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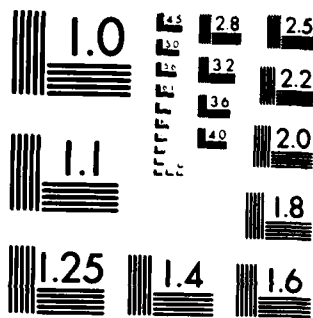
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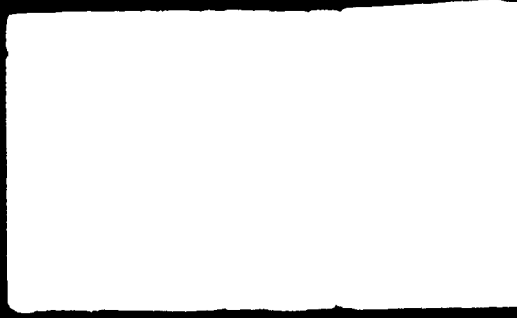
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By

A. S. Hedayat and D. Majumdar

Department of Mathematics, Statistics and Computer Science
University of Illinois, Chicago

March 1983

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REPORT DOCUMENTATION PAGE		READ INSTRUCTIONS BEFORE COMPLETING FORM
1. REPORT NUMBER AFOSR-TR- 83- 0462	2. GOVT ACCESSION NO. AD A129322	3. RECIPIENT'S CATALOG NUMBER
4. TITLE (and Subtitle) A COLLECTION OF A-OPTIMAL DESIGNS FOR CONTROL- TEST TREATMENT COMPARISONS, I		5. TYPE OF REPORT & PERIOD COVERED TECHNICAL
7. AUTHOR(s) A.S. Hedayat and D. Majumdar		6. PERFORMING ORG. REPORT NUMBER
9. PERFORMING ORGANIZATION NAME AND ADDRESS Department of Mathematics, Statistics, and Computer Science, University of Illinois, P.O. Box 4348, Chicago IL 60680		8. CONTRACT OR GRANT NUMBER(s) AFOSR-80-0170
11. CONTROLLING OFFICE NAME AND ADDRESS Mathematical & Information Sciences Directorate Air Force Office of Scientific Research Bolling AFB DC 20332		10. PROGRAM ELEMENT, PROJECT, TASK AREA & WORK UNIT NUMBERS PE 61102F; 2304/A5
14. MONITORING AGENCY NAME & ADDRESS (if different from Controlling Office)		12. REPORT DATE MAR 83
		13. NUMBER OF PAGES 33
		15. SECURITY CLASS. (of this report) UNCLASSIFIED
		15a. DECLASSIFICATION/DOWNGRADING SCHEDULE
16. DISTRIBUTION STATEMENT (of this Report) Approved for public release; distribution unlimited.		
17. DISTRIBUTION STATEMENT (of the abstract entered in Block 20, if different from Report)		
18. SUPPLEMENTARY NOTES		
19. KEY WORDS (Continue on reverse side if necessary and identify by block number) Control-test treatment comparisons; A-optimal designs; BTIB designs; Augmented BIB designs.		
20. ABSTRACT (Continue on reverse side if necessary and identify by block number) A-optimal designs for comparing v test treatments with a control in b blocks of size k each are considered. 111 such designs are given when the parameters are in the range: $2 \leq k \leq 8$, $k \leq v \leq 30$, $v \leq b \leq 50$.		

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ABSTRACT

A-optimal designs for comparing v test treatments with a control in b blocks of size k each are considered. 111 such designs are given when the parameters are in the range: $2 \leq k \leq 8$, $k \leq v \leq 30$, $v \leq b \leq 50$.

KEY WORDS: Control-test treatment comparisons; A-optimal designs; BTIB designs; Augmented BIB designs.



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1. INTRODUCTION

The problem of comparing several treatments with a control has wide applications. Whenever one examines the possibility of replacing an existing system by one of several recently developed systems, the central question is whether the gain is commensurate with the enormous expenditures that are generally associated with such a change. For example, we may want to consider replacing some equipment, a computer or some drugs. In all such cases the principal concern is whether the existing system (control) results in significantly inferior performance vis-a-vis some of the new ones. A change involving large financial expenditures may be justified only when the answer is truly in the affirmative. There are plenty of such examples in industrial, agricultural and biological investigations.

The theory of optimal designs, as developed by Kiefer and other researchers, has traditionally concentrated on developing good designs for drawing inferences on a set of mutually orthogonal treatment contrasts. Nothing much is known, however, for the case when the contrasts are not mutually orthogonal as in the case of comparing the control with other treatments. Till recently, there was little literature for this important problem of comparing treatments with a control. Some of the recent developments relevant to our study are Bechhofer and Tamhane (1981), Constantine (1983), Majumdar and Notz (1983) and Ture (1982). Other notable references include Bechhofer (1969), Bechhofer and Nocturne (1972), Bechhofer and Tamhane (1983),

Bechhofer and Turnbull (1971), Cox (1958), Dunnett (1955,1964), Notz and Tamhane (1983), Pesek (1974) and Robson (1961).

