}^{t_f} [\hat{x}^T(t) Q(t) \hat{x}(t) + u^T(t) R(t) u(t)] dt$$

$$+ 4\alpha \hat{x}^T(t_f) S \Sigma(t_f) S \hat{x}(t_f) + 8\alpha \hat{x}^T(t_f) S \eta_1(t_f)$$

$$+ 8\alpha \int_{t_o}^{t_f} \hat{x}^T(t) Q(t) \eta_1(t) dt\} ,$$

where $\eta_1(t)$ evolves from

$$\frac{d}{dt}\eta_1(t) = A(t)\eta_1(t) + \Sigma(t)Q(t)\hat{x}(t), \; t \in \mathcal{J} \tag{53}$$

with

$$\eta_1(t_o) = 0. \tag{54}$$

By defining an augmented state vector this criterion may be rewritten as

$$\min_u E\left\{ \begin{bmatrix} \hat{x}(t_f) \\ \eta_1(t_f) \end{bmatrix}^T \begin{bmatrix} S + 4\alpha S\Sigma(t_f)S & 4\alpha S \\ 4\alpha S & 0 \end{bmatrix} \begin{bmatrix} \hat{x}(t_f) \\ \eta_1(t_f) \end{bmatrix} \right.$$

$$+ \int_{t_o}^{t_f} \begin{bmatrix} \hat{x}(t) \\ \eta_1(t) \end{bmatrix}^T \begin{bmatrix} Q(t) & 4\alpha Q(t) \\ 4\alpha Q(t) & 0 \end{bmatrix} \begin{bmatrix} \hat{x}(t) \\ \eta_1(t) \end{bmatrix}$$

$$\left. + u^T(t)R(t)u(t)dt \right\}$$

where the augmented state vector evolves from

$$\frac{d}{dt} \begin{bmatrix} \hat{x}(t) \\ \eta_1(t) \end{bmatrix} = \begin{bmatrix} A(t) & 0 \\ \Sigma(t)Q(t) & A(t) \end{bmatrix} \begin{bmatrix} \hat{x}(t) \\ \eta_1(t) \end{bmatrix} + \begin{bmatrix} B(t) \\ 0 \end{bmatrix} u(t)$$

$$+ \begin{bmatrix} W(t) \\ 0 \end{bmatrix} \nu(t), \quad t \epsilon J \tag{55}$$

with

$$\begin{bmatrix} \hat{x}(t_o) \\ \eta_1(t_o) \end{bmatrix} = \begin{bmatrix} x_o \\ 0 \end{bmatrix}. \tag{56}$$

However, the solution of an accessible state problem with this structure is well known [11] and is given by

$$u(t) = -R^{-1}(t)[B^T(t) \;\; 0]M(t) \begin{bmatrix} \hat{x}(t) \\ \eta_1(t) \end{bmatrix}, \; t \epsilon J \tag{57}$$

where M(t) is a solution of

$$\frac{d}{dt} M(t) = -M(t) \begin{bmatrix} A(t) & 0 \\ \Sigma(t)Q(t) & A(t) \end{bmatrix}$$

$$- \begin{bmatrix} A^T(t) & Q(t)\Sigma(t) \\ 0 & A^T(t) \end{bmatrix} M(t)$$

$$- \begin{bmatrix} Q(t) & 4\alpha Q(t) \\ 4\alpha Q(t) & 0 \end{bmatrix}$$

$$+ M(t) \begin{bmatrix} B(t)R^{-1}(t)B^T(t) & 0 \\ 0 & 0 \end{bmatrix} M(t) \text{ , } t\varepsilon\mathcal{J} \qquad (58)$$

$$\text{with } M(t_f) = \begin{bmatrix} S + 4\alpha S\Sigma(t_f)S & 4\alpha S \\ 4\alpha S & 0 \end{bmatrix} \qquad (59)$$

Notice that the resulting feedback structure is linear, dynamical, and physically realizable. A block diagram of the overall system is shown in Figure 1, where $M_{11}(t)$ and $M_{12}(t)$ are the obvious n x n submatrices of $M(t)$. The next section contains a numerical example with accompanying performance analysis to illustrate the procedure and properties of the solution.

