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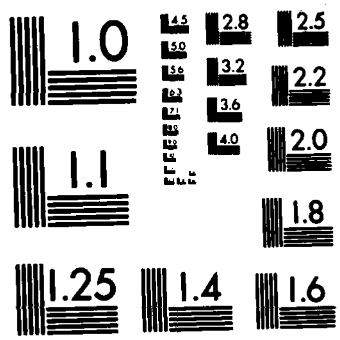

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SOME NONLINEAR PROBLEMS OF WATER WAVES IN A CHANNEL

M. C. Shen

Mathematics Research Center
University of Wisconsin—Madison
610 Walnut Street
Madison, Wisconsin 53706

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ABSTRACT

On the basis of some approximate equations derived for nonlinear water waves in a channel of variable cross section, several problems of interest, which have attracted much attention recently, will be taken up in this report. First we shall consider the breaking of an acceleration wave moving toward a shoreline in a general channel. Next the so-called infinite mass dilemma, which arises in the study of the development of a solitary wave in a channel of variable cross section, will be resolved. Finally we shall use an approximation method to study the fission of solitons in a general channel and justify it.

AMS (MOS) Subject Classifications: 76B15, 76B25

Key Words: Nonlinear water waves, breaking point, infinite mass dilemma, fission of solitons

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SIGNIFICANCE AND EXPLANATION

Consider a wave in an open channel moving toward a shoreline. Under physically reasonable assumptions, we shall determine whether the wave will break, and the location of the break if it does. In recent years, there has also been growing interest in the development of a solitary wave in an open channel of variable cross section. It is found that a shelf is generally formed behind the solitary wave. In some studies, the shelf is found to extend to infinity. This is the so-called infinite mass dilemma. In other words, infinite mass would be created or annulled by a perturbation on the solitary wave. We shall resolve this dilemma by showing that the shelf can only be finite. On the other hand, if a solitary wave propagates from one uniform cross section of a channel to another, the solitary wave may split into several solitons. We shall use an approximation method to find a criterion for the fission of n solitons in a general channel and show that the approximation method indeed yields an approximate solution to the exact solution of the governing equation.

The responsibility for the wording and views expressed in this descriptive summary lies with MRC, and not with the author of this report.

SOME NONLINEAR PROBLEMS OF WATER WAVES IN A CHANNEL

M. C. Shen

1. INTRODUCTION

In this report we shall study several interesting problems of nonlinear water waves in a channel of variable cross section by means of some approximate equations derived from the hydrodynamical equations. For finite amplitude waves, a system of shallow water equations will be given. For small amplitude waves, we shall use a K-dV equation with variable coefficients derived in a previous report (Zhong and Shen, 1982).

One of the fascinating phenomena of finite amplitude water waves in a channel concerns the breaking of a wave moving toward a shoreline, the development of a bore and the movement of the shoreline after the bore reaches it. For the two-dimensional case corresponding to a rectangular channel of variable depth, the bore run-up problem was studied by Keller, Levine and Whitham (1960), Ho and Meyer (1962), and Shen and Meyer (1963a,b) on the basis of shallow water equations (Stoker, 1957). Later Gurtin (1975) derived a criterion for the breaking of an acceleration wave in a two-dimensional channel, and his result was extended by Jeffrey and Mvungi (1980) to the case of a rectangular channel of variable width and depth. We shall make use of the shallow water equations for a general channel and generalize Gurtin's result to predict the breaking point of an acceleration wave. Needless to say, the use of shallow-water equations for the study of bore propagation may be open to criticism. The issue would be settled if we knew the precise conditions for the validity of these approximate equations. Up to date, the shallow water equations for a two-dimensional channel with analytical initial data have been justified by Kano and Nishida (1979), and for a nearly uniform channel with a priori assumptions on the free surface by Berger (1976). At present we may accept shallow water equations as model equations, and the bore run-up problem for a general channel certainly deserves further investigation.

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Recently the problem of the development of a solitary wave in a channel of variable cross section has attracted much attention, and there have been discussions on the so-called infinite mass dilemma, which arises from the formation of a shelf behind the solitary wave. If the shelf were extended to infinity, then infinite mass would be created or annulled by a perturbation on the solitary wave. A study of this problem may be found in Miles (1979) and Knickbocker and Newell (1980), based upon the K-dV equation for a rectangular channel of variable depth or width (Kakutani, 1971; Johnson, 1973; Shuto, 1974). As an extension of the existence results due to Kato (1975, 1980), a global existence theorem will be established here for the solution of the K-dV equation with variable coefficients. It follows that the shelf can only be finite if it is formed behind the solitary wave.

