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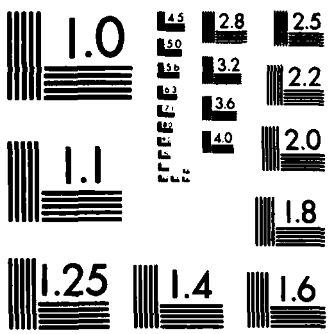

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A CONDITION NUMBER FOR DIFFERENTIABLE CONVEX INEQUALITIES

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ABSTRACT

For a system of differentiable convex inequalities, a new bound is given for the absolute error in an infeasible point in terms of the absolute residual. By using this bound a condition number is defined for the system of inequalities which gives a bound for the relative error in an infeasible point in terms of the relative residual.

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SIGNIFICANCE AND EXPLANATION

Many important problems can be reduced to finding a feasible point to a system of inequalities. When a feasible point is not readily available or when only an "approximate" feasible point is available, one is interested in determining the error in a given infeasible point. In this work we show how such error can be bounded.

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A CONDITION NUMBER FOR DIFFERENTIABLE CONVEX INEQUALITIES

O. L. Mangasarian

1. Introduction

For a system of linear equalities

$$(1.1) \qquad Ax = b$$

where A is a given $n \times n$ real nonsingular matrix and b is a given nonzero vector in the n-dimensional real Euclidean space R^n, the norm of the inverse $\|A^{-1}\|$ and the condition number $\|A^{-1}\|\,\|A\|$ provide the following useful bounds for the absolute error $\|x-\bar{x}\|$ in terms of the residual $\|Ax-b\|$, and for the relative error $\frac{\|x-\bar{x}\|}{\|\bar{x}\|}$ in terms of the relative residual $\frac{\|Ax-b\|}{\|b\|}$ [1,6]

$$(1.2) \qquad \|x-\bar{x}\| \le \|A^{-1}\|\,\|Ax-b\|$$

$$(1.3) \qquad \frac{\|x-\bar{x}\|}{\|\bar{x}\|} \le \|A^{-1}\|\,\|A\|\,\frac{\|Ax-b\|}{\|b\|}$$

Here, x is any point in R^n, $\bar{x}$ is the exact solution $A^{-1}b$ and $\|\cdot\|$ denotes a vector norm on R^n or its subordinate matrix norm [1,6]. The condition number $\|A^{-1}\|\,\|A\|$, which depends on the specific norm employed and which is never less than 1, provides a very useful stability measure for the system (1.1). It is the purpose of this work to obtain a corresponding number for the system of inequalities

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$$g(x) \leq 0 \tag{1.4}$$

where $g\colon R^n \to R^m$ is differentiable and convex on R^n. The key to obtaining a condition number for (1.4) is the definition of a quantity that plays the same role as that of $\|A^{-1}\|$ for (1.1) and which would provide an error bound similar to (1.2). In [2] Hoffman extended the bound (1.2) to a system of linear inequalities, and in [5] a new explicit expression was derived for that bound for a system of linear inequalities and equalities. In [7] Robinson extended the bound (1.2) to a system of convex inequalities that define a <u>bounded</u> feasible region with a nonempty interior. In Section 2 of this paper we shall extend the bound (1.2) to a system of <u>differentiable</u> convex inequalities which satisfy a constraint qualification, but <u>without</u> any boundedness assumption on the feasible region. The diameter of the bounded feasible region which appears linearly in Robinson's bound [7, equation (4)] does not appear in our bound (2.3). In Section 3 of this paper we employ the results of Section 2 to obtain a condition number for (1.4) and thereby extend the relative error bound (1.3) to a system of differentiable convex inequalities satisfying a constraint qualification.

We briefly describe now the notation and some of the basic concepts used in this work. For a vector x in the n-dimensional real Euclidean space R^n, $|x|$ and x_+ will denote the vectors in R^n with components $|x|_i = |x_i|$ and $(x_+)_i = \max\{x_i, 0\}$, $i=1,2,\ldots,n$, respectively. For a norm $\|x\|_\beta$ on R^n, $\|x\|_{\beta*}$ will denote the dual norm on R^n, that is $\|x\|_{\beta*} = \max\limits_{\|y\|_\beta=1} xy$, where xy denotes the scalar product. The generalized Cauchy-Schwarz inequality $|xy| \leq \|x\|_\beta \|y\|_{\beta*}$, for x