There is a practical need for a catalog of efficient designs for comparing two or more test treatments with a control under various models and settings. When there is no nuisance parameter in the model there is one general easy solution. This is indicated in Section 2. Cases in which there are one or more nuisance parameters in the model are very difficult to deal with. To avoid complications in design and its analysis later we should run, if possible, the experiment under a designed condition in which the number of nuisance parameters have been reduced to a minimum. Thus as a first step in this direction we need efficient designs when there is precisely one nuisance parameter in the model for each response. This falls in the category of block designs. In response to this later need this paper provides 111 efficient (formally defined as A-optimal designs in Section 2) block designs for cases in which the block size (k), the number of test treatments (v) and the number of blocks (b) are in the following ranges

$$2 \leq k \leq 8$$

$$2 \leq v \leq 10 \quad \text{and } v = 12, 13, 14, 15, 16, 21, 25$$

$$v \leq b \leq 50$$

After formally introducing the concept of A-optimal design for comparing a control with v test treatments in Section 2, we shall then explain the detailed description of our methods of construction. In Section 3, we shall give a table of our 111 A-optimal designs.

Sufficient description is provided for the construction of each design. In Section 4, we shall summarize our findings and points out some interesting observations.

2. A-OPTIMAL DESIGNS FOR COMPARING TEST TREATMENTS WITH A CONTROL

The objective of an experiment is to compare v treatments called test treatments with a special treatment called the control. Consider first the situation where all the experimental units are homogenous as far as the response to the treatments is concerned.

If y_{ij} denotes the j th observation related to treatment i , then we assume the model

$$y_{ij} = \mu + \tau_i + e_{ij} \quad (2.1)$$

$j = 1, \dots, n_i$; $i = 0, 1, \dots, v$, 0 denoting the control. In (2.1) μ denotes the general mean, τ_i the effect of treatment i and e_{ij} the random error which is assumed to have

$$E(e_{ij}) = 0, V(e_{ij}) = \sigma^2, \text{Cov}(e_{ij}, e_{i'j'}) = 0. \quad (2.2)$$

If we are given n experimental units, then the problem is to determine an optimal allocation of the $v+1$ treatments to these units. Since we are primarily interested in comparing treatments with a control, we should look at the contrasts

$$\tau_0 - \tau_1, \dots, \tau_0 - \tau_v.$$

If the best linear unbiased estimator of $\tau_0 - \tau_i$ is denoted by $\hat{\tau}_0 - \hat{\tau}_i$, then

$$V(\hat{\tau}_0 - \hat{\tau}_i) = \sigma^2 (1/n_0 + 1/n_i)$$

To determine the optimal allocation, one sensible criterion is to minimize $\sum_{i=1, \dots, v} v(\hat{\tau}_0 - \hat{\tau}_i)$. Thus we determine $n_0, n_1, \dots, n_v$ which minimizes

$$\sum_{i=1, \dots, v} (1/n_0 + 1/n_i)$$

subject to the restriction

$$n_0 + \sum_{i=1, \dots, v} n_i = n.$$

In other words, we minimize

$$v(n - \sum_{i=1, \dots, v} n_i)^{-1} + \sum_{i=1, \dots, v} n_i^{-1} \quad (2.3)$$

over positive integers $n_1, \dots, n_v$ satisfying

$$v \leq n_1 + \dots + n_v < n. \quad (2.4)$$

If the optimal solution to (2.3) and (2.4) is denoted by n_i^0 , then the optimal n_0 is

$$n_0^0 = n - \sum_{i=1, \dots, v} n_i^0.$$

In case $n = r \sqrt{v}(\sqrt{v} + 1)$, where r is an integer, one can show, using some calculus that the optimal allocation is given by

$$n_1^0 = n_2^0 = \dots = n_v^0 = r, n_0^0 = r \sqrt{v}. \quad (2.5)$$

The allocation (2.5) can also be shown to minimize $\text{Max}_{1 \leq i \leq v} v(\hat{\tau}_0 - \hat{\tau}_i)$, the maximum variance of the estimators of the contrasts $\tau_0 - \tau_1, \dots, \tau_0 - \tau_v$.

Now suppose that all the experimental units are not homogenous to start with, but can be grouped into b sets (called blocks) of k units each. We suppose

$$k < v + 1 \quad (2.6)$$

which means that we are in the incomplete block design situation since including the control, there are $v+1$ treatments. Here the model is

$$Y_{ijl} = \mu + \tau_i + \beta_j + e_{ijl} \quad (2.7)$$

$i = 0, 1, \dots, v$; $j = 1, \dots, b$; $l = 1, \dots, n_{ij}$; $n_{ij} = 0, 1, 2, \dots$. There is no observation Y_{ijl} if $n_{ij}=0$. β_j is the block effect of the j th block, and all the other symbols have meanings similar to those in equation (2.1). Let us denote

$$\psi = (\tau_0 - \tau_1, \dots, \tau_0 - \tau_v)'$$

the vector of control - test treatments contrasts. Our main objective is to allocate the treatments $0, 1, \dots, v$ to the blocks in a way that allows the best possible inference on ψ . In other words, if $\hat{\psi}$ is the best linear unbiased estimator, then we want to 'minimize' the variance-covariance matrix $V(\hat{\psi})$, in some sense. Among the three most standard methods of minimizing $V(\hat{\psi})$, viz., E-optimality which minimizes the maximum eigenvalue of $V(\hat{\psi})$; D-optimality, which minimizes the determinant of $V(\hat{\psi})$; and A-optimality which minimizes the trace of $V(\hat{\psi})$, A-optimality is the only statistically meaningful method in this situation. In this paper we shall concentrate on designs which are A-optimal, i.e., which minimize trace $V(\hat{\psi})$.

