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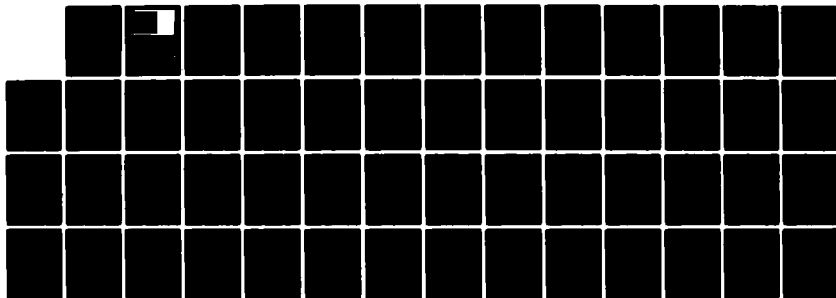
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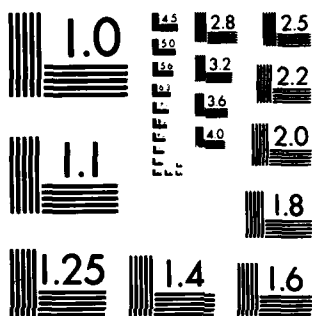
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A VARIATIONAL APPROACH TO
SURFACE SOLITARY WAVES

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ABSTRACT

Two-dimensional flow of an incompressible, inviscid fluid in a region with a horizontal bottom of infinite extent and a free upper surface is considered. The fluid is acted on by gravity and has a non-diffusive, heterogeneous density which may be discontinuous. It is shown that the governing equations allow both periodic and single-crested progressing waves of permanent form, the analogues, respectively, of the classical cnoidal and solitary waves. These waves are shown to be critical points of flow related functionals and are proved to exist by means of a variational principle.

AMS (MOS) Subject Classifications: 35J20, 35J60, 76B25, 76C99

Key Words: Solitary wave, surface wave, heterogeneous fluid, cnoidal wave, critical point, symmetrization, bifurcation

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SIGNIFICANCE AND EXPLANATION

The research in experimental and theoretical hydrodynamics in the last few decades has indicated that solitary waves play a special role in the evolution of general disturbances in fluids. Still, the investigation of solitary waves and, in particular, the use of variational principles associated with these waves is far from complete. While variational principles for surface waves in fluids of constant density have been discussed in the literature, the existence proofs given here appear to be the first rigorous use of critical point theory to obtain surface waves. Moreover, we treat a class of density profiles not heretofore included in an exact theory.

In this report we treat a two-dimensional flow of an incompressible, inviscid fluid in a region with a horizontal bottom of infinite extent and a free upper surface. The fluid is acted on by gravity and has a non-diffusive, variable density which may be discontinuous. It is shown by means of a variational principle that the governing equations allow both periodic and single-crested progressing waves of permanent form, the analogues, respectively, of the classical cnoidal and solitary waves. The solitary waves are obtained from periodic ones as the periods grow unboundedly. All of the waves obtained have elevated streamlines and have speeds greater than the critical speed associated with the ambient density. Further, the amplitudes are shown to be exponentially decreasing away from the crest.

The responsibility for the wording and views expressed in this descriptive summary lies with MRC, and not with the author of this report.

A VARIATIONAL APPROACH TO SURFACE SOLITARY WAVES

R. E. L. Turner

INTRODUCTION

This paper is concerned with two-dimensional flow of an incompressible, inviscid fluid in a region with a horizontal bottom of infinite extent and a free upper surface. The fluid is acted on by gravity and has a non-diffusive variable density which may be discontinuous. It is shown that the governing equations allow both periodic and single-crested progressing waves of permanent form, the analogues, respectively, of the classical cnoidal and solitary waves. Moreover, solitary waves are shown to arise from periodic ones as the period grows unboundedly. A survey of earlier work on steady waves in stratified fluids and references to the literature are given in [1] and [2]. The work on surface waves in fluids of constant density has a much longer history, going back to the middle of the nineteenth century; see [2], [3], [4] for references and accounts of the development of the subject.

The problem treated here is close to that examined by Ter-Krikorov [5] who treats a smoothly varying density, decreasing with height, and allows a free or fixed upper surface. He shows that from each vertical mode of a linearized flow problem there is bifurcation to a wave of arbitrarily prescribed horizontal period, including that of "infinite period", i.e., a solitary wave. The methods used are close to the perturbation technique of Friedrichs and Hyers [6] who gave an alternate proof to that of Lavrentiev [7] for the existence of small amplitude surface solitary waves. The techniques used here are variational in nature. They are an outgrowth of the work of Bona, Bose, and Turner [2] on smoothly stratified flows in regions with fixed upper and lower boundaries and are particularly close to the methods used by the author in [1], wherein we considered two fixed boundaries, but allowed a discontinuous density. The present paper is based on the observation that the free surface can be treated as an additional discontinuity at which

the density drops to zero. The estimates and proofs required here are similar to those in [1], differing mainly in the behavior near the free surface. To shorten the length of the presentation we will frequently make reference to proofs in [1], pointing out how the free surface is accommodated.

Variational principles satisfied by free surface flows of constant density have been described by a number of authors (cf. e.g. [8]-[13] and their lists of references). In [9], [10] and [13] a case is made for the use of dynamically invariant quantities in a variational characterization of a flow as a step toward a treatment of stability in the spirit of Liapunov. Similar ideas have been carried through for the Korteweg de Vries and other model equations (of [14], [15], [16]). It appears difficult to base a rigorous proof of the existence of steady wave solutions of the Euler equations on the dynamic principle given in [13]. Here we obtain waves using a different principle and, to our knowledge, ours is the first rigorous use of a variational procedure to obtain periodic and solitary surface waves. Garabedian gives a critical point principle for periodic surface waves, but it appears that his appeal to Morse theory needs further justification. It can be shown that the functional he uses is not uniformly positive definite at the origin, as claimed. The principle used here is Lagrangian in character and reduces to a constrained variational problem. The separate functionals used are not constants of time dependent motions and so the method does not immediately suggest a means for establishing stability. However, we feel that a further understanding of the variational structure of the problem will be useful. It should be noted that in a related problem, that of vortex flow, principles allied to dynamics have been successfully used to establish the existence of steady flows (cf. [17] and references given there).

Here a steady wave will correspond to a critical point of a "displacement" functional on a manifold of prescribed "kinetic energy" R . For the classical solitary wave what corresponds here to R almost certainly takes values in a finite range $0 < R < \hat{R} < \infty$ (the continua found in [4] have this property). Thus it is to be expected that with variable density the range of R will be finite. This limitation is reflected in a lack of coercivity in the analytical problem derived here. Our method of treatment involves

introducing artificial coercivity and combining this with estimates on the size of the solution of the altered problem. The outcome for the original problem is a restriction $0 < R < \tilde{R}$ on allowable energies with the size of $\tilde{R}$ buried among elliptic estimates. While it is difficult to compare the size of the solutions obtained here with those of Ter-Krikorov, they must both be considered to have small amplitude. However, the variational approach, with improved estimates, could provide finite amplitude waves. Apart from treating less regular densities than considered in [5] we can show that in the presence of a free surface there are always waves of elevation with amplitudes decreasing away from the crest. On the other hand, Ter-Krikorov's techniques give an explicit asymptotic form for the wave near a bifurcation from a parallel flow, though not uniformly in the horizontal variable, and are applicable to bifurcation from higher vertical modes, not covered by the treatment here. It should be noted that in the case of constant density it has been shown that families of cnoidal and solitary waves exist which include small amplitude waves and the Stokes wave with a sharp crest having a 120° opening (cf. [4], [18], [19]).

The organization of the present paper is as follows. In section 1 the physical situation is described and the relevant mathematical equations set down. The analytical problems are posed and the main results are described in theorem 1.2. The remainder of the paper is devoted to establishing these results. In section 2 a variational formulation is given and an "extended" problem with artificial coercivity is solved. In section 3 estimates for the solution of the extended problem are given. Section 4 contains estimates which establish that solutions of the extended problem, when restricted to have small energy, solve the original flow problem in the periodic case. Section 5 deals with exponential decay of wave amplitudes away from the crest and the existence of solitary waves as the limit of periodic ones when the periods increase indefinitely.

1. STEADY FLOWS WITH A FREE SURFACE

Here we briefly describe the passage from a physical model of wave motion to a boundary value problem for a partial differential equation. For a more complete discussion we refer to [1] and [2]. Consider a heterogeneous, incompressible fluid acted upon by gravity and restrict attention to flows which are two-dimensional. That is, assume all quantities depend only on a horizontal coordinate, a vertical coordinate, in the direction of gravity, and on time. The fluid is further assumed to be inviscid and non-diffusive, the latter property to be elaborated in the following paragraphs. While our interest is in wave patterns which progress horizontally at a fixed velocity c , we can remove the time dependence by considering Cartesian coordinates referred to a moving crest of a wave. It is then possible to seek a steady flow in a region which is independent of time. The region will have to be determined as part of the solution of the problem. However, we do assume the flow is over an infinite horizontal bottom. Let x be a coordinate in the horizontal boundary and y , a vertical coordinate chosen so that the bottom boundary is at $y = -1$ and so that the acceleration due to gravity is represented by $(0, -g)$ with $g > 0$. The fluid is assumed to have a free surface

$$y = h(x) > -1 \quad (1.1)$$

which is to be determined. To begin with we require h to be continuous and satisfy

$$\lim_{x \rightarrow \pm \infty} h(x) = 0 \quad (1.2)$$

The fluid is then assumed to occupy the region

$$F_h = \{(x, y) | -1 < y < h(x), -\infty < x < \infty\} \quad (1.3)$$

For $(x, y) \in F_h$ let

$$\mathbf{q} = (U, V) \quad (1.4)$$

where $U(x, y)$ and $V(x, y)$, respectively, are the horizontal and vertical components of fluid velocity in a moving frame. A flow is sought which is steady in the moving frame so the Euler equations take the form

$$\rho(\mathbf{q} \cdot \nabla)\mathbf{q} = -\nabla p - g\rho\mathbf{e}_2 \text{ in } F_h \quad (1.5)$$

where $\rho = \rho(x, y)$ is the density, p is the pressure, and $\mathbf{e}_2 = (0, 1)$. The condition of

incompressibility becomes

$$\operatorname{div} \mathbf{q} = 0 \quad (1.6)$$

and the condition of nondiffusivity entails

$$\mathbf{q} \cdot \nabla p = 0 \quad (1.7)$$

throughout F_h . Supplementing equations (1.5), (1.6), and (1.7) we have the following boundary conditions. The fluid should not penetrate the horizontal bottom; i.e.,

$$V(x, -1) = 0, \quad -\infty < x < \infty. \quad (1.8)$$

On the free surface $S = \{(x, y) | y = h(x), -\infty < x < \infty\}$ there are two conditions: a kinematic condition requiring the velocity to be tangential and a second requiring the pressure to be zero; i.e.

$$-h_x U + V = 0 \quad \text{on } S \quad (1.9)$$

and

$$p = 0 \quad \text{on } S. \quad (1.10)$$

Here h_x denotes a derivative. Finally, conditions must be specified at $x = \pm\infty$. We can specify a velocity distribution at infinity and here take the simplest case of a wave propagating in fluid which, in "laboratory" coordinates, is at rest at infinity. For coordinates based in a wave moving to the left with velocity c the condition becomes

$$\lim_{x \rightarrow \pm\infty} (U, V) = (c, 0) \quad (1.11)$$

Further, the density in the "undisturbed region" is specified and we take it to depend only on y . Thus

$$\lim_{x \rightarrow \pm\infty} \rho(x, y) = \rho(y) \quad (1.12)$$

where $\rho(y)$ is a decreasing function of y for $-1 < y < 0$, normalized to satisfy $\rho(0) = 1$.

The problem that will occupy us is to find $c, h, \mathbf{q}, \rho$ and p satisfying (1.1), (1.2), (1.5)-(1.12). The regularity required of the functions will be made more precise in what follows. For now we continue the discussion at a formal level. In particular, we

assume to begin with that ρ is continuously differentiable although we ultimately want to allow discontinuous densities.