and y in R^n, follows immediately from this definition of the dual norm. For $1 \leq p, q \leq \infty$ and $\frac{1}{p} + \frac{1}{q} = 1$ the p-norm $(\sum_{i=1}^{n} |x_i|^p)^{\frac{1}{p}}$ and the q-norm are dual norms on R^n [3]. If $\|\cdot\|_\beta$ is a norm on R^n, we shall, with a slight abuse of notation, let $\|\cdot\|_\beta$ also denote the corresponding norm on R^m for $m \neq n$. For an $m \times n$ real matrix A, A_i denotes the ith row and $A_{\cdot j}$ denotes the jth column, while $\|A\|_\beta$ denotes the matrix norm [1,6] subordinate to the vector norm $\|\cdot\|_\beta$, that is $\|A\|_\beta = \max\limits_{\|x\|_\beta = 1} \|Ax\|_\beta$. The consistency condition $\|Ax\|_\beta \leq \|A\|_\beta \|x\|_\beta$ follows immediately from this definition of a matrix norm. We shall also use $\|\cdot\|$ to denote an arbitrary vector norm and its subordinate matrix norm. A vector norm $\|\cdot\|$ on R^n is said to be monotonic if and only if $|x_i| \leq |y_i|$, $i=1,\ldots,n$, implies that $\|x\| \leq \|y\|$ [6]. The p-norms $1 \leq p \leq \infty$, are all monotonic norms [6]. A vector of ones in any real Euclidean space will be denoted by e. For a differentiable function $g\colon R^n \to R^m$, $\nabla g(x)$ will denote the $m \times n$ Jacobian matrix evaluated at the point x in R^n. For a subset $I \subset \{1,2,\ldots,m\}$, $g_I(x)$ or $g_{i \in I}(x)$ will denote those components of $g_i(x)$ such that $i \in I$. Similarly $\nabla g_I(x)$ will denote the rows $(\nabla g(x))_i$ of $\nabla g(x)$ such that $i \in I$.

2. An Absolute Error Bound for Differentiable Convex Inequalities

We shall use the approach of [5] to obtain a bound (2.3) for the absolute error $\|x-p(x)\|_\gamma$, where x is any infeasible point for (1.4) and $p(x)$ is some feasible point for (1.4), in terms of the absolute residual $\|g(x)_+\|_\beta$. The constant $\mu_{\beta\gamma}$ relating these two quantities plays a similar role for the differentiable convex inequalities (1.4) as $\|A^{-1}\|$ does for the linear system (1.1).

2.1 Theorem Let g be a differentiable convex function from R^n into R^m, let $S°$ and S defined by

$$S°:=\{x|g(x)<0,\ x\in R^n\}\subset S:=\{x|g(x)\leq 0,\ x\in R^n\} \tag{2.1}$$

be nonempty and let g satisfy the following asymptotic constraint qualification

(2.2) For each sequence of points $\{x^i\}\subset S$ such that $g_I(x^i)=0$ and $\nabla g_I(x^i)\neq 0$ for $I\subset\{1,2,\ldots,m\}$, each accumulation point $\overline{\nabla g}_I$ of the sequence $\{\nabla g_I(x^i)\}$ when the latter is bounded, and of the subsequence $\{\nabla g_I(x^i)/\|\nabla g_I(x^i)\|\}$ when $\{\nabla g_I(x^i)\}$ is unbounded, satisfies

$$\overline{\nabla g}_I z > 0 \quad \text{for some} \quad z\in R^n$$

Then for each x in R^n there exists a point $p(x)$ in S such that

$$\|x-p(x)\|_\gamma \leq \mu_{\beta\gamma}\|g(x)_+\|_\beta \tag{2.3}$$

where $\mu_{\beta\gamma}$ is a constant independent of x and defined by

(2.4) $$\mu_{\beta\gamma} := \alpha_{\gamma\infty} \sup_{w,p} \{\|w\|_{\beta*} \mid p \in S,\ w \geq 0,\ wg(p) = 0,\ \|w\nabla g(p)\|_1 = 1\}$$

where $\|\cdot\|_{\beta*}$ is the dual norm to $\|\cdot\|_\beta$ and $\alpha_{\gamma\infty}$ is the positive constant relating the γ-norm and ∞-norm by

$$\|z\|_\gamma \leq \alpha_{\gamma\infty} \|z\|_\infty \quad \text{for all} \quad z \in R^n$$