Though some design specialists have talked of block designs for control - test treatment comparisons, a systematic study have probably only begun since the paper by Bechhofer and Tamhane (1981). The reader is also referred to their paper for a survey of the literature. Bechhofer and Tamhane (1981) defined a class of designs, known as Balanced Test treatment Incomplete Block (BTIB) designs, and discussed some optimal properties of these designs for setting simultaneous confidence bounds for the elements of ψ . A BTIB design is an incomplete

block design in which each test treatment appears in the same block which the control the same number of times ($=\lambda_0$) and any pair of test treatments appear together in the same block the same number of times ($=\lambda_1$). Using the notation n_{ij} introduced along with (2.7) we may also define a BTIB design by the following relations.

$$\sum_{j=1, \dots, b} n_{0j} n_{ij} = \sum_{j=1, \dots, b} n_{0j} n_{1j} = \lambda_0 (\text{say}), \text{ for } i = 1, \dots, v$$

$$\sum_{j=1, \dots, b} n_{ij} n_{i'j} = \sum_{j=1, \dots, b} n_{1j} n_{2j} = \lambda_1 (\text{say}), \text{ for } i \neq i' = 1, \dots, v$$

Note that the usual BIB design is a BTIB design with $n_{ij}=0,1$ and $\lambda_0=\lambda_1$.

The problem of determining optimal designs was considered by Majumdar and Notz (1983). It was shown that if a BTIB design with the following properties exist, then it is A-optimal.

$$(i) \ n_{ij} = 0 \text{ or } 1, \ i = 1, \dots, v, \ j = 1, \dots, b$$

$$\sum_{j=1, \dots, b} n_{ij} = \sum_{j=1, \dots, b} n_{1j}, \ i = 1, \dots, v$$

$$(ii) \ n_{01} = \dots = n_{0M} = L + 1, \ n_{0M+1} = \dots = n_{0b} = L$$

where $a=L$ and $e=M$ minimizes

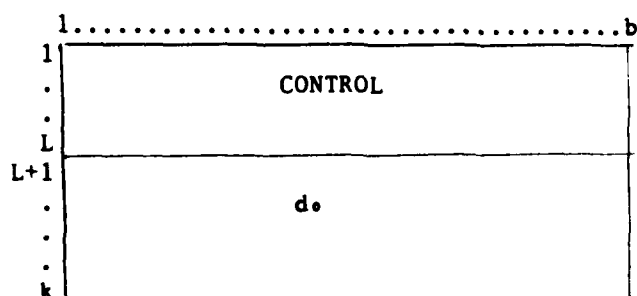
$$g(a, e) = (v-1)^2 [bvk(k-1) - (ba+e)(kv-v+k) + (ba^2 + 2ae+e)]^{-1} + [k(ba+e) - (ba^2 + 2ae+e)]^{-1} \quad (2.8)$$

among integers $a = 0, \dots, [k/2]$, $e = 0, \dots, b-1$, $[k/2]$ being the largest integer not greater than $k/2$. Note that for each v , b and k , there is obviously an A-optimal design, but this result covers only those cases for which a BTIB design with properties (i) and (ii) exist. In addition, it was also shown that this A-optimal design minimizes

$$\text{Max}_{1 \leq i \leq v} \hat{V}(\tau_0 - \tau_i).$$

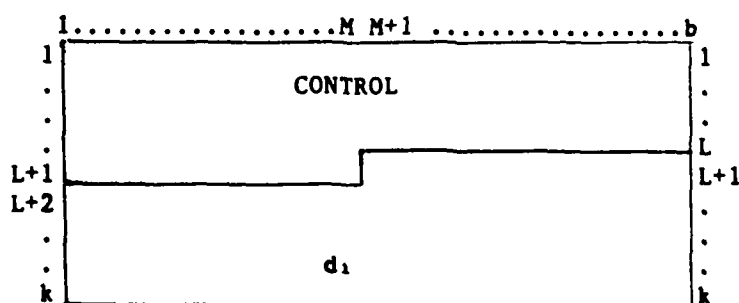
A BTIB design with properties (i) and (ii) can essentially be of only two types:

(1) Rectangular type.



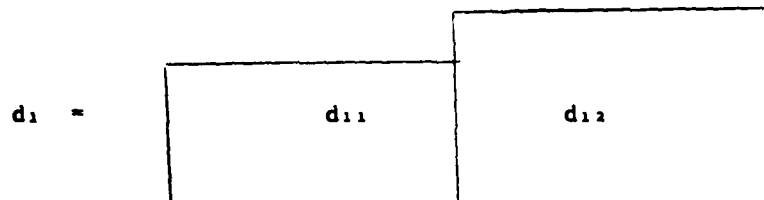
This arises when $M=0$. Here each block has L controls. If we consider columns as blocks, then the design looks as above, where d_0 is a binary design in b blocks of size $(k-L)$ each, in test treatments $1, \dots, v$. From (i) and (ii) it follows that d_0 is a BIB design. Here 'rectangular' refers to d_0 . A design of this type will be referred to as a R type, for brevity.

(2) Step type.