To study the fine structure of the shelf behind a solitary wave, we shall consider the solitary wave moving from one uniform cross section of a channel to another through a transition section. This problem, in fact, has a long history. Using a system of Boussinesq equations, Madsen and Mei (1969) studied numerically the problem of a solitary wave propagating from a channel of constant depth to a shoal of constant but smaller depth through a transition region with mild slope. They found that the solitary wave is disintegrated into a train of solitons of decreasing amplitudes, in qualitative agreement with experiments. An analytical investigation of the shoal-induced fission of solitons was made by Tappert and Zabusky (1971). They used the WKB approximation to describe the transition of the solitary wave and the inverse scattering method for the K-dV equation with constant coefficients by Muira et al. (1974), and obtained a criterion for the fission of n solitons. Johnson (1973) derived a K-dV equation with variable coefficients for a channel of variable depth and found the same criterion by a similar method. Recently Djordjevic and Redekopp (1978) considered the disintegration of internal solitary waves and Zhong and Shen (1982) extended the previous results to the case of a symmetric triangular channel. As a simple consequence of the existence results just obtained, the approximation method used is shown to be valid if the distance to cover the transition section is sufficiently small.

In Section 2, we derive the shallow water equations, and study the breaking of a wave in a general channel. In Section 3 we establish the global existence theorem for the K-dV equation with variable coefficients and resolve the infinite mass dilemma. Finally the approximation method is used and justified for the study of fission of solitons in a general channel in Section 4.

2. SHALLOW WATER EQUATIONS AND THE BREAKING OF A WAVE

We consider the irrotational motion of an inviscid, incompressible fluid of constant density under gravity in a channel with a boundary defined by $h^*(x^*,y^*,z^*) = 0$, where z^* is positive upward and x^* is in the longitudinal direction (Fig. 1). The governing equations are

$$\nabla^* \cdot \bar{q}^* = 0 \, , \tag{2.1}$$

$$\nabla^* \times \bar{q}^* = 0 \, , \tag{2.2}$$

$$\rho(\bar{q}^*_{t^*} + \bar{q}^* \cdot \nabla \bar{q}^*) = -\nabla p^* + \bar{g} \, , \tag{2.3}$$

subject to the boundary conditions

$$\eta^*_{t^*} + \bar{q}^* \cdot \nabla^* z^* = 0, \quad \text{at} \quad z^* = -z^* + \eta^*(t^*,x^*,y^*) = 0 \, , \tag{2.4}$$

$$p^* = 0 \, , \tag{2.5}$$

$$\bar{q}^* \cdot \nabla^* h^* = 0, \quad \text{at} \quad h^* = 0 \, . \tag{2.6}$$

Here $\nabla^* = (\partial/\partial x^*, \partial/\partial y^*, \partial/\partial z^*)$, $\bar{q}^* = (u^*,v^*,w^*)$ is the velocity, t^* is the time, $\bar{g} = (0,0,-g)$ is the constant gravitational acceleration, ρ is the constant density, p^* is the pressure, and $z^* = \eta^*$ is the equation of the free surface. To derive the shallow water equations, we make the following assumptions. The channel boundary is convex, sufficiently smooth, and varies slowly in the longitudinal direction; the magnitude of the transverse velocities is much smaller than that of the longitudinal velocity. In a manner as suggested by Friedrichs (1948), we introduce nondimensional variables

$$t = \beta^{-1/2} t^*/(H/g)^{1/2}, \quad (x,y,z) = (\beta^{-1/2} x^*/H, y^*/H, z^*/H) \, ,$$

$$\eta = \eta^*/H, \quad h = h^*/H, \quad (u,v,w) = (u^*/(gH)^{1/2}, \beta^{1/2} v^*/(gH)^{1/2}, \beta^{1/2} w/(gH)^{1/2}) \, ,$$

where $\beta^{1/2} = L/H$ and L and H are respectively the horizontal and transverse length scales. In terms of them, (2.1) to (2.6) become