<u>Proof</u> We note first that $\mu_{\beta\gamma}$ as defined by (2.4) is finite under the asymptotic constraint qualification (2.2). For if not then there exists a sequence $\{(w^i, p^i)\}$ such that $w^i \neq 0$ and $\|w^i\| \to \infty$, $p^i \in S$, $w^i \geq 0$, $w^i g(p^i) = 0$ and $\|w^i \nabla g(p^i)\|_1 = 1$. Since $w^i g(p^i) = 0$, $w^i \geq 0$, $g(p^i) \leq 0$, it follows that $g_I(p^i) = 0$ and $w^i_{j \notin I} = 0$ for some fixed nonempty subset I of $\{1,\ldots,m\}$ and for a subsequence $\{(w^i,p^i)\}_{i\in L}$ of $\{(w^i,p^i)\}$. Hence $\{\|w^i_I\|\}_{i\in L} \to \infty$, $\|w^i_I \nabla g_I(p^i)\|_{i\in L} = 1$ and $\nabla g_I(p^i) \neq 0$. For the case when $\{\nabla g_I(p^i)\}_{i\in L}$ is bounded the subsequences $\{\nabla g_I(p^i)\}_{i\in L}$ and $\{w^i_I/\|w^i_I\|\}_{i\in L}$ have respective accumulation points $\overline{\nabla g}_I$ and $\bar{w}_I$. When $\{\nabla g_I(p^i)\}_{i\in L}$ is unbounded the subsequences $\{\nabla g_I(p^i)/\|\nabla g_I(p^i)\|\}_{i\in L}$ and $\{w^i_I/\|w^i_I\|\}_{i\in L}$ have respective accumulation points $\overline{\nabla g}_I$ and $\bar{w}_I$. In either case, since $\|w^i_I \nabla g_I(p^i)\|_{i\in L} = 1$ and $\{\|w^i_I\|\}_{i\in L} \to \infty$, it follows that

$$\bar{w}_I \overline{\nabla g}_I = 0,\ \bar{w}_I \geq 0,\ \|\bar{w}_I\| = 1$$

This however contradicts the asymptotic constraint qualification (2.2) that $\overline{\nabla g}_I z > 0$ for some z in R^n. Consequently $\mu_{\beta\gamma}$ is finite.

Now for any x in R^n not in S define $p(x)$ as the projection of x on S using the ∞-norm. Hence $p(x)$ and some $\delta(x) > 0$ solve the following convex programming problem

$$(2.5) \qquad \min_{p,\delta} \{\delta \mid -e\delta \leq p-x \leq e\delta,\ g(p) \leq 0,\ p \in R^n,\ \delta \in R\}$$

Note that since S° is nonempty by assumption, the interior of the feasible region of (2.5) is also nonempty and hence $p(x)$, $\delta(x)$ and some $u(x)$, $v(x)$ and $w(x)$ satisfy the following Karush-Kuhn-Tucker conditions [4] for (2.5) where the explicit dependence of p, δ, u, v and w on x has been dropped for simplicity

$$1 - e(u+v) = 0,\ u - v + w\nabla g(p) = 0,\ (u,v,w) \geq 0$$

$$-e\delta \leq p - x \leq e\delta,\ g(p) \leq 0$$

$$wg(p) = 0,\ u_j(p-x-e\delta)_j = 0,\ v_j(-e\delta-p+x)_j = 0,\ j=1,\ldots,n$$

From the last two equalities it follows that $u_j v_j = 0$ for $j=1,\ldots,n$, because

$$(p-x-e\delta)_j + (-e\delta-p+x)_j = -2\delta < 0$$

Consequently

$$(2.6) \qquad p \in S,\ wg(p) = 0,\ w \geq 0,\ \|w\nabla g(p)\|_1 = 1$$

Hence

$$\begin{aligned} 0 < \|x-p\|_\infty &= \delta \\ &= \delta + u(p-x-e\delta) + v(-e\delta-p+x) + wg(p) \\ &= x(v-u) + wg(p) - w\nabla g(p)p \\ &= w\nabla g(p)x + wg(p) - w\nabla g(p)p \\ &\leq wg(x) \qquad \text{(By convexity of } g \text{ and } w \geq 0) \\ &\leq wg(x)_+ \end{aligned}$$

$$\leq \|w\|_{\beta^*} \|g(x)_+\|_\beta \quad \text{(By Canchy-Schwarz inequality)}$$

$$\leq \frac{\mu_{\beta\gamma}}{\alpha_{\gamma\infty}} \|g(x)_+\|_\beta \quad \text{(By (2.4) and (2.6))}$$

Hence

$$\|x-p\|_\gamma \leq \alpha_{\gamma\infty} \|x-p\|_\infty \leq \mu_{\beta\gamma} \|g(x)_+\|_\beta \qquad \square$$

2.2 <u>Remark</u> The asymptotic constraint qualification (2.2) is merely used as a sufficient condition for the finiteness of $\mu_{\beta\gamma}$ as defined by (2.4). Hence Theorem 2.1 can be stated with the constraint qualification (2.2) replaced by the <u>assumption</u> that the supremum of (2.4) defining $\mu_{\beta\gamma}$ is finite.