This arises when $M > 0$, d_1 is a design in b blocks in the test

treatments $1, 2, \dots, v$. M blocks in d_1 have $(k-L-1)$ units each while $b-M$ blocks have $k-L$ units each. 'Step' refers obviously to d_1 . One may further decompose $d_1 = d_{11} \cup d_{12}$, or pictorially,



d_{11} and d_{12} each being binary and proper. Using (i) and (ii), it is easily seen that the test treatments in d_{11} are equireplicated and so are in d_{12} . Moreover, if B_i denotes the set of blocks in d_i ($i=1,2$), then λ_1 of (2.7) can be expressed as

$$\lambda_1 = \sum_{j \in B_1} n_{ij} n_{i'j} + \sum_{j \in B_2} n_{ij} n_{i'j}, \quad (2.9)$$

the same for each i and i' with $i \neq i' = 1, \dots, v$. Also, λ_0 of (2.7) is a given by

$$\lambda_0 = (L+1) \sum_{j \in B_1} n_{ij} + L \sum_{j \in B_2} n_{ij} \quad (2.10)$$

an expression independent of $i = 1, \dots, v$. A design of the type will be referred to us as S type for brevity.

Let us define $r = \sum_{j \in d_1} n_{1j}$ and $\delta = \sum_{j \in d_{11}} n_{1j}$ in a S type design. It is easy to see that for the existence of a BTIB design satisfying (i) and (ii), a necessary condition is that there exists nonnegative integers r , δ and λ_2 for which the following conditions hold:

$$\begin{aligned}
 vr &= b(k - L) - M \\
 v\delta &= M(k - L - 1) \\
 (v - 1)\lambda_2 &= \delta(k - L - 2) + (r - \delta)(k - L - 1).
 \end{aligned}
 \tag{2.11}$$

When we are looking at an R type design, $M = 0$ and hence δ is always 0.

In search of an A-optimal design with parameters v , b and k , we first obtain L and M by minimizing $g(a, e)$ in equation (2.8). The next step is to verify the necessary conditions (2.11). If they are not satisfied then there is no BTIB design according to our specifications and the method fails to produce an A-optimal design. On the other hand, if the necessary conditions are satisfied then there is hope that a BTIB design exists, and this will be A-optimal for the set $\{v, b, k\}$ we started with. There is no known method which settles the question of the existence and eventual construction (if it exists) of a BTIB design for a general set of parameters. So usually one studies each case individually. However, sometimes the parameters $\{v, b, k, L, M\}$ allow a quick solution. For example, if $M=0$, then we are looking for an R type design for which d_0 is a BIB design. In this case, an appeal to some known result often settles the question. Usually, the BIB design may be obtained from tables listed in Fisher and Yates (1963), Rao (1961), Hall (1967) and others. It is interesting to note that the d_1 part of a S design is a design of index λ_2 as defined by Bose and Shrikhande (1959), with two different block sizes $k-L$ and $k-L-1$. In general, one can think of several ways of constructing an S design, but since they depend heavily on the parameters v , b , k , L , M , we shall discuss the step designs we come across individually in the next section.

Let us now give two examples of A-optimal designs, one each of R and S types:

Example 1 When $b=14$, $v=7$, $k=4$; $L=1$, $M=0$. Denoting test treatments by a, b, c, d, e, f, g and the control by -, the following R type design is A-optimal:

```

- - - - -
a b c d e f g a b c d e f g
b c d e f g a b c d e f g a
d e f f a b c d e f f a b c

```

Example 2 When $b=18$, $v=6$, $k=5$; $L=1$, $M=6$. Denoting test treatments by a, b, c, d, e, f and the control by -, the following S type design is optimal:

```

- - - - -
- - - - - a a a a a a a a b b b b
a a a b c c b b b b c d c c c c d
b d b d d e c e c d d e e d d d e e
c e f e f f d f e f e f f e f f f

```

It may be noticed in the tables in the next section that a majority of the A-optimal designs is R type with exactly one control in each block. Constantine (1983) showed that in the class of block design in b blocks of size k each and v test treatments, containing exactly one control in each block, the R type designs, which are BIB designs augmented by a control in each block is A-optimal. Even though many of these designs are A-optimal in the wider class of all block designs with

parameters v , b , k , there are instances where such designs fail to be A-optimal. Consider for example, $v=9$, $b=48$, and $k=7$. The design number 104 given in the next section is an A-optimal design. However, a BIB design with $v=9$, $b=48$, and $k=6$ augmented by a control in each block is not A-optimal since it was not given by our method which produces all A-optimal binary BTIB designs whenever it produces one. Another counterexample can be found in Constantine (1983).

Notz and Tamhane (1983) and Ture (1982) have considered the problem of constructing BTIB designs which are binary in test treatments. Ture (1982) also looks at designs which are 'approximately' A-optimal.

Before concluding this section we would like to remark that the Fisher's inequality, viz.,

$$b \geq v+1,$$

which is valid for a BIB design based on b blocks and $v+1$ treatments is not necessarily true for an A-optimal design in this setting. One can, however, show quite easily, using the standard technique of computing the rank of the incidence matrix, that

$$b \geq v.$$

This inequality cannot be improved, in general, since a symmetric BIB design augmented by a control in each block can be A-optimal, as when $v=b=k=3$, in design number 32 in the next section. However, for a S type design one can show that $b \geq v+1$.

3. A CATALOG OF A-OPTIMAL DESIGNS FOR COMPARING TEST TREATMENTS WITH A CONTROL

Using the technique pointed out in Section 2, we searched for the parameters of BTIB designs which made them optimal within the following practical constraints,

$$2 \leq k \leq 8$$

$$k \leq v \leq 30$$

$$v \leq b \leq 50.$$

There are precisely 111 sets of parameters satisfying the sufficiency condition for A-optimality and the necessity conditions for BTIB designs. We were fortunate to be able to construct all the corresponding BTIB designs.

