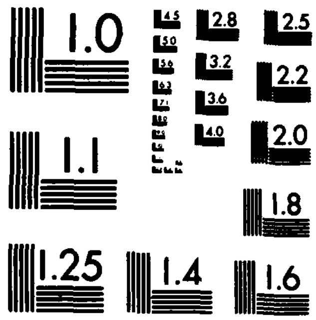
AD-A128 509 ON A PROBLEM CONCERNING SPACINGS(U) NORTH CAROLINA UNIV 1/1
AT CHAPEL HILL CENTER FOR STOCHASTIC PRECESSES S CHENG
FEB 83 TR-27 AFOSR-TR-83-0420 F49620-82-C-0009
UNCLASSIFIED F/G 12/1 NL

END
DATE
FILMED
DTIC



MICROCOPY RESOLUTION TEST CHART
NATIONAL BUREAU OF STANDARDS-1963-A

CENTER FOR STOCHASTIC PROCESSES

Department of Statistics
University of North Carolina
Chapel Hill, North Carolina

ON A PROBLEM CONCERNING SPACINGS

Shihong Cheng

TECHNICAL REPORT #27

February 1983

UNCLASSIFIED

SECURITY CLASSIFICATION OF THIS PAGE (When Data Entered)

REPORT DOCUMENTATION PAGE

READ INSTRUCTIONS BEFORE COMPLETING FORM

1. REPORT NUMBER AFOSR-TR- 83-0420 / 2. GOVT ACCESSION NO. A128509	3. RECIPIENT'S CATALOG NUMBER
4. TITLE (and Subtitle) ON A PROBLEM CONCERNING SPACINGS	5. TYPE OF REPORT & PERIOD COVERED TECHNICAL
	6. PERFORMING ORG. REPORT NUMBER TR #27
7. AUTHOR(s) Shihong Cheng	8. CONTRACT OR GRANT NUMBER(s) F49620-82-C-0009
9. PERFORMING ORGANIZATION NAME AND ADDRESS Department of Statistics, University of North Carolina, Chapel Hill NC 27514	10. PROGRAM ELEMENT, PROJECT, TASK AREA & WORK UNIT NUMBERS PE61102F; 2304/A5
11. CONTROLLING OFFICE NAME AND ADDRESS Mathematical & Information Sciences Directorate, Air Force Office of Scientific Research, Bolling AFB DC 20332	12. REPORT DATE FEB 83
	13. NUMBER OF PAGES 12
14. MONITORING AGENCY NAME & ADDRESS(if different from Controlling Office)	15. SECURITY CLASS. (of this report) UNCLASSIFIED
	15a. DECLASSIFICATION/DOWNGRADING SCHEDULE

16. DISTRIBUTION STATEMENT (of this Report)

Approved for public release; distribution unlimited.

17. DISTRIBUTION STATEMENT (of the abstract entered in Block 20, if different from Report)

18. SUPPLEMENTARY NOTES

19. KEY WORDS (Continue on reverse side if necessary and identify by block number)

Spacings; exact distribution; limiting distribution; Fibonacci distribution; almost sure convergence.

20. ABSTRACT (Continue on reverse side if necessary and identify by block number)

SEE REVERSE

DD FORM 1 JAN 73 1473

UNCLASSIFIED

SECURITY CLASSIFICATION OF THIS PAGE (When Data Entered)

UNCLASSIFIED

SECURITY CLASSIFICATION OF THIS PAGE(When Data Entered)

ITEM #20, CONTINUED:

Abstract: Let $\{U_n\}$ be an i.i.d. sequence uniformly distributed on $[0,1]$ and $U_1^{(n-1)} \le \ldots \le U_{n-1}^{(n-1)}$ be the order statistics of $U_1, \ldots, U_{n-1}$. Then $S_i^{(n)} = U_i^{(n-1)} - U_{i-1}^{(n-1)}$, $i=1, \ldots, n$ (Define $U_0^{(n-1)} = 0$, $U_n^{(n-1)} = 1$) are called the spacings divided by $U_1, \ldots, U_{n-1}$. Denote $M_n = \max_{1 \le i \le n-1} S_i^{(n)} \wedge S_{i+1}^{(n)}$. The exact and limiting distribution of M_n is determined. It is also proved that

$$\lim_n 2nM_n/\log n = 1 \quad \text{a.s.}$$

which refines a result of Chow, Geman and Wu.