The conditions (1.6) and (1.7) imply that there is a pseudo-stream-function

$\psi = \psi(x, y)$ such that

$$\frac{\partial \psi}{\partial y} = \rho^{1/2} U, \quad -\frac{\partial \psi}{\partial x} = \rho^{1/2} V \quad (1.13)$$

and that ρ is a function of ψ , $\rho(\psi)$. From Bernoulli's theorem the total head is constant along streamlines, hence on level sets of ψ . Thus

$$H = p + \frac{1}{2} \rho(U^2 + V^2) + \rho g y = H(\psi) \quad (1.14)$$

Eliminating p , using the two components of the vector equations (1.5), one finds that ψ satisfies the equation

$$\Delta \psi(x, y) + g y \frac{d\rho}{d\psi} = \frac{dH}{d\psi} \quad (1.15)$$

(cf. Dubreil-Jacotin [20], Long [21] and Yih [22]). If the density ρ is specified as a function of ψ and if the dynamics are specified by giving $H(\psi)$, then (1.15) is a semilinear elliptic equation for $\psi(x, y)$. Any solution of (1.15) gives rise to a solution of (1.5) with U, V and p obtained from (1.13) and (1.14). For now we leave aside the question of boundary conditions on the top and bottom of F_h and examine the implications of (1.11) and (1.12).

In the search for a solitary wave, a disturbance which should be of essentially finite extent, it is natural to ask that for large x , $\psi(x, y)$ should approach a pseudo-stream-function corresponding to a flow with velocity $(c, 0)$ in a stream of density $\rho(y)$. Thus, letting

$$\dot{\psi}(y) = c \int_0^y \rho^{1/2}(s) ds \quad (1.16)$$

we require

$$\psi(x, y) - \dot{\psi}(y) \rightarrow 0 \quad \text{as } |x| \rightarrow \infty, \quad (1.17)$$

The density ρ (we use the same symbol) associated with the stream coordinate ψ , compatible with that already specified at $x = \pm\infty$, is

$$\rho(\psi) = \rho(\dot{\gamma}(\psi)) \quad (1.18)$$

where $\dot{\gamma}(\psi)$ is the function inverse to $\dot{\psi}(y)$. The expression (1.18) is taken as the

definition of $\rho(\psi)$. Similarly, the total head associated with the stream coordinate ψ is taken to be

$$H(\psi) = p(\dot{y}(\psi)) + \frac{1}{2} \rho(\psi) c^2 + \rho(\psi) g \dot{y}(\psi) \quad (1.19)$$

where p is hydrostatic:

$$p = -g \int_0^y \rho(s) ds. \quad (1.20)$$

With $\rho(\psi)$ and $H(\psi)$ given we can now seek a solution $\psi(x, y)$ of (1.15). Note that for any constant c , $\dot{\psi}(y)$ (cf. 1.16) is a solution of (14). We call it a trivial solution.

The flows examined here will have free surfaces and discontinuities in velocity occurring along certain streamlines. In order to deal with a problem in a fixed domain and to confine irregular behavior to coordinate lines we replace (1.15) by an equation for y in semi-Lagrangian independent coordinates x and ψ . The interest here is in flows without reversal; i.e. with $U > 0$ or $\frac{\partial \psi}{\partial y} > 0$ and so an inversion yielding $y(x, \psi)$ is reasonable. Corresponding to (1.15) is the equation

$$-\frac{\partial}{\partial x} \left(\frac{y}{y_\psi} x \right) + \frac{\partial}{\partial \psi} \left(\frac{1 + y^2}{y^2} x \right) + g \frac{d\rho}{d\psi} y = \frac{dH}{d\psi} \quad (1.21)$$

(cf. [1], [5]), obtainable using the relations

$$\psi_x + \psi_y y_x = 0, \quad \psi_y y_\psi = 1 \quad (1.22)$$

Naturally, $y = \dot{y}(\psi)$, the inverse of $\dot{\psi}(y)$, is a solution of (1.21) and we refer to it as trivial. Corresponding to (1.17) is a condition

$$y(x, \psi) - \dot{y}(\psi) \rightarrow 0 \quad \text{as } |x| \rightarrow \infty, \quad (1.23)$$

which is imposed uniformly in $\psi_{-1} < \psi < 0$ where

$$\psi_{-1} = \dot{\psi}(-1) = c \int_0^{-1} \rho^{1/2}(s) ds. \quad (1.24)$$

The condition

$$y(x, \psi_{-1}) = -1 \quad (1.25)$$

replaces (1.8), and from (1.22) it is clear that (1.9) will be satisfied merely by choosing

$$h(x) = y(x, 0). \quad (1.26)$$

Since $\rho(U^2 + V^2) = (1 + y_x^2)y_\psi^{-2}$, the pressure condition at the fluid surface becomes

$$p(x, 0) = \text{constant} \quad (1.27)$$

where

$$p(x, \psi) = H(\psi) - \frac{1}{2} \left(\frac{1 + y_x^2}{y_\psi^2} \right) - \rho(\psi) g y(x, \psi), \quad (1.28)$$

using (1.14). Suppose that a continuously differentiable $\rho(y)$ is given and that for any fixed speed c , $\rho(\psi)$ and $H(\psi)$ are defined by (1.18) and (1.19). It is then possible to interpret the foregoing equations and conditions in a classical sense. For a density ρ with possible discontinuities a meaning must be given to equation (1.21) and we do that next.

Let $C^{k, \beta}$ on a domain consist of functions with continuous derivatives through order k , each satisfying a Hölder condition with exponent β . For a more complete description of spaces see [1], section 1. Suppose $\rho(y)$, given initially, satisfies

- i) ρ is nonincreasing on $[-1, 0]$
- ii) $\rho \in C^{1, \beta}$ for some $\beta > 0$ except for jumps (1.29)
at points $\eta^1 : -1 < \eta^1 < \eta^2 < \dots < \eta^N < 0$
where ρ is continuous from the right.

The corresponding function $\rho(\psi)$ from (1.18) will have discontinuities at points

$\psi^j = \psi(\eta^j)$, $j = 1, 2, \dots, N$. Extending the domain of ψ^j to $\eta^0 = -1$ and $\eta^{N+1} = 0$ (so that $\psi^0 = \psi(\eta^0) = \psi_{-1}$) we let

$$D^j = \mathbb{R} \times (\psi^j, \psi^{j+1}), \quad j = 0, 1, \dots, N \quad (1.30)$$

and $D = \mathbb{R} \times (\psi_{-1}, 0)$. For $0 < \alpha < 1$ let

$$M^\alpha = \{y | y \in C^{0,1}(D), y_x \in C^\alpha(D), y \in C^{1,\alpha}(D^j), \quad (1.31)$$

$$\text{and } y_\psi > 0 \text{ in } D^j, \quad j = 1, 2, \dots, N\}$$

Since (1.21) has a divergence form, the notion of a weak solution where ρ and H are smooth is described in a standard way (cf. [1] (1.13)). A weak solution can also be

defined with the presence of discontinuities in ρ and H and is equivalent to the following definition which stresses a physical aspect of the flow.

Definition 1.1. We call $y \in M^a$ a solution of (1.21) if and only if

- i) y is a weak solution of (1.21) in D_j for $j = 1, 2, \dots, N$.
- (ii) The pressure $p(x, \psi)$ computed from (1.28) in each D_j , $j = 0, 1, 2, \dots, N$ is the restriction to D_j of a continuous function p defined on $\bar{D}$.

The solitary wave problem can be posed as follows.

Problem P_∞ . Find a $c > 0$ and, for some $a > 0$, a nontrivial (1.32)
function $y \in M^a$, satisfying (1.21), (1.23), (1.25) and (1.27).

In the course of solving the solitary wave problem P_∞ we will solve the corresponding problem for periodic waves.

Problem P_k . Find a $c > 0$ and a nontrivial $y \in M^a$, y a (1.33)
 $2k$ periodic function of x , satisfying (1.21), (1.25) and (1.27).

The interpretation of a solution of problem P_∞ in terms of a solitary wave hinges on the condition (1.23), requiring the flow to be "trivial" at $x = \pm\infty$. While the physical interpretation of a solution of problem P_k , for finite k , is tenuous, we will nevertheless find solutions which are $2k$ periodic in x and show that they are exponentially decaying from crest to trough. Thus they might be viewed as a train of waves with quiescent zones of approximately trivial flow between the crests. In any case, the periodic formulation is a convenient analytical device.

To analyze the problems P_k for $k < \infty$ and to expand the class of density profiles, we associate with any density function ρ described by (1.29) a family of smooth densities as follows. For a δ_0 which is positive but smaller than $\eta^j - \eta^{j-1}$ for $j = 1, 2, \dots, N$

and for each δ , $0 < \delta < \delta_0$, let $\rho_\delta(y)$ be a nonincreasing function of class $C^{1,\beta}$ such that

$$\rho_\delta(y) = \rho(y) \text{ for } y \notin \bigcup_{j=1}^N (\eta^j - \delta, \eta^j) \quad (1.34)$$

and set $\rho_0(y) = \rho(y)$. The manner in which the jump at η^j is approximated by a smooth transition is immaterial to our estimates and so we need not specify ρ_δ on the intervals $(\eta^j - \delta, \eta^j)$, $j = 1, 2, \dots, N$. The estimates in the subsequent sections give a uniform picture of the transition from smooth to discontinuous densities.

In order to summarize the main results of the paper in a convenient form it is necessary to anticipate a transformation to be introduced in section 2 and give some additional notation. A rescaled stream variable is given by

$$\eta(\psi) = \int_0^\psi \frac{ds}{c \rho^{1/2}(y(s))} \quad (1.35)$$

(cf. 2.14) so that at $x = x^\infty$ the streamline with label η has height η . The expression

$$w(x, \eta) = y(x, \psi(\eta)) - \eta$$

(cf. 2.2) represents, for each x , the displacement from its position in a trivial flow, of the streamline which has height η at $x = x^\infty$ (under the condition 1.23). The equation satisfied by w , with x and η as independent coordinates, is given by (2.12) (where $x = x_1$ and $\eta = x_2$). Its formal linearization about $w = 0$ is

$$\left. \begin{aligned} \frac{c^2}{2} \left[\frac{\partial}{\partial x} \rho(\eta) \frac{\partial z}{\partial x} + \frac{\partial}{\partial \eta} \rho(\eta) \frac{\partial z}{\partial \eta} \right] &= g \frac{d\rho}{d\eta} z \text{ in } \mathbb{R} \times (-1, 0) \\ z &= 0 \text{ at } \eta = -1; \quad c^2 \frac{\partial z}{\partial \eta} = gz \text{ at } \eta = 0 \end{aligned} \right\} \quad (1.36)$$

(cf. 4.22), the last condition arising (as we shall see in proposition 2.1) from the Bernoulli condition (1.27). The Sturm-Liouville problem obtained from (1.36) by letting $z = z(\eta)$ has a least eigenvalue $\lambda = g/c^2$. The corresponding value of c , denoted c_δ to indicate its dependence on δ through $\rho = \rho_\delta$, is the largest speed (correspondingly, the lowest spectral point) associated with (1.36). This speed is referred to in the hydrodynamics literature as the "critical" speed or the speed with which infinitesimal long waves travel. Its relevance here is that the waves we obtain have propagation speeds which

are "supercritical", i.e., they are larger than c_0 . A full discussion of these ideas has been given by Benjamin [23].

The main results of the paper are contained in the following theorem. Its proof, and explicit estimates related to the waves obtained, are spread over the ensuing sections. In particular we refer to propositions 2.1, 2.2, 4.4, 4.6, 4.8 and 5.2 together with remarks 2.3, 4.9 and 5.3. Note that a discontinuous density is allowed when the parameter $\delta = 0$, while a solitary wave corresponds to the case $k = \infty$, by our convention.

Theorem 1.2. There are positive numbers $\tilde{R}$, $\tilde{k}(R)$, and α such that for $0 < \delta < \delta_0$, $0 < R < \tilde{R}$, and $\tilde{k}(R) < k < \infty$ the problem P_k for ρ_δ has a nontrivial solution y in M^α corresponding to a speed c with the following properties

- 1) $c > c_0(1 - C_1 R^{4/3})^{-1/2}$ i.e. c is "supercritical"
- 2) y has period $2k$ in x (for $k < \infty$).

The streamline displacement w satisfies

$$3) \int_{-1}^0 \int_{-k}^k \rho(\eta) \frac{|Vw|^2}{1 + w_\eta} dx d\eta = R^2$$

$$4) w(x, \eta) > 0 \text{ for } -1 < \eta < 0.$$

$$5) w(x, \eta) = w(-x, \eta) \text{ and for } 0 < x < x' < k$$

$$w(x, \eta) > w(x', \eta).$$

$$6) |w| < C_2 \exp[-\beta x] \text{ and } |Vw| < C_3 \exp[-\beta x] \text{ on } 0 < x < k \text{ for a } \beta > 0.$$

The quantities $\tilde{R}$, $\tilde{k}$, α , C_1 , C_2 and C_3 depend on ρ in (1.29); C_2 and C_3 also depend on R and β .