For each triplet (v, b, k) the values of L and M (see the pictorial representation of the design given in Section 2) were obtained by minimizing the function $g(a, e)$, given in (2.8), using a computer. The computer then verified the necessary conditions in (2.11).

Table 3.1 summarizes 111 cases for which these necessary conditions were met, whenever $M=0$ the corresponding design will be R type and it will be S type otherwise. This is indicated in the table. In the last column of this table we have indicated how to obtain the d_0 or d_1 portion of the design, as the case may be. For those cases in which d_1 could not be described completely in the limited space we give the entire layout in the Section 3.3. For completeness, we also give the layouts of d_0 which happen to be BIB designs in Section 3.2.

Throughout this section the symbol $v\Sigma k$ denote all $\binom{v}{k}$ blocks of size k in v test treatments. This is clearly a BIB design - which will be referred to as a complete design.

3.1 A Table of A-optimal Designs.

TABLE 3.1

A Catalogue of A-optimal Designs

Design number	V	b	k	L	M	Type	d_0 or d_1 part (The layout of the test treatments in this design)
1	2	2	2	1	0	R	1 copy of $2\Sigma_1$
2	2	3	2	0	2	S	d_{11} is 1 copy of $2\Sigma_1$, d_{12} is 1 copy of $2\Sigma_2$
3	2	5	2	0	4	S	d_{11} is 2 copies of $2\Sigma_1$, d_{12} is 1 copy of $2\Sigma_2$
4	2	7	2	0	6	S	d_{11} is 3 copies of $2\Sigma_1$, d_{12} is 1 copy of $2\Sigma_2$
5	2	9	2	0	8	S	d_{11} is 4 copies of $2\Sigma_1$, d_{12} is 1 copy of $2\Sigma_2$
6	2	10	2	0	8	S	d_{11} is 4 copies of $2\Sigma_1$, d_{12} is 2 copies of $2\Sigma_2$
7	2	12	2	0	10	S	d_{11} is 5 copies of $2\Sigma_1$, d_{12} is 2 copies of $2\Sigma_2$
8	2	14	2	0	12	S	d_{11} is 6 copies of $2\Sigma_1$, d_{12} is 2 copies of $2\Sigma_2$
9	2	16	2	0	14	S	d_{11} is 7 copies of $2\Sigma_1$, d_{12} is 2 copies of $2\Sigma_2$
10	2	17	2	0	14	S	d_{11} is 7 copies of $2\Sigma_1$, d_{12} is 3 copies of $2\Sigma_2$
11	2	19	2	0	16	S	d_{11} is 8 copies of $2\Sigma_1$, d_{12} is 3 copies of $2\Sigma_2$
12	2	21	2	0	18	S	d_{11} is 9 copies of $2\Sigma_1$, d_{12} is 3 copies of $2\Sigma_2$
13	2	24	2	0	20	S	d_{11} is 10 copies of $2\Sigma_1$, d_{12} is 4 copies of $2\Sigma_2$
14	2	26	2	0	22	S	d_{11} is 11 copies of $2\Sigma_1$, d_{12} is 4 copies of $2\Sigma_2$
15	2	28	2	0	24	S	d_{11} is 12 copies of $2\Sigma_1$, d_{12} is 4 copies of $2\Sigma_2$
16	2	31	2	0	26	S	d_{11} is 13 copies of $2\Sigma_1$, d_{12} is 5 copies of $2\Sigma_2$
17	2	33	2	0	28	S	d_{11} is 14 copies of $2\Sigma_1$, d_{12} is 5 copies of $2\Sigma_2$
18	2	35	2	0	30	S	d_{11} is 15 copies of $2\Sigma_1$, d_{12} is 5 copies of $2\Sigma_2$
19	2	36	2	0	30	S	d_{11} is 15 copies of $2\Sigma_1$, d_{12} is 6 copies of $2\Sigma_2$
20	2	38	2	0	32	S	d_{11} is 16 copies of $2\Sigma_1$, d_{12} is 6 copies of $2\Sigma_2$
21	2	40	2	0	34	S	d_{11} is 17 copies of $2\Sigma_1$, d_{12} is 6 copies of $2\Sigma_2$