V. PERFORMANCE CRITERIA - A NUMERICAL EXAMPLE

Consider a scalar system described by (1) - (9) with

$$J = [0, 1], \tag{60}$$

$$A(t) = B(t) = C(t) = x_0 = 1, \; t\varepsilon[0, 1], \tag{61}$$

$$\Sigma_0 = 0 \tag{62}$$

$$\Xi(t) = 0.25, \quad t\varepsilon[0, 1], \tag{63}$$

and

$$\Theta(t) = 0.35, \quad t\varepsilon[0, 1]. \tag{64}$$

We select (11) as the measure of system performance with

$$S = 0, \tag{65}$$

and

$$Q(t) = R(t) = 1, \quad t\varepsilon[0, 1]. \tag{66}$$

We will use as our performance criteria,

$$\min E\{\kappa_{1|F} + \alpha\kappa_{2|F}\}$$

The resulting feedback structure for the above system matrices is given by (57) - (59).

A plot of mean and variance of the performance measure is shown in Fig. 2. As expected, for an increase in α one observes both an increase in the mean and a decrease in the variance. However, there is yet an additional characteristic of these plots which is just as significant. In the region of small α, a given change in α results in a relative change in the variance which is greater than the corresponding relative change in the mean. Hence, for small α one obtains a mean only slightly larger than the minimum mean, while the resulting variance is considerably less than the maximum variance occurring at $\alpha = 0$. Of course, general conclusions cannot be drawn here, but our philosophy is supported.

In order to provide a more complete statistical description of system performance, the probability density functions of J for $\alpha = 0$, $\alpha = 0.5$, and $\alpha = 2.0$ are shown in Fig. 3.

For purposes of evaluating state regulation and control effort, we will now examine the resulting behavior of J_x and J_u given in (12) and (13) where we assume that

$$X(t) = U(t) = 1, \ t \varepsilon [0, 1], \tag{67}$$

Plots of mean and variance of both J_x and J_u are shown, respectively, in Figs. 4 and 5. An interesting characteristic of these curves is that for increasing α, both the mean and variance of the state regulation decrease. One obviously has to pay a price for this good performance, and this is seen in the control effort where both the mean and variance increase with increasing α. Figs. 6 and 7 show the probability density functions of J_x and J_u, respectively, for the values $\alpha = 0$, $\alpha = 0.5$, and $\alpha = 2.0$. The real value of these density curves is that for each value of α, one can see probabilistically the resulting trade-off between state regulation and control effort.

In an actual design the design engineer would iterate through the procedure, varying α, S, Q(t), and R(t) at each iteration as needed, until desirable performances, J_x and J_u, are obtained.