UNCLASSIFIED

SECURITY CLASSIFICATION OF THIS PAGE(When Data Entered)

ON A PROBLEM CONCERNING SPACINGS

Shihong Cheng
Peking University
and
University of North Carolina

Abstract: Let $\{U_n\}$ be an i.i.d. sequence uniformly distributed on [0,1] and $U_1^{(n-1)} \le \ldots \le U_{n-1}^{(n-1)}$ be the order statistics of $U_1, \ldots, U_{n-1}$. Then $S_i^{(n)} = U_i^{(n-1)} - U_{i-1}^{(n-1)}$, $i=1,\ldots, n$ (Define $U_0^{(n-1)} = 0$, $U_n^{(n-1)} = 1$) are called the spacings divided by $U_1, \ldots, U_{n-1}$. Denote $M_n = \max_{1\le i\le n-1} S_i^{(n)} \wedge S_{i+1}^{(n)}$. The exact and limiting distribution of M_n is determined. It is also proved that

$$\lim_n 2nM_n/\log n = 1 \quad \text{a.s.}$$

which refines a result of Chow, Geman and Wu.

Keywords: Spacings, exact distribution, limiting distribution, Fibonacci distribution, almost sure convergence.

1. Introduction

Let $\{U_n, n=1,2,\ldots\}$ be an i.i.d. sequence uniformly distributed on $[0,1]$, and $U_1^{(n)} \le \ldots \le U_n^{(n)}$ be the order statistics of $U_1,\ldots, U_n$. The random variables $S_i^{(n+1)} = U_i^{(n)} - U_{i-1}^{(n)}$, $i=1,\ldots, n+1$ are called the spacings divided by $U_1,\ldots, U_n$, where $U_0^{(n)} \triangleq 0$, $U_{n+1}^{(n)} \triangleq 1$. The maximum of spacings plays an important part in nonparametric problems. Its exact and asymptotic behavior has been studied by many authors (See Darling [4], Pyke [8], Slud [9], Devroye [5] and so on). Write $W_i^{(n+1)} = S_i^{(n+1)} \wedge S_{i+1}^{(n+1)}$, $i=1,\ldots, n$. The behavior of $M_{n+1} = \max\limits_{1\le i\le n} W_i^{(n+1)}$ is important in cross-validated kernel density estimation. (See Chow, Geman and Wu [3] and Marron [7]). In this paper we give the exact distribution of M_n in section 2. In section 3 we discuss the behavior of a certain distribution function, which will be called the Fibonacci distribution function. Chow, Geman and Wu [3] have shown that there exists a constant C such that

$$(1.1) \quad P(nM_n/\log n \le C \ \text{i.o.}) = 0 ,$$

where "i.o." means infinitely often. In section 4, we will refine (1.1) by showing that

$$(1.2) \quad P(\lim_n 2nM_n/\log n = 1) = 1 .$$

The limiting distribution of M_n is also discussed in this section.

2. The exact distribution of M_n.

Let $Y_1,\ldots, Y_n$ be n random variables whose joint distribution function is $F(y_1,\ldots, y_n)$. We call $Y_1,\ldots, Y_n$ exchangeable random variables if $F(y_{i_1},\ldots, y_{i_n}) = F(y_1,\ldots, y_n)$ for any permutation $\{i_1,\ldots, i_n\}$ of $\{1,\ldots, n\}$. Given $y\in R$, let

$$A_{j_1\cdots j_k} = \{Y_j>y,\ j\in\{j_1,\ldots, j_k\};\ Y_j\le y,\ j\bar{\in}\{j_1,\ldots, j_k\}\},\ 1\le j_1<\ldots<j_k\le n .$$

If $Y_1,\ldots, Y_n$ are exchangeable, the probabilities of the events $A_{j_1\ldots j_k}$ will be the same. Denote

$$(2.1)\quad F^{(k)}(y) = P(A_{j_1\ldots j_k})\ ,\ 1\le j_1<\ldots<j_k\le n$$

and $F^{(0)}(y) = F(y_1,\ldots, y_n)$.

