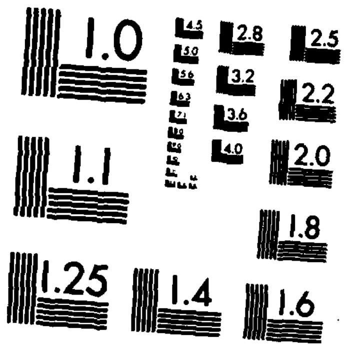
AD-A130 186 SOLUTION MANIFOLDS AND SUBMANIFOLDS OF PARAMETRIZED EQUATIONS AND THEIR D..(U) PITTSBURGH UNIV PA INST FOR COMPUTATIONAL MATHEMATICS AND APP.. J P FINK ET AL. 1/1

UNCLASSIFIED JAN 83 ICMA-83-59 N00014-80-C-0455 F/G 12/1 NL

END



MICROCOPY RESOLUTION TEST CHART
NATIONAL BUREAU OF STANDARDS-1963-A

(15)

INSTITUTE FOR COMPUTATIONAL MATHEMATICS AND APPLICATIONS

Technical Report ICMA-83-59

Solution Manifolds and Submanifolds of Parametrized Equations and Their Discretization Errors[1)]

by

James P. Fink and Werner C. Rheinboldt

Department of Mathematics and Statistics

University of Pittsburgh

Technical Report ICMA-83-59

Solution Manifolds and Submanifolds of Parametrized Equations and Their Discretization Errors[1)]

by

James P. Fink and Werner C. Rheinboldt

JAN 83

Institute for Computational Mathematics and Applications
Department of Mathematics and Statistics
University of Pittsburgh
Pittsburgh, PA 15261

1) This work was in part supported by the U.S. Air Force Office of Scientific Research under Grant 80-0176, the National Science Foundation under Grant MCS-78-05299, and the Office of Naval Research under Contract N-00014-80-C-0455.

lambda

Abstract

sub 0

The paper concerns solution manifolds of nonlinear parameter-dependent equations (1) $F(u,\lambda) = y_0$ involving a Fredholm operator F between (infinite-dimensional) Banach spaces $X = Z \times \Lambda$ and Y, and a finite-dimensional parameter space Λ. Differential-geometric ideas are used to discuss the connection between augmented equations and certain one-dimensional submanifolds produced by numerical path-tracing procedures. Then, for arbitrary (finite) dimension of Λ, estimates of the error between the solution manifold of (1) and its discretizations are developed. These estimates are shown to be applicable to rather general nonlinear boundary-value problems for partial differential equations.

lambda lambda lambda

1. Introduction

Considerable attention has been directed toward nonlinear parameter-dependent equations of the form

$$F(u,\lambda) = y_0, \tag{1.1}$$

where F is a mapping from a Banach space $X = Z \times \Lambda$ to a Banach space Y, Z represents a state space, and Λ an m-dimensional parameter space, $1 \leq m < \infty$. Under appropriate conditions on F, the set of regular solutions of (1.1) forms an m-dimensional manifold in X and it is of interest to obtain both analytical and computational information about this manifold.

Since Z and hence X are usually infinite-dimensional, any computational examination of (1.1) requires the introduction of finite-dimensional approximations. Such approximations raise questions about the resulting error between the solution manifolds of (1.1) and its discretized versions.

The estimation of this error between solution manifolds has seen relatively limited activity, and most of that has been directed toward the case of a one-dimensional parameter space Λ [3], [4], [6]. Since all numerical procedures for analyzing the solution manifold of an equation of the form (1.1) involve tracing one-dimensional submanifolds, consideration of this special case is, in a sense, reasonable. However, it is generally not easy to construct a correct picture about an m-dimensional manifold from information obtained on certain one-dimensional submanifolds. Accordingly, it is certainly desirable to derive error estimates for the general case of an m-dimensional parameter space.

