FRANK J. SEILER RESEARCH LABORATORY

FJSRL-TR-83-0005 MAY 1983

OPTIMAL PERIODIC CONTROL

FINAL REPORT

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PROJECT 2305-F2-67

AIR FORCE SYSTEMS COMMAND
UNITED STATES AIR FORCE

This document was prepared by the Guidance and Control Division, Directorate of Aerospace-Mechanics Sciences, Frank J. Seiler Research Laboratory, United States Air Force Academy, Colorado Springs, CO. The research was conducted under Project Work Unit Number 2305-F2-67, Optimal Periodic Control Applications in Aerospace Research. Lt Colonel Richard T. Evans was the Project Scientist in charge of the work.

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This report has been reviewed by the Commander and is releasable to the National Technical Information Service (NTIS). At NTIS it will be available to the general public, including foreign nations.

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SECURITY CLASSIFICATION OF THIS PAGE (When Data Entered)

REPORT DOCUMENTATION PAGE		READ INSTRUCTIONS BEFORE COMPLETING FORM
1. REPORT NUMBER FJSRL-TR-83-0005	2. GOVT ACCESSION NO. AD-A130114	3. RECIPIENT'S CATALOG NUMBER
4. TITLE (and Subtitle) Optimal Periodic Control		5. TYPE OF REPORT & PERIOD COVERED Final Report Oct 1980 - Dec 1982
		6. PERFORMING ORG. REPORT NUMBER
7. AUTHOR(s) Richard T. Evans, Lt Col, USAF		8. CONTRACT OR GRANT NUMBER(s)
9. PERFORMING ORGANIZATION NAME AND ADDRESS Frank J. Seiler Research Laboratory (AFSC) USAF Academy Colorado Springs, CO 80840		10. PROGRAM ELEMENT, PROJECT, TASK AREA & WORK UNIT NUMBERS DRS 61102F 2305-F2-67
11. CONTROLLING OFFICE NAME AND ADDRESS		12. REPORT DATE May 1983
		13. NUMBER OF PAGES 31
14. MONITORING AGENCY NAME & ADDRESS(if different from Controlling Office)		15. SECURITY CLASS. (of this report) UNCLASSIFIED
		15a. DECLASSIFICATION/DOWNGRADING SCHEDULE

16. DISTRIBUTION STATEMENT (of this Report)

Distribution Unlimited

17. DISTRIBUTION STATEMENT (of the abstract entered in Block 20, if different from Report)

18. SUPPLEMENTARY NOTES

19. KEY WORDS (Continue on reverse side if necessary and identify by block number)

Cyclic (periodic) processes
Optimization
Optimal Control
Bifurcation Theory
Periodic Orbits
Periodic Solutions to Diff Equs
Riccati Equations
Hamiltonian Systems
Optimal Cruise of Aircraft

20. ABSTRACT (Continue on reverse side if necessary and identify by block number)

This final report is a summary of research performed in optimal periodic control at the Frank J. Seiler Research Laboratory. Most of the work has been presented and published during the course of the research.

The development, clarification, and correction of general theory for optimal periodic control was a significant contribution of this research. Second variation relationships were determined by exploiting properties of autonomous Hamiltonian systems. A new necessary condition was developed and an existing

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SECURITY CLASSIFICATION OF THIS PAGE (When Data Entered) sufficiency condition corrected and expanded. Several useful algorithms were designed from the second variation relationships found enabling the development of a systematic computational technique for determining solutions to the optimal periodic control problem. The technique was verified on an illustrative problem that was formulated to characterize the complexities of this class of problem. Locally optimizing flight paths to minimize fuel per range for aircraft were also obtained using this technique on a point mass model during cruise.

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INTRODUCTION AND BACKGROUND

Optimal periodic control has become an increasingly important area of research as evidenced by the greater numbers of published papers, theses, and dissertations resulting from government grants, academic interest and industrial sponsorship. Examples of periodic or cyclic processes abound in nature, ranging from the rhythmic pulse of living creatures to the perpetual orbits of celestial bodies. Many are examples of an optimization process of nature. In contrast, there are numerous engineering systems designed to operate in a steady state conditions. In many of these systems performance could be significantly improved by some form of cyclic operation. Chemical plant process control was one of the first to be investigated for improvement compared to steady state performance[1,2,3]. This motivated the first paper on optimal periodic control[4] and led to its subsequently rapid theoretical development[5,6,7,8]. Survey papers 9, 10, and 11 summarize major results through 1975.

