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ON CONFIDENCE INTERVALS FOR CYCLIC REGENERATIVE
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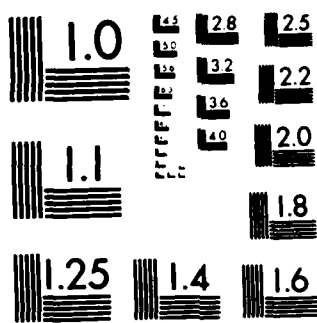
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ON CONFIDENCE INTERVALS
FOR CYCLIC REGENERATIVE PROCESSES

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MATHEMATICS RESEARCH CENTER

ON CONFIDENCE INTERVALS FOR CYCLIC REGENERATIVE PROCESSES

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ABSTRACT

We study precise conditions under which the cyclic regenerative confidence intervals of Sargent and Shantihikumar are asymptotically valid. We also obtain an optimal way of implementing the cyclic regenerative variance reduction technique, and obtain a sufficient condition under which the procedure yields a lower variance than that of the standard regenerative method.

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SIGNIFICANCE AND EXPLANATION

Simulation is a commonly used method of analysis for studying complex stochastic systems. Often, the parameter of interest to the simulator can be estimated by more than one quantity. When more than one estimator exists, it is desirable to use the more stable estimate, namely the one with the lesser variance.

In this paper, we consider a class of stochastic processes which enjoy cyclic regenerative structure - such systems often arise, for example, in analysis of queues. We study a family of estimators recently introduced by Sargent and Shantihikumar and determine precise conditions under which the estimators are asymptotically valid. We also obtain a closed-form solution for the minimum variance estimate in the family, and prove that this estimator will often be superior to the standard regenerative estimator for the simulation.

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ON CONFIDENCE INTERVALS FOR CYCLIC REGENERATIVE PROCESSES

Peter W. Glynn

1. Introduction

Recently, Sargent and Shantihikumar [5] developed an interesting new variance reduction technique designed to exploit the stochastic structure associated with a cyclic regenerative process. Our purpose here is to study precise conditions under which the confidence intervals proposed in [5] are asymptotically valid. This analysis will provide us with the side benefit of obtaining an optimal way of implementing the variance reduction procedures introduced there. To be precise, we will obtain the minimum variance estimate in the class of estimates proposed in [5]. We will also determine conditions under which the minimum variance estimate achieves a lower variance than that of the standard regenerative method (see Crane and Lemoine [4] for a description of the standard procedure).

We will use the convention that assumptions in force throughout the entire paper will be prefixed by A (eg. A1) whereas all others will be prefixed by B. We can now state our basic assumptions for the problem:

A1. $\{X_n : n \geq 0\}$ is a regenerative process with regenerative times

$0 = T_0 < T_1 < \dots$ satisfying $E\tau_1 < \infty$, where $\tau_n = T_n - T_{n-1}$.

A2. f is a real-valued function such that $EY_n(|f|) \stackrel{\Delta}{=} E\{|f(X_{T_{n-1}})| + \dots + |f(X_{T_n-1})|\} < \infty$.

A3. There exist random times $\{\alpha_{n,i} : n \geq 0, 0 \leq i \leq t\}$ such that

$T_{n-1} = \alpha_{n,0} < \alpha_{n,1} < \dots < \alpha_{n,t} = T_n$ and for which $\{(Y_{n,i}, \tau_{n,i}) :$

$n \geq 1\}$ are independent and identically distributed (i.i.d.) random

vectors (r.v.'s) for $1 \leq i \leq t$, where $\tau_{n,i} = \alpha_{n,i} - \alpha_{n,i-1}$ and

$$Y_{n,i} = f(X_{\alpha_{n,i-1}}) + \dots + f(X_{\alpha_{n,i}-1}).$$

Assumptions A1 and A3 basically define the notion of a t-phase cyclic regenerative process. We will also suppose that the simulator possesses the following knowledge:

A4. $EY_{1,i}$, $E\tau_{1,i}$ are known for $i \in D$.