$$u_x + v_y + w_z = 0 \, , \tag{2.7}$$

$$\beta u_y = v_x, \quad \beta u_z = w_x, \quad v_z = w_y \, , \tag{2.8}$$

$$u_t + uu_x + vu_y + wu_z + p_x = 0 \, , \tag{2.9}$$

$$v_t + uv_x + vv_y + wv_z + \beta p_y = 0 \, , \tag{2.10}$$

$$w_t + uw_x + vw_y + ww_z + \beta(p_z + 1) = 0 \, , \tag{2.11}$$

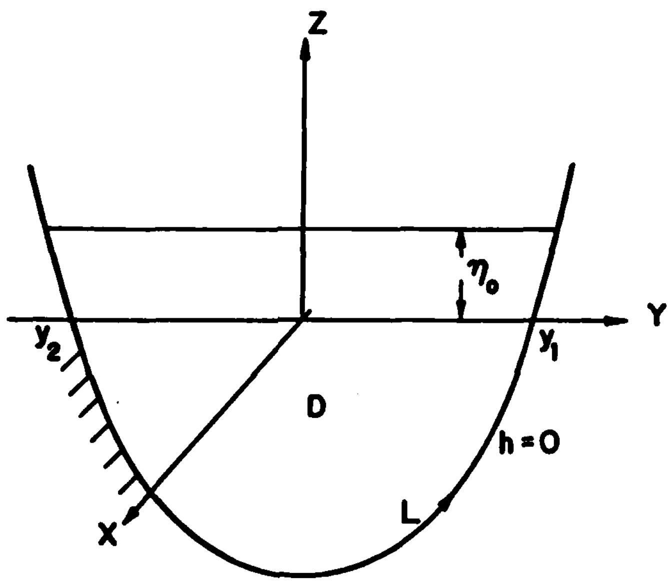

Fig. 1. A CROSS SECTION OF THE CHANNEL

$$\left.\begin{aligned} \eta_t + u\eta_x + v\eta_y - w &= 0 \\ p &= 0 \end{aligned}\right\} \text{at} \quad z = \eta \qquad \begin{matrix}(2.12)\\(2.13)\end{matrix}$$

$$uh_x + vh_y + wh_z = 0 \quad \text{at} \quad h = 0 \ . \tag{2.14}$$

Assume that u, v, w, p and η possess an asymptotic expansion of the form

$$\phi \sim \phi_0 + \beta^{-1}\phi_1 + \beta^{-2}\phi_2 + \dots , \tag{2.15}$$

and substitute (2.15) in (2.7) to (2.14). The equations for the zeroth approximation are

$$u_{0x} + v_{0y} + w_{0z} = 0 \ . \tag{2.16}$$

$$u_{0y} = u_{0z} = 0 , \tag{2.17}$$

$$u_{0t} + u_0u_{0x} + p_{0x} + v_0u_{0y} + w_0u_{0z} = 0 , \tag{2.18}$$

$$p_{0y} = 0, \quad p_{0z} = -1 , \tag{2.19}$$

$$\left.\begin{aligned} \eta_{0t} + u_0\eta_{0x} + v_0\eta_{0y} - w_0 &= 0 \\ p_0 &= 0 \end{aligned}\right\} \text{at} \quad z = \eta_0 , \qquad \begin{matrix}(2.20)\\(2.21)\end{matrix}$$

$$u_0h_x + v_0h_y + w_0h_z = 0 \quad \text{at} \quad h = 0 \ . \tag{2.22}$$

As seen from (2.17), (2.19) and (2.21), u_0 is a function of t,x only, and

$$p_0 = -z + \eta_0 , \tag{2.23}$$

which implies η_0 is also a function of t,x only. It follows from (2.17), (2.18) and (2.23) that

$$u_{0t} + u_0u_{0x} + \eta_{0x} = 0 \ . \tag{2.24}$$

Now we integrate (2.16) over a cross section D of the channel, apply the divergence theorem and make use of (2.20) and (2.22) to obtain

$$\iint_D (v_{0y} + w_{0z})dydz = -u_{0x}A(t,x) =$$

$$= -u_0 \int_L h_x(h_y^2 + h_z^2)^{-1/2}ds + (\eta_{0t} + u_0\eta_{0x})B(t,x) \ .$$

By rearranging the terms, we have

$$\eta_{0t} + u_0\eta_{0x} + u_{0x}A(t,x)/B(t,x) - [u_0/B(t,x)] \int_L h_x(h_g^2 + h_z^2)^{-1/2}ds = 0 , \tag{2.25}$$

where $A(t,x)$ is the area, $B(t,x)$ is the width and L is the wetted boundary, of the cross section D (Fig. 1). (2.24) and (2.25) form a system of nonlinear equations, which may be used to model bore formation and its subsequent development in a channel of variable cross section.

In the following we extend Gurtin's method to the case of a genersl channel. The assumptions made are the following:

(1) u_0, η_0 are continuous,

(2) the first and second derivatives of u_0 and η_0 possess at most jump discontinuities,

(3) $u_0 = \eta_0 = 0$ ahead of the wave.

Denote the value of a function f immediately behind the wave front by f^-. Hereafter we also drop the subscript 0. From assumptions (1), (2), we have

$$u^- = \eta^- = 0 \ . \tag{2.26}$$

By total differentiation,

$$u_t^- = -cu_x^- \ , \quad \eta_t^- = -c\eta_x^- \ , \tag{2.27}$$

where C is the speed of the wave front. From (2.24), (2.25) and (2.26), it follows that

$$u_t^- + \eta_x^- = 0 \ , \quad \eta_t^- + u_x^- A^-/B^- = 0 \ . \tag{2.28}$$

Comparing (2.27) and (2.28), we have

$$u_t^- = c^{-1}\eta_x^- \ , \quad C = (A^-/B^-)^{1/2} \ . \tag{2.29}$$

Now we differentiate (2.24) with respect to t and (2.25) with respect to x, and evaluate the equations behind the wave front. When we eliminate η_{tx}^- and make use of the expression