Lemma 2.1

Suppose that $Y_1,\ldots, Y_n$ are exchangeable. Then

$$(2.2)\quad P(\max_{1\le i\le n-1}(Y_i\wedge Y_{i+1})\le y) = \sum_{k=0}^{[(n+1)/2]}\binom{n-k+1}{k} F^{(k)}(y)\ ,$$

where $F^{(k)}(y)$ is defined by (2.1) and $[x]$ is the integer part of x.

Proof. The event $\{\max_{1\le i\le n-1}(Y_i\wedge Y_{i+1})\le y\}$ means that there is no index i such that $\{Y_i>y\}$ and $\{Y_{i+1}>y\}$ happen simultaneously. Hence this event is the union of E_k, $k=0,1,\ldots, [(n+1)/2]$, where E_k is the event "there are k integers $1\le j_1<\ldots<j_k\le n$ which do not contain two consecutive integers, such that the event $A_{j_1\ldots j_k}$ happens." We obtain

$$P(\max_{1\le i\le n-1}(Y_i\wedge Y_{i+1})\le y) = \sum_{k=0}^{[(n+1)/2]} P(E_k)$$

since $\{E_k\}$ are disjoint events. Since $Y_1,\ldots, Y_n$ are exchangeable, it follows that

$$P(E_k) = \binom{n-k+1}{k} F^{(k)}(y)\ ,$$

where $\binom{n-k+1}{k}$ is the number of k-element subsets that can be selected from the set $\{1,\ldots, n\}$ and that do not contain two consecutive integers (see [1], Chapter 3).

We first find the exact distribution of M_n. Define

$$(x)_+ = \begin{cases} x & x>0 \\ 0 & x\le 0 \end{cases}\ .$$

We have

Theorem 2.2

$$(2.3)\quad P(M_n\le x) = \sum_{k=0}^{[(n+1)/2]}\binom{n-k+1}{k}\sum_{t=0}^{n-k}(-1)^t\binom{n-k}{t}\{[1-(k+t)x]_+\}^{n-1}$$

Proof. It is known that

$$P(S_1^{(n)}>x_1,\ldots,\ S_n^{(n)}>x_n) = [(1-\sum_{i=1}^{n} x_i)_+]^{n-1} ,$$

where $x_1,\ldots,\ x_n$ are nonnegative numbers (see Devroye [5]). Hence the spacings $S_1^{(n)},\ldots,\ S_n^{(n)}$ are exchangeable, and Lemma 2.1 can be used in this case. Notice that

$$P(S_1^{(n)}>x,\ldots,\ S_k^{(n)}>x,\ S_{k+1}^{(n)}\le x,\ldots,\ S_n^{(n)}\le x)$$

$$= P(S_1^{(n)}>x,\ldots,\ S_k^{(n)}>x) - \sum_{t=1}^{n-k} (-1)^{t-1} \sum_{k+1\le j_1<\ldots<j_t\le n} P(S_1^{(n)}>x,\ldots,\ S_k^{(n)}>x,\ S_{j_1}^{(n)}>x,\ldots,S_{j_t}^{(n)}>$$

$$= \sum_{t=0}^{n-k} (-1)^t \binom{n-k}{t} \{[1-(k+t)x]_+\}^{n-1} ,$$

so that the theorem is proved.

3. The Fibonacci distribution function.

Let $X_n \sim G(x)$, $n=1,2,\ldots$ be an i.i.d. sequence and $Z_n = \max_{1\le i\le n-1} (X_i \wedge X_{i+1})$. From (2.2) we have

Lemma 3.1

$$(3.1)\quad P(Z_n\le x) = \sum_{k=0}^{[(n+1)/2]} \binom{n-k+1}{k} G^{n-k}(x)[1-G(x)]^k .$$

Now define the Fibonacci distribution function by

$$(3.2)\quad F_n^*(x) = \begin{cases} 0 & x<0 \\ \sum_{k=0}^{[(n+1)/2]} \binom{n-k+1}{k}\, x^{n-k}(1-x)^k & 0\le x<1 . \\ 1 & x\ge 1 \end{cases}$$

The name Fibonacci was chosen because

$$F_0 \overset{\Delta}{=} 0 \quad , \quad F_1 = F_2 = 1 \quad , \quad F_{n+2} = \sum_{k=0}^{[(n+1)/2]} \binom{n-k+1}{k} \quad , \quad n=1,2,\ldots$$

is the sequence of Fibonacci numbers. By using the generating function method for finding the values of Fibonacci numbers, the sum $g_n \overset{\Delta}{=} \sum_{k=0}^{[n/2]} \binom{n-k}{k} \alpha^k$ can be found as follows.