A5. The simulator can sample independently from each of the distributions

$(Y_{1,i}, \tau_{1,i})$, $i \in F \stackrel{\Delta}{=} \{1, \dots, t\} \setminus D$.

Under A1 and A2, $\sum_{k=0}^n f(X_k)/n \rightarrow r = EY_1(f)/E\tau_1$ a.s. (see [4] for a proof).

The goal of the simulator is to obtain confidence intervals for r .

2. A Central Limit Theorem

In the setting of a cyclic regenerative process, the practitioner must decide on a sampling order before initiating the simulation. To be precise, the simulator must assign, for each $n \geq 1$, an integer m_n from $G = F \cup \{0\}$. The practitioner then simulates the sequence of independent r.v.'s $\{(W_n, x_n) : n \geq 1\}$, where (W_n, x_n) is sampled from the distribution of $(Y_{1, m_n}, \tau_{1, m_n})$ if $m_n \in F$ and from that of $(Y_1(f), \tau_1)$ if $m_n = 0$ (independent sampling is possible on account of A5). Put $\omega_{n,i} = \{j < n : m_j = i\}$ and let $k_{n,i}$ be the cardinality of $\omega_{n,i}$ for $i \in G$. The natural point estimate for r is given by

$$r_n = (\sum_{i \in G} \bar{Y}_{n,i} + \sum_{i \in D} EY_{1,i}) / (\sum_{i \in G} \bar{\tau}_{n,i} + \sum_{i \in D} E\tau_{1,i})$$

where $\bar{Y}_{n,i} = \sum_{j \in \omega_{n,i}} W_j / k_{n,i}$, $\bar{\tau}_{n,i} = \sum_{j \in \omega_{n,i}} x_j / k_{n,i}$. The a.s. convergence

of r_n to r is ensured by:

A6. either i.) $k_{n,i} \rightarrow \infty$ if $i \in F$, $k_{n,0} \equiv 0$ or ii.) $k_{n,i} \rightarrow \infty$ if $i \in G$.

To obtain a confidence interval for r , we need a central limit theorem (CLT)

- such behaviour is guaranteed by:

B1. $0 < \sigma_1^2 \triangleq \sigma^2(Y_{1,i} - r\tau_{1,i}) < \infty$ for $i \in F$,
 $0 < \sigma^2 \triangleq \sigma^2(Y_1(f) - r\tau_1) < \infty$.

Theorem 1: Under B1, there exist constants a_n such that

$a_n(r_n - r) \Rightarrow N(0,1)$, where $N(0,1)$ is a unit normal r.v.

Proof: We shall prove the result under A6 i.), the proof under A6 ii.) being similar. We view the problem in terms of a triangular array of r.v.'s by setting

$$U_{n,j} = (W_j - rx_j + \beta_1)/k_{n,i}$$

if $n_j = i$, where $\beta_1 = rE\tau_{1,i} - EY_{1,i}$. Set $U_n = \sum_{j \in n} U_{n,j}$ and observe that

$$s_n^2 \triangleq \sigma^2(U_n) = \sum_{i \in F} \sigma_1^2/k_{n,i}.$$

Then, the triangular array $\{U_{n,j}/s_n\}$ satisfies Lindeberg's condition since for any $\epsilon > 0$,

$$\begin{aligned} & \sum_{j=1}^n E\{U_{n,i}^2/s_n^2; U_{n,i}^2 > \epsilon^2 s_n^2\} \\ &= \sum_{i \in F} E\{Z_{1,i}^2/k_{n,i} s_n^2; Z_{1,i}^2 > \epsilon^2 s_n^2 k_{n,i}\} \\ &< \sum_{i \in F} E\{Z_{1,i}^2/\sigma_1^2; Z_{1,i}^2 > \epsilon^2 k_{n,i} \sigma_1^2\} \rightarrow 0 \end{aligned}$$