$$c^2 u_{xx}^- - u_{tt}^- = c^2 d(u_x^-)/dx - c\,du_t/dx$$

to obtain

$$-2Cd(\eta_x^-)dx^{-1} + (\eta_x^-)^{-1}[C' - \Gamma^-(B^-C)] + 3C^{-1} = 0 \ .$$

where

$$\Gamma = \int_L h_x (h_y^2 + h_z^2)^{-1/2} ds \ .$$

Hence,

$$\eta_x^- = a_0 c^{-1/2}\left[\left(\tfrac{3}{2}\right)a_0 \int^x c^{-5/2} \exp \int_x^{x'} \Gamma^-(2A^-)^{-1}dx' + 1\right]_x^{-1} \exp \int_{x_0}^x \Gamma^-(2A^-)^{-1}dx \ , \tag{2.20}$$

where a_0 is the initial value of η_x at $x = x_0$. We call $x = \ell$ a shoreline if $A^-(\ell) = 0$ but $B^-(\ell) \neq 0$, and let

$$I(x) = \left(\frac{3}{2}\right) \int_{x_0}^{x} c^{-5/2} \exp \int_{x_0}^{x^1} \Gamma^-(2A^-)^{-1} dx^- .$$

Suppose $a_0 < 0$. If $I(\ell) = \infty$, then $\eta_x^- = \infty$ and the wave breaks before it reaches the shoreline. If $I(\ell) \neq \infty$, then either the wave breaks before it reaches the shoreline or it breaks at the shoreline. Next suppose $a_0 > 0$. If $I(\ell) \neq \infty$, then the wave breaks at the shoreline. Otherwise if $I(\ell) = \infty$, we evaluate the limit of η_x^- given by (2.30) as $x \to \ell$ and obtain

$$\lim_{x\to\ell} \eta_x^- = \left(\frac{2}{3}\right) [-(d^-)'/4 + \Gamma^-/(2B^-)]_{x=\ell} , \tag{2.31}$$

where $d^- = A^-/B^-$. Hence the wave will never break if $(d^-)'$ is finite at $x = \ell$. However for channels of variable cross section the equilibrium water surface may converge to a point and this case is also of interest. Assume again $a_0 > 0$, $I(\ell) = \infty$. Of $B^-(\ell) = d^-(\ell) = 0$, $(d^-)'$ is finite at $x = \ell$, and $h(x,y,z) = -z + g(x,y) = 0$, we have

$$\Gamma^- = \int_L h_x(h_y^2 + h_z^2)^{-1/2} ds = \int_{-b_1}^{b_2} g_x dy ,$$

where $y = -b_1, b_2$ are the endpoints of the width $B^-(x)$. It follows from (2.31) that

$$\lim_{x\to\ell} \eta_x^- = \left(\frac{2}{3}\right) [-(x^-)'/4 + g_x/2]_{x=\ell} ,$$

and the wave will never break.

3. K-dV EQUATION AND THE INFINITE MASS DILEMMA

We only sketch the derivation of the K-dV equations for a channel of variable cross section and the details may be found in Zhong and Shen (1982). We first introduce the non-dimensional variables

$$t = \beta^{-3/2} t^*/(H/g)^{1/2}, \qquad (x,y,z) = (\beta^{-3/2} x^*/H, y^*/H, z^*/H) ;$$

η, h, p and (u,v,w) are the same as before. The method used here is a specialization of the procedure developed by Shen and Keller (1973). Assume that u,v,w,p and η as functions of t,x,y and z also depend explicitly upon a new variable

$$\xi = \beta S(t,x) ,$$

where S, a function of t and x only, will be a phase function, and that they possess an asymptotic expansion of the form

$$\phi(\xi,t,x,y,z,\beta) \sim \phi_0 + \beta^{-1}\phi_1 + \beta^{-2}\phi_2 + \cdots .$$

The zeroth approximation is assumed to be given by

$$(u_0,v_0,w_0) = 0, \qquad p_0 = -z, \qquad \eta_0 = 0 .$$

The equations for the first approximation determine a Hamilton-Jacobi equation for S. Let

$$k = S_x, \qquad \omega = -S_t .$$

Then

$$\omega = kG(x), \qquad G(x) = \pm[a(x)/b(x)]^{1/2} , \tag{3.1}$$

where $a(x)$ is the area of a cross section D_0, and $b(x)$ is the width of D_0, of water at rest (Fig. 1). (3.1) may be solved by the method of characteristics and the corresponding characteristics equations (Courant and Hilbert, 1962) are

$$dt/d\sigma = \mu \quad dx/d\sigma = \mu G(x), \quad dk/d\sigma = -k\mu G'(x), \quad d\omega/d\sigma = dS/d\sigma = 0 . \tag{3.2}$$