This is the topic of the present paper. The definition of the errors under consideration depends critically on the choice of the local coordinates on the manifolds. After some preliminaries in Section 2, these local coordinates are the topic of Section 3. Then in Sections 4,5,6 we consider in some detail the idea behind the numerical procedures used to trace paths on a manifold. Included here are the role played by augmented equations and their use in characterizing points on the manifold.

Then in Section 7 we extend the error estimates developed in [4] to the general situation of m-dimensional solution manifolds. These results allow for the study of approximations of such manifolds directly, without recourse to paths. The arguments of Section 7 depend on the validity of a certain stability condition whose implications and various reformulations are the topic of Section 8. Finally, two examples, in Section 9, illustrate the theory and show, in particular, that it applies to a very wide class of boundary-value problems for partial differential equations.

2. Preliminaries

Throughout this paper, we shall use the following assumption:

(A) $F: E \subset X \to Y$ is a C^r-Fredholm mapping with $r \geq 1$ and index $m \geq 1$ from an open subset E of a Banach space X into a Banach space Y.

As usual, a point $x \in E$ is a regular point of F if $DF(x)$ is surjective. For any $y_0 \in F(E)$, our interest centers on the regular-solution set

$$M = \{x \in E: \ x \text{ regular}, \ F(x) = y_0\}, \tag{2.1}$$

of the equation

$$F(x) = y_0. \tag{2.2}$$

The fundamental result concerning the structure of M is contained in the following theorem proved in [4].

Theorem 2.1: For any $y_0 \in F(E)$, the regular solution set M is a relatively open, m-dimensional C^r-manifold in X.

The notion of parametrization introduced in [4] extends easily to the manifold M. However, we modify the terminology somewhat in order to be consistent with the customary language of differentiable manifolds and bifurcation theory.

For any $x_0 \in M$, let $T_{x_0}M$ denote the tangent space of M at x_0. In our setting, we can identify $T_{x_0}M$ with $\ker DF(x_0)$. Let $X = V \oplus T$ be any splitting of X such that $\dim T = m$ and $V \cap T_{x_0}M = \{0\}$. With any such splitting, it is always possible to define an isomorphism $A: Y \to V$ of Y onto V. In particular, the splitting $X = V \oplus T_{x_0}M$ with any complementary subspace V of $T_{x_0}M$ and the isomorphism $A = (DF(x_0)|V)^{-1}$ has important applications in the reduced-basis technique (see [5]).

The following theorem is an immediate generalization of the corresponding result in [4]. Its proof involves an application of the implicit function theorem.

<u>Theorem 2.2</u>: Let $y_0 \in F(E)$, $X = V \oplus T$ be a splitting as above at a given point $x_0 \in M$, and $A: Y \to V$ an isomorphism of Y onto V. Then there exist an open ball $J \subset T$ with $0 \in J$, an open neighborhood $U \subset X$ of x_0, and a unique C^r-function $\eta: J \to Y$ such that $\eta(0) = 0$ and

$$M \cap U = \{x \in X: \; x = x(t) \equiv x_0 + t + A\eta(t), \; t \in J\}.$$

This theorem justifies the use of the word parametrization in [4]. However, since we wish to reserve the word parameter for another concept, we have to carry the result of Theorem 2.2 somewhat further.

<u>Corollary 2.3</u>: Under the conditions of Theorem 2.2, there exist an open ball $J_0 \subset J$ with $0 \in J_0$ and an open neighborhood $U_0 \subset U$ of x_0 such that $t \mapsto x(t) \equiv x_0 + t + A\eta(t)$ is a C^r-diffeomorphism of J_0 onto $M \cap U_0$.

<u>Proof</u>: From $F(x(t)) = 0$ for $t \in J$, it follows that $DF(x_0)x'(0) = 0$ and hence, since T and V are complementary subspaces, that $x'(0): T \to T_{x_0}M$ is an isomorphism. Let W be any complementary subspace of $T_{x_0}M$ in X and define

$$\phi: \; J + W \subset X \to X, \quad \phi(x) = x(t) + w, \quad x = t + w \in J + W.$$

Then we have

$$\phi'(0)\xi = x'(0)\tau + \omega, \quad \xi = \tau + \omega \in T_{x_0}M \oplus W,$$

and thus $\phi'(0)$ is an isomorphism of X. By the inverse function theorem, ϕ maps a neighborhood of 0 C^r-diffeomorphically onto a neighborhood of 0. Hence, because of $\phi(t + 0) = x(t)$, the result follows.