as $n \rightarrow \infty$, here $Z_{1,i} = Y_{1,i} - r\tau_{1,i} + \beta_1$ (in the inequality, we used $s_n^2 > \sigma_1^2/k_{n,i}$). Since $EU_{n,j} = 0$, it follows by Lindeberg's theorem (see Chung [3], p. 205) that $U_n/s_n \Rightarrow N(0,1)$. Hence,

$$\left(\sum_{i \in F} \bar{Y}_{n,i} - r\bar{\tau}_{n,i} + \beta_1 \right) / s_n \Rightarrow N(0,1).$$

But $\sum_{i \in F} \beta_1 = \sum_{i \in D} EY_{1,i} - rE\tau_{1,i}$. Thus, using the fact that

$$\hat{\tau}_n \triangleq \sum_{i \in F} \bar{\tau}_{n,i} + \sum_{i \in D} E\tau_{1,i} \rightarrow E\tau_1 \text{ a.s. and the converging-together lemma}$$

(Billingsley [2], p.25) proves that

$$a_n(r_n - r) \Rightarrow N(0,1)$$

where $a_n = E\tau_1/s_n$. ||

In a simulation application, one needs to estimate the constants a_n .

For the estimation, we need to add an additional hypothesis:

$$B2. E(Y_{1,i}^2 + \tau_{1,i}^2) < \infty \text{ for } i \in F, E(Y_1^2(f) + \tau_1^2) < \infty.$$

Notice that if $Y_{1,i} = r\tau_{1,i} + \theta_1$ where $E\tau_{1,i}^2 = \infty$ and $0 < E\theta_1^2 < \infty$, then B1 is satisfied but not B2.

Corollary 1: Under B1-B2, there exist estimates $\hat{a}_n$ such that

$$\hat{a}_n(r_n - r) \Rightarrow N(0,1).$$

Proof: By the converging together lemma, this follows from Theorem 1 if we obtain estimates $\hat{a}_n$ such that $\hat{a}_n/a_n \rightarrow 1$ a.s. Under A6 i.) an appropriate candidate for $\hat{a}_n$ is $\hat{\tau}_n/\hat{s}_n$ where $\hat{s}_n^2 = \sum_{i \in F} \hat{\sigma}_{n,i}^2/k_{n,i}$ and

$$\hat{\sigma}_{n,i}^2 = \sum_{j \in w_{n,i}} (w_j - r_n x_j)^2/k_{n,i} - \left(\sum_{j \in w_{n,i}} w_j - r_n x_j/k_{n,i} \right)^2.$$

But $|1 - \hat{a}_n^2/a_n^2| < \sum_{i \in F} |1 - E^2 \tau_{1,n,i}^2/\tau_{n,i}^2| \rightarrow 0$ a.s. A similar proof works under

A6 ii.) ||

The CLT of Corollary 1 can be used to construct confidence intervals for r . The half-width of a $100(1-\delta)\%$ confidence interval for r , based on a sample of size n , will be $z_\delta/\hat{a}_n$, where z_δ solves $P\{N(0,1) < z_\delta\} = 1 - \delta/2$.

3. Another Central Limit Theorem

To analyse the degree of variance reduction of a method, one needs to compare the half-width of competing intervals generated in a given amount of simulation time. In our context, this is accomplished by constructing intervals based on $(W_1, x_1), \dots, (W_{l(N)}, x_{l(N)})$, where $l(N) = \max\{k : x_1 + \dots + x_k < N\}$.

To base a CLT on a random number $l(N)$ of independent r.v.'s requires control on the growth of the $k_{n,i}$'s:

B3. if $k_{n,i} \rightarrow \infty$, then $k_{n,i}/n \rightarrow c_i$. If c_i, c_j are both zero, then $k_{n,j}/k_{n,i} \rightarrow \gamma_{ij} > 0$.