Where μ is a proportionality factor. We choose $\mu = 1$ so that $\sigma = t$. The solutions of (3.2) determine a one-parameter family of bicharacteristics called rays,

$$x = x(t,\sigma_1)$$

where σ_1 is constant along a ray. The equations for the second approximation determine a K-dV equation with variable coefficients

$$m_0\eta_{1t} + m_1\eta_{1x} + m_2\eta_1 + m_3\eta_1\eta_{1\xi} + m_4\eta_{1\xi\xi\xi} = 0 . \tag{3.3}$$

Here

$$m_0 = 2b(x) , \tag{3.4}$$

$$m_1 = 2a(x)/G(x) , \tag{3.5}$$

$$m_2 = -[G(x)]^{-1} \int_{L_0} h_x(h_y^2 + h_x^2)^{-1/2} ds - G^{-2}(x)G'(x)a(x) , \tag{3.6}$$

$$m_3 = 3k[G(x)]^{-1}b(x) - \omega^{-1}[\phi_y(t,x,y_2,0) - \mu_y(t,x,y_1,0)] , \tag{3.7}$$

$$m_4 = \omega^{-1} \iint_{D_0} (\nabla\phi)^2 dydz , \tag{3.8}$$

L_0 is the wetted boundary of D_0, $y = y_1, y_2$ are the endpoints of the width of D_0, and ϕ is the solution of the following Neumann problem

$$\nabla^2\phi = k^2 \quad \text{in} \quad D_0 ,$$

$$\phi_z = \omega^2 \quad \text{at} \quad z = 0 ,$$

$$\phi_y h_y + \phi_z h_z = 0 \quad \text{at} \quad L_0 .$$

Since from (3.2)

$$d/d\sigma = \partial/\partial t + G(x)\ \partial/\partial x, \quad dx/d\sigma = G(x) ,$$

along a ray, we may express (3.3) in terms of σ and ξ

$$m_0\eta_{1\sigma} + m_2\eta_1 + m_3\eta_1\eta_{1\xi} + m_4\eta_{1\xi\xi\xi} = 0 ,$$

or in terms of x and ξ

$$m_1\eta_{1x} + m_2\eta_1 + m_3\eta_1\eta_{1\xi} + m_4\eta_{1\xi\xi\xi} = 0 . \tag{3.9}$$

To be definite, we choose

$$G(x) = +[a(x)/b(x)]^{1/2} ,$$

$$S = -t + \int_0^{\psi} [G(z)]^{-1} dz ,$$

which is a solution of (3.2) and it follows that

$$\omega = 1, \quad k = G^{-1}(x) .$$

We note that other choices of S are also possible. For rectangular and triangular channels, the coefficients given in (3.4) to (3.8) can be explicitly evaluated (Zhong and Shen, 1982).

1) Rectangular channel

Let $d(x)$ and $b(x)$ be the variable depth and width respectively

$$m_0 = 2b(x),\quad m_1 = 2b(x)d^{1/2}(x),\quad m_2 = b'(x)d^{1/2}(x) + d'(x)b(x)d^{-1/2}(x)/2\ ,$$
$$m_3 = 3d^{-1}(x)b(x),\quad m_4 = (1/3)b(x)d(x)\ .$$

2) Triangular channel

Let the two sides of a cross section D_0 be defined by $z = \mu_1(x)y - d(x)$, $z = -\mu_2(x)y - d(x)$, where $\mu_i(x) = d(x)/b_i(x)$, $i = 1,2$.

$$m_0 = 2[b_1(x) + b_2(x)] = 2b(x),\quad m_1 = \sqrt{2}\ d^{1/2}(x)b(x)\ ,$$

$$m_2 = \sqrt{2}\ d^{-1/2}(x)[b'(x)d(x) + d'(x)b(x)/2]/2,\quad m_3 = 5d^{-1}(x)b(x)\ ,$$

$$m_4 = [d^{-1}(x)/4][b(x)d^2(x) + (b_1^3(x) + b_2^3(x)/3]\ .$$

In both cases, $m_1 > 0$, $m_3 > 0$, $m_4 > 0$ if $d(x) > 0$, $b(x) > 0$. For general cases, as seen from (3.5), (3.8) $m_1 > 0$, $m_4 > 0$ where we choose $G(x) > 0$, but the sign of m_3 given by (3.7) is not obvious. We shall assume that $m_3 \neq 0$ is the case.