The research encompassed by this report was stimulated by a controversy[12,13,14] over an aerospace problem. The nonoptimality of steady state cruise for an aircraft with respect to fuel efficiency was shown in 1976 by Speyer[15]. However, subsequent effort[16] to find a locally optimizing solution to this problem failed using standard optimization computational techniques such as steepest ascent and conjugate gradient methods. This led Speyer and Evans to the formulation of a minimum state optimal periodic control problem that would generate periodic solutions. A locally optimizing solution to this new problem was obtained in the form of an asymptotic expansion about a small parameter.

This analytical result provided excellent agreement with numerical results obtained in a parallel effort[17]. Further numerical study of this problem and the development of corrections and additions to the general theory of optimal periodic control followed[18]. The results of this work provided a better understanding of the source of failures previously encountered in attempts to solve the aircraft cruise problem.

This report describes a continuation of this research conducted at the Frank J. Seiler Research Laboratory under Work Unit 2305-F2-67. The principal results of this effort have been presented at several conferences and have been published in the open literature. The major accomplishments obtained are summarized in the remainder of this report under three headings: (1) A Second Variation Condition; (2) Computational Techniques; and (3) Optimal Aircraft Cruise.

This research has been a collaboration of work by the principal investigator, Richard T. Evans, Lt Col, USAF; Professor Jason L. Speyer, The University of Texas at Austin; and two of his graduate students, David E. Walker and David P. Dannemiller. The work by Professor Speyer and his students was partially sponsored by the National Science Foundation Grant ECS7918246. The activities of the students also contributed significantly to their Master's degrees.

A complementary research effort was sponsored by AFOSR Grant Number 77-3158 during the period 1 October 1976 to 31 January 1982. Personnel associated with this work included: Principal Investigator, Professor Elmer G. Gilbert, The University of Michigan; Arthur E. Frazho, post-doctoral researcher; and PhD students Daniel T. Lyons and Dennis S. Bernstein. See Professor Gilbert's Final Report[19] for a summary and a complete bibliography of their work in optimal periodic control.

OBJECTIVES OF RESEARCH

The stated objective of this research was to develop the theory and computational technique for optimizing the flight path of an aircraft with respect to fuel consumption (maximize range for a given amount of fuel) during the cruise segment of flight. It then was intended to apply these tools to a point mass model of an aircraft and determine a locally optimizing cyclic cruise flight path in a proof of principle demonstration.

Their are numerous potential applications for this research. The more obvious Air Force benefits include: extend the range of an air vehicle with a fixed amount of fuel; reduce its fuel requirements for a given range thereby increasing its load capability; and increase its endurance allowing it to remain aloft longer. The improvements of periodic cruise flight paths appear to be most suited for remotely piloted aircraft or cruise missile type applications. However, in many emergency or back-up operations, it also would be quite feasible for manned systems.

PROBLEM FORMULATION AND FIRST ORDER NECESSARY CONDITIONS

The optimal periodic control problem consists of minimizing the performance criterion

$$J(u(\cdot), x_0, \tau) = \frac{1}{\tau}\int_0^{\tau} L(x(t),u(t))dt, \qquad (1)$$

with respect to the period $\tau \in T \equiv (0,\infty)$, the p-vector control functions $u(\cdot) \in U$, where U is defined in Assumption 2 below, and the initial states $x(0) = x_0 \in R^n$, subject to the time-invariant dynamical equations

$$\dot{x}(t) = f(x(t),u(t)), \qquad (2)$$

with the periodic boundary conditions

$$x(0) = x(\tau). \tag{3}$$

Note that both the integrand of the performance index and the dynamical equations are time-invariant.

<u>Assumption 1:</u> $f(\cdot,\cdot)$ and $L(\cdot,\cdot)$ and their derivatives up to second order are assumed to be continuous with respect to both arguments.

<u>Assumption 2:</u> $U \equiv \{u(\cdot): \ u(\cdot)$ is piecewise continuous in the interval $[0,\infty)$ and $||u(\cdot)||_\infty \equiv \sup_{t\in[0,\infty)} |u(t)| < \infty$ where

$$|u(t)| \equiv (\sum_{i=1}^{p} u_i^2(t))^{1/2}, \ u(t) \in R^p\}$$

<u>Definition:</u> A piecewise continuous function $f(\cdot)$ has a period if there exists a minimum $\tau \in T$ such that

$$f(\tau) = f(0) \tag{4}$$

This minimal τ is called the period of $f(\cdot)$.

Remark: This definition excludes constant $f(\cdot)$.