22	2	42	2	0	36	S	d_{11} is 18 copies of $2\Sigma 1$, d_{12} is 6 copies of $2\Sigma 2$
23	2	43	2	0	36	S	d_{11} is 18 copies of $2\Sigma 1$, d_{12} is 7 copies of $2\Sigma 2$
24	2	45	2	0	38	S	d_{11} is 19 copies of $2\Sigma 1$, d_{12} is 7 copies of $2\Sigma 2$
25	2	47	2	0	40	S	d_{11} is 20 copies of $2\Sigma 1$, d_{12} is 7 copies of $2\Sigma 2$
26	2	50	2	0	42	S	d_{11} is 21 copies of $2\Sigma 1$, d_{12} is 8 copies of $2\Sigma 2$
27	3	12	2	0	9	S	d_{11} is 3 copies of $3\Sigma 1$, d_{12} is 1 copy of $3\Sigma 2$
28	3	24	2	0	18	S	d_{11} is 6 copies of $3\Sigma 1$, d_{12} is 2 copies of $3\Sigma 2$
29	3	36	2	0	27	S	d_{11} is 9 copies of $3\Sigma 1$, d_{12} is 3 copies of $3\Sigma 2$
30	3	48	2	0	36	S	d_{11} is 12 copies of $3\Sigma 1$, d_{12} is 4 copies of $3\Sigma 2$
31	4	18	2	0	12	S	d_{11} is 3 copies of $4\Sigma 1$, d_{12} is 1 copy of $4\Sigma 2$
32	3	3	3	1	0	R	1 copy of $3\Sigma 2$
33	3	6	3	1	0	R	2 copies of $3\Sigma 2$
34	3	9	3	1	0	R	3 copies of $3\Sigma 2$
35	3	12	3	1	0	R	4 copies of $3\Sigma 2$
36	3	15	3	1	0	R	5 copies of $3\Sigma 2$
37	3	18	3	1	0	R	6 copies of $3\Sigma 2$
38	3	21	3	1	0	R	7 copies of $3\Sigma 2$
39	3	24	3	1	0	R	8 copies of $3\Sigma 2$
40	3	27	3	1	0	R	9 copies of $3\Sigma 2$
41	3	30	3	1	0	R	10 copies of $3\Sigma 2$
42	3	33	3	1	0	R	11 copies of $3\Sigma 2$
43	3	36	3	1	0	R	12 copies of $3\Sigma 2$
44	3	39	3	1	0	R	13 copies of $3\Sigma 2$
45	3	42	3	1	0	R	14 copies of $3\Sigma 2$
46	3	45	3	1	0	R	15 copies of $3\Sigma 2$

47	3	48	3	1	0	R	16 copies of 3Σ2
48	4	6	3	1	0	R	1 copy of 4Σ2
49	4	12	3	1	0	R	2 copies of 4Σ2
50	4	18	3	1	0	R	3 copies of 4Σ2
51	4	24	3	1	0	R	4 copies of 4Σ2
52	4	30	3	1	0	R	5 copies of 4Σ2
53	4	36	3	1	0	R	6 copies of 4Σ2
54	4	42	3	1	0	R	7 copies of 4Σ2
55	4	48	3	1	0	R	8 copies of 4Σ2
56	5	10	3	1	0	R	1 copy of 5Σ2
57	6	37	3	0	33	S	S1 in Sec. 3.3
58	8	40	3	0	32	S	S2 in Sec. 3.3
59	9	24	3	0	18	S	S3 in Sec. 3.3
60	4	4	4	1	0	R	1 copy of 4Σ3
61	4	8	4	1	0	R	2 copies of 4Σ3
62	4	12	4	1	0	R	3 copies of 4Σ3
63	4	16	4	1	0	R	4 copies of 4Σ3
64	4	20	4	1	0	R	5 copies of 4Σ3
65	5	10	4	1	0	R	1 copy of 5Σ3
66	5	20	4	1	0	R	2 copies of 5Σ3
67	5	30	4	1	0	R	3 copies of 5Σ3
68	5	40	4	1	0	R	4 copies of 5Σ3
69	5	50	4	1	0	R	5 copies of 5Σ3
70	6	10	4	1	0	R	R1 in Sec. 3.2
71	6	20	4	1	0	R	1 copy of 6Σ3

72	6	30	4	1	0	R	Union of the d _o parts of Designs 70 & 71 in this table
73	6	40	4	1	0	R	2 copies of 6Σ3
74	6	50	4	1	0	R	Union of the d _o parts of Designs 70 & 73 in this table
75	7	7	4	1	0	R	R2 in Sec. 3.2
76	7	14	4	1	0	R	2 copies of the d _o part of Design 75 in this table
77	7	21	4	1	0	R	3 copies of the d _o part of Design 75 in this table
78	7	28	4	1	0	R	4 copies of the d _o part of Design 75 in this table
79	7	35	4	1	0	R	5 copies of the d _o part of Design 75 in this table
80	7	42	4	1	0	R	6 copies of the d _o part of Design 75 in this table
81	7	49	4	1	0	R	7 copies of the d _o part of Design 75 in this table
82	9	12	4	1	0	R	R3 in Sec. 3.2
83	9	24	4	1	0	R	2 copies of the d _o part of Design 82 in this table
84	9	36	4	1	0	R	3 copies of the d _o part of Design 82 in this table
85	9	48	4	1	0	R	4 copies of the d _o part of Design 82 in this table
86	6	18	5	1	6	S	S4 in Sec. 3.3
87	6	36	5	1	12	S	S5 in Sec. 3.3
88	7	35	5	1	7	S	d ₁₁ is R2 in Sec. 3.2, d ₁₂ is complement of 4 copies of R2.
89	9	18	5	1	0	R	R4 in Sec. 3.2
90	9	36	5	1	0	R	2 copies of the d _o part of Design 89 in this table
91	10	15	5	1	0	R	R5 in Sec. 3.2
92	10	30	5	1	0	R	2 copies of the d _o part of Design 91 in this table
93	10	45	5	1	0	R	3 copies of the d _o part of Design 91 in this table
94	12	33	5	1	0	R	R6 in Sec. 3.2
95	13	13	5	1	0	R	R7 in Sec. 3.2
96	13	26	5	1	0	R	2 copies of the d _o part of Design 95 in this table