We give now a simple example illustrating the above theorem which is not covered by Robinson's result [7, equation (4)] because the feasible region is unbounded.

2.3 <u>Example</u> $S := \{x \mid x \in R^2,\ x_2 \geq e^{x_1},\ x_1 \geq 0\}$.

It is easy to verify that the assumptions of Theorem 2.1 are satisfied with $g_1(x) := e^{x_1} - x_2$, $g_2(x) = -x_1$ and that

$$\mu_{\infty\infty} := \sup_{w,p} \{\|w\|_1 \mid p \in S,\ w \geq 0,\ wg(p) = 0,\ \|w\nabla g(p)\|_1 = 1\} = 2$$

Consequently for each x in R^2 there exists a $p(x)$ in S such that

$$\|x-p(x)\|_\infty \leq 2 \left\| \begin{matrix} (e^{x_1} - x_2)_+ \\ (-x_1)_+ \end{matrix} \right\|_\infty$$

The bound $\mu_{\infty\infty} = 2$ is sharp here, for take the sequence of points $x_1 = -t$, $x_2 = 1 - 2t$ with t a nonnegative number converging to zero.

Then

$$\frac{\|x-p(x)\|_\infty}{\|g(x)_+\|_\infty} = \frac{\|-t, -2t\|_\infty}{e^{-t}-1+2t} = \frac{2t}{e^{-t}-1+2t}$$

which approaches 2 as t approaches 0.

3. A Relative Error Bound for Differentiable Convex Inequalities

We extend now the relative error bound (1.3) to convex differentiable inequalities by using the error bound (2.3). We will again use the approach of [5] and will need the following simple lemma established there.

3.1 Lemma [5, Lemma 2] Let $\|\cdot\|_\beta$ be a monotonic norm on R^m and let a, b be in R^m. Then $a \leq b$ implies that $\|(a)_+\|_\beta \leq \|b\|_\beta$.

The following is a direct consequence of the above lemma.

3.2 Lemma Let $g\colon R^n \to R^m$ be differentiable and convex on R^n and let $\|\cdot\|_\beta$ be a monotonic norm on R^m. Then

$$\|g(0)_+\|_\beta \leq \|\nabla g(0)p\|_\beta \quad \text{for} \quad g(p) \leq 0$$

Proof By the convexity of g, $0 \geq g(p) \geq g(0) + \nabla g(0)p$. Hence applying Lemma 3.1 to $g(0) \leq -\nabla g(0)p$ we obtain the desired inequality. □

3.3 Theorem (Condition number bound) Let the assumptions of Theorem 2.1 hold, let $g(0)_+ \neq 0$ and let $\|\cdot\|_\beta$ be a monotonic norm on R^m. Then for each x in R^n there exists a $p(x)$ in S such that

$$(3.1) \qquad \frac{\|x-p(x)\|_\beta}{\|p(x)\|_\beta} \leq \mu_{\beta\beta}\,\|\nabla g(0)\|_\beta\,\frac{\|g(x)_+\|_\beta}{\|g(0)_+\|_\beta}$$

where $\mu_{\beta\beta}$ is defined by (2.4), and $\mu_{\beta\beta}\|\nabla g(0)\|_\beta$ defines the condition number of (1.4).

Proof For each x in R^n there exists a $p(x)$ in S such that

$$\frac{\|x-p(x)\|_\beta}{\|p(x)\|_\beta} \le \mu_{\beta\beta} \frac{\|g(0)_+\|_\beta}{\|p(x)\|_\beta} \frac{\|g(x)_+\|_\beta}{\|g(0)_+\|_\beta} \qquad \text{(By Theorem 2.1)}$$

$$\le \mu_{\beta\beta} \frac{\|\nabla g(0)p(x)\|_\beta}{\|p(x)\|_\beta} \frac{\|g(x)_+\|_\beta}{\|g(0)_+\|_\beta} \qquad \text{(By Lemma 3.2)}$$

$$\le \mu_{\beta\beta} \|\nabla g(0)\|_\beta \frac{\|g(x)_+\|_\beta}{\|g(0)_+\|_\beta} \qquad \square$$

For Example 2.3, it is easy to verify that $\|\nabla g(0)\|_\infty = 2$ and hence the condition number for the example, using the ∞-norm, is: $\mu_{\infty\infty} \|\nabla g(0)\|_\infty = 4$.

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