2. A VARIATION FORMULATION

In this section it is assumed that the density is smooth and given by (1.34) for $\delta > 0$, but the subscript is often omitted. The equation (1.21) is formally the Euler equation for the functional

$$\Phi(y) = \iint_D \left[\frac{1}{2} \left(\frac{1+y^2}{y_\psi} \right) + g\rho'(\psi) \frac{y^2}{2} - H'(\psi)y \right] dx d\psi, \quad (2.1)$$

where primes denote derivatives. That is, the condition that Φ has a critical point (zero derivative) at the function y is expressed by (1.21). This will be made precise at a later stage. Note that the first term in the integrand in (2.1) is merely the Dirichlet integral in the new coordinates. As noted in section 1, $y = \dot{y}(\psi)$ is a solution of (1.21); thus it is formally a critical point of Φ . If $y(x, \psi)$ is another solution of (1.21) then

$$w(x, \eta) = y(x, \dot{\psi}(\eta)) - \eta \quad (2.2)$$

is formally a critical point of

$$\tilde{G}(w) = \iint_\Omega \left[\frac{1}{2} c^2 \rho(\eta) \frac{|w_\eta|^2}{1+w_\eta} + g\rho'(\eta) \frac{w^2}{2} \right] dx d\eta \quad (2.3)$$

where

$$\Omega = \{(x, \eta) | -\infty < x < \infty, -1 < \eta < 0\} \quad (2.4)$$

(cf. [1], §2). Here η is the stream coordinate introduced in (1.35) and w , the vertical streamline displacement. We emphasize that for a nontrivial flow $w \neq 0$, in general; i.e., only at $x = \pm\infty$ does one require that the streamline with label η have height η . Note that with the new scaling $\rho(\eta)$ is the same function introduced at the outset, describing the density as a function of height.

Just below the free upper surface of the flow the density has the positive value 1. While the usual model for surface waves implicitly takes $\rho = 0$ in the atmosphere, retaining only the Bernoulli pressure condition, it appears necessary for a workable variational principle in the present context to explicitly incorporate a drop in density at the upper surface. Starting with ρ (or ρ_0) define

$$\dot{\rho}(\eta) = \begin{cases} \rho(\eta), & -1 < \eta < 0 \\ 0, & \eta = 0 \end{cases} \quad (2.5)$$

The change in ρ at the one point $\eta = 0$ will not alter the first term in the integrand in (2.3), but will alter the second term. Define

$$G(w) = \iint_{\Omega} \left[\frac{1}{2} c \rho(\eta) \frac{|\nabla w|^2}{1+w_\eta} + g \dot{\rho}'(\eta) \frac{w^2}{2} \right] dx d\eta. \quad (2.6)$$

Equivalently,

$$G(w) = \tilde{G}(w) - \int_{-\infty}^{\infty} g \frac{w^2(x,0)}{2} dx. \quad (2.7)$$

We turn now to the periodic version of the wave problem. Let

$$\Omega_k = \{(x, \eta) \mid |x| < k, -1 < \eta < 0\} \quad (2.8)$$

and

$$G(w) = \int_{\Omega_k} \left[\frac{1}{2} c^2 \rho(\eta) \frac{|\nabla w|^2}{1+w_\eta} + g \dot{\rho}'(\eta) \frac{w^2}{2} \right] \quad (2.9)$$

It will often be convenient to use the notation $x_1 = x$, $x_2 = \eta$, $p_1 = \frac{\partial w}{\partial x_1}$ and $f_1 = \frac{\partial f}{\partial p_1}$ for $i = 1, 2$ where

$$f(p_1, p_2) = \frac{1}{2} \frac{p_1^2 + p_2^2}{1 + p_2} \quad (2.10)$$

Still proceeding formally, one verifies that if w (assumed to vanish when $x_2 = -1$) is a critical point of G , then with $\lambda = g/c^2$,

$$\int_{\Omega_k} \rho(x_2) f_1(\nabla w) \cdot x_{x_1} = \lambda \int_{\Omega_k} \dot{\rho}'(x_2) w x_2 \quad (2.11)$$

for allowable "variations" z . Here and in the sequel a repeated index is understood to be summed over $\{1, 2\}$. The equation (2.11) is merely a weak form of

$$\begin{cases} \frac{\partial}{\partial x_1} \rho(x_2) f_1(\nabla w) = \lambda p'(x_2) w & \text{in } \Omega_k \\ w = 0 & \text{at } x_2 = -1 \\ \rho(x_2) f_2(\nabla w) = \lambda w & \text{at } x_2 = 0. \end{cases} \quad (2.12)$$

We interrupt the reformulations of the problem at this point to show that a nonzero smooth solution of (2.12) gives rise to a nontrivial solution of problem P_k .

Proposition 2.1. Suppose $w \in C^2(\Omega) \cap C^1(\bar{\Omega})$. If w , with $w_\eta > -1$, satisfies (2.12) for some $\lambda > 0$, then (cf. 2.2) y defined by

$$y(x, \psi) = \eta(\psi) + w(x, \eta(\psi)) \quad (2.13)$$

where

$$\eta(\psi) = \int_0^\psi \frac{ds}{c p^{1/2}(y(s))} \quad (2.14)$$

and

$$c = g/\sqrt{\lambda} \quad (2.15)$$

satisfies equation (1.21) and the conditions (1.25) and (1.27).

Proof. The correspondence between the elliptic equation (2.12) for w in Ω and the equation (1.21) for y in D is shown exactly as in section 2 of [1]. Also the assertion that y takes the value -1 when $\psi = \psi_{-1}$ follows trivially. To see that the pressure condition (1.27) is satisfied start with

$$f_2(\nabla w) = \lambda w$$

at $x_2 = 0$ (note $p(0) = 1$) and express the relation in the coordinates x, η :

$$\frac{w_\eta + \frac{1}{2} w_\eta^2 - \frac{1}{2} w_x^2}{(1 + w_\eta)^2} - \frac{g}{c^2} w = 0 \quad (2.16)$$

for $\eta = 0$. To compute the pressure from (1.28) the value of H for $\eta = 0$ is needed.

From (1.19)

$$H|_{\psi=0} = p(0) + \frac{1}{2} c^2 + g\psi(0)$$

$$= \frac{1}{2} c^2$$

From the relations (1.16) and (2.2), evaluated where $\eta = 0$ (or $\psi = 0$) and $p(0) = 1$,

$$w_x = y_x$$

and

$$w_\eta = y_\psi \cdot \psi_\eta - 1$$

$$= y_\psi \cdot c - 1.$$

Hence the expression (1.28) for the pressure becomes

$$p(x,0) = \frac{1}{2} c^2 - \frac{1}{2} \frac{1 + w_x^2}{(1 + w_\eta)^2} c^2 - gw$$

$$= 0$$

when (2.16) is used, completing the proof of the proposition.

To obtain nontrivial solutions of Problem P_k ($k < \infty$) for a density which is smooth on $-1 < \eta < 0$ it will suffice to obtain nonzero periodic solutions of (2.12). The case of discontinuities in density on $-1 < \eta < 0$ and the resolution of problem P_∞ for solitary waves will be handled through limiting procedures.

The equation (2.11) can be expressed as

$$F'(w) = \lambda B'(w) \quad (2.17)$$

where

$$F(w) = \int_{\Omega_k} \rho(x_2) f(\nabla w)$$

$$B(w) = - \int_{\Omega_k} \rho'(x_2) \frac{w^2}{2} \quad (2.18)$$

and the primes in (2.17) denote derivatives (still to be defined in a suitable space). A tempting approach to solving equation (2.17) is to consider

$$\sup_{F(w)=\text{const.}} B(w)$$

and obtain λ as a Lagrange multiplier associated with an extremal w . However, B is unbounded on level sets of F . To see this consider the case that $p \equiv 1$ and let

$$w = \frac{1 + \eta}{\sqrt{|x| + |\eta|^{1/2}}}$$

on Ω_k . Clearly $w(x,0) \notin L^2$; however, a simple estimate shows $F(w) < \infty$. Nevertheless, as shown in [1] for a similar problem without a free surface, the variational approach can be salvaged by altering f in the integrand of F where $|w| > r$, for some positive r , and then showing that a solution obtained by a variational procedure and having a suitably restricted "energy" satisfies $|w| < r$. The approach here is similar, but we also take account of the free upper surface by an altered functional.

We now proceed to define a substitute for equation (2.17) as a step to obtaining the results in theorem 1.2. We refer the reader to section 2 of [1] for proofs of some of the assertions to be made. We are still considering a family of densities ρ_δ , $0 < \delta < \delta_0$, (cf. (1.34)) which are smooth on $-1 < \eta < 0$ and which could reflect a rapid change in density at certain levels in a fluid. A similar "smoothing" for $\hat{\rho}$ near $\eta = 0$ will be useful, purely as an analytical device. Let $\tau = \tau(\eta)$ be a nonincreasing C^2 function on $(-\infty, 0)$ which is zero for $\eta < -1$ and which satisfies $\tau(0) = -1$. For each ϵ , $0 < \epsilon < \delta_0$ let $\tau_\epsilon(\eta) = \tau(\eta/\epsilon)$ and define

$$\hat{\rho}_{\delta, \epsilon}(\eta) = \rho_\delta(\eta) + \tau_\epsilon(\eta) \quad (2.19)$$

so that $\hat{\rho}_\delta(\eta) = \lim_{\epsilon \rightarrow 0} \hat{\rho}_{\delta, \epsilon}$. As earlier, the manner in which $\hat{\rho}_{\delta, \epsilon}$ decreases from $\rho_\delta(-\epsilon)$ to 0 on $[-\epsilon, 0]$ is not important, though we do require that $\hat{\rho}_{\delta, \epsilon}' < \rho_\delta'$. Now extend ρ_δ to $0 < \eta < 1$ as an even function and $\hat{\rho}_{\delta, \epsilon}$ to $0 < \eta < 1$ as an odd function, retaining the same symbols for the extended functions. The extended functions will be in class $C^{1, \beta}$ if $\rho_\delta'(0) = 0$; otherwise ρ_δ is merely Lipschitz continuous at 0. We also require the function s defined by

$$s(\eta) = -\text{sgn}(\eta) \quad (2.20)$$

where the signum of 0 is 0. Now, suppose $\zeta = \zeta(t)$ is a smooth decreasing cutoff function, equal to 1 for $0 < t < 1$ and equal to 0 for $t > 2$. Replacing $f(p_1, p_2)$ in (2.18) will be

$$a(x_2, p_1, p_2) = \xi_r \frac{p_1^2 + p_2^2}{2(1 + s(x_2)p_2)} + (1 - \xi_r) \left(\frac{p_1^2 + p_2^2}{2} \right) \quad (2.21)$$

where $\xi_r(p_1, p_2) = \zeta\left(\frac{p_1^2 + p_2^2}{r^2}\right)$ so that $a = (p_1^2 + p_2^2)/2$ when $p_1^2 + p_2^2 > 2r^2$. Let $a_1 = \partial a / \partial p_1$ and $a_{ij} = \partial^2 a / \partial p_i \partial p_j$. The function a is globally convex in (p_1, p_2) , uniformly in x_2 , for r sufficiently small, and satisfies the following inequalities (proved exactly as in [1], lemma 2.1).

$$\left. \begin{aligned} \frac{1}{2} \sigma_1(p_1^2 + p_2^2) &< a(p_1, p_2) < \frac{1}{2} \sigma_2(p_1^2 + p_2^2) \\ \sigma_3(p_1^2 + p_2^2) &< a_1 p_1 + a_2 p_2 < \sigma_4(p_1^2 + p_2^2) \\ a_1^2 + a_2^2 &< \sigma_5(a_1 p_1 + a_2 p_2) \\ v(\zeta_1^2 + \zeta_2^2) &< a_{ij} \zeta_1 \zeta_j \end{aligned} \right\} \quad (2.22)$$

Here σ_i for $1 \leq i \leq 5$ and v are positive constants independent of x_2 , ϵ , p_1 and p_2 .