Let

$$\tau = \int_0^x m_4(x')m_1^{-1}(x')dx'\ , \tag{3.10}$$

$$A = m_3(x)m_4^{-1}(x)\eta_1\ . \tag{3.11}$$

In terms of τ and A, (3.9) becomes

$$A_\tau + AA_\xi + A_{\xi\xi\xi} = H(\tau)A,\quad \tau > 0,\quad -\infty < \xi < \infty\ , \tag{3.12}$$

subject to

$$A(0,\xi) = A_0(\xi),\quad -\infty < \xi < \infty\ , \tag{3.13}$$

where

$$h(\tau) = -m_2(x)m_4^{-1}(x) - m_1(x)m_4^{-2}(x)m_3(x)[m_4(x)/m_3(x)]'\ .$$

A global existence theorem for (3.12) and (3.13) can be easily obtained by extending the existence results for the K-dV equation with constant coefficients due to Kato (1975, 1980). Assume $H(\tau)$ is continuous and let $H^s(-\infty,\infty)$ denote the Sobolev space of order s of the L^2-type. Since H9z)A satisfies the conditions (f1), (f2) in Kato (1975), we have the following local existence result (Kato, 1980).

Theorem 1.

(3.12) subject to (3.13) with $A_0 \in H^s$, $s > 2$ has a unique solution

$$A \in C[0,T';H^s] \quad C^1[0,T';H^{s-3}], \quad A(0,\xi) = A_0(\xi) ,$$

for some $T' > 0$ and $A(t)$ depends upon A_0 continuously in the H^s-norm.

Hereafter we shall denote an H^s-norm by $\|\cdot\|_s$ and an L^2-norm by $\|\cdot\|$. To show that there exists a global solution in $[0,T]$ for any $T > 0$, we need a regularity result and an a priori estimate for $\|A\|_2$, which may be obtained by means of the first three conservation laws for the K-dV equation with constant coefficients (Lions, 1969).

Theorem 2.

If $A \in C[0,T;H^s]$ is a solution to (3.12) with $s > 2$ and if $A(0) = A_0 \in H^{s'}$ with $s' > s$, then $A \in C[0,T;H^{s'}]$ with the same T.

This theorem is a simple extension of Kato's result (1980) and we omit the proof.

Lemma 1.

Suppose for any $T > 0$ and τ in $[0,T]$, a solution of (3.12) $A \in C[0,T;H^2]$ exists and $A(0) = A_0 \in H^2$. Then

$$\|A(\tau)\|_2 < \phi_{2,T} \; (\|A(0)\|_2) ,$$

where $\phi_{2,T}(\cdot)$ depending upon T is a monotone increasing function with $\phi_{2,T}(0) = 0$.

Proof: Assume first $A, A_0 \in H^4$. We multiply (3.12) by A and integrate with respect to ξ from $-\infty$ to ∞ to obtain, for τ in $[0,T]$,

$$d\|A\|^2/d\tau = 2 \int_{-\infty}^{\infty} H(\tau)A^2 d\xi .$$

It follows that

$$\|A\|^2 = \|A(0)\|^2 \exp 2 \int_0^{\tau} H(\tau')d\tau' ,$$

and

$$\|A\| < \phi_{0,T}(\|A(0)\|) , \qquad (3.14)$$

where

$$\phi_{0,T}(\|A(0)\|) = \|A_0\|\exp(MT) ,$$

$$M = \sup_{0<\tau<T} |H(\tau)| .$$

Next we multiply (3.12) by $A^2 + 2A_{\xi\xi}$ and integrate again to obtain

$$d/d\tau \int_{-\infty}^{\infty} (A^3/3 - A_\xi^2)d\xi = H(\tau) \int_{-\infty}^{\infty} (A^3 - 2A_\xi^2)d\xi .$$

Then we integrate the above equation with respect to τ from 0 to τ to obtain

$$\|A_\xi\|^2 < \|A_\xi(0)\|^2 + (1/3)\|A(0)\|^3_{L^3} + (1/3)\|A\|^3_{L^3}$$

$$+ 2\int_0^\tau |H(\tau')| \int_{-\infty}^{\infty} A_\xi^2 d\xi d\tau' + \int_0^\tau |H(\tau')| \int_{-\infty}^{\infty} |A|^3 d\xi d\tau' .$$

Since

$$\|A\|^3_{L^3} < 2\|A_\xi\|^{1/2}\|A\|^{5/2} < \|A_\xi\|^2 + C_0$$

where $C_0 = 2\phi_{0,T}^{5/2}$, we have

$$\left(\tfrac{2}{3}\right) \|A_\xi\|^2 < \left(\tfrac{4}{3}\right) \|A_\xi(0)\|^2 + \left(\tfrac{2}{3}\right) \|A(0)\|^{5/2} + C_0/3 + MTC_0 + 3M \int_0^\tau \|A_\xi\|^2 d\tau' .$$