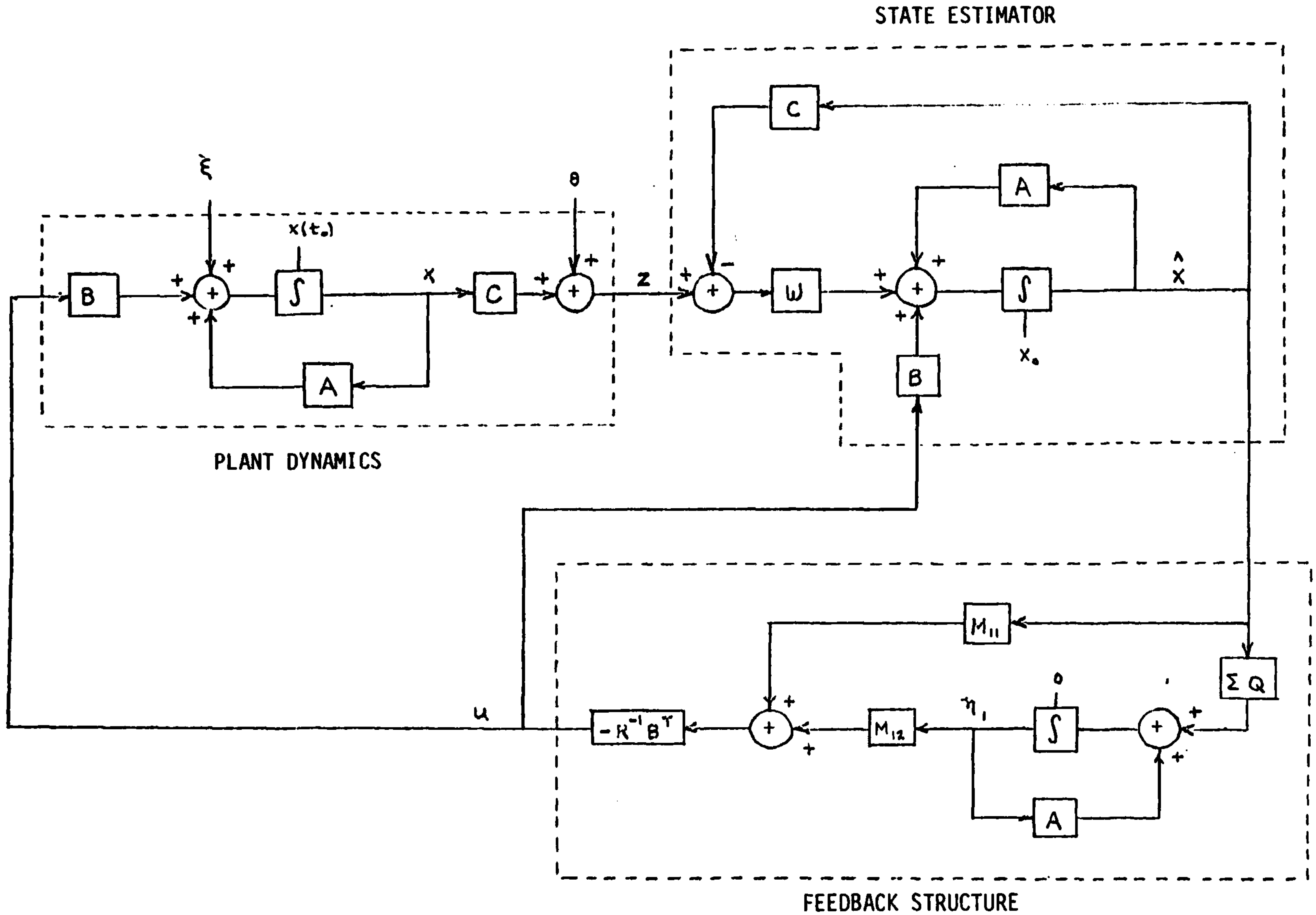

Fig. 1 Block diagram of the overall system resulting from the criteria min $E\{\kappa_{1|F} + \alpha\kappa_{2|F}\}$.