Let

$$\hat{\Omega}_k = \{(x_1, x_2) \mid |x_1| < k, |x_2| < 1\} \quad (2.23)$$

and set $\hat{\Omega}_\infty = \hat{\Omega}$. Let C_k denote the C^∞ functions on $\hat{\Omega}$ which are $2k$ periodic in x_1 , and have support not containing points where $x_2 = \pm 1$. The symbol C_k° denotes the elements of C_k which are even with respect to x_1 and x_2 . Since the functions in C_k and C_k° vanish when $x_2 = -1$, the Poincaré inequality guarantees that the expression

$$|v|_k = \left(\int_{\hat{\Omega}_k} |v_w|^2 \right)^{1/2} \quad (2.24)$$

provides a norm, and the completions of the respective spaces in the norm are denoted by

$H_k = H_k(\hat{\Omega})$ and $H_k^\circ = H_k^\circ(\hat{\Omega})$. The symbols H_k and H_k° will also be used in later

sections to denote the restrictions to Ω of functions in those spaces. Thus a function in $H_k^0(\Omega)$ is even in x_1 . For $w \in H_k^0(\hat{\Omega})$ define the functionals

$$\hat{A}(w) = \int \rho_\delta(x_2) a(x_2, \nabla w) \quad (2.25)$$

and

$$\hat{B}(w) = - \int \rho_{\delta, \epsilon}'(x_2) \frac{w^2}{2} \quad (2.26)$$

where integration is over $\hat{\Omega}_k$ here and in the remainder of this section. The notation $\hat{A}', \hat{B}'$ will be used for the Frechet derivatives of the functionals in $H_k^0(\hat{\Omega})$.

Our subsequent program is briefly described as follows. We show that for each $R > 0$ the equation (2.27) below has a solution (λ, w) with $w = w_{k, \delta, \epsilon}$ positive in $\hat{\Omega}_k$ and normalized by $\hat{A}(w) = 2R^2$. Restricting attention to the lower region Ω_k we show that for ϵ converging to zero through a suitable subsequence we obtain a function w which is smooth on Ω_k and satisfies (3.23), essentially the restriction of (2.27) to Ω_k for $\epsilon = 0$. For R suitably restricted, $|\nabla w| < r$ and we can show that (λ, w) satisfies the original equation (2.12). All estimates obtained are uniform in $\delta > 0$ and $k > 0$. Taking limits of solutions as these parameters vary we obtain the desired wave forms. Until we consider limits involving the parameters k, δ , and ϵ , we selectively suppress them. It is assumed that $k < \infty$ until Remark 5.3. Through the penultimate paragraph of section 4 it is assumed that $\delta > 0$, while for the remainder of this section, it is also assumed that $\epsilon > 0$.

Proposition 2.2. For each $R > 0$ the problem

$$\hat{A}'(w) = \lambda \hat{B}'(w) \quad (2.27)$$

has a solution (λ, w) satisfying $\lambda > 0$, $w \in H_k^0(\hat{\Omega})$, $\hat{A}(w) = 2R^2$, and $w > 0$ in $\hat{\Omega}$. The function w is characterized by

$$\hat{B}(w) = \sup_{\substack{v \in H_k^0 \\ \hat{A}(v) = 2R^2}} \hat{B}(v^+) \quad (2.28)$$

where $v^+ = \max(0, v)$.

Proof. The proof proceeds as that of Theorem 2.1 of [1] with H_k^0 replacing H_k . The variational procedure leads to $M(\phi) = 0$ for all $\phi \in H_k^0(\hat{\Omega}_k)$ and for a suitable $\lambda > 0$,

where M is defined by

$$M(\phi) = \int [\rho(x_2) a_1(x_2, \nabla w) \frac{\partial \phi}{\partial x_1} + \lambda \rho'(x_2) w^+ \phi]. \quad (2.29)$$

The derivatives of a are easily seen to satisfy $a_1(x_2, -p_1, p_2) = -a_1(x_2, p_1, p_2)$ and $a_1(-x_2, p_1, -p_2) = a_1(x_2, p_1, p_2)$, while a_2 has the opposite parity, i.e., even in p_1 and odd in (x_2, p_2) . Further ρ and ρ' are even in x_2 . As a consequence one finds that M annihilates all functions ϕ which are odd in x_1 or in x_2 or in both. Then M is zero on all test functions and the remainder of the proof follows as before.

Remark 2.3. According to [24], Theorem 6.3, the smoothness of w , restricted to the original region Ω , is limited only by the smoothness of ρ . Thus with $\rho_\delta \in C^{1,\beta}$, w is of class $C^{2,\beta}$ in each subdomain of Ω .

Lemma 2.4. The multiplier λ occurring in proposition 2.2 satisfies

$$C_1 < \lambda < C_2 \quad (2.30)$$

where C_1, C_2 depend on the total variation of $\rho(\eta)$.

Proof. Since $\rho(0) = 1$, it follows from (2.27) and (2.22) that

$$\lambda = \frac{\langle \hat{A}'(w), w \rangle}{\langle \hat{B}'(w), w \rangle} > C \frac{\int |\nabla w|^2}{-\int \rho' w^2} \quad (2.31)$$

Since $-\int \rho' w^2 = 2 \int \rho w \frac{\partial w}{\partial \eta} < \rho(-1) \int w^2 + \left(\frac{\partial w}{\partial \eta}\right)^2 < C\rho(-1) \int |\nabla w|^2$ a lower bound depending on $\rho(-1)$ results.

In a similar fashion

$$\lambda = \frac{\langle \hat{A}'(w), w \rangle}{\langle \hat{B}'(w), w \rangle} < \frac{2\sigma_4 \sigma_1^{-1} \hat{A}(w)}{\hat{B}(w)} = \frac{4\sigma_4 \sigma_1^{-1} R^2}{\hat{B}(w)} \quad (2.32)$$

follows from (2.22), (2.27), and the characterization of w . The last quotient in (2.32) can only become larger if w is replaced by any function $z \in H_k^0$ having $\hat{A}(z) = 2R^2$. Let

$z = \gamma(1 - |\eta|)$, $\gamma > 0$. From (2.22) it follows that $\gamma^2 \sigma_{1k} < \hat{\lambda}(z) < \gamma^2 \sigma_{2k\rho(-1)}$. Now if γ is chosen to achieve $\hat{\lambda}(z) = 2R^2$, a simple computation using (2.32) shows $\lambda < C_2$, with C_2 depending upon $\rho(-1)$.

3. ESTIMATES FOR THE EXTENDED PROBLEM

In this section some additional notation will be useful. When $x_2 < 0$ the expression for a in (2.21) reduces to a function of p_1 and p_2

$$a(p_1, p_2) = \xi_r \frac{p_1^2 + p_2^2}{2(1 + p_2)} + (1 - \xi_r) \frac{p_1^2 + p_2^2}{2}. \quad (3.1)$$

A functional similar to (2.25), but associated with Ω_k , is defined by the expression

$$\lambda(w) = \int_{\Omega_k} \rho_\delta(x_2) a(\nabla w). \quad (3.2)$$

Let $\zeta = \zeta(x_1)$ be a cutoff function, i.e. an element of $C_0^\infty(\mathbb{R})$ with range in $[0, 1]$. Let

$$\Omega'' = \{(x_1, x_2) \in \hat{\Omega} \mid \zeta = 1\} \quad (3.3)$$

and

$$\Omega' = \{(x_1, x_2) \in \hat{\Omega} \mid \zeta > 0\}. \quad (3.4)$$

We make a standing assumption that $|\zeta'| < 2$, thus restricting the nested domains

$\Omega'' \subset \Omega'$ somewhat, but avoiding a dependence on ζ' in the estimates we'll make, which will be the typical interior type, relative to the variable x_1 . The constants occurring in the estimates will be denoted by C , possibly with a subscript or superscript, or in the case of a Hölder exponent, by the letter α . These numbers will depend upon the maximum density $\rho(-1)$, upon the positions of the discontinuities in ρ in (1.29), and upon the size of ρ' where it is continuous, but will be independent of ϵ, δ , and the period $2k$. The estimates also depend on $a(x_2, p_1, p_2)$ and its derivatives with respect to p_1 and p_2 , but in an inessential way (cf. [1], section 3). By lemma 2.4 we can also absorb the dependence on λ into the constants referred to. Having indicated that the parameters C and α depend on the given density ρ we will usually not display the dependence. In general we still suppress the parameters ϵ, δ and k . Throughout this section $\delta > 0$ and $k < \infty$; at the end we display the ϵ dependence and let ϵ approach zero. The immediate aim is to obtain estimates of w and its derivatives in terms of

integrals of $|\nabla w|^2$ over subregions of Ω_k . It follows from (2.22) that

$$\int_{\Omega_k} |\nabla w|^2 < C \Lambda(w) \quad (3.5)$$

and thus the various norms of w can be estimated in terms of the size of $R^2 = \Lambda(w)$.

Lemma 3.1. There is an $\alpha > 0$ such that the solution w in proposition 2.2 satisfies

$$\|w\|_{C^\alpha(\Omega^n)}^2 < C \int_{\Omega'} |\nabla w|^2 \quad (3.6)$$

Proof. This is immediate from the known results on elliptic equations ([25], Theorem 8.29), for w satisfies

$$\frac{\partial}{\partial x_1} \tilde{a}_{1j} \frac{\partial w}{\partial x_j} = \lambda((\rho w)_{x_2} - \rho w_{x_2}) \quad (3.7)$$

with $\tilde{a}_{1j} = \rho \int_0^1 a_{1j}(x_2, t \nabla w) dt$.

Remark. The symbol α occurring in subsequent results and in Theorem 1.2 should be understood to be the smaller of the exponents occurring in lemmas 3.1 and 3.2.

Lemma 3.2. Let w be the function occurring in Theorem 2.2 and let $v = w_{x_1}$. Then there is an $\alpha > 0$ such that

$$\| \nabla v \|_{L^2(\Omega^n)}^2 < C_1 \int_{\Omega'} |\nabla w|^2 \quad (3.8)$$

and

$$\|v\|_{C^\alpha(\Omega^n)}^2 < C_2 \int_{\Omega'} |\nabla w|^2. \quad (3.9)$$

Proof. This lemma combines lemmas 3.1 and 3.2 of [1], and the proofs are essentially the same. First one shows

$$\int_{\Omega^n} |\nabla v|^2 < C' \int_{\Omega'} |\nabla w|^2 \quad (3.10)$$

for $\Omega'' \subset \Omega^n \subset \Omega'$ using a cutoff function adapted to Ω'' and Ω' and a test function

$\phi = \frac{\partial}{\partial x_1} \zeta^2 v$ with (2.27) (equivalently (2.29) with $w^+ = w$). Inequality (3.9) is then a consequence of theorem 8.29 of [25], completing the proof.

In addition to a global estimate of ∇v in L^2 the following local estimate will be used. For convenience we'll assume $0 < \alpha < 1/2$ in Lemma 3.2.

Lemma 3.3. Let $v = w_{x_1}$ be as in the previous lemma and suppose $\lambda(w) = R^2$. Let $\bar{x} = (\bar{x}_1, \bar{x}_2)$ be a point in $\hat{\Omega}$ and $B_\sigma \subset \hat{\Omega}$ the ball of radius $\sigma < 1/4$ centered at $\bar{x}$. Then

$$\left(\int_{B_\sigma} |\nabla v|^2 \right)^{1/2} \leq C R' \sigma^\alpha \quad (3.11)$$

where

$$R' = \int_{-1}^1 \int_{\bar{x}_1-1}^{\bar{x}_1+1} |w|^2 dx_1 dx_2$$

and α is the exponent from lemma 3.2.