Theorem 2: Under B1 and B3, $a_{l(N)}(x_{l(N)} - r) \Rightarrow N(0,1)$ where the a_n 's are the constants of Theorem 1.

Proof: We assume we are dealing with A6 i.), the proof for A6 ii.) being similar. Suppose then that c_i is minimal for $i = s$. Then, by B1 and B3,

$$(3.1) \quad k_{n,s}^{1/2}(\bar{Z}_{n,i} : i \in F) \Rightarrow \hat{N}$$

where $\hat{N}$ is a multivariate normal r.v. with (possibly) singular components

$(\bar{Z}_{n,i} = \sum_{j \in F} z_{n,j}/k_{n,i})$ - in fact, it is easy to obtain a weak invariance

principle version of (3.1)

Put $S_n = x_1 + \dots + x_n$ and observe that

$$\begin{aligned} S_n/n &= \sum_{i \in F} \sum_{j \in F} x_j / n \\ &= \sum_{i \in F} \left(\sum_{j \in F} x_j / k_{n,i} \right) \cdot k_{n,i}/n \\ &\quad + \sum_{i \in F} c_i E\tau_{1,i} \text{ a.s.} \end{aligned}$$

by B3. But $S_{l(N)} < N < S_{l(N)+1}$ so

$$S_{l(N)}/l(N) < N/l(N) < S_{l(N)+1}/l(N)$$

and thus, by "squeezing" $N/l(N)$, we obtain the result that

$N/l(N) \rightarrow \sum_{i \in F} c_i E\tau_{1,i}$ a.s. Then, using the weak invariance version of (3.1)

and the random time change results of [2], p. 146, we have that

$$k_{l(N),s}^{1/2} \sum_{i \in F} \bar{z}_{l(N),i}/\underline{\sigma} \Rightarrow N(0,1)$$

where $\underline{\sigma}^2 = \sum_{i \in F} c_s \sigma_i^2 / c_i$ (if $c_s = c_i = 0$, set $c_s/c_i = \gamma_{is}$). Another

application of the converging together lemma shows that

$$\underline{a}_{l(N)}(r_{l(N)} - r) \Rightarrow N(0,1)$$

where $\underline{a}_n^2 = k_{n,s} E^2 \tau_1 / \underline{\sigma}^2$. But B3 guarantees that $\underline{a}_n / a_n \rightarrow 1$, yielding the theorem. ||

Again, in terms of the confidence interval problem, one needs to estimate $a_{l(N)}$. The following corollary follows immediately from Theorem 2, and the fact that $\hat{a}_n / a_n \rightarrow 1$ a.s.

Corollary 2: Under B1-B3, $\hat{a}_{l(N)}(r_{l(N)} - r) \Rightarrow N(0,1)$ where the $\hat{a}_n$'s are the estimators of Corollary 1.

Finally, we can often re-write the CLT of Theorem 2 in another form.

If c_s is positive, then $k_{n,s}/nc_s \rightarrow 1$, so that we obtain the following result.

Corollary 3: Assume B3 holds with all c_i 's positive. Then, under B1, there exists $\bar{\sigma}$ such that $\sqrt{N}(r_{l(N)} - r)/\bar{\sigma} \Rightarrow N(0,1)$. Also, under B1-B2, there exist estimators $\bar{\sigma}_N$ such that $\sqrt{N}(r_{l(N)} - r)/\bar{\sigma}_N \Rightarrow N(0,1)$.