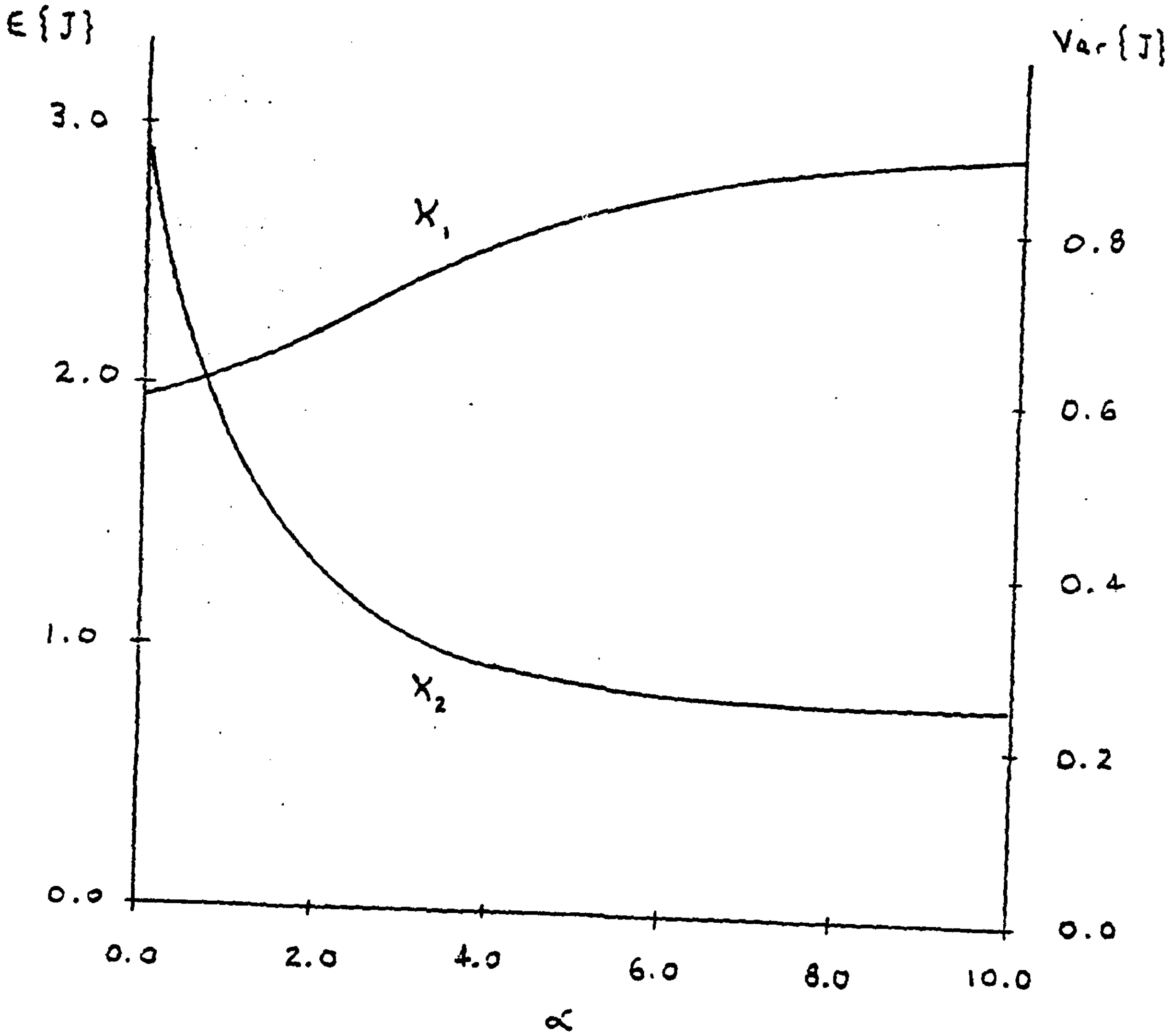

Fig. 2. Mean and variance of J vs. α.