Proof. We use the ideas of [25], chapter 12. Let $\tilde{\sigma}$ be a radial coordinate with respect to an origin at $\bar{x}$ and let $\psi = \psi(\tilde{\sigma}) > 0$ be a C^∞ function with support in $B_{2\sigma}$ (suppose $B_{2\sigma} \subset \hat{\Omega}$; otherwise extend w to be odd and ρ even about $x_2 = \pm 1$, obtaining a weak solution of an equation on a larger region). Suppose that $\psi \equiv 1$ on B_σ and that $|\nabla \psi| \leq 2/\sigma$. Let $h(x_2) = -\lambda \rho(x_2)$ and note that $|h|$ has a bound depending on the maximum density $\rho(-1)$. If ϕ is any test function, (2.27) yields

$$\int \rho a_1(x_2, \nabla w) \frac{\partial}{\partial x_1} \phi = \int \left(\frac{\partial}{\partial x_2} (hw) - h \frac{\partial w}{\partial x_2} \right) \phi.$$

With $\gamma = v(\bar{x})$ and $\phi = \frac{\partial}{\partial x_1} [\psi^2(v - \gamma)]$ the last equation can, after integration by parts, be written as

$$- \int \left(\frac{\partial}{\partial x_1} \rho a_1 \right) \frac{\partial}{\partial x_1} [\psi^2(v - \gamma)] = \int \left(\frac{\partial}{\partial x_1} hw \right) \frac{\partial}{\partial x_2} [\psi^2(v - \gamma)] - \int hw_{x_2} \frac{\partial}{\partial x_1} [\psi^2(v - \gamma)]$$

where integration is over $B_{2\sigma}$ unless indicated otherwise. Or,

$$\begin{aligned}
& - \int \rho a_{ik} \frac{\partial v}{\partial x_k} \left[2\psi_{x_1} (v - \gamma) + \psi^2 \frac{\partial v}{\partial x_1} \right] \\
& = \int h w_{x_1} \left[2\psi_{x_2} (v - \gamma) + \psi^2 \frac{\partial v}{\partial x_2} \right] - \int h w_{x_2} \left[2\psi_{x_1} (v - \gamma) + \psi^2 \frac{\partial v}{\partial x_1} \right]
\end{aligned}$$

Since $\rho > 1$, $|h| < C'$; $a_{ik} > \nu$, as a quadratic form, and $|a_{ik}| < C''$,

$$\begin{aligned}
\int \psi^2 |\nabla v|^2 & \leq C \int 2|\nabla v| \psi |\nabla \psi| |v - \gamma| \\
& + C' \int 2|\nabla v| \psi |\nabla \psi| |v - \gamma| + C' \int \psi^2 \left(|w_{x_1} \frac{\partial v}{\partial x_2}| + |w_{x_2} \frac{\partial v}{\partial x_1}| \right)
\end{aligned}$$

Using the inequality $2ab \leq Ca^2 + \frac{1}{C} b^2$ to absorb terms involving ∇v into the left member of the last inequality and to combine terms involving ∇w we arrive at

$$\int \psi^2 |\nabla v| \leq C_1 \int |\nabla \psi|^2 (v - \gamma)^2 + C_2 \int \psi^2 |\nabla w|^2 \quad (3.12)$$

Now let $\zeta = \zeta(x_1)$ be a cutoff function which is equal to 1 on $B_{2\sigma}$ and vanishes for $x_1 = \bar{x}_1 - 1$. Then

$$w_{x_2}(x_1, x_2) = \int_{\bar{x}_1 - 1}^{x_1} \frac{\partial}{\partial s} \zeta(s) w_{x_2}(s, x_2) ds$$

so

$$\begin{aligned}
\iint_{B_{2\sigma}} \psi^2 w_{x_2}^2 & \leq \int_{x_1 = \bar{x}_1 - 2\sigma}^{\bar{x}_1 + 2\sigma} \int_{x_2 = \bar{x}_2 - 2\sigma}^{\bar{x}_2 + 2\sigma} \psi^2 \left(\int_{\bar{x}_1 - 1}^{x_1} (\zeta' w_{x_2} + \zeta w_{x_2 x_1}) ds \right)^2 \\
& \leq C' \int_{\bar{x}_1 - 2\sigma}^{\bar{x}_1 + 2\sigma} \left[\int_{\bar{x}_2 - 2\sigma}^{\bar{x}_2 + 2\sigma} \int_{\bar{x}_1 - 1}^{\bar{x}_1 + 1} (w_{x_2}^2 + \zeta^2 w_{x_2 s}^2) ds dx_2 \right] dx_1 \leq C'' (R')^2 \sigma \quad (3.13)
\end{aligned}$$

where use has been made of (3.8). Since, by (3.9), $v = w_{x_1}$ has a C^α norm bounded by a multiple of R' , it follows from (3.12) that

$$\int \psi^2 |\nabla v|^2 \leq C_1 \left(\frac{2}{\sigma} \right)^2 (R')^2 \sigma^{2\alpha} \cdot \sigma^2 + C_2 (R')^2 (\sigma + \sigma^2).$$

It is assumed that $0 < \alpha \leq 1/2$ so (3.11) follows.

Recall the notation $\eta^1 < \dots < \eta^N$ for the points of discontinuity of the density ρ given in (1.29).

Lemma 3.4. Let w be the solution occurring in proposition 2.2 and let r be the cutoff parameter associated with a in (2.21). Then there is a constant C such that for

$$-1 < x_2 < 0$$

$$w_{x_2}(x_1, x_2) < 2rp(\eta^N)/\rho(0) + CR' \quad (3.14)$$

while for $-1 < x_2 < -\varepsilon < 0$

$$-w_{x_2}(x_1, x_2) < 2rp(\eta^N)/\rho(0) + CR' \quad (3.15)$$

where

$$R' = \int_{-1}^0 \int_{x_1-2}^{x_1+2} |\nabla w(s, t)|^2 ds dt \quad (3.16)$$

Proof. If $-1 < x_2 < \eta^N/2$ the result follows by using a comparison argument in the region

$$\Omega' = \{(s, t) \mid |s - x_1| < 1, x_2 < t < 0\}$$

The fact that $\dot{\rho}'(t) < \rho'(t)$ allows the proof of lemma 3.3 of [1] to be carried over with constants depending on the width $|\eta^N|$.

For $1/2 \eta^N < x_2 < 0$, the inequality (3.14) is obtained from a comparison argument on a region where the second variable is between η^N and x_2 . Let $d = |\eta^N - x_2|$. To simplify notation let (x_1, x_2) be the origin of new coordinates (x'_1, x'_2) and then omit the primes to obtain a region

$$\tilde{\Omega} = \{(x_1, x_2) \mid |x_1| < 1, -d < x_2 < 0\}$$

We continue to use the expression $\rho(x_2)$ for the density in the new coordinates and let $-q$ ($q > 0$) be a lower bound for ρ_{x_2} on $\tilde{\Omega}$. Note that a in (2.21) is independent of spatial coordinates in the region under consideration. Let Q be defined by

$$Qw = \frac{\partial}{\partial x_1} \rho(x_2) a_1(\nabla w)$$

and observe that for the solution w , $Qw = \lambda \dot{\rho}' w < 0$. If u is constructed so that $u(0, 0) = w(0, 0)$, $u < w$ on $\partial \tilde{\Omega}$, and $Qu > 0$ in $\tilde{\Omega}$, then according to theorem 9.2 of [25], $u < w$ in $\tilde{\Omega}$. It will then follow that $w_{x_2}(0, 0) < u_{x_2}(0, 0)$. For u take

$$u(x_1, x_2) = w(0,0) + w_{x_1}(0,0)x_1 + A_\alpha g + A_3 e \quad (3.17)$$

where

$$g(x_1, x_2) = \operatorname{Re}[-(x_2 + ix_1)]^{1+\alpha}$$

(α from lemma 3.2) and

$$e(x_2) = \int_0^{x_2} (sd^{-1} + 2\rho(-d)/\rho(s))ds$$

If $|u_{x_2}|$ is larger than the cutoff r then (cf. (2.21))

$$Qu = \frac{\partial}{\partial x_1} \rho(x_2) \frac{\partial u}{\partial x_1} = \rho_{x_2} A_\alpha g_{x_2} + A_3 d^{-1}$$

since g is harmonic in $\tilde{\Omega}$. Then since $|g_{x_2}| < (1+\alpha)(1+d^2)$,

$$A_3 > qA_\alpha(1+\alpha)(d+d^3) \quad (3.18)$$

will insure that $Qu > 0$ on $\tilde{\Omega}$. Since

$$u_{x_2} > A_\alpha g_{x_2} + A_3(x_2 d^{-1} + 2\rho(-d)/\rho(x_2)) > -A_\alpha(1+\alpha)(1+d^2) + A_3,$$

having

$$A_3 - A_\alpha(1+\alpha)(1+d^2) > r \quad (3.19)$$

suffices to give $|u_{x_2}| > r$. The term $w(0,0) + w_{x_1}(0,0)x_1$ is of order R' according to

lemmas 3.1 and 3.2 and since g is negative and of order $|x_1|^{1+\alpha}$ when $x_2 = 0$ it will suffice, as in [1], lemma 3.3, to choose $A_\alpha > C'R'$ to have $u < w$ on that part of $\partial\tilde{\Omega}$

where $x_2 = 0$. As in the lemma cited, $u < w$ will also be satisfied where $x_1 = \pm 1$ and

$\bar{x}_2 < x_2 < 0$ for some $\bar{x}_2 \in [-d, 0)$, with such a choice of A_α . On the remainder of the

boundary of $\tilde{\Omega}$, i.e. where $x_2 < \bar{x}_2$, $e < \int_0^{x_2} 1 \cdot ds < \bar{x}_2 < 0$. Since the first three terms on the right side of (3.17) are of order R' , a choice of $A_3 > C''R'$ will make $u < 0$ on

this remaining portion (recall $w > 0$, so $u < w$). The coefficient A_3 can be increased

if necessary to satisfy (3.18) and (3.19), and can ultimately be chosen to be of size

$r + CR'$. Since $u_{x_2}(0,0) = 2A_3\rho(-d)\rho^{-1}(0)$ (in the new coordinates) a bound for

$w_{x_2}(0,0)$ follows; namely, the upper bound for w_{x_2} contained in (3.14). The bound from

below follows in a like manner, using the fact that $\rho'_{\delta,\varepsilon} = \rho'_\delta$ for $x_2 < -\varepsilon$, and hence can be estimated independently of ε .

Lemma 3.5. Define β by

$$\beta(\varepsilon, v, w) = \begin{cases} \int_{\Omega_k} \left(\frac{\partial}{\partial x_k} \tau_\varepsilon \right) v w, & \varepsilon > 0 \\ \int_{\Gamma_k} v(x_1, 0) w(x, 0) dx_1, & \varepsilon = 0 \end{cases} \quad (3.20)$$

where τ_ε is the function occurring in (2.19) and $\Gamma_k = \{(x_1, 0) \mid |x_1| < k\}$. Suppose $\varepsilon_j \rightarrow 0$, $w_j \rightarrow w$ in H_k , and $v_j \rightarrow v$ in H_k for $j = 1, 2, \dots$, the convergence in H_k being weak. Then

$$\lim_{j \rightarrow \infty} \beta(\varepsilon_j, w_j, v_j) = \beta(0, w, v) \quad (3.21)$$

Proof. An integration by parts produces the expression

$$\beta(\varepsilon, v, w) = \beta(0, v, w) - \int_{\Omega_k} \tau_\varepsilon \left(v \frac{\partial w}{\partial x_2} + w \frac{\partial v}{\partial x_2} \right). \quad (3.22)$$

There is a continuous linear map taking an element of H_k to its trace in $H^{1/2}(\Gamma_k)$ ([26], theorem 9.4). Since this last space embeds compactly into $L^2(\Gamma_k)$, the trace of w_j (or v_j) converges strongly in $L^2(\Gamma_k)$. The weakly convergent elements w_j and v_j lie in a bounded set in H_k , from which it follows that $\left(\frac{\partial}{\partial x_2} v_j \right) w_j + v_j \frac{\partial}{\partial x_2} w_j$ lies in a bounded set in $L^p(\Omega_k)$ for any $p < 2$. Since τ_{ε_j} converges to zero in L^r for any $r < \infty$, it follows from Hölder's inequality that the integral term in (3.22), evaluated with $\varepsilon = \varepsilon_j$, $v = v_j$, and $w = w_j$, converges to zero as $j \rightarrow \infty$.

Proposition 3.6. For each $\delta > 0$, $k > 0$ and $R > 0$ there is a $\lambda > 0$ and

$w \in H_k^0(\Omega) \cap C^2(\Omega) \cap C^{0,1}(\bar{\Omega})$ with $w > 0$ and $\Lambda(w) = R^2$ such that

$$\Lambda'(w) = \lambda B'(w)$$

in the notation (2.18) and (3.2). That is

$$\int_{\Omega_k} \rho_{\delta,1} (\nabla w) \phi_{x_1} = \lambda \int_{\Omega_k} \rho_{\delta}^0 w \phi \quad (3.23)$$

for all $\phi \in H_k(\Omega)$. The function w is an extremal for the problem

$$\sup_{\substack{A(u)=R^2 \\ u \in H_k^0(\Omega)}} (-1) \int_{\Omega_k} \rho_{\delta}^0(x_2) \frac{(u^+)^2}{2} \quad (3.24)$$

Moreover, with the restrictions that $B_0 \subset \Omega$ and that $\hat{\Omega}$ is replaced by Ω in definitions (3.3) and (3.4), the estimates (2.30), (3.6), (3.8), (3.9), and (3.11) hold uniformly in δ and k , as well as

$$\|w\|_{x_2, L^\infty(\Omega)} < 2rp(\eta^N)/\rho(0) + CR' \quad (3.25)$$

with R' defined as in (3.16).