Lemma 3.2

For any n=0,1,2,...,

(3.3) $g_n = \{[1+(1+2\alpha)/\beta][(1+\beta)/2]^n + [1-(1+2\alpha)/\beta)][(1-\beta)/2]^n\}/2$,

where $\beta = (1+4\alpha)^{1/2}$.

Proof. For convenience, let $\binom{n}{k} = 0$ if $k<0$ or $n<k$. Then it follows that

$$\binom{n-k+1}{k} = \binom{n-k}{k-1} + \binom{n-k}{k} \quad , \quad k=0,1,\ldots, [(n+1)/2] ,$$

and therefore that

$$g_{n+1} = \alpha g_{n-1} + g_n \quad , \quad n=1,2,\ldots .$$

Hence we obtain

$$P(x) - [xP(x)+\alpha x^2 P(x)] = 1 + \alpha x \quad , \text{ i.e.}$$

$$P(x) = (1+\alpha x)/(1-x-\alpha x^2) ,$$

where $P(x) \overset{\Delta}{=} \sum_{n=0}^{\infty} g_n x^n$ is the *generating function* of the sequence $\{g_n\}$. Expand $P(x) = (1+\alpha x)/(1-x-\alpha x^2)$ into a power series:

$$P(x) = \{[1+(1+2\alpha)/\beta]/[1-x(1+\beta)/2] + [1-(1+2\alpha)/\beta]/[1-x(1-\beta)/2]\}/2$$

$$= \sum_{n=0}^{\infty} \{[1+(1+2\alpha)/\beta][(1+\beta)/2]^n + [1-(1+2\alpha)/\beta][(1-\beta)/2]^n\}x^n/2 .$$

Comparing the above series with the definition of $P(x)$, we have (3.3), to complete the proof.

Now the Fibonacci d.f. can be represented as

(3.4) $F_n^*(x) = x^n g_{n+1}((1-x)/x) \quad , \quad 0<x<1$,

where $g_{n+1}((1-x)/x)$ is the value of g_{n+1} at $\alpha = (1-x)/x$. We discuss the asymptotic behavior of the Fibonacci distribution function as follows.

Theorem 3.3

If $x_n \in (0,1)$, n=1,2,... is a sequence such that, as $n \to \infty$, (3.5) $ny_n^3 \to 0$,

where $y_n = 1 - x_n$, then we have

(3.6) $F_n^*(X_n) = [1+0(y_n)+0(ny_n^3)]\exp[-ny_n^2]$.

Proof. Write

(3.7) $F_n^*(x_n) = u_n(x_n)[1-v_n(x_n)/u_n(x_n)]/(2x_n)$,

where

$$u_n(x_n) = [1+(1+2y_n/x_n)/(1+4y_n/x_n)^{1/2}]\{x_n[1+(1+4y_n/x_n)^{1/2}]/2\}^{n+1}$$

$$v_n(x_n) = [1-(1+2y_n/x_n)/(1+4y_n/x_n)^{1/2}]\{x_n[1-(1+4y_n/x_n)^{1/2}]/2\}^{n+1} .$$