The inverse of the mapping $x: J_0 \to M \cap U_0$ is a chart or coordinate mapping at x_0 of the manifold M and such mappings are often said to define a system of local coordinates on M. For that reason, we call a splitting $X = V \oplus T$ such that $\dim T = m$ and $V \cap T_{x_0}M = \{0\}$ a <u>coordinate splitting</u> for M at $x_0 \in M$, and we refer to T as a <u>coordinate space</u>.

3. Parameters as Coordinates

It often happens in applications that certain quantities are naturally identified as parameters. This means that there is an intrinsic splitting $X = Z \oplus \Lambda$ of X, where Λ is an m-dimensional parameter-space and Z represents a state space. Such a splitting of X will be called a parameter splitting. It is natural to attempt to use the parameter space Λ as the coordinate space T of a coordinate splitting, and the question arises when this is possible.

In order to provide an answer to this question, we suppose for the remainder of this section that a parameter splitting $X = Z \oplus \Lambda$ is available. Moreover, for a given $y_0 \varepsilon F(E)$ let M denote the corresponding regular-solution manifold (2.1) and x_0 any point of M.

The suitability of Λ as the coordinate space $\dot{T}$ of a coordinate splitting depends on the subspace

$$Z_0 = Z \cap T_{x_0}M. \tag{3.1}$$

If $Z_0 = \{0\}$, then clearly $X = Z \oplus \Lambda$ is a coordinate splitting for the manifold M at x_0. But if $Z_0 \neq \{0\}$ then $T_{x_0}M$ has a nontrivial component in Z and hence we cannot use all of Λ as the coordinate space T. Minimally we should exchange Z_0 for a part of Λ not meeting $T_{x_0}M$.

To this end, let Π be the natural projection of X onto Λ along Z and define the space

$$\Lambda_0 = \Pi T_{x_0}M. \tag{3.2}$$

Moreover, the following nota[illegible] wi[illegible] be convenient: If W is any Banach space and $W_0 \subset W$ a closed subspace which splits W, then $W \ominus W_0$ shall denote any closed complementary subspace of W_0 in W.

Some important properties about the subspaces Z_0 and Λ_0 are contained in the following lemma.

<u>Lemma 3.1</u>:

(i) $\dim \Lambda_0 = m - \dim Z_0$.

(ii) $\Lambda_0 = \Lambda \cap (Z + T_{x_0}M)$.

(iii) $Z \oplus \Lambda_0 = Z + T_{x_0}M$.

(iv) $[(Z \ominus Z_0) \oplus (\Lambda \ominus \Lambda_0)] \cap T_{x_0}M = \{0\}$.

<u>Proof</u>: The proof involves only elementary notions about subspaces. Clearly (i) needs no elaboration. To prove (ii), let $\lambda \in \Lambda_0$. Then $\lambda = z + u$ for some $u \in T_{x_0}M$ and any $z \in Z$, whence $\lambda \in \Lambda \cap (Z + T_{x_0}M)$. Conversely, if the latter inclusion holds for some λ then $\lambda = z + u$ for some $z \in Z$, $u \in T_{x_0}M$ and $\lambda = \Pi\lambda = \Pi u \in \Lambda_0$. In (iii) the containment $Z \oplus \Lambda_0 \subset Z + T_{x_0}M$ is obvious in view of (ii). To show the reverse, let $x = z + u$, $z \in Z$, $u \in T_{x_0}M$. Then there exist $z' \in Z$, $\lambda \in \Lambda$ such that $x = z' + \lambda$, whence $\lambda = (z-z') + u \in Z + T_{x_0}M$ and thus $\lambda \in \Lambda_0$ by (ii) or $x \in Z \oplus \Lambda_0$. Finally, let u belong to the set on the left of (iv) and $u = z + \lambda$, $z \in Z$, $\lambda \in \Lambda$. Then $\lambda = -z + u \in Z + T_{x_0}M = Z \oplus \Lambda_0$, whence $\lambda \in \Lambda_0$ and $\lambda \in \Lambda_0 \cap (\Lambda \ominus \Lambda_0) = \{0\}$. As a result we have $u = z \in Z \ominus Z_0 \subset Z$ and therefore $u \in Z \cap T_{x_0}M = Z_0$ which implies that $x \in Z_0 \cap (Z \ominus Z_0) = \{0\}$.