design parameter

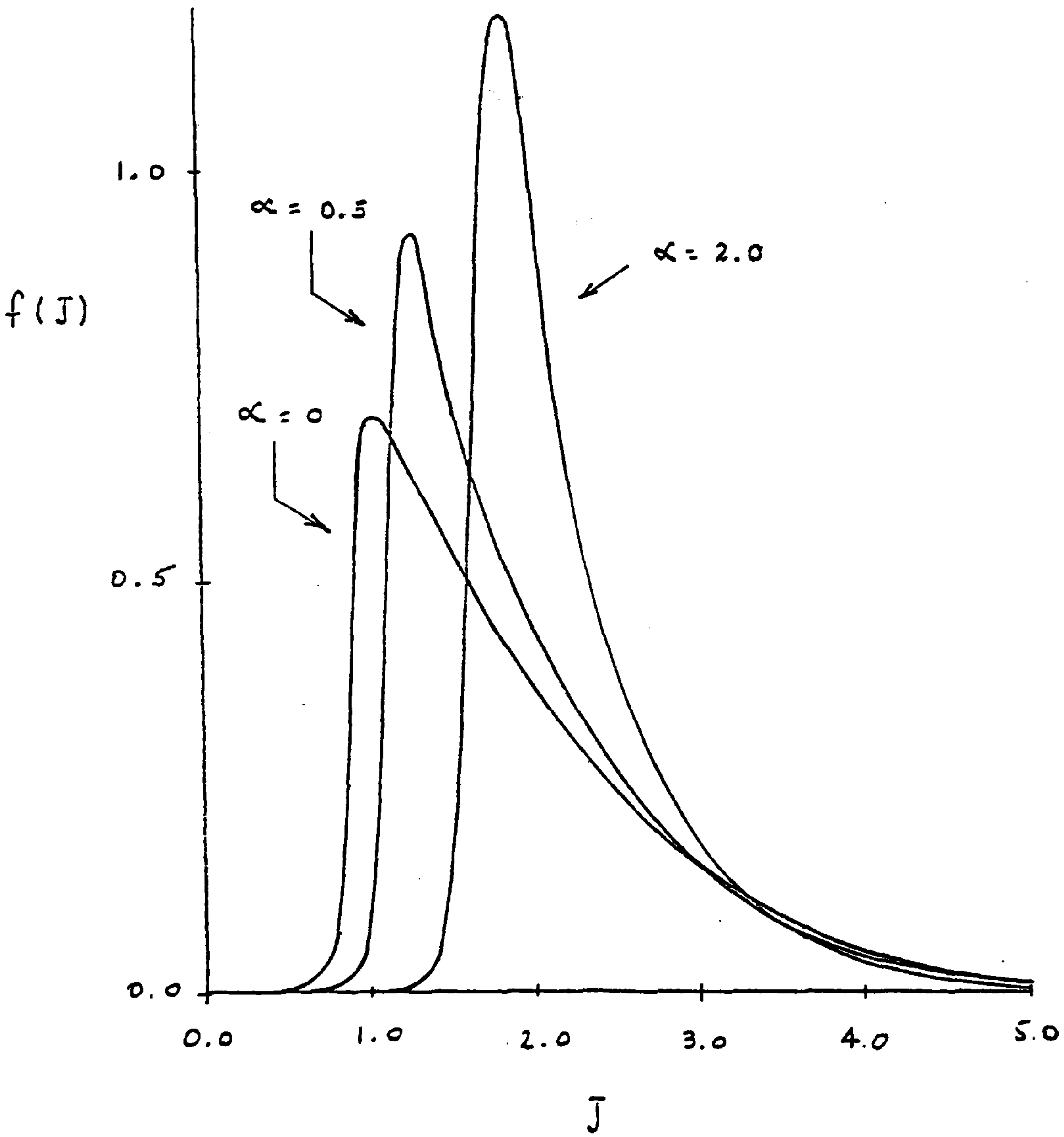

Fig. 3. Probability density function of J for α=0, α=0.5 and α=2.0.

$$J = x^T(t_f) S x(t_f) + \int_{t_0}^{t_f} \left[x^T(t) Q(t) x(t) + u^T(t) R(t) u(t) \right] dt$$