Noticing that

$$(1+2y_n/x_n)/(1+4y_n/x_n)^{1/2}$$

$$= (1+2y_n/x_n)(1-2y_n/x_n+0(y_n^2)) = 1 + 0(y_n^2) ,$$

we know $v_n(x_n)/u_n(X) = 0(y_n^2)$, and therefore

$$1 - v_n(x_n)/u_n(x_n) = 1 + 0(y_n^2) .$$

Hence to prove (3.6), we need only to show that

$$u_n(X_n)/(2x_n) = (1+0(y_n))(1+0(ny_n^3))\exp(-ny_n^2)$$

(See (3.7)). This follows from

$$u_n(x_n)/(2x_n) = (1+0(y_n^2))\{x_n[1+(1+4y_n/x_n)^{1/2}]/2\}^n[1+(1+4y_n/x_n)^{1/2}]/2$$

$$= (1+0(y_n))\{x_n[1+(1+2y_n/x_n-2y_n^2/x_n+0(y_n^3))]/2\}^n$$

$$= (1+0(y_n))(1-y_n^2+0(y_n^3))^n$$

$$= (1+0(y_n))\exp\{-ny_n^2+0(ny_n^3)\}$$

$$= (1+0(y_n))(1+0(ny_n^3))\exp\{-ny_n^2\} ,$$

completing the proof of theorem 3.3.

4. The asymptotic behavior of M_n

Unless otherwise stated, X_n, $n=1,2,\ldots$ will be an i.i.d. exponential sequence in this section.

Lemma 4.1

Let $T_n = \sum_{i=1}^n X_i$. The spacings $(S_1^{(n)},\ldots, S_n^{(n)})$ are distributed as $(X_1/T_n,\ldots, X_n/T_n)$.

Proof. See Pyke [8].

Lemma 4.2

For all $x>0$, the following inequalities hold:

(4.1) $P(T_n/n-1>x) \le \exp[-nx^2(1-x)/2]$

(4.2) $P(T_n/n-1<-x) \le \exp(-nx^2/2)$.

Proof. See Devroye [5].

Lemma 4.3

Let $\{\xi_n\}$, $\{\eta_n\}$ be r.v. sequences. If there exist $a_n>0$, b_n such that

$$P(\xi_n \le a_n x+b_n) \overset{\subsetneq}{\to} \Psi(x)$$

$$\eta_n/n \overset{P}{\to} 1 \quad \text{and} \quad (\eta_n/n-1)b_n/a_n \overset{P}{\to} 0 ,$$

then

$$P(\xi_n/\eta_n \le (a_n x+b_n)/n) \overset{\subsetneq}{\to} \Psi(x) .$$

Proof. The sequences $\{(\xi_n-b_n)/a_n\}$ and $\{(\xi_n-b_n\eta_n/n)/a_n\}$ have the same limiting distribution since $(\eta_n/n-1)b_n/a_n \to 0$ in probability. The sequences $\{(\xi_n-b_n\eta_n/n)/a_n\}$ and $\{(n\xi_n/\eta_n-b_n)/a_n\}$ have the same limiting distribution since $\eta_n/n \to 1$ in probability. Hence lemma 4.3 is proved.

Theorem 4.4

For any $x\in R$

(4.3) $\lim_{n} P(M_n \le x/2n + \log n/2n) = \exp[-\exp(-x)]$.

Proof. By using lemma 4.1, (4.3) is equivalent to

(4.4) $\lim_{n} P(Z_n/T_n \le x/2n + \log n/2n) = \exp[-\exp(-x)]$.

By using lemma 4.2, it can be shown that

$$(T_n/n-1)\log n \xrightarrow{P} 0 \ .$$

Hence by using lemma 4.3, (4.4) holds if we show that

(4.5) $\lim_{n} P(Z_n \le (x+\log n)/2) = \exp[-\exp(-x)]$.

From (3.1) and (3.2), we have

$$P(Z_n \le (x+\log n/2) = F_n^*(x_n) \ ,$$

where $x_n = 1 - \exp[-(x+\log n)/2]$. It is easy to check that $ny_n^3 \to 0$, $y_n \to 0$, $ny_n^2 \to \exp(-x)$.
Hence (4.5) follows from (3.6), completing the proof of (4.3).

Remark 1. Using the methods of section 3, we can show that Gnedenko's theorems (See [6]) for the i.i.d. case will still be valid for the sequence $\{X_i \wedge X_{i+1}\}$ where $\{X_i\}$ is i.i.d. but not necessarily exponential. The same statement may also follow from Watson [10].

Remark 2. It is easy to show, from theorem 4.4, that

(4.6) $2nM_n/\log n \xrightarrow{P} 1$.