Proof. Let $w^\varepsilon \in H_k^0(\hat{\Omega})$ denote a solution obtained from proposition 2.2 for $0 < \varepsilon < \delta_0$ and let $\Omega^\varepsilon = \{(x_1, x_2) \in \hat{\Omega} \mid x_2 < -\varepsilon\}$. From (3.5) and lemma 3.1 we know that, independently of δ, k and ε , the functions w^ε lie in bounded sets in $H_k(\hat{\Omega})$ and $C^\alpha(\hat{\Omega})$. Likewise from lemmas 3.2 and 3.4 the derivatives $w_{x_1}^\varepsilon$ lie in a bounded set in $C^\alpha(\hat{\Omega})$ and the derivatives $w_{x_2}^\varepsilon$ are bounded in $L^\infty(\Omega^\varepsilon)$. In addition, since $\rho_{\delta, \varepsilon}$ is in $C^{1, \beta}(-1, 0)$, it follows from quasilinear elliptic theory ([24], theorem 6.3, p. 283) that for each $\eta > 0$, $w^\varepsilon \in C^{2, \beta}(\Omega^\eta)$ with bounds depending on δ , but uniform in ε as ε approaches 0.

Let ε take the values 2^{-j} , $j = 1, 2, \dots$, and let w^j denote the corresponding solution. By the Arzela-Ascoli theorem a subsequence $w_{1,1}, w_{1,2}, \dots$ converges in $C^0(\hat{\Omega})$ and in $C^2(\Omega^{1/2})$ to a function w . A further subsequence $w_{2,1}, w_{2,2}, \dots$ converges in $C^2(\Omega^{1/4})$ to w . Continuing, we find the usual diagonal sequence $w_j = w_{j,j}$ converging to w in $C^2(\Omega^\eta)$ for each $\eta > 0$. Moreover, the estimates cited at the outset of the proof give bounds on $w \in C^0(\Omega)$ and on $\frac{\partial w}{\partial x_1} \in C^\alpha(\Omega)$ as well as the estimate

$$\|w\|_{x_2, L^\infty(\Omega)} < 2rp(\eta^N)/\rho(0) + CR'.$$

It follows that $w \in C^2(\Omega)$ and has an extension to $\bar{\Omega}$ which is in $C^{0,1}(\bar{\Omega})$. Clearly, then, $w \in H_k^0(\Omega)$.

Next, it follows from (3.5) that $\|w_j\|_{H_k}^2 < CA(w_j) = CR^2$ and so, for a further subsequence (also denoted w_j), it can be assumed that w_j converges to w weakly in $H_k(\Omega)$ and strongly in $L_2(\Omega_k)$. Since $a(p_1, p_2)$ is a convex function, the functional $\Lambda(w)$ is also. Then the set of u for which $\Lambda(u) < R^2$ is convex and, as w_j converges weakly, $\Lambda(w) < R^2$. Let $\hat{\rho}_{\delta, \epsilon}$ with $\epsilon = \epsilon_j$ be denoted by $\hat{\rho}_j$, let $2b_j = \int \hat{\rho}_j^2 (w_j)^2$, and let $2\tilde{b} = \int \hat{\rho}^2 w^2$ (here and in the remainder of the proof all integrals are taken over Ω_k unless otherwise indicated). It is then immediate from the known convergence of w_j and lemma 3.5 that $b_j \rightarrow \tilde{b}$ as $j \rightarrow \infty$.

Let the extreme value in (3.24) be denoted by b . First we show that $\tilde{b} < b$ is impossible. If $b - \tilde{b} = d > 0$ then from the characterization (3.24) there is a $u \in H_k^0(\Omega)$ satisfying $\Lambda(u) = R^2$ and $\int \hat{\rho}_\delta^2 (u^+)^2 / 2 > b - d/3$. Then from lemma 3.5, for all sufficiently small positive ϵ , $\int \hat{\rho}_{\delta, \epsilon}^2 (u^+)^2 / 2 > b - d/2$. But then since w^ϵ is an extremal for the problem with $\epsilon > 0$, $\int \hat{\rho}_{\delta, \epsilon}^2 (w^\epsilon)^2 / 2 > b - d/2$. This is incompatible with having b_j converge to $\tilde{b} = b - d$ as $j \rightarrow \infty$. We have shown that $\Lambda(w) < R^2$ and $\int \hat{\rho}_\delta^2 w^2 / 2 > b$. However, neither of these inequalities can be strict without contradicting the characterization (3.24). In particular, if $\Lambda(w) < R^2$, one easily shows that $\Lambda(tw) = R^2$ for some $t > 1$ with a corresponding supremum larger than b in (3.24).

As regards the equation (3.23), since w^ϵ is even in x_2 , we conclude from (2.27) that for each $\epsilon > 0$

$$\int \rho_\delta(x_2) a_1(\nabla w^\epsilon) \frac{\partial \phi}{\partial x_1} = \lambda^\epsilon \int \hat{\rho}_{\delta, \epsilon}^2 w^\epsilon \phi \quad (3.26)$$

for all $\phi \in H_k(\Omega)$. It is easy to verify that for $i = 1$ or 2 ,

$$|a_i(p_1, p_2)| \leq C(|p_1| + |p_2|) \quad (3.27)$$

(cf. [1] lemma 2.1) and thus for $\phi \in C^{0,1}(\Omega_k)$ the integral of $\rho_\delta a_1 \phi_{x_1}$ over $\Omega_k - \Omega^n$ will, by Hölder's inequality, be $O(n^{1/2})$ uniformly as $\epsilon \rightarrow 0$. Thus with $\epsilon = \epsilon_j$, the left hand member of (3.26) converges to the left hand member of (3.23) as $j \rightarrow \infty$. Since

$\lambda_j^{e_j}$ satisfies the inequalities (2.30) it can be assumed, without loss of generality, that $\lambda_j^{e_j}$ converges to a value λ satisfying (2.30). It then follows from lemma 3.5 that the right hand member of (3.26) converges to the corresponding term in (3.23). Thus (3.23) holds for $\phi \in C^{0,1}$ and the extension to $\phi \in H_k$ follows by continuity.

We next show that (3.8) holds for the w just obtained. Let $v_j = (w_j)_{x_1}$ and $v = w_{x_1}$. From (3.8) it follows that for $\gamma > \eta > 0$

$$\int_{\Omega' \cap \Omega^\gamma} |\nabla v_j|^2 < C_1 \int_{\Omega' \cap \Omega^\eta} |\nabla w_j|^2 + C_1 \int_{\Omega' \setminus \Omega^\eta} |\nabla w_j|^2$$

and hence for each $\gamma > 0$,

$$\int_{\Omega' \cap \Omega^\gamma} |\nabla v|^2 < C_1 \int_{\Omega'} |\nabla w|^2 \quad (3.28)$$

provided

$$\lim_{\eta \rightarrow 0} \overline{\lim}_{j \rightarrow \infty} \int_{-k}^k \int_{-\eta}^0 |\nabla w_j|^2 = 0 \quad (3.29)$$

If (3.29) does not hold, then from (2.22) and the observed convergence of A_ϵ to A , uniformly on bounded sets in H_k , it will follow that there is a sequence $\eta_m \rightarrow 0$, $m = 1, 2, \dots$ such that

$$\overline{\lim}_{j \rightarrow \infty} \int_{-k}^k \int_{-\eta_m}^0 \rho_a(\nabla w_j) > (R^+)^2 > 0 \quad (3.30)$$

for all m . But then

$$\int_{-k}^k \int_{-1}^{\eta_m} \rho_a(\nabla w) = \lim_{j \rightarrow \infty} \int_{-k}^k \int_{-1}^{\eta_m} \rho_a(\nabla w_j) < R^2 - (R^+)^2 \quad (3.31)$$

for all m . If m approaches infinity (3.31) yields $A(w) < R^2$, which has been ruled out. It follows that (3.28) holds for each $\gamma > 0$ and hence with the integral taken over all of Ω' , completing the proof of (3.8). The other inequalities listed in the proposition are done similarly.

Since $w \in C^2(\Omega)$ the following result is immediate from (3.23) and the strong maximum principle (cf. (3.7)).

Corollary 3.7. The function w in the previous proposition satisfies

$$\frac{\partial}{\partial x_1} \rho_\delta(x_2) a_1(\nabla w) = \lambda \rho'_\delta(x_2) w \quad (3.32)$$

in Ω and is positive there.

4. A RETURN TO THE ORIGINAL PROBLEM

Here we show that the solution (λ, w) of $A'(w) = \lambda B'(w)$ given by proposition 3.6 is, for suitably restricted R , a solution of the "physical" problem (2.12). We also obtain additional estimates which complete the assertions of theorem 2.2 (except exponential decay) in the case $\delta > 0$ and $k < \infty$.

In section 3 estimates were derived for w and $v = w_{x_1}$. The norms of w and v in C^α and of ∇v in L^2 on a region $\Omega' \subset \Omega$ were estimated in terms of R' , the L^2 norm of ∇w on a larger region $\Omega' \subset \Omega$ (cf. proposition 3.6). In addition in (3.11) the L^2 norm of ∇v on a ball $B_\sigma \subset \Omega$ was shown to be of order $R\sigma^\alpha$ and in (3.25) an L^∞ bound was given for w_{x_2} in terms of r and R' . Our next step is to obtain bounds on w_{x_2} in terms of R' alone. We require a preliminary lemma. Recall that A in (3.2) involves a from (2.21) with a cutoff parameter r .

Lemma 4.1. Let w be the function occurring in Proposition 3.6. There exist positive constants r_1 and R_0 such that if A is defined using a cutoff $r < r_1$ and $A(w) = R^2 < R_0^2$, then $v = w_{x_1}$ satisfies

$$\int_{S''} |\nabla v_{x_1}|^2 < C \int_{S'} |w|^2 \quad (4.1)$$

where

$$\begin{aligned} S'' &= \{(x_1, x_2) \mid b_1 < x_1 < b_2, -1 < x_2 < \eta^N/2\}, \\ S' &= \{(x_1, x_2) \mid b_1 - 1 < x_1 < b_2 + 1, -1 < x_2 < \eta^N/4\}, \end{aligned} \quad (4.2)$$

$$S' \subset \Omega_k.$$

Proof. This lemma is a counterpart to lemma 3.4 of [1] and the proof is similar. One starts with a test function $\phi = \frac{\partial}{\partial x_1} \zeta^2 v_{x_1}$ in (3.22). Here, however, the cutoff function is taken to be 1 on S'' and to vanish outside of S' given by (4.2). Now that ζ depends on x_2 as well as on x_1 ,

$$\frac{\partial}{\partial x_2} \zeta^2 v_{x_1} = \zeta^2 \frac{\partial}{\partial x_2} v_{x_1} + 2\zeta \zeta_{x_2} v_{x_1}.$$

In the earlier case just cited, the term with ζ_{x_2} was not present, but the new term is easily incorporated into the estimates in terms of a constant involving the gap $|\eta^N|$.

Lemma 4.2. Let w and R_0 be as in the previous lemma. Then there is a positive constant $\tilde{r} < r$, such that if A is defined using $r = \tilde{r}$, and $\Omega'' \subset \Omega' \subset \Omega_K$,

$$\|w_{x_2}\|_{L^\infty(\Omega'')} \leq CR' \quad (4.3)$$

where

$$R' = \left(\int_{\Omega'} |Vw|^2 \right)^{1/2} \quad (4.4)$$

Moreover, if $\tilde{\Omega}$ is a closed region not containing any points of discontinuity of ρ in (1.29) then

$$\|w_{x_2}\|_{C^\alpha(\Omega'' \cap \tilde{\Omega})} \leq C'R' \quad (4.5)$$

for all sufficiently small δ where α is the exponent from lemma 3.2.