optimize on J

$$J_u = \int_{t_0}^{t_f} u^T(t) U(t) u(t) \, dt$$

may be identity

$$J_x = \int_{t_0}^{t_f} x^T(t) X(t) x(t) \, dt$$

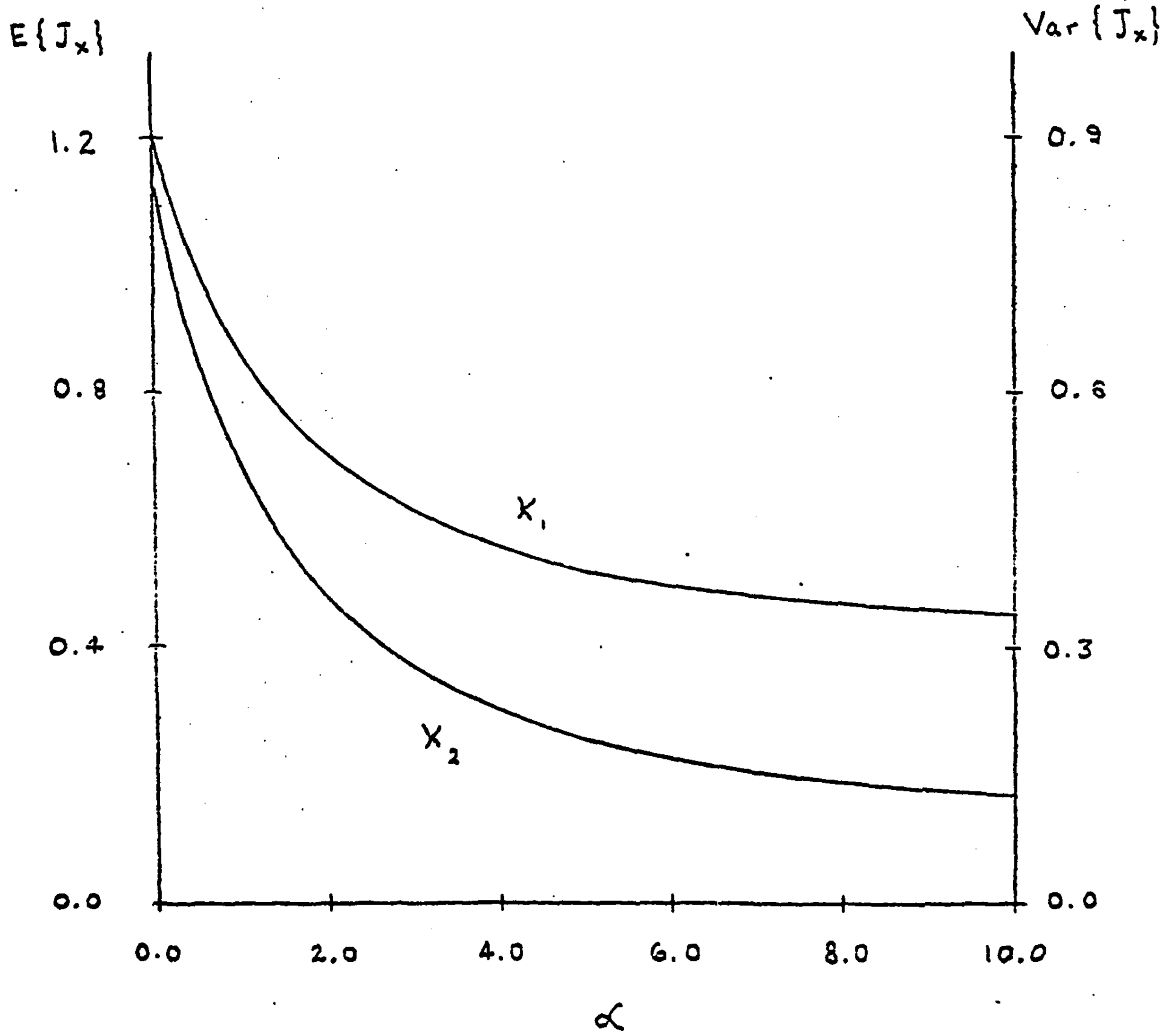

Fig. 4. Mean and variance of J_x vs. α.

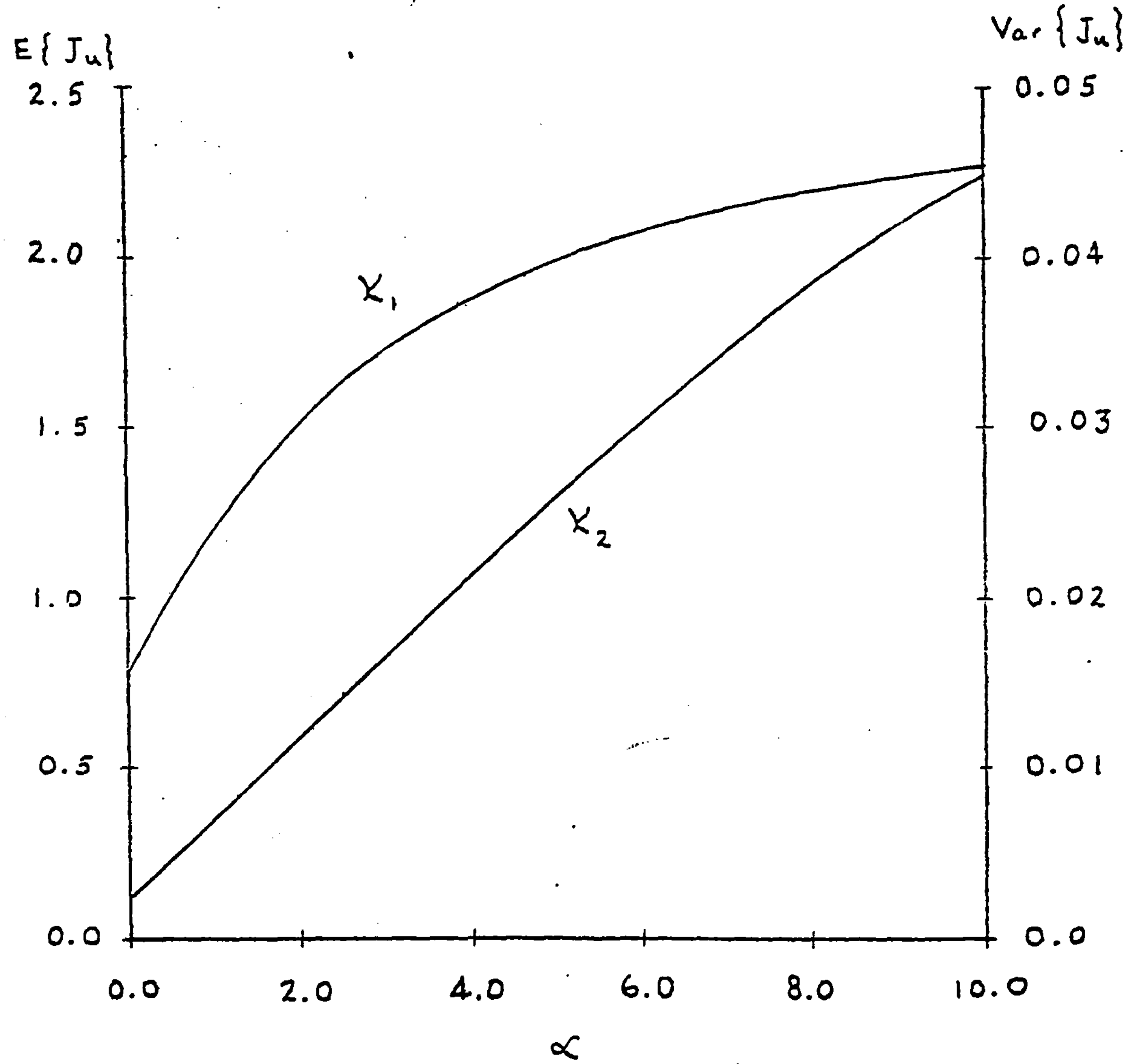

Fig. 5. Mean and variance of J_u vs. α.