The above facts can be combined to give the desired theorem on coordinate splittings.

<u>Theorem 3.2</u>: In addition to assumption (A), let $X = Z \oplus \Lambda$ be a parameter splitting of X, $y_0 \varepsilon F(E)$, and x_0 a point of the corresponding regular-solution manifold M. Let $Z_0 = Z \cap T_{x_0}M$ and $\Lambda_0 = \Pi T_{x_0}M$, where Π is the projection onto Λ along Z. Then $X = [(Z \ominus Z_0) \oplus (\Lambda \ominus \Lambda_0)] \oplus [Z_0 \oplus \Lambda_0]$ is a coordinate splitting for M at x_0.

This theorem is illustrated in the next section.

4. Parameters and Integral Manifolds

All currently available numerical methods for analyzing a solution manifold of a parametrized equation involve the computational trace of one-dimensional submanifolds by means of a continuation process. These one-dimensional submanifolds are usually defined by combinations of the natural parameters with one degree of freedom and the specification of a starting point. In differential-geometric terms, this means that we compute specific integral manifolds of certain 1-distributions on the manifold.

Before we define these notions explicitly, an example to illustrate the situation may be in order. Consider the buckling of a spring system discussed in [9 ; pp. 301-305]. After a suitable scaling, the total energy of the system is

$$U(p,q,\lambda,\nu,\gamma) = (1-p)^2 + \frac{1}{2}\gamma(2q)^2 + 2\lambda p\cos q + \nu p\sin q,$$

where p,q are displacements (the state variables) and the parameters λ,ν,γ represent applied forces and a spring constant, respectively. The spring constant γ is intrinsically positive. As a result, the equilibrium equation is

$$F(x) = \begin{bmatrix} -2(1-p) + 2\lambda\cos q + \nu\sin q \\ \\ 4\gamma q - 2\lambda p\sin q + \nu p\cos q \end{bmatrix} = 0, \tag{4.1}$$

where $F\colon E \subset \mathbb{R}^5 \to \mathbb{R}^2$ is a C^∞-mapping on $E = \{x = (p,q,\lambda,\nu,\gamma)^T \in \mathbb{R}^5 : \gamma > 0\}$. It is easly verified that all points of E are regular for F and that $0 \in F(E)$. Consequently, the equilibria of the system correspond to points on the three-dimensional manifold $M = \{x \in E: F(x) = 0\}$. The natural parameter splitting $X = Z \oplus \Lambda$ associated with the parameters λ, ν, γ is obtained by setting

$$(4.2) \qquad Z = \{(p,q,0,0,0)^T\}, \quad \Lambda = \{(0,0,\lambda,\nu,\gamma)^T\}.$$

As indicated before, we introduce now some combination of parameter values with one degree of freedom. The simplest approach is to fix two of the above three parameters, say $\nu = 0$, $\gamma = \frac{1}{8}$. This is equivalent to the introduction of the augmented equation

$$(4.3) \qquad G(x) \equiv \begin{bmatrix} F(x) \\ \nu \\ \gamma \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ \frac{1}{8} \end{bmatrix},$$

where $G\colon E \subset \mathbb{R}^5 \to \mathbb{R}^4$ is still of class C^∞ on E, but now not all points of E are regular for G. For example, at $x_0 = (\frac{1}{2}, 0, \frac{1}{2}, 0, \frac{1}{8})^T \varepsilon E$, $DG(x_0)$ has rank 3 whereas, of course, the rank of $DF(x_0)$ is 2.