Proof. The estimate (4.3) for $x_2 < \eta^N/2$ is shown as in lemma 3.5 of [1] with one small change. In the present context the function w_{x_2} need not vanish on each line $x_1 =$ constant. However, since $\|w\|_{L^\infty} \leq C'R'$, w_{x_2} is of order R' at some point $(x_1, \tilde{x}_2)$ for each x_1 , by the mean value theorem, and this suffices for the argument. An argument given in the lemma cited also shows that the oscillation of w_{x_2} over a distance d satisfies

$$\text{osc}(w_{x_2}) \leq CR'd^\alpha \quad (4.6)$$

in the region where $x_2 < \eta^N/2$, where C does depend on a bound for $|\rho_{x_2}|$ in the region (and hence is independent of δ , for small δ , in a region $\tilde{\Omega}$ containing no discontinuity of ρ). In the present context lemma 4.1 is used.

Now consider the region

$$\Omega^0 = \{(x_1, x_2) | \eta^N/2 < x_2 < 0\}$$

and recall that $|\rho_{x_2}|$ is bounded in Ω^0 , independently of δ . From Corollary 3.7, w satisfies the equation (3.31) in Ω^0 ; that is,

$$\rho(x_2) \sum a_{ij}(\nabla w)_{x_i x_j} + \rho' a_2(\nabla w) = \lambda \rho' w. \quad (4.7)$$

If $B_\sigma \subset \Omega^0 \cap \Omega^*$ is any disc of radius σ , then from (4.7) it follows that

$$\int_{B_\sigma} |w_{x_2 x_2}|^2 = \int_{B_\sigma} \frac{1}{a_{22}} \left[- \sum_{2 \leq i+j \leq 3} a_{ij} w_{x_i x_j} - \frac{\rho'}{\rho} a_2(\nabla w) + \lambda \frac{\rho'}{\rho} w \right]^2 \quad (4.8)$$

The expressions $|a_{ij}/a_{22}|$ and $|\rho'/\rho|$ are bounded above in Ω^0 so the expression on the right in (4.8) can, according to proposition 3.6, be estimated using lemmas 3.1 to 3.3 together with inequalities (3.13) and (3.27). The result is

$$\int_{B_\sigma} |w_{x_2 x_2}|^2 \leq C(R')^2 (\sigma^{2\alpha} + \sigma) \quad (4.9)$$

where α (assumed to satisfy $0 < \alpha < 1/2$) is from lemma 3.2. The estimate (4.9) together with lemma 3.3 shows that

$$\int_{B_\sigma} |\nabla w_{x_2}|^2 \leq C(R')^2 \sigma^{2\alpha}$$

and hence by Morrey's estimate ([25], p. 158, combined with the Schwarz inequality) the oscillation of w_{x_2} over a distance d in Ω^0 satisfies (4.6). This estimate in Ω^0 combined with the estimates for $x_2 < \eta^N/2$ yield the assertions of the lemma.

From lemmas 3.1, 3.2 and 4.2 the following result is immediate.

Corollary 4.3. Under the conditions of the previous lemma

$$|w|_{C^{0,1}(\Omega)} \leq CR' \quad (4.10)$$

and

$$|w|_{C^{1,\alpha}(\Omega^* \cap \tilde{\Omega})} \leq C'R' \quad (4.11)$$

In what follows the notation (2.18) for $F(w)$ and $B(w)$ will be used and for the remainder of this section an integration is over Ω_k unless otherwise indicated.

Proposition 4.4. There is an $R_1 > 0$ depending on the density ρ in (1.29) such that for each δ in $(0, \delta_0)$, $k > 0$, and R in $(0, R_1)$ there is a $\lambda > 0$ and a nonnegative $w \in H_k^0(\Omega) \cap C^2(\Omega) \cap C^{1,\alpha}(\bar{\Omega})$ with $F(w) = R^2$ and

$$F'(w) = \lambda B'(w). \quad (4.12)$$

The pair (λ, w) is a solution of the problem (2.12), i.e.

$$\frac{\partial}{\partial x_1} \rho(x_2) f_1(\nabla w) = \lambda \rho' w \text{ in } \Omega_k \quad (4.13)$$

and

$$\left. \begin{aligned} w &= 0 \text{ on } x_2 = -1 \\ \rho(x_2) f_2(\nabla w) &= \lambda w \text{ on } x_2 = 0 \end{aligned} \right\} \quad (4.14)$$

where $\rho = \rho_\delta$ (cf. (1.34)). The estimates (4.10) and (4.11) from Corollary 4.3 hold uniformly in δ and k .

Proof. From lemmas 3.2 and 4.2 there is a positive R_1 such that for $R < R_1$, $|w| < \tilde{r}$. Thus $\Lambda(w) = F(w)$ and $a_i(\nabla w) = f_i(\nabla w)$ for $i = 1, 2$ so that (4.12) follows from Proposition 3.6 and (4.13) from Corollary 3.7. Naturally $w = 0$ for $x_2 = -1$ and all estimates for λ and w persist. In particular, from Corollary 4.3 it follows that w has a uniquely defined C^α extension to the line $x_2 = 0$. We assume the extension is made and can then explicitly express the boundary condition which is implicit in the weak equation (4.12).

Let $g = g(x_1)$ be an arbitrary $2k$ periodic, C^∞ function and for $0 < s < 1$ let $H(x_2) = \max[1 + s^{-1}x_2, 0]$. Using (4.12) to write $\langle F'(w), \phi \rangle = \lambda \langle B'(w), \phi \rangle$ with $\phi = g(x_1)H(x_2)$ one obtains

$$\begin{aligned} & \int_{-k}^k \int_{-s}^0 \rho(x_2) [f_1(\nabla w) g_{x_1} H + f_2(\nabla w) g s^{-1}] \\ &= \lambda \int_{-k}^k \left[\int_{-s}^0 \rho'(x_2) w g H dx_2 + w(x_1, 0) g(x_1) \right] dx_1. \end{aligned}$$

Since $\rho(x_2)$ approaches 1 as x_2 approaches 0 and ρ' is bounded near $x_2 = 0$, the limit as $s \rightarrow 0$ gives

$$\int_{-k}^k f_2(\nabla w(x_1, 0))g(x_1)dx_1 = \lambda \int_{-k}^k w(x_1, 0)g(x_1)dx.$$

Since g is arbitrary, the remainder of condition (4.14) follows and the proof is complete.

The material in section 4 of [1] shows that in the case of a fixed upper surface, the streamline displacement w can be assumed to be even in x for each η and nonincreasing in x as x runs from 0 to k . That is, an extremal function w for the appropriate variational principle can be replaced by its symmetrization (decreasing rearrangement) $\hat{w}$ without destroying the extremal character (cf. [1], [27]). Here again the proofs carry over with some minor modifications which are furnished by the following result. The functional N defined below is part of the functional Φ in (2.1) and enters in the proof of lemma 4.3 of [1]. Using evenness, it will suffice to consider $\psi < 0$.

Lemma 4.5. Let $u = u(x, \psi)$ be a nonnegative piecewise linear function defined on

$D_k = [-k, k] \times [\psi_{-1}, 0]$ and suppose u has a continuous extension to $\mathbb{R} \times [\psi_{-1}, 0]$ which is $2k$ periodic in x . Let $y(x, \psi) = \hat{y}(\psi) + u(x, \psi)$ where $\hat{y}(\psi)$ is the inverse function to (1.16) and assume $y_\psi > 0$ a.e.. Define

$$N(y) = \int_{D_k} \frac{1 + y_x^2}{y_\psi} dx d\psi \quad (4.15)$$

so that

$$\langle N'(\hat{y}), u \rangle = \int_{D_k} \frac{u}{y_\psi^2} dx d\psi \quad (4.16)$$

Then if $\hat{u}(x, \psi)$ and $\hat{y}(x, \psi)$ are the symmetrizations of $u(x, \psi)$ and $y(x, \psi)$, respectively, it follows that

$$\langle N'(\hat{y}), u \rangle = \langle N'(\hat{y}), \hat{u} \rangle \quad (4.17)$$

and

$$N(\hat{y}) \leq N(y). \quad (4.18)$$

Proof. One property relating u and $\hat{u}$ is

$$\int_{-k}^k u(x, \psi) dx = \int_{-k}^k \hat{u}(x, \psi) dx \quad (4.19)$$

for each ψ ([27], Note A) so that

$$\int_{D_k} g(\psi) u(x, \psi) dx d\psi = \int_{D_k} g(\psi) \hat{u}(x, \psi) dx d\psi \quad (4.20)$$

for any integrable g . Hence

$$\int_{D_k} \frac{1}{y_\psi^2} u_\psi = \int_{-k}^k \frac{1}{y_\psi^2} u dx \Big|_{\psi=-1}^0 - \int_{D_k} \frac{\partial}{\partial \psi} \left(\frac{1}{y_\psi^2} \right) u,$$

from which (4.17) follows (in fact (4.17) is used with $u(x_1, \psi_{-1}) \equiv 0$).

The integral defining N , when expressed in the variables x and y , is a Dirichlet integral and so (4.18) reduces to showing

$$\int_{S_y} |\nabla \hat{\psi}|^2 dx dy < \int_{S_y} |\nabla \psi|^2 dx dy \quad (4.21)$$

where

$$S_y = \{(x, y) \mid |x| < k, -1 < y < y(x, 0)\}$$

and $S_{\hat{y}}$ is defined analogously. The methods of Polya and Szego ([27], Note A) can easily be adapted to the case at hand. For periodic functions one shows that the area of the surface $y(x, \psi)$ over D_k is at least as large as the area of the surface $\hat{y}(x, \psi)$ over D_k . Expressed in (x, y) coordinates this yields

$$\int_{S_y} \sqrt{1 + |\nabla \hat{\psi}|^2} dx dy < \int_{S_y} \sqrt{1 + |\nabla \psi|^2} dx dy.$$

Applying the last inequality to $t\psi$ and $t\hat{\psi}$ for $t > 0$ gives

$$\int_{S_y} \left[1 + \frac{1}{2} t^2 |\nabla \hat{\psi}|^2 + o(t^2) \right] < \int_{S_y} \left[1 + \frac{1}{2} t^2 |\nabla \psi|^2 + o(t^2) \right]. \quad (4.22)$$

Replacing u by y in (4.19) one sees that S_y and $S_{\hat{y}}$ have equal areas. Thus the contribution of the "1" in each integrand of (4.22) can be omitted. If the remaining inequality is divided by $\frac{1}{2} t^2$ and t is allowed to approach zero, the inequality (4.21) is the result.

Proposition 4.6. There is a function w satisfying the conclusions of proposition 4.4 which, in addition, satisfies $w = \hat{w}$ and $w > 0$ where $-1 < x_2 < 0$.

Proof. As noted, the assertion that one can take $w = \hat{w}$ in proposition 2.2 and thereafter follows from the arguments of [1], section 4 together with lemma 4.5. The positivity of w in Ω_k follows from corollary 3.7. On Γ_k we have w nonincreasing for $0 < x < k$ and $f_2(\nabla w) = \lambda w$. Since $w_{x_1}(0, k) = 0$, having $w(0, k) = 0$ would entail $f_2(0, w_{x_2}) = 0$ at $(0, k)$. Since $f_2 = a_2$ for the solution w it follows from (2.22) that $w_{x_2}(0, k) = 0$, violating the strong maximum principle.

The linearization of (4.13), (4.14) about $w = 0$ is the problem

$$\left. \begin{aligned} \frac{\partial}{\partial x_1} \rho(x_2) \frac{\partial z}{\partial x_1} &= \hat{\lambda} \rho'(x_2) z \text{ in } \Omega_k \\ z &= 0 \text{ on } x_2 = -1 \\ \frac{\partial z}{\partial x_2} &= \hat{\lambda} z \text{ on } x_2 = 0 \end{aligned} \right\} \quad (4.23)$$

The lowest eigenvalue (denoted μ) for (4.23) can easily be shown, through separation of variables, to be positive and to correspond to a function of x_2 alone, $\xi = \xi(x_2) > 0$ which solves the Sturm-Liouville problem obtained by omitting x_1 dependence in (4.23). It also provides an extremal for

$$\sup_{\int_0^1 \rho \xi_n^2 = 1} \left[\int_0^1 \rho' \xi^2 + \xi^2(0) \right], \quad (4.24)$$

the analogue of (3.24). The solution ξ can be obtained by a shooting method or by using (4.24).

Lemma 4.7. The "lowest" eigenfunction ξ for (4.23) satisfies $\xi'(x_2) > 0$.