The first order necessary conditions for optimality derived from the calculus of variations are:

$$\dot{x} = H_\lambda^T = f(x,u), \tag{5}$$

$$\dot{\lambda} = -H^T, \tag{6}$$

$$0 = H_u, \tag{7}$$

$$x(0) = x(\tau), \tag{8}$$

$$\lambda(0) = \lambda(\tau), \tag{9}$$

$$H(\tau) - J(u(\bullet), x_0, \tau) = 0, \tag{10}$$

where $\lambda(t) \in R^n$ is the Lagrange multiplier that adjoins the system constraints (2) to the performance index, $L(x,u)$, forming the variational Hamiltonian defined as

$$H(x,u,\lambda) = L(x,u) + {}^{T}f(x,u). \tag{11}$$

It is also assumed that the Legendre-Clebsch condition is met in strong form along the external path, i.e.,

$$H_{uu} > 0. \tag{12}$$

Any periodic solution to the two point boundary value problem, equations (5) through (9) is an extremum of the problem. The condition (10) relating the Hamiltonian and the performance index, evaluated along the optimal path is the special condition for testing the optimal period, first derived by Horn and Lin[4].

A SECOND VARIATION CONDITION

Solutions that satisfy the first order necessary conditions, equations (5) through (10) and (12) are examined for local optimality by second variation tests, such as the Jacobi necessary condition. A very useful form of the Jacobi test, developed by Bittanti et. al.[7] and extended by Gilbert and Bernstein[20], shows whether or not a static solution is locally optimal. This test was used by Speyer[15] and by Breakwell and Shoaee[21] to show that static cruise for many aircraft models is not fuel minimizing. However, this test provides insufficient information to determine the optimality of cyclic or periodic solutions.

An important result of the research covered by this report is the development of a variational theory for testing periodic solutions. This work was presented by Speyer[22] at the 1981 Joint Automatic Control Conference in Charlottesville, VA and will be published this fall in the IEEE Transactions on Automatic Control[23]. Two earlier papers, clarified and extended by this effort, are Brittanti, et. al.[6]. who considered the problem for fixed period, and Chang[5], who extended the work to free period. The new results are summarized in the remaining paragraphs of this section.

Properties of autonomous Hamiltonian systems and their related monodromy matrix are used to establish relationships essential to developing the second variational conditions for optimality. The monodromy matrix is the transition matrix for the state equations (5) through (9) evaluated over one period. It is shown that the monodromy matrix has at least two unity eigenvalues and that two of them are coupled (in the same Jordan block). The eigenvector associated with one of the unity eigenvalues is tangent to the state space orbit described by the related extremal solution. The generalized eigenvector associated with the other unity eigenvalue defines the direction of a one-dimensional family of orbits which varies continuously with the Hamiltonian. Determining this direction was a key factor in developing the algorithm used for computing extremal solutions to the optimal periodic control problem described in the next section.

Another important relationship derived from this effort involves classifying the eigenvalues that can result for real values of the Riccati variable. The existence of a real-valued solution to the Riccati differential equation is a well known second variation condition. When the solution to the Riccati equation is periodic it can be expressed in the form

$$P\Phi_{12}P + P\Phi_{11} - \Phi_{22}P - \Phi_{21} = 0, \tag{13}$$

where P is a vector of initial conditions defining the periodic solution and the Φ's are square partitions of the monodromy matrix. The canonical similarity transformation of the monodromy matrix gives

$$L\,\Phi L^{-1} = \begin{bmatrix} \Phi_{11} + \Phi_{12}P & \Phi_{12} \\ 0 & \Phi_{22} - P\Phi_{12} \end{bmatrix}, \tag{14}$$

where the identity (13) is used to obtain the zero element and the similarity transform matrix is specified as,

$$L = \begin{bmatrix} I & 0 \\ -P & I \end{bmatrix}.$$

The eigenvalues of this transformed matrix (14) are those of the monodromy matrix due to their invariance through a similarity transformation. Because of the zero element on the off-diagonal of the transformed matrix, the eigenvalues of the submatrices on the diagonal are the same as for the entire matrix.

Using the symplectic property of the monodromy matrix, which is also preserved through the similarity transformation, the following important relationship is obtained,

$$\left[\Phi_{11} + \Phi_{12}P\right]^{-1} = \left[\Phi_{22} - P\Phi_{12}\right]^{T}. \tag{15}$$

The significance of this equation is that it strongly restricts the eigenvalues of the monodromy matrix that correspond to real-valued Riccati variable elements since the elements of the monodromy matrix must also be real-valued for a physically realizeable system. Recall also the matrix property that the determinant of a matrix is equal to the product of its eigenvalues. Considering eigenvalues of magnitude one, the following result can be stated:

A necessary condition that the Riccati variable matrix, P, be real-valued is that there be no distinct eigenvalues of the monodromy matrix on the unit circle.