Now we turn our attention to proving (1.2). It is easy to see that (1.2) is equivalent to the following equations:

(4.7) $P(\limsup_{n} 2nM_n/\log n \le 1) = 1$,

(4.8) $P(\liminf_{n} 2nM_n/\log n \ge 1) = 1$.

But (4.7) and (4.8) are equivalent to the statement that for any $\delta > 0$,

(4.9) $P(2nM_n/\log n > 1+\delta \text{ i.o.}) = 0$, (4.10) $P(2nM_n/\log n \le 1-\delta \quad \text{i.o.}) = 0$.

Lemma 4.5

Equation (4.10) holds.

Proof. Let $A_n = \{2nM_n/\log n \le 1-\delta\}$, n=1,2,... . By the Borel-Cantelli lemma, to prove (4.10), it is sufficient to show that

$$(4.10) \quad \sum_{n=1}^{\infty} P(A_n) < \infty .$$

Using lemma 3.1 and 3.2, we have

$$\begin{aligned}(4.11) \quad P(A_n) &= P(Z_n \le (1-\delta)T_n \log n/(2n)) \\ &\le P(Z_n \le (1-\delta)(1+\varepsilon_n)\log n/2) + P(T_n/n > 1+\varepsilon_n) \\ &\le P(Z_n \le (1-\delta/2)\log n/2) + \exp(-n\varepsilon_n^2/4),\end{aligned}$$

where $\varepsilon_n = 2n^{-(1-\delta/2)/2}$ and therefore $(1-\delta)(1+\varepsilon_n) \le 1 - \delta/2$, $1-\varepsilon_n > 1/2$ if n is sufficiently large. By letting $x_n = 1 - \exp[-(1-\delta/2)\log n/2]$, it is easy to check that $ny_n^3 \to 0$, $ny_n^2 = n^{\delta/2}$, and therefore theorem 3.3 can be used for this case. Hence

$$\begin{aligned}&P(Z_n \le (1-\delta/2)\log n/2) \\ &= (1+0(1))\exp(-n^{\delta/2}) .\end{aligned}$$

Now from (4.11), we have

$$P(A_n) \le C\exp(-n^{\delta/2})$$

where C is a constant. Then (4.10) follows since $\sum_{n=1}^{\infty} \exp(-n^{\varepsilon})$ converges for any $\varepsilon>0$, completing the proof of lemma 4.5.

To prove (4.9) we need the following lemma. which is stronger than the Borel-Cantelli lemma.

Lemma 4.6

Let $\{A_n\}$ be a sequence of events with $\lim_n P(A_n) = 0$. If either $\sum_{n=1}^{\infty} P(A_n^c A_{n+1}) < \infty$ or $\sum_{n=1}^{\infty} P(A_n A_{n+1}^c) < \infty$, then $P(A_n \text{ i.o.}) = 0$.

Proof. See Barndorff-Neilsen [2].

Lemma 4.7

Equation (4.9) holds.

Proof. Let $u_n = (1+\delta)\log n/2$, $\tilde{u}_n = (1+\delta)\log n/(2n)$ and $A_n = \{M_n > \tilde{u}_n\}$. By using lemma 4.6, to prove (4.9) we need only to show

$$(4.12) \quad \sum_{n=1}^{\infty} P(A_n A_{n+1}^c) < \infty$$

Writing $E_i^{(n)}$ for the set $\{S_i^{(n)} \wedge S_{i+1}^{(n)} > \tilde{u}_n,\ S_j^{(n)} \wedge S_{j+1}^{(n)} \le \tilde{u}_{n+1},\ j \neq i\}$ and noticing that $\{\tilde{u}_n\}$ is nonincreasing, we have

$$P(A_n A_{n+1}^c) = P(M_n > \tilde{u}_n,\ M_{n+1} \le \tilde{u}_{n+1})$$

$$= \sum_{i=1}^{n-1} P(E_i^{(n)} \cap \{U_i^{(n-1)} - \tilde{u}_{n+1} \le U_n \le U_i^{(n-1)} + \tilde{u}_{n+1}\})$$

$$= \sum_{i=1}^{n-1} \int_{E_i^{(n)}} P(U_i^{(n-1)} - \tilde{u}_{n+1} \le U_n \le U_i^{(n-1)} + \tilde{u}_{n+1} \mid U_1, \ldots, U_{n-1})\,dP$$