97	13	39	5	1	0	R	3 copies of the de part of Design 95 in this table
98	16	20	5	1	0	R	R8 in Sec. 3.2
99	16	40	5	1	0	R	2 copies of the de part of Design 98 in this table
100	16	48	6	1	0	R	R9 in Sec. 3.2
101	21	21	6	1	0	R	R10 in Sec. 3.2
102	21	42	6	1	0	R	2 copies of the de part of Design 101 in this table
103	25	30	6	1	0	R	R11 in Sec. 3.2
104	9	48	7	1	36	S	d ₁₁ is 2 copies of R4, d ₁₂ is complement of R3 in Sec. 3.2
105	14	35	7	1	14	S	S6 in Sec. 3.3
106	15	16	7	1	6	S	S7 in Sec. 3.3
107	8	28	8	2	0	R	complement of 1 copy of 8Σ2
108	9	12	8	2	0	R	complement of R3 in Sec. 3.2
109	9	24	8	2	0	R	2 copies of the de part of Design 108 in this table
110	9	36	8	2	0	R	3 copies of the de part of Design 108 in this table
111	9	48	8	2	0	R	4 copies of the de part of Design 108 in this table

3.2 R Type Designs

An R type design is a BIB design in the v test treatments in b blocks of size $k-L$ each (called d_e) augmented by L controls in each block. In this section we give an exact layout of all the BIB designs d_e used in Table 3.1. The test treatments are denoted by alphabets. A BIB in v test treatments in b blocks of size of k , each test treatment replicated r times and with each pair of treatments appearing in λ blocks will be denoted by $BIB(v, b, r, k, \lambda)$. In Table 3.1, These layouts are referred as R1-R11. The layouts of d_e 's are as follows:

R1 : $BIB(6, 10, 5, 3, 2)$

	a	a	a	a	a	b	b	b	c	c
d_e :	b	b	c	d	d	c	d	e	d	e
	e	f	e	e	f	d	e	f	f	f

R2 : $BIB(7, 7, 3, 3, 1)$

	a	b	c	d	e	f	g
d_e :	b	c	d	e	f	g	a
	d	e	f	g	a	b	c

R3 : BIB(9, 12, 4, 3, 1)

a b c a d g a b c a b c
d.: d e f b e h f d e e f d
g h i c f i h i g i g h

R4 : BIB(9, 18, 8, 4, 3)

a b c d e f g h i a b c d e f g h i
b c d e f g h i a b c d e f g h i a
d.: c d e f g h i a b e f g h i a b c d
e f g h i a b c d g h i a b c d e f

R5 : BIB(10, 15, 6, 4, 2)

a b a a a b c e e a b c a b c
b c c b e f d f d f d h g e d
d.: c f g d f h e g i d g i h g f
e i j h i j h h j j i j i j g

R6 : BIB(12, 33, 11, 4, 3)

```

a b c d e f g h i j k a b c d e f g h i j k
b c d e f g h i j k a c d e f g h i j k a b
d e f g h i j k a b c h i j k a b c d e f g
h i j k a b c d e f g i j k a b c d e f g h

a b c d e f g h i j k
b c d e f g h i j k a
d e f g h i j k a b c
l l l l l l l l l l

```

R7 : BIB(13, 13, 4, 4, 1)

```

a b c d e f g h i j k l m
b c d e f g h i j k l m a
d e f g h i j k l m a b c
j k l m a b c d e f g h i

```

R8 : BIB(16, 20, 5, 4, 1)

```

d h f j h i j k l m n o p e g b c a b c
h i i k k l m n o p e e g a a c d d f f
m n j p l m n o p a b g d f b h i j k l
o p o e g e g a b c d c f h i j k l m n

```

R9 : BIB(16, 48, 15, 5, 4)

```

a b c d e f g h i j k l m n o p
b c d e f g h i j k l m n o p a
d.: c d e f g h i j k l m n o p a b
e f g h i j k l m n o p a b c d
h i j k l m n o p a b c d e f g

a b c d e f g h i j k l m n o p
b c d e f g h i j k l m n o p a
f g h i j k l m n o p a b c d e
i j k l m n o p a b c d e f g h
k l m n o p a b c d e f g h i j

a b c d e f g h i j k l m n o p
b c d e f g h i j k l m n o p a
d e f g h i j k l m n o p a b c
h i j k l m n o p a b c d e f g
l m n o p a b c d e f g h i j k

```

R10 : BIB(21, 21, 5, 5, 1)

```

d e f g h i j k l m n o p q r s t u a b c
g h i j k l m n o p q r s t u a b c d e f
d.: h i j k l m n o p q r s t u a b c d e f g
m n o p q r s t u a b c d e f g h i j k l
o p q r s t u a b c d e f g h i j k l m n

```

R11 : BIB(25, 30, 6, 5, 1)

```

a b c e f g l i j k l m r o p
e f g i j k r m t o p q x s t
de: k l m o p q h s b u v w f y n
x y n d d h a a c c e e g g i
y n d h a b c b e f g f i j k

q r s t u v w x y n h a d c h
u v w x y n b c d e f g b i j
i j d l h a o p q r s t u v w
j k k m m o p q r s t u v w x
l m l o p q r s t u v w x y n