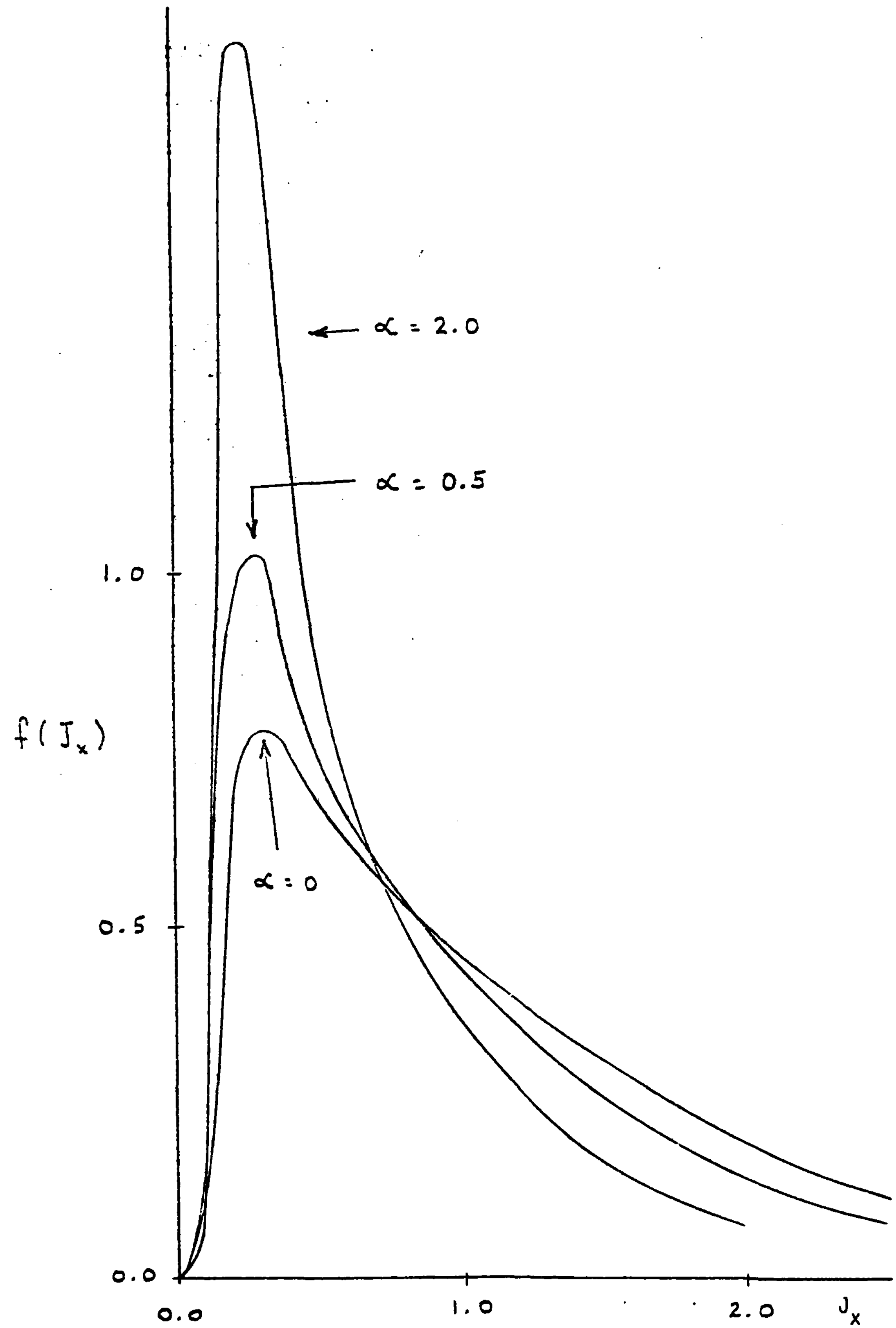

Fig. 6. Probability density functions of J_x for $\alpha=0$, $\alpha=0.5$, $\alpha=2.0$.