$$\le 2\tilde{u}_n \sum_{i=1}^{n-1} P(E_i^{(n)})$$

$$\le 2\tilde{u}_n P(M_n > \tilde{u}_n)$$

$$= 2\tilde{u}_n P(Z_n > u_n T_n/n)$$

$$\le 2\tilde{u}_n P(Z_n > (1-\varepsilon_n)u_n) + 2\tilde{u}_n P(T_n/n < 1-\varepsilon_n)$$

for any $\varepsilon_n > 0$. Furthermore, using lemma 4.2, we obtain that

$$\tilde{u}_n P(T_n/n < 1-\varepsilon_n) \le (1+\delta)(\log n)\exp(-n\varepsilon_n^2/2)/n\ ,$$

and therefore can choose $\varepsilon_n \to 0$ such that

$$\sum_{n=1} \tilde{u}_n P(T_n/n < 1-\varepsilon_n) < \infty\ .$$

Hence to show (4.12), it is sufficient to prove

$$(4.13) \quad \sum_{n=1}^{\infty} \tilde{u}_n P(Z_n > (1+\delta/2)\log n/2) < \infty$$

since $(1-\varepsilon_n)(1+\delta) > 1 - \delta/2$ and therefore

$$P(Z_n < (1-\varepsilon_n)u_n) \le P(Z_n > (1+\delta/2)\log n/2)$$

for large enough n. Let $x_n = 1 - \exp[-(1+\delta/2)\log n/2] = 1 - n^{-(1+\delta/2)/2}$, $y_n = 1 - x_n$. Then we have $ny_n^3 = o(y_n)$ and therefore (3.6) can be rewritten as

$$F_n^*(x_n) = [1+O(y_n)]\exp(-ny_n^2) \ .$$

Now (4.13) follows from the fact that

$$P(Z_n > (1+\delta/2)\log n/2) = 1 - F_n^*(x_n)$$

$$\le C_1[1-\exp(-n^{-\delta/2})] + C_2 n^{-(1+\delta/2)/2}$$

$$\le C_3 n^{-\delta/2} + C_2 n^{-(1+\delta/2)/2}$$

(C_1, C_2, C_3 are constants) and the fact that the series $\sum_{n=1}^{\infty} \log n/n^{1+\varepsilon}$ converges for any $\varepsilon>0$. This completes the proof of lemma 4.7.

Combining lemma 4.5 and 4.1, we obtain

Theorem 4.8

Equation (1.2) holds.

Acknowledgement. This paper was completed during the author's visit to the University of North Carolina at Chapel Hill. The author is grateful to Dr. J.S. Marron for posing the question discussed in this paper. The author also thanks Dr. Marron and Professor M.R. Leadbetter for helpful discussions during the course of this research.

References

[1] Aigner, M. (1980). Combinatorial Theory. Springer-Verlag.

[2] Barndorff-Nielsen, O. (1961). On the rate of growth of the partial maxima of a sequence of independent identically distributed random variables. Math. Scand. 9 383-394.

[3] Chow, Y.S., Geman, S., and Wu, L.D. (1983). Consistent cross-validated density estimation. Ann. Statist. 11 (to appear).

[4] Darling, D.A. (1953). On a class of problems related to the random division of an interval. Ann. Math. Statist. 24 239-253.

[5] Devroye, L. (1981). Laws of the iterated logarithm for order statistics of uniform spacings. Ann. Prob. 9 860-867.

[6] Gnedenko, B. (1943). Sur la distribution limite de terme maximum d'un série aléatoire. Ann. Math. 44 423-453.

[7] Marron, J.S. (1983). An asymptotically efficient solution to the bandwidth problem of kernel density estimation. NC Inst. Stat. Mimeo Series #1518.

[8] Pyke, R. (1965). Spacings, J. Roy. Statist. Soc. 27 395-436.

[9] Slud, E. (1978). Entropy and maximal spacings for random partitions. Z. Wahrsch. Verw. Geb. 41 341-352.

[10] Watson, G.S. (1954). Extreme value in samples from m-dependent stationary stochastic processes. Ann. Math. Statist. 25 798-800.