Proof: The result is obvious, upon identifying $\bar{\sigma}^2, \bar{\sigma}_N^2$. Under A6 i.),

$$\bar{\sigma}^2 = (\sigma^2 / E^2 \tau_1) \cdot \left(\sum_{i \in F} c_i E\tau_{1,i} / c_s \right) \text{ and } \bar{\sigma}_N^2 =$$

$$\left(\sum_{i \in F} c_s \hat{\sigma}_{l(N),i}^2 / c_i \hat{\tau}_{l(N)}^2 \right) \cdot \left(\sum_{i \in F} c_i \bar{\tau}_{l(N),i} / c_s \right). \quad ||$$

4. Optimal Confidence Intervals

We now wish to investigate the amount of variance reduction over the standard regenerative method that is accomplished by using the intervals proposed in Section 3. Let $v(N)$, $v(\vec{c}, N)$ be the half-widths of $100(1-\delta)\%$ confidence intervals based on simulating N time units and using the standard regenerative interval and the interval of Corollary 2, respectively (we write $v(\vec{c}, N)$ to reflect dependence on $\vec{c} = (c_i)$). The following result may be found in [4].

Lemma 1: Under B1-B2, $N^{1/2}v(N) \rightarrow z_\delta \sigma / (E\tau_1)^{1/2} \triangleq z_\delta \tilde{\sigma}$ a.s.

In view of Lemma 1, the next lemma shows that it is never optimal to allow $k_{n,i}$ to tend to ∞ in such a way that $k_{n,i}/n \rightarrow 0$.

Lemma 2: Suppose B1-B3 hold and $k_{n,s} \rightarrow \infty$ with $c_s = 0$. Then $N^{1/2}v(N, \vec{c}) \rightarrow \infty$ a.s.

Proof: The assertion is equivalent to proving that $N/a_{l(N)}^2 \rightarrow \infty$. But

$$\begin{aligned} N/a_{l(N)}^2 &= N \hat{s}_{l(N)}^2 / \hat{\tau}_{l(N)}^2 \\ &> N \hat{\sigma}_{l(N),s}^2 / k_{l(N),s} \hat{\tau}_{l(N)}^2 \\ &= (N/l(N)) \cdot (l(N)/k_{l(N),s}) \cdot (\hat{\sigma}_{l(N),s}^2 / \hat{\tau}_{l(N)}^2) \rightarrow \infty \text{ a.s. } || \end{aligned}$$

Thus, in our search for optimal intervals, we need only consider the case where all c_i 's are positive. This allows Corollary 3 to be applied to obtain a second cyclic regenerative interval with half-length $\kappa(N, \vec{c})$ (say). The following result follows from the proofs of Theorem 2 and Corollary 3.

Lemma 3: Suppose B1-B3 hold with all c_i 's positive. Then, $v(N, \vec{c})/\kappa(N, \vec{c}) \rightarrow 1$ a.s. Furthermore, under A6 i.),

$$N^{1/2}v(N, \vec{c}) \rightarrow z_\delta \left(\sum_{i \in F} \sigma_i^2 / c_i \right)^{1/2} \left(\sum_{i \in F} c_i E\tau_{1,i} \right)^{1/2} / E\tau_1 \text{ a.s.}$$

and under A6 i.)

$$N^{1/2} v(N, \hat{c}) \rightarrow z_\delta (\sigma^2/c_0 + \sum_{i \in F} \sigma_i^2/c_i)^{1/2} (c_0 E\tau_1 + \sum_{i \in F} c_i E\tau_{1,i})^{1/2} / 2E\tau_1 \text{ a.s.}$$

We are now in a position to determine the optimal constants $\hat{c}_i$.

Theorem 3: Assume B1-B2 hold. If $\sigma^2 < (\sum_{i \in F} \sigma_i (E\tau_{1,i})^{1/2})^2$, then no

variance reduction is possible via the cyclic intervals of Section 3.