Remark: The satisfaction of this condition does not guarantee that P exists for all starting times over the interval of a period. The solution of the Riccati differential equation is still required to ensure that there are no finite escape times.

A second variational sufficiency condition for weak local optimality of cyclic processes can also be stated. The following condition extends and clarifies previous statements[5,6] of the condition:

For the periodic control problem described by (1) through (3) and assumptions 1 and 2, $(u^o(\cdot), x_0^o, \tau^o) \in U \times R^n \times T$ forms a weak local minimum if;

(i) the first order necessary conditions (5) through (10) are satisfied,

(ii) the strong form of the Legendre-Clebsch condition (12) is satisfied,

(iii) there exists a real valued bounded symmetric matrix solution to the Riccati differential equation on $0 \leq t \leq \tau^o$ satisfying the periodic condition $P(0) = P(\tau^o)$,

(iv) there are no eigenvalues of the monodromy matrix on the unit circle except for the two coupled unity (+1) eigenvalues associated with the velocity vector $\dot{y}(0)$ where $\delta\tau(H)/\delta H \neq 0$ ensures this coupling, and

(v) the eigenvalues of the monodromy matrix off the unit circle are distinct.

Remarks:

1. The requirement for earlier statements of this sufficiency condition[5,6] that the matrix $\Phi_z(\tau,0) \equiv \Phi_{11} + \Phi_{12}P$ have no unity eigenvalue is never satisfied. Furthermore, it is required in condition (iv) that the remaining eigenvalues of $\Phi_z(\tau,0)$ not be on the unit circle.

2. Condition (v) is a form of the strongly positive condition which is totally lacking in the previous statements of this sufficiency condition.

COMPUTATIONAL TECHNIQUES

Although periodic optimal control problems had formed an important class of practical problems, few numerical investigations had been reported through 1980. Initial experimentation indicated poor convergence behavior for first order optimization schemes relative to a class of aircraft cruise problems.[16] It has been recognized that this was due in part to the shallow curvature of the cost criterion, partly due to the lack of sensitivity of first order methods, and finally due to the great difficulty in closing the solutions (satisfying the periodicity requirements).

In order to develop a better understanding of the difficulties encountered in earlier numerical investigations of the optimal cyclic aircraft cruise problem, an illustrative, minimum state, optimal periodic control problem was formulated. An analytical solution to this problem was first obtained[17] using a perturbation method most frequently credited to Lindstedt and Poincare. The solution can be expressed in the form of an asymptotic series expansion about a small parameter which also can be written in the form of a Fourier series expansion. This result captures an interesting characteristic of the solutions that satisfy the first order necessary conditions, equations (5) through (9). That is, they form a set of solutions varying continuously in amplitude and period. The results of this study were presented by Evans[17] at the 1979 Joint Automatic Control Conference in Denver, CO..

The initial numerical study[18] of the illustrative problem showed that the approximate analytical solution was quite good. It verified the infinity of solutions that satisfy equations (5) through (9) and form a continuous set or family. However, it was also discovered that an infinity of families of solutions were found to exist. The families intersect at common solutions called bifurcation points. An illustration of these results is given later in the section. It should be noted that much work has been accomplished by dynamicists determining periodic solutions to a set of first order differential equations. The work by Henon[24] and by Contopoulos[25] has been invaluable in this research and it provides a detailed characterization of solution families and bifurcation points.

The results from the numerical investigation of the illustrative problem were presented by Evans[26] at the 2nd International Federation of Automatic Control Applications of Nonlinear Programming and Optimization at Oberpfaffenhofen, West Germany in September 1980. The emphasis here was placed on identifying the richness and complexity of the solutions to this type of control problem. Characteristics of solutions and of families of solutions were examined in some detail. The associated paper was published in the Conference Proceedings.

The computer program that was used for the initial study was an adaption of one developed by Broucke[27] to find periodic solutions to 4th order dynamic systems. Search methods about a known solution were used to find new solutions of the family. As indicated in an earlier section the direction of the family can be predicted from the generalized eigenvector associated with the second unity eigenvalue of the monodromy matrix. A shooting method, using these predicted starting values, integrates the Euler-Lagrange equations (5) through (9) to obtain periodic solutions in convergent iterations. The improved shooting method and additional results from the numerical investigation of the problem was presented by Evans[28] at the 20th IEEE Conference on Decision and Control at San Diego in December 1981. The associated paper was published in the proceedings to the conference.