It follows that

$$\|A\|_1^2 < C_{1,T}(\|A_0\|_1) + \left(\tfrac{9}{2}\right) M \int_0^\tau \|A\|_1^2 d\tau' ,$$

where $C_{1,T}(\cdot)$ is a monotone increasing function with $C_{1,T}(0) = 0$. By Gronwall's inequality,

$$\|A\|_1 < C_{1,T}^{1/2}\exp(9MT/4) = \phi_{1,T}(\|A\|_1) . \qquad (3.15)$$

Similarly, we multiply (3.12) by $A^3 + 3A_\xi + 6AA_\xi + (18/5)A_{\xi\xi\xi\xi}$, integrate the resulting equation and make use of

$$\|A\|^4_{L_4} < 2\|A_\xi\|\|A\|^3 ,$$

and (3.14), (3.15) to obtain

$$\|A\|_2 < \phi_{2,T}(\|A(0)\|_2) .$$

For $A, A_0 \in H^2$, we may approximate A_0, A by sequences of smoother functions, make use of the result of local continuous dependence, and theorem 4 to complete the proof. By Lemma 1, Theorems 1 and 2 we have the global existence theorem.

Theorem 3. For any $T > 0$, (3.12) possesses a unique solution $A \in C[0,T;H^s] \cap C^1[0,T;H^{s-3}]$ satisfying $A(0,\xi) = A_0 \in H^s$, $s > 2$. A depends upon A_0 continuously in H^s norm.

It is evident from the above theorem that if we prescribe $A_0 = \alpha \operatorname{sech}^2 \beta\xi$ at $x = 0$, then $\lim_{\xi \to \pm\infty} A = 0$ for any $\tau > 0$. If there is a shelf created behind the solitary wave, it can never be extended to infinity in finite τ.

4. FISSION OF SOLITONS

We consider an open channel with two uniform sections S_0 for $x < 0$ and S_1 for $x > x_1$ connected by a transition section for $0 < x < x_1$, and assume that the transitions at $x = 0$ and $x = x_1$ are sufficiently smooth so that Theorem 3 holds, and that $x_1 > 0$ is sufficiently small. Let

$$\eta_1 = m(x)B, \quad m(x) = \exp - \int_0^x m_2(x')m_1^{-1}(x')dx' \ . \tag{4.1}$$

In terms of B, (3.9) becomes

$$B_x + P(x)BB_\xi + (1/6)Q(x)B_{\xi\xi\xi} = 0 \ , \tag{4.2}$$

where

$$P(x) = m_3(x)m(x)m_1^{-1}(x), \quad (1/6)Q(x) = m_4(x)m_1^{-1}(x) \ . \tag{4.3}$$

In S_1 and S_2, P and Q are constants and we may transform (4.2) to the standard form of the K-dV equation with constant coefficients except the change of a sign (Muira et al., 1974). Let

$$U = PB/Q, \quad \zeta = Qx/6 \ , \tag{4.4}$$

and (4.2), in terms of U and ζ, becomes

$$U_\zeta + 6UU_\xi + U_{\xi\xi\xi} = 0 \ . \tag{4.5}$$

Suppose we envisage a solitary wave moving from $x = -\infty$ toward $x = +\infty$ as a progressive solution of (4.2) in S_1, and would like to find the condition under which fission of solitons will take place after the solitary wave moves into S_2.

A progressive wave solution of (4.5) is given by

$$U = (c/2)\mathrm{sech}^2[(c)^{1/2}(\xi - c\zeta)/2] \ . \tag{4.6}$$

In othr words, if we prescribe U at $\zeta = 0$ in the form of (4.6), U will be a solution of (4.5) for all ζ. Assume

$$B = B_0 \ \mathrm{sech}^2\alpha\xi \quad \text{at} \quad x = 0 \ . \tag{4.7}$$

From (4.1), we have

$$\eta_1 = B_0 m(0)\mathrm{sech}^2\alpha\xi \quad \text{at} \quad x = 0 \ ,$$

where $m(x) = m(0)$ for $x < 0$. Then (4.4) implies

$$U = [P(0)B_0/Q(0)] \operatorname{sech}^2 \alpha\xi \quad \text{at} \quad \zeta = 0 , \tag{4.8}$$

where $P(x) = P(0)$, $Q(x) = Q(0)$ for $x < 0$. In comparing (4.8) with (4.6), A_0 and α must satisfy the following conditions

$$P(0)B_0/Q_0 = c/2, \qquad c^{1/2}/2 = \alpha ,$$

so that (4.7) is the initial condition for a progressive solution of (4.5). Hence it follows that

$$P(0)B_0/[2Q(0)\alpha^2] = 1 . \tag{4.9}$$

After the solitary wave moves into S_2, we neglect the effect of the transition section. With (4.4) given by

$$U = P(x_1)B/Q(x_1), \quad \zeta = Q(x_1)x/6 , \tag{4.10}$$

where $P9x) = P(x_1)$; $Q(x) = Q(x_1)$ for $x > x_1$, (4.5) may be solved by the inverse scattering method, and the corresponding eigenvalue problem is posed by the equation