```

3.3 S Type Designs

An S type design consists of a part d_1 which is made up entirely of the v test treatments augmented by L or $L+1$ controls in each block. d_1 is the union of d_{11} , an arrangement of test treatments in blocks of size $k-L-1$ and d_{12} in blocks of size $k-L$. When $k = 2$, d_{11} and d_{12} can only be multiples of the complete design. But when $k > 2$, construction of d_{11} and d_{12} becomes more involved. The various d_1 's in this section were constructed by different techniques. Design no. 59 in Table 3.1 (see S3 later in this section) was obtained from Majumdar and Notz (1983). In design no. 88 in Table 3.1, d_{11} is BIB(7,7,3,3,1) and d_{12} is BIB(7,28,12,3,4). In design no. 104 in Table 3.1, d_{11} is BIB(9,36,20,5,10) and d_{12} is BIB(9,12,8,6,5). In

design no. 106 (see S7 later in this section) d_{12} is the dual of a BIB(10,15,6,4,2). In design no. 105 (see S6 in this section) d_1 was constructed by removing the entire 14 replications of one arbitrarily chosen treatment from BIB(15,35,14,6,5). This gives the required layout in 35 blocks, 14 with size 5 and the rest with size 6, in 14 test treatments.

The layouts for d_1 for the remaining S type designs (S1, S2, S4, S5 and d_{11} of S7) was constructed without relying on any design available in the literature. Let us use the symbol $\lambda_{ij}(d)$ to denote the number of blocks containing the pair of test treatments i and j in a design d . In d_1 , $\lambda_{ij}(d_1)$ is a constant, independent of i and j . Thus, if we decide to construct d_{12} first (since this is usually the larger part), then this can be done quite quickly using several distinct $\lambda_{ij}(d_{12})$'s. However, this means that there should be many distinct $\lambda_{ij}(d_{11})$'s too; and these are now fixed since $\lambda_{ij}(d_{11}) + \lambda_{ij}(d_{12}) = \lambda_{ij}(d_1)$, a constant. Constructing a d_{11} gets more and more difficult as the number of distinct $\lambda_{ij}(d_{11})$'s gets large. Thus it is usually helpful to start with a minimum number of distinct $\lambda_{ij}(d_{12})$'s - usually 2 - so that there are also a minimum number of distinct $\lambda_{ij}(d_{11})$'s. The same principle is helpful if we decide to construct d_{11} first.

Denoting the test treatments by alphabets, we give the layouts for d_1 's for $k > 2$ as follows. They are referred as S1-S7 in Table 3.1.

S1 :

a a b c
d₁₂: b d d e
c e f f

d₁₁ is 2 copies of 6Σ2 design with the following three additional blocks

a b c
f e d

S2 :

a a a b b c c d
d₁₂: b d f d e e g f
c e g g h f h h

d₁₁ is a 8Σ2 design with the following additional blocks

a b c e
h f d g

S3 :

a a b c d e
d1: a a a a b b b b c c c d d e f f g g b f c d e h
c d e h d e g h e g i f i f h i h i i g f h g i

S4 :

a a a a a a a a b b b b
d1: a a a b c c b b b b c d c c c c d
b d b d d e c e c d d e e d d d e e
c e f e f f d f e f e f f f e f f f

S5 :

a a a a a a b b b b c d
d11: b b c c c e c c d e d e
d e d d f f e f f f e f

a a a a a a a a a a a a a a a b b b b b c c
d12: b b b b b b b b b c c c c d d c c c c c d d d
c c c c c d d d d e d d e e e e d d d d e e e e
d e e f f e e f f f e f f f f f e e f f f f f f

S6 :

a b c d e f g h i j k l m n c d e f g a b
 h i j k l m n a b c d e f g e f g a b c d
 d1: i j k l m n h g a b c d e f h i j k l m n
 j k l m n h i f g a b c d e i j k l m n h
 l m n h i j k d e f g a b c k l m n h i j

 c d e f g a b a b c d e f g
 d e f g a b c e f g a b c d
 f g a b c d e f g a b c d e
 h i j k l m n h i j k l m n
 i j k l m n h i j k l m n h
 k l m n h i j k l m n h i j

S7 :

a b c a b c a a b c
 a a b d d e e f g e f g b d d d
 b c c h i f h h h i j k c f e e
 d1: g f e m j g j i i l l l d g g f
 k j i n k h k k j n m m h i j k
 o n m o l l m n o o o n l m n o

4. CONCLUDING REMARKS

In this paper we have provided 111 A-optimal designs for comparing a control with 2 to 25 test treatments in blocks of sizes from 2 to 8. These designs were obtained by using a single method. Not surprisingly, a single method is incapable of producing optimal designs for all parameter combinations. Clearly there is a great need to develop methods to cover the yawning gaps in the catalog.

The structure of A-optimal designs present a few unexpected outcomes, rather contrary to conventional intuition. For example, if $v=k=2$ and $b=10$, one might imagine that it is impossible to improve on the design consisting of 5 copies of

- -
a b

since our interest is only in control-test treatment comparisons. However Table 3.1 (design no. 6) shows that the A-optimal design is

- - - - - a a
a a a a b b b b b

Thus it is an interesting problem to completely understand the rationale for the complex structure of these designs. One more notable feature is that the augmented BIB designs are often A-optimal, but not always so, as pointed out in Section 2.

For some values of the parameter v , Table 3.1 contains no design at all. For these we have to look at larger values of k and/or b to obtain an optimal design by our method. For example, when $v=11$, $k=2$, the first

design was obtained when $b=209$. For the optimal design $L=0$ and $M=99$.

5. ACKNOWLEDGMENTS

This research was supported by Grant AFOSR 80-0170 and by a Faculty Summer Fellowship from the University of Illinois at Chicago.

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