The remainder of this section expresses important concepts and key relationships associated with the development of the computational technique employed during this research effort. First, identification of solutions is most easily accomplished by association with their initial conditions since a set of initial conditions identifies a unique solution. An important characteristic of a periodic solution is the number of axis crossings in the same direction that occurs during one period of the solution for a particular variable of the problem. This is a distinguishing characteristic of a family of solutions. In most cases, the number of axis crossings is the same for all solutions of the family. This can be used as a program check to verify that a new solution belongs to the family that was intended to be followed.

After one solution has been obtained, the initial conditions for the next solution on the family can be projected. With the definition,

$$\Psi \equiv y(0) - y(\tau) \tag{16}$$

a small change in Ψ due to a variation in the initial conditions y(0) and the period (τ) gives the following results:

$$d\,\Psi = \delta y(0) - \delta\, y(\tau) - \dot{y}(\tau)\, d\tau \tag{17}$$

$$-d\Psi = [\dot{y}(\tau) \quad \Phi - I] \begin{bmatrix} d\tau \\ \delta y(0) \end{bmatrix} \tag{18}$$

Two elements must be fixed to use $d\Psi$ to predict new guesses since (1) $\dot{y}(\tau)$ is proportional to a column in $\Phi - I$, and (2) $\Phi - I$ becomes singular as a solution is approached. Removing corresponding columns of $[\dot{y}(\tau) \quad \Phi - I]$ eliminates indeterminancy resulting in

$$-d\,\Psi = \Omega\delta z \tag{19}$$

Taking the pseudo inverse of Ω allows computation of new starting conditions from δz for determining a new solution of the family. A less cumbersome predictor, such as a curve fitting interpolator, is suggested after several solutions have been obtained.

To illustrate the computational technique some results of the minimum state, optimal periodic control problems will be used. The Euler Lagrange equations derived from the first order conditions for this problem are

$$\dot{x}_1 = x_2 , \tag{20}$$

$$\dot{x}_2 = -\frac{\lambda_2}{b} \tag{21}$$

$$\dot{\lambda}_1 = -x_1 , \tag{22}$$

$$\dot{\lambda}_2 = x_2 - x_2^3 - \lambda_1 , \tag{23}$$

$$\text{and} \quad x(0) = x(\tau) , \quad \lambda(0) = \lambda(\tau). \tag{24}$$

For this problem a static solution exists; i.e., $x_1 = x_2 = \lambda_1 = \lambda_2 = 0$. Starting from this solution, a family of solutions can be determined as in Figure 1. The solutions are represented by initial conditions. For this illustration $x_2(0) = \lambda_1(0) = 0$ for all points graphed. The initial value of x_1 is represented on the graph. The last condition is determined by the relationship of the other states and the Hamiltonian evaluated at the initial time. The state relationships (x_1 vs x_2) for several solutions are superimposed on the graph of the family and centered at points corresponding to their respective initial conditions. The scale of the x_1 vs. x_2 plots are all the same.

As indicated before, an infinity of families of solutions exist. Plots similar to the previous one are depicted in Figures 2,3, and 4 for three additional families. Each plot is to the same scale as in Figure 1. Note the number of axis crossings in the various examples. For a single family the number crossings are generally the same. The family that emanates from the static solution is called the principal family; all others are branch families. The solution identified by c in Figures 1 through 4 represents the solution of the respective family that also satisfies the optimal period conditions, equation (10).

Several additional levels of branching are shown in the detail of Figure 5. Note that families branch only in "stable" areas of the family. Here stability refers to no eigenvalues of the monodromy matrix existing outside the unit circle. Bifurcation points (branch points of the families) are dense in the stable regions of the family.