Proof. One integration of the ordinary differential equation for $\xi = \xi(x_2)$ corresponding to (4.23) gives

$$\rho(0)\xi'(0) - \rho(x_2)\xi'(x_2) < 0.$$

Since $\xi'(0) = \mu\xi(0) > 0$, $\xi'(x_2) > 0$.

Since lemma 4.7 implies $\int_{-1}^1 \rho \xi_{x_2}^3 > 0$, it is possible to use a trial function $z = c_0 \xi(x_2) e^{-c_1|x|}$ in (3.24), as in [1], section 5, to show the following.

Proposition 4.8. There are positive constants $\tilde{R}$ and $k_1(R)$ such that for $0 < \delta < \delta_0$, $0 < R < \tilde{R}$ and $k > k_1(R)$, the pair (λ, w) in proposition 4.3, chosen in conformity with proposition 4.6, satisfies

$$\lambda < \mu(1 - CR^{4/3}) \quad (4.25)$$

and

$$\|w\|_{L^\infty} > C'R^{4/3} \quad (4.26)$$

where C and C' depend upon ρ in (1.29)

Note that the inequality

$$\int_{\Omega''} |\nabla w|^2 < C \max_{\Omega'} w^2 \quad (4.27)$$

which is used in obtaining (4.26) from (4.25) is valid in the present context and shown exactly as in lemma 3.6 of [1].

Remark 4.9. The results up to this point, in particular propositions 2.1, 4.3, 4.4 and 4.6, contain the assertions of theorem 1.2 (except exponential decay) for the case in which density "transitions" take place over intervals of width $\delta > 0$ and in which there is a finite period $2k$ in the horizontal direction. All estimates obtained are uniform in δ

and k for each "energy" $R > 0$ and so the limiting case of discontinuous density ($\delta = 0$), including the pressure condition in definition 1.1 (ii), follows by arguments strictly paralleling those given for theorem 8.2 of [1].

5. EXPONENTIAL DECAY AND SOLITARY WAVES

The aim in this section is to show that w obtained from propositions 4.4 and 4.6 has exponential decay on $0 < x < k$, as does the gradient of w . The estimate will involve L^∞ bounds on the gradient and we remark that in order to get satisfactory estimates for w_{x_2} up to $x_2 = 0$ we had first to pass to a limit, letting ϵ approach zero in the expression $\dot{\rho}_{\delta, \epsilon}$, here denoted $\dot{\rho}_\epsilon$. While there is more than one way to exhibit exponential decay we find it convenient to reintroduce the extended domain $\hat{\Omega}$, the function $\dot{\rho}_\epsilon$ from (2.19) and the expression a from (2.21).

Lemma 5.1. Let μ_ϵ be the lowest eigenvalue of

$$\frac{\partial}{\partial x_2} \rho \frac{\partial z}{\partial x_2} = \tilde{\lambda} \dot{\rho}_\epsilon z, \quad -1 < x_2 < 1 \quad (5.1)$$

$$z(-1) = z(1) = 0$$

then

$$\mu_\epsilon > \mu - C\epsilon^{1/2} \quad (5.2)$$

where μ is the lowest eigenvalue of (4.23).

Proof. Express the odd function $\dot{\rho}_\epsilon$ as $\rho + \tau_\epsilon$ on $-1 < x_2 < 0$ as in (2.19) and let $\xi(x_2) > 0$ be an eigenfunction for the problem (4.23) corresponding to μ , normalized so that $\int_0^1 \rho(x_2)(\xi')^2 = 1$. Then since $\rho > 1$, $\int_0^1 (\xi')^2 < 1$, $|\xi(x_2) - \xi(0)| < \left(\int_0^{x_2} (\xi')^2\right)^{1/2} |x_2|^{1/2}$, and $|\xi^2(x_2) - \xi^2(0)| < C_1 |x_2|^{1/2}$. Extend ξ to be even in x_2 . The variational characterization of μ_ϵ enables us to conclude that

$$\begin{aligned} \mu_\epsilon &> - \left[\int_{-1}^1 \rho \xi'^2 + \int_{-1}^1 \tau_\epsilon' \xi'^2 \right] \\ &> - \left[\int_{-1}^1 \rho \xi'^2 + \int_{-1}^1 \tau_\epsilon' (\xi^2(0) - C_1 \epsilon^{1/2}) \right] = \mu - C\epsilon^{1/2} \end{aligned}$$

proving (5.2)

If λ is fixed with $\lambda < \mu$, then for all sufficiently small $\epsilon > 0$, $\lambda < \mu_\epsilon$ according to (5.2). Estimates involving the eigenfunctions w_n , $n = 1, 2, \dots$, and corresponding eigenvalues γ_n for

$$\left. \begin{aligned} \frac{d}{dx_2} \rho \frac{dz}{dx_2} + \lambda \rho_\epsilon' z &= \gamma \rho z \\ z(1) = z(-1) &= 0 \end{aligned} \right\} \quad (5.3)$$

are done exactly as in lemma 7.1 of [1], merely by using the fact that $|\rho_\epsilon| < \rho$ for $-1 < x_2 < 1$. The Green's function for

$$L_\lambda^\epsilon = -\frac{\partial}{\partial x_1} \rho \frac{\partial z}{\partial x_1} + \lambda \rho_\epsilon' z \quad (5.4)$$

with zero boundary conditions at $x_2 = \pm 1$ has the form

$$G(x - x', \eta, \eta') = \sum_{n=1}^{\infty} \frac{e^{-\gamma_n^{1/2} |x-x'|}}{2\gamma_n^{1/2}} w_n(\eta) w_n(\eta') \quad (5.5)$$

with $\gamma_1 > C_0(\mu_\epsilon - \lambda)$.

Proposition 5.2. The function w from proposition 4.6 satisfies

$$\begin{aligned} |w(x, \eta)| &< C e^{-\beta x} \\ |\nabla w(x, \eta)| &< C' e^{-\beta x} \end{aligned} \quad (5.6)$$

for $0 \leq x \leq k$, for any β satisfying $0 < \beta < \gamma_1^{1/2}$. The constants C and C' depend on ρ in (1.29), R , and β .

Proof. Define $V = (V_1, V_2)$ where

$$V_i(x_2, p_1, p_2) = \rho(p_1 - a_i(x_2, p_1, p_2)) \quad (5.7)$$

for $i = 1, 2$ and $a_i = \partial a / \partial p_i$ as before. The function V_1 has the same parity as a_1 (cf. the proof of proposition 2.2); i.e., a_1 is odd in p_1 and even in (x_2, p_2) , while V_2 has the opposite parity. Both V_1 and V_2 are of order $p_1^2 + p_2^2$ near $p_1 = p_2 = 0$ and vanish for $p_1^2 + p_2^2 > 2\tilde{r}^2$.

Suppose w from proposition 4.6 is extended to be even on $\hat{\Omega}$ and then extended to be odd about $x_2 = \pm 1$. Convolution of w with respect to a mollifying kernel which is symmetric in x_1 and x_2 and has support in the ball of radius ϵ at $(0,0)$ will produce a family of C^2 functions z_ϵ which are even in x_2 , satisfy $\hat{z}_\epsilon = z_\epsilon$, and converge to w in $H_k(\hat{\Omega})$ as ϵ approaches zero. Since $|\nabla w| < \tilde{r}$, it can also be assumed that $|\nabla z_\epsilon| < \tilde{r}$. As in lemma 7.3 of [1] it is easily seen that

$$|z_\epsilon(x_1, x_2)| < \frac{C R^{2/3}}{x_1^{1/3}} \quad (5.8)$$

on $0 < x_1 < k$ where C depends on ρ from (1.29).

Now let $w^\epsilon \in H_k$ be the weak solution of

$$L_\lambda^\epsilon w^\epsilon = \operatorname{div} V(x_2, \nabla z_\epsilon). \quad (5.9)$$

Given the parities of V_1 and V_2 it is easy to verify that $w_-^\epsilon(x_1, x_2) = w^\epsilon(-x_1, x_2)$ is also a weak solution of (5.9) and since L_λ^ϵ is coercive, $w_-^\epsilon = w^\epsilon$, i.e. w^ϵ is even in x_1 . Likewise w^ϵ is even in x_2 and $2k$ periodic in x_1 . For $j = 0, 1, 2, \dots, 2k-1$ define

$$\tilde{b}_j = \int_{B_j} |\nabla w^\epsilon|^2, \quad b_j = \int_{B_j} |\nabla z_\epsilon|^2 \quad (5.10)$$

where $B_j = \{(x_1, x_2) \in \hat{\Omega} \mid j < x_1 < j+1\}$. For a positive integer n less than k let $\tilde{b}$ denote the sequence $\{\tilde{b}_j\}$ and b , the sequence $\{b_j\}$ for $n < j < 2k-n-1$. Now if β is a real number satisfying $0 < \beta < \gamma_1^{1/2}$, it follows as in the proof of lemma 7.4 of [1], (one can smooth the signum function in $a(x_2, p_1, p_2)$ and then pass to a limit) that

$$\tilde{b}_j < \tau_j(b) \quad (5.11)$$

for $n+1 < j < k$ where

$$\tau_j(b) = c_1(b_{j-1} + b_j + b_{j+1})^2 + c_2 \left[\sum_{i=n+1}^{2k-n-1} e^{-\beta|i-j|} b_i + q e^{-\beta(j-n)} \right] \quad (5.12)$$

and

$$q = c_3 e^{-\beta n} + c_4 n^{-2/3}.$$

The solution map taking z_ϵ to w^ϵ is continuous from $H_k(\hat{\Omega})$ to $H_k(\hat{\Omega})$ and thus as $\epsilon \rightarrow 0$ and z_ϵ converges to w in H_k , w^ϵ converges to a function w^0 in H_k . The weak form of (5.9) requires

$$\int_{\hat{\Omega}_k} \left[\rho \frac{\partial w^\epsilon}{\partial x_1} \frac{\partial \phi}{\partial x_1} + \lambda \hat{\rho}_\epsilon^1 w^\epsilon \phi \right] = \int_{\hat{\Omega}_k} v_1(x_2, \nabla z_\epsilon) \frac{\partial \phi}{\partial x_1}$$

for all $\phi \in H_k(\hat{\Omega})$, where $\hat{\rho}_\epsilon$ is still assumed to be odd in x_2 . Now, letting $\epsilon \rightarrow 0$ and using lemma 3.5 to define $\int \hat{\rho}_0^1 w^0 \phi$ as the limit of $\int \hat{\rho}_\epsilon^1 w^\epsilon \phi$, one concludes that

$$\int_{\hat{\Omega}_k} \left[\rho \frac{\partial w^0}{\partial x_1} \frac{\partial \phi}{\partial x_1} + \lambda \hat{\rho}_0^1 w^0 \phi \right] = \int_{\hat{\Omega}_k} v_1(x_2, \nabla w) \frac{\partial \phi}{\partial x_1}. \quad (5.13)$$

But from proposition 3.6 it follows that (5.13) is satisfied if w^0 is replaced by w (extended evenly in x_2). Letting $w^0 - w = z$ it follows that

$$\int_{\hat{\Omega}_k} \rho |\nabla z|^2 + \lambda \hat{\rho}_0^1 z^2 = 0. \quad (5.14)$$

Since the coercivity of L_λ^ϵ is uniform for all small ϵ , it follows that $z = 0$. That is, w^ϵ and z_ϵ both converge to w in H_k as ϵ approaches 0. But then

$$b_j < \tau_j(b) \quad (5.15)$$

where b_j is now associated with $w^0 = w$.

The inequality (5.15) will imply there is an n , independent of k , such that $b_j < C e^{-28j}$ for $n < j < k$ according to lemma 7.2 of [1], provided the sequence b_j meets two other conditions. The first is a symmetry condition $b_{k+i} = b_{k-i-1}$ (this shift in index should appear in [1]; the proof is almost identical) which follows from the evenness and periodicity of w . The second is a decay $b_j < C_0 j^{-2/3}$ which follows from lemmas 3.6 and 7.3 of [1], trivially adapted to the present context. The exponential decay of b_j implies exponential decay of w and ∇w according to lemmas 3.1, 3.2 and 4.2. This completes the proof.

Remark 5.3. The resolution of problem S (1.32), the solitary wave problem, is carried out by letting the period $2k$ approach infinity just as in theorems 8.3 and 8.3' of [1] for $\delta = 0$ or $\delta > 0$, respectively. The assertions of theorem 1.2 concerning exponential decay and the case $k = +\infty$ are thus covered, completing the proof of the theorem.

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