$$d^2\psi/d\xi^2 + [U(0,\xi) + \lambda]\psi = 0, \quad -\infty < \xi < \infty, \tag{4.11}$$

where by (4.4) and (4.7),

$$U(0,\xi) = [P(x_1)B_0/Q(x_1)]\operatorname{sech}^2\alpha\xi . \tag{4.12}$$

From the known results (Tappert and Zabusky 1971; Johnson 1973; Muira et al. 1974), U consists of n solitons for large ζ if

$$P(x_1)B_0/[Q(x_1)\alpha^2] = n(n + 1), \quad n = 1,2,\ldots \tag{4.13}$$

$$\lambda m/\alpha^2 = -m^2, \qquad m = 1,2,\ldots,n .$$

and the amplitude of the m^{th} soliton is given by

$$U_m = -2\lambda_m, \qquad m = 1,2,\ldots,n .$$

We may eliminate B_0/α^2 from (4.9) and (4.13) to obtain

$$\frac{P(x_1)Q(0)}{P(0)Q(x_1)} = n(n + 1)/2 \qquad n = 1,2,\ldots , \tag{4.14}$$

which yields a relationship among $P(0)$, $P(x_1)$, $Q(0)$, $Q(x_1)$ for the fission of n solitons to take place. For rectangular and triangular channels, $P(x)$ and $Q(x)$ can be explicitly expressed in terms of channel width $b(x)$ and depth $d(x)$, and (4.15) assumes rather simple forms.

(1) Rectangular channel.

Let $d(x)$, $b(x)$ assume constant values d_0, b_0 in S_0 and d_1, b_1 in S_1. It is found that

$$m(x) = b^{-1/2}(x)d^{-1/4}(x), \quad P(x) = \left(\frac{3}{2}\right)d^{-1/4}(x)b^{-1/2}(x), \quad \left(\frac{1}{6}\right)Q(x) = \left(\frac{1}{6}\right)d^{-1/2} .$$

(4.14) now becomes

$$(d_1/d_0)^{-9/4}(b_1/b_0)^{-1/2} = n(n+1)/2, \quad n = 1,2,\dots . \tag{4.15}$$

If $b_1 = b_0$, (4.15) reduces to the criterion obtained by Tappert and Zabusky (1971), and Johnson (1973).

(2) Symmetric triangular channel.

Let the two sides of the channel be defined by $z = \pm\mu(x)y - d(x)$, where $\mu(x) = 2d(x)/b(x)$ and let $d(x)$, $b(x)$ assume constant values d_0, b_0 and d_1, b_1 in S_0 and S_1 respectively. Then (4.14) reduces to (Zhong and Shen, 1982)

$$[(\gamma_0^2 + 12)/(\gamma_1^2 + 12)](\gamma_0/\gamma_1)^{-1/2}(d_1/d_0)^{-11/4} = n(n+1)/2, \quad n = 1,2,\dots ,$$

where $\gamma_i = b_i/d_i$, $i = 0,1$.

In solving the K-dV equation (4.5) with constant coefficients by the inverse scattering method, we tacitly assumed that the initial condition used in (4.11) was given by (4.12) and the effect of the transition section upon the evolution of the solitary wave was completely neglected. However, if x_1 is sufficiently small, then by the continuity of A and its continuous dependence on A_0 as obtained in Theorem 3, the approximation made here is justified. Let $\eta_1(x_1,\xi)$ be the solution of (3.9) at $x = x_1$ with the initial condition $\eta_1 = \eta_1(0,\xi) \in H_s$, $s > 2$ and $\eta_1^*(x,\xi)$ and $\eta_1(x,\xi)$ are respectively the solutions of (3.9) for $x > x_1$ with initial conditions $\eta_1^*(x_1,\xi) = \eta_1(0,\xi)$, $\eta_1 = \eta_1(x_1,\xi)$. Assume that the coefficients in (3.3) are sufficiently smooth. We state the result as

Theorem 4.

For any given $X > 0$ and $\varepsilon > 0$, there exists $\delta > 0$ such that

$$\|\eta_1^*(x,\xi) - \eta_1(x,\xi)\|_s < \varepsilon \quad \text{for} \quad x_1 < \delta, \quad x \in [0,X], \quad \text{and} \quad s > 2 .$$

We note that the theorem cannot tell what happens as $x \to \infty$. This concerns the global stability of a solution of the K-dV equation, which is an interesting problem for further study.

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On the basis of some approximate equations derived for nonlinear water waves in a channel of variable cross section, several problems of interest, which have attracted much attention recently, will be taken up in this report. First we shall consider the breaking of an acceleration wave moving toward a shoreline in a general channel. Next the so-called infinite mass dilemma, which arises in the study of the development of a solitary wave in a channel of variable cross section, will be resolved. Finally we shall use an approximation method to study the fission of solitons in a general channel and justify it.

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