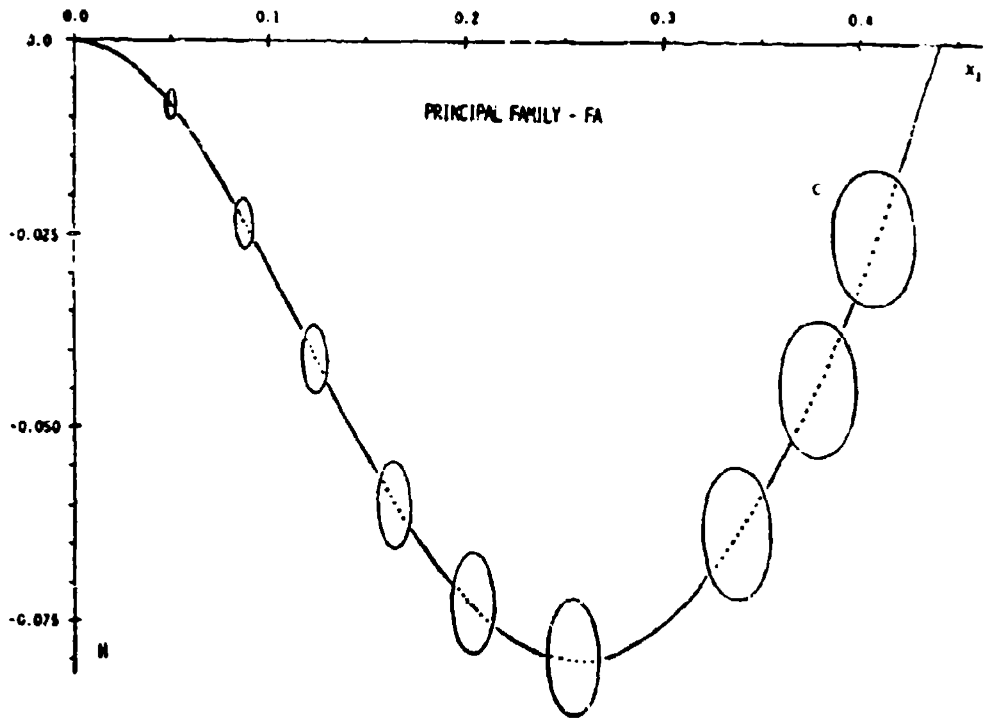

Figure 1. Variation of Periodic Solution Along Principal Family

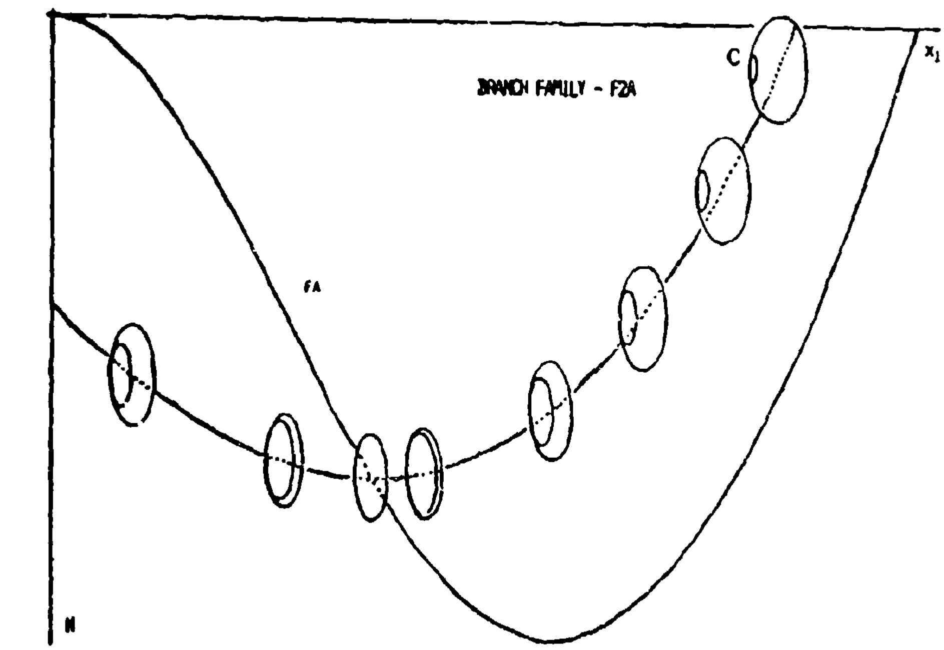

Figure 2. Variation of Periodic Solution Along Branch Family F2A

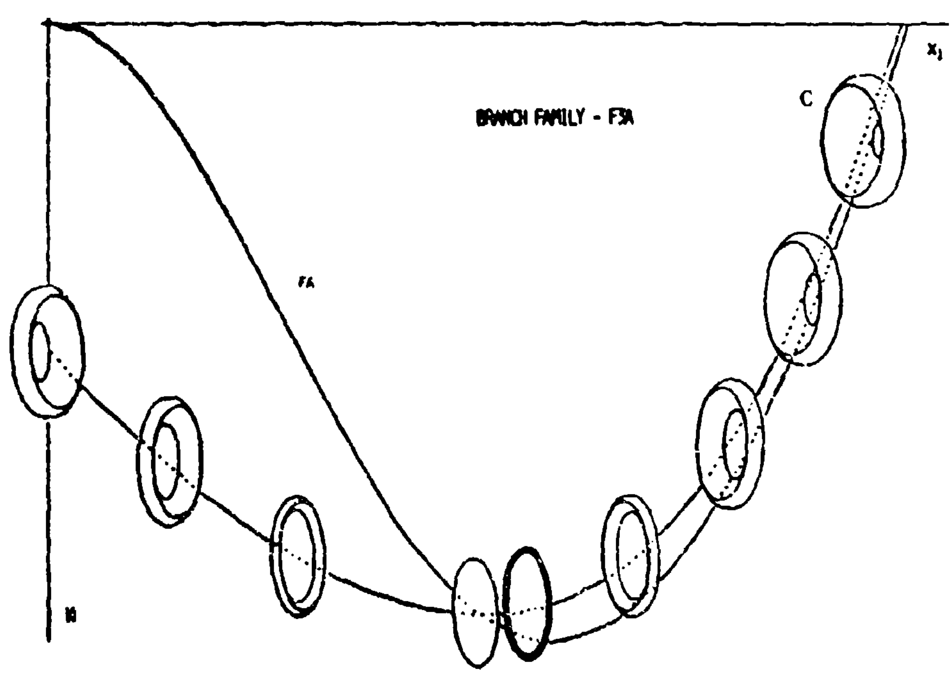

Figure 3. Variation of Periodic Solution Along Branch Family F3A

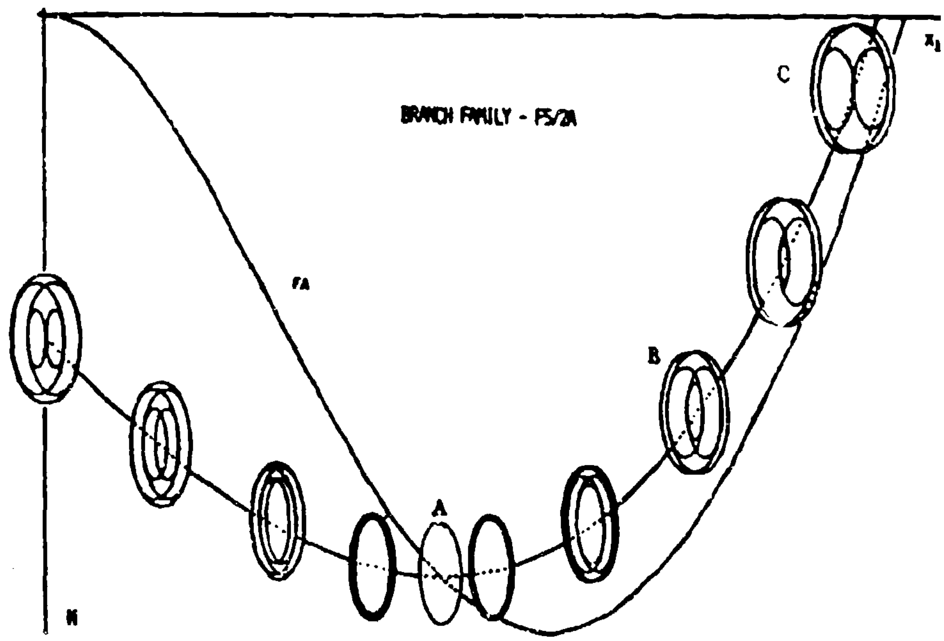

Figure 4. Variation of Periodic Solution Along Branch Family F5/2A

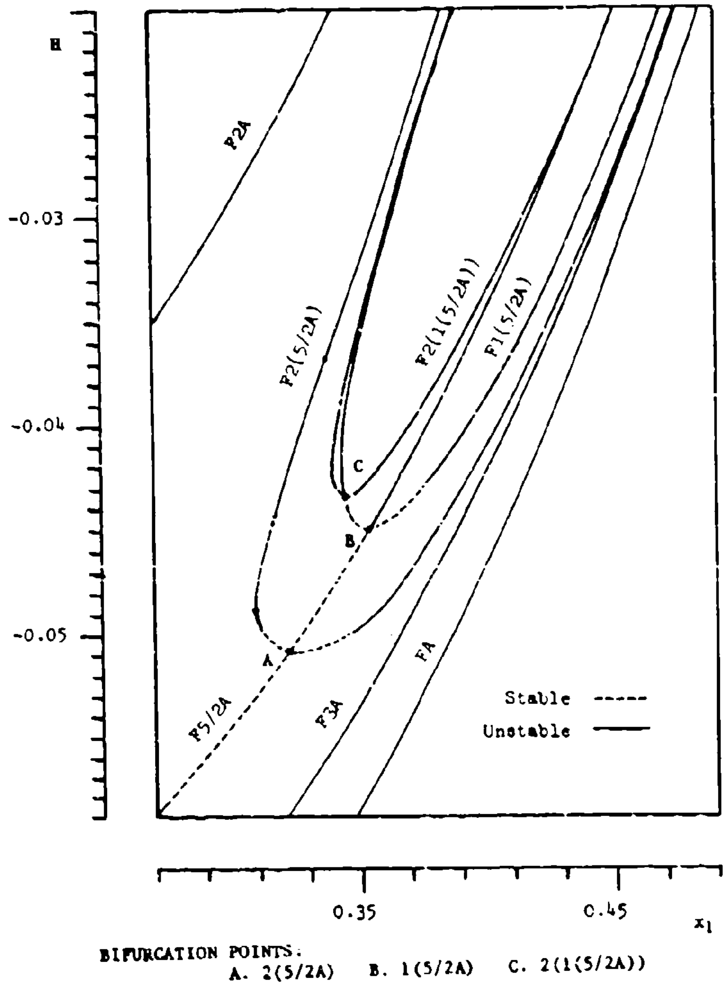

Figure 5. Branch Families (Detail)

The purpose of this detail is intended to identify some characteristics of the solutions and to emphasize the necessity of developing a systematic approach for investigating optimal periodic control problems. The amplitude and period of solutions along a family vary continuously. However, sharp differences in the amplitude and period of solutions of different families generally exist even though initial conditions for both may vary only slightly. See reference 18 for identifying bifurcation points and branch families.

Summarizing the computational technique; first, a shooting method is used to find a closed periodic path which satisfies the first order necessary conditions (the Euler Lagrange equation (5) through (9), except for the transversality condition (10) associated with free period). Then a one-dimensional family of periodic solutions is constructed using the generalized eigenvector or a curve fitting interpolator to predict initial conditions of additional solutions. Finally, the family is traversed in the direction of decreasing cost criterion until the optimal period condition (10) is satisfied.

OPTIMAL AIRCRAFT CRUISE