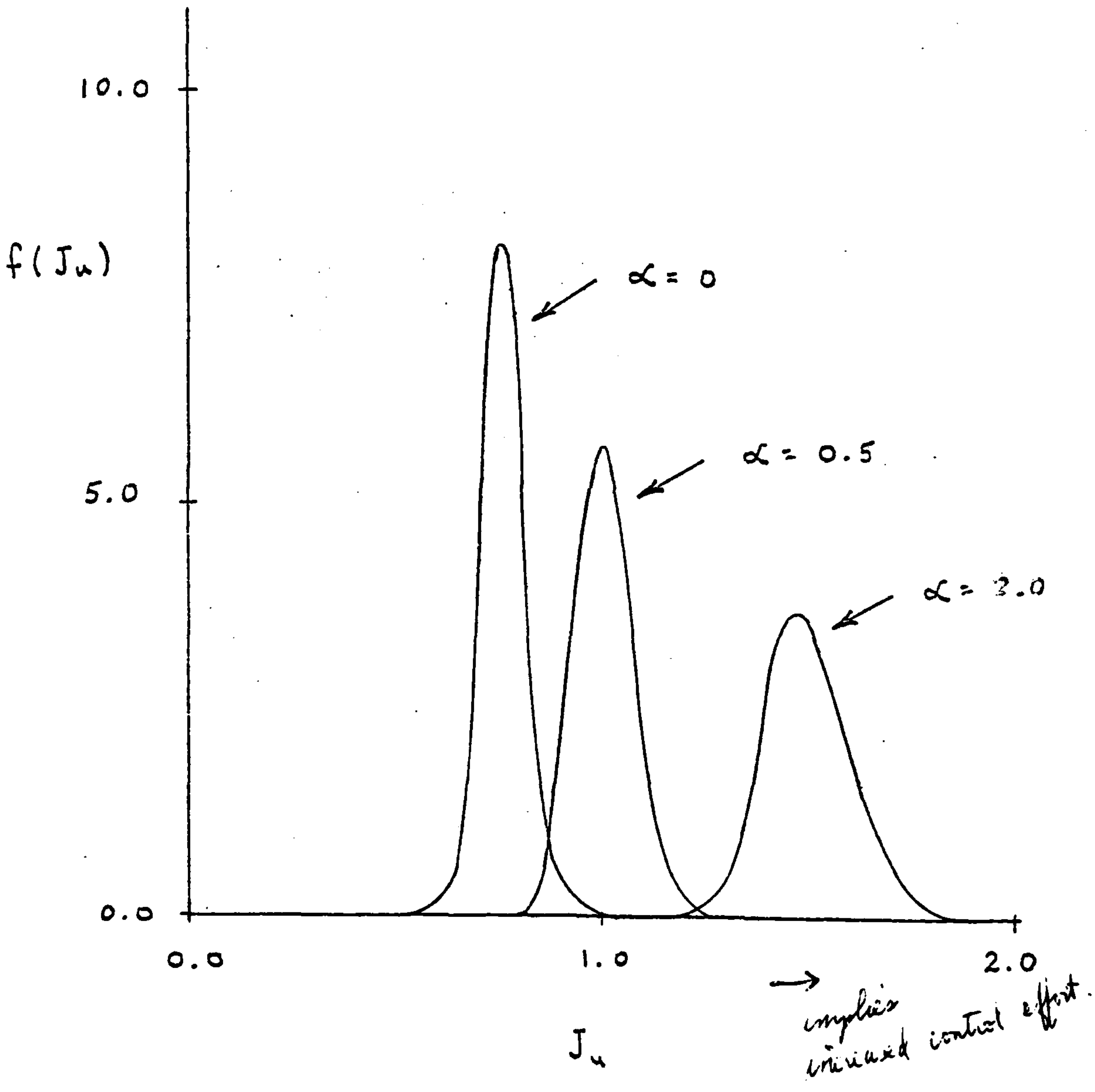

Fig. 7. Probability density functions of J_u for α=0, α=0.5, α=2.0.

VI. CONCLUSION

The new class of LQG performance criteria developed should enhance the applicability of the LQG theory. Experimentation with the proposed design procedure should be carried out. There are details on controllability and properties of equation solution, etc. that we have chosen not to address here.

The line of thinking that has led to these results may also prove to be useful in the estimation area,particularly with regard to the development of "robust" linear estimators.

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