Otherwise, the maximal reduction is obtained via the cyclic interval of

Section 3 in which $k_{n,i}/n \rightarrow \hat{c}_i$ for $i \in F$, where

$$(4.1) \quad \begin{aligned} \hat{c}_i &= \alpha \sigma_i / (E\tau_{1,i})^{1/2} \\ \alpha &= (\sum_{i \in F} \sigma_i / (E\tau_{1,i})^{1/2})^{-1} \end{aligned}$$

The percentage variance reduction achieved is then

$$100(1 - \sum_{i \in F} \sigma_i (E\tau_{1,i})^{1/2} / \sigma)^2 \%$$

Proof: By Lemma 3, it is clear that the optimal interval possible via a cyclic method of type A6 i.) is obtained by choosing $k_{n,i}/n \rightarrow \hat{c}_i$ for $i \in F$, where $\hat{c}_i$ solves the optimization problem

$$(4.2) \quad \begin{aligned} &\text{minimize } (\sum_{i \in F} \sigma_i^2/c_i) (\sum_{i \in F} c_i E\tau_{1,i}) / E^2\tau_1 \\ &\text{subject to } \sum_{i \in F} c_i = 1, c_i > 0 \end{aligned}$$

Application of the method of Lagrange multipliers to this problem (see Avriel

[1], p. 48) show that a minimal $\hat{c}$ must satisfy

$$(4.3) \quad -\sigma_i^2 (\sum_{i \in F} c_i E\tau_{1,i}) / c_i^2 + E\tau_{1,i} (\sum_{i \in F} \sigma_i^2/c_i) + \lambda = 0$$

for each $i \in F$ and some constant λ . Multiplying the i 'th equation of (4.3)

by c_i and adding all the resulting equations proves that $\lambda = 0$. Equation

(4.3) shows that

$$\hat{c}_i^2 = \eta \sigma_i^2 / E\tau_{1,i}$$

for some η . The proportionality factor η is determined by $\sum_{i \in F} c_i = 1$. It is easily checked that c_i , as given, is the minimum desired, with minimal value

$$g^2 = \left(\sum_{i \in F} \sigma_i(E\tau_{1,i})^{1/2} \right)^2 / E^2 \tau_1.$$

A similar analysis for cyclic intervals of type A6 ii.) shows that the minimal possible value for the analog of (4.2) is given by

$$\begin{aligned} & (\sigma(E\tau_1)^{1/2} + \sum_{i \in F} \sigma_i(E\tau_{1,i})^{1/2})^2 / 4E^2 \tau_1 \\ & = (\tilde{\sigma}/2 + g/2)^2 > \min(\tilde{\sigma}^2, g^2) \end{aligned}$$

which shows that intervals of type A6 ii.) can never achieve lower asymptotic half-width than the better of the standard or cyclic (of type A6 i.)) regenerative intervals. The other assertions of the theorem are trivial. ||

This theorem suggests that the practitioner should execute a small "pilot run" to obtain approximate values for c_i . If the "pilot run" suggests a variance reduction over the standard method, the simulator should construct a sampling order which ensures that $k_{n,i}/n \rightarrow c_i$ for $i \in F$, and then employ the cyclic regenerative method.

We conclude with a sufficient condition that guarantees that the cyclic regenerative method achieves a variance reduction over the standard procedure.

Lemma 4: If B1-B3 holds, then $g^2 < \tilde{\sigma}^2$ if

$$\text{cov}(Y_{1,i} - r\tau_{1,i}, Y_{1,j} - r\tau_{1,j}) > 0 \text{ for } 1 \leq i, j \leq t.$$

Proof: Since g^2 is minimal for (4.2),

$$\begin{aligned} g^2 & < \left(\sum_{i \in F} \sigma_i^2 \right) \left(\sum_{i \in F} E\tau_{1,i} \right) / E^2 \tau_1 \\ & < \left(\sum_{i \in F} \sigma_i^2 \right) / E\tau_1 < \sigma^2 \left(\sum_{i \in F} Y_{1,i} - r\tau_{1,i} \right) / E\tau_1 < \tilde{\sigma}^2 \end{aligned}$$

the last two inequalities by the covariance condition. ||

We caution that $\underline{g}^2 > \tilde{\sigma}^2$ is possible if the $y_{1,i} - r\tau_{1,i}$ are negatively correlated.

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