Fuel efficient cruise trajectories for aircraft have been a subject of continuous theoretical interest and are becoming one of practical interest as well. Since the steady state cruise path is not minimizing[15] for most point mass aircraft models, the objective is to obtain the periodic paths that are minimizing. There appear to be two underlying mechanisms for producing periodic paths. The first mechanism is the mismatch in the regions of velocity and altitude where the aircraft is aerodynamic and propulsion efficient. This is the mechanism behind chattering (or relaxed steady state) cruise. There also is a potential and kinetic energy interchange which is optimal for fuel interchange which is optimal for fuel performance. The need for substantial kinetic energy seems to be the reason for the velocity threshold found earlier[15].

Recent work in this area includes the Master's Thesis of Walker[29] and Dannemiller[30]. Both applied the techniques summarized in this report to investigate the optimal periodic cruise of a hypersonic cruiser. The results of this work were also presented by Speyer[31] at the AIAA Guidance and Control Conference at Danvers, MA, in August 1980.

A point mass model of an atmospheric vehicle operating in the hypersonic region was used to investigate the fuel improvement from the steady state cruise path obtained by modulating the flight path. The fuel improvement obtained was due solely to a potential-kinetic energy interchange which was indicated by a frequency type second variational analysis of the steady state cruise for the flat earth model. A family of solutions was generated for both the flat and spherical earth models. By applying the second variational sufficiency conditions for periodic processes, only one flight path which involves the flat earth model was found to be locally minimizing. The improvement of the periodic cruise over the steady state cruise for this example is 4.2%. No locally minimizing path for the spherical earth model was found. Nevertheless, the periodic extremal cruise paths found did improve fuel performance over their respective steady state cruise paths by as much as 4.5%.

RECOMMENDATIONS

The three principal objectives of this research have been satisfactorily achieved. Even though the minimum state illustrative problem has been exhaustively studied, further investigation has merit. Verification of new theory or new computational techniques are more easily accomplished with the reduced state problem. Additional relationships may be exhibited by further investigation of out-of-plane solutions, eigenvector directions at bifurcation points, and other solutions that also might satisfy all first and second order conditions.

Now that a locally optimizing periodic flight path has been found for one model, potential applications should be examined. More realistic aircraft models should be developed and studied. Certainly, the feasibility of subjecting the aircraft to cyclic control must be considered, in particular the cycling on and off of its engines.

A related area of research is associated with quasi-periodic solutions which may provide better performance in some instances than the periodic solutions. A considerable amount of research effort in this area has been expended by statistical dynamicists.

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