The point $x_0 = (\frac{1}{2}, 0, \frac{1}{2}, 0, \frac{1}{8})^T$ is an example of a point where the natural parameter splitting (4.2) of the original problem is not a coordinate splitting and Theorem 3.2 comes into play. For this particular x_0, we have that

$$T_{x_0}M = \ker DF(x_0) = \{(p,q,-p,0,\gamma)^T\},$$

and hence that

$$(4.4) \qquad \begin{aligned} Z_0 &= Z \cap T_{x_0}M = \{(0,q,0,0,0)^T\}, \\ \Lambda_0 &= \Pi T_{x_0}M = \{(0,0,\lambda,0,\gamma)^T\}. \end{aligned}$$

By Theorem 3.2, we see that

$$
\begin{aligned}
X &= [(Z \ominus Z_0) \oplus (\Lambda \ominus \Lambda_0)] \oplus [Z_0 \oplus \Lambda_0] \\
&= \{(p,0,0,\nu,0)^T\} \oplus \{(0,q,\lambda,0,\gamma)^T\}
\end{aligned}
\tag{4.5}
$$

is one possible coordinate splitting for M at x_0.

Generally, let M denote any finite-dimensional Hausdorff manifold of class C^r, $r \geq 1$. We again denote the tangent space of M at a point $x \in M$ by T_xM and write TM for the tangent bundle. Recall that a <u>vector field</u> of class C^ρ, $1 \leq \rho < r$, on a relatively open subset $M_0 \subset M$ is a C^ρ-mapping $\xi: M_0 \to TM$ such that $\xi(x) \in T_xM$ for each $x \in M_0$. Moreover, a <u>1-distribution</u> Δ of class C^ρ on M_0 is defined by the following two properties:

(i) Δ is a mapping $\Delta: M_0 \to TM$ such that Δ_x is a one-dimensional subspace of T_xM for each $x \in M_0$.

(ii) For each $x_0 \in M_0$, there exist an open neighborhood U of x_0 in X and a vector field $\xi: M_0 \cap U \to TM$ of class C^ρ such that $\Delta_x = \text{span}\,\{\xi(x)\}$ for all $x \in M_0 \cap U$.

A one-dimensional submanifold N of M_0 is an <u>integral manifold</u> of Δ on M_0 if for every $x \in N$ the tangent space T_xN is equal to Δ_x. For an introduction to these concepts see, for example, [11].

For our simple example, let $E_0 \subset E$ be the open set of all regular points of G and set $M_0 = M \cap E_0$. Then $\Delta: M_0 \to TM$, $\Delta_x = \ker DG(x)$, $x \in M_0$, is a mapping which associates with each regular point $x \in M_0$ a one-dimensional subspace of the tangent space T_xM. In this case, it happens to be possible to define a global vector field on M_0 which satisfies the second condition of a 1-distribution. In fact, for any $x \in E_0$ a vector $\xi(x) \in \mathbb{R}^5$ is uniquely defined by

$$(4.6) \qquad DG(x)\xi(x) = 0, \quad ||\xi(x)||_2 = 1, \quad \det\begin{bmatrix} DG(x) \\ \xi(x)^T \end{bmatrix} > 0,$$

and it is readily verified that the mapping $x \mapsto \xi(x)$ is of class C^{r-1} on E_0 and hence a C^{r-1}-vector field on M_0. This shows that Δ is indeed a 1-distribution of class C^{r-1} on M_0.

The point $x_0 = (\frac{1}{2}, 0, \frac{1}{2}, 0, \frac{1}{8})^T$ turns out to be a singular point of the vector field ξ. In fact, for the value of $\gamma = \frac{1}{8}$, the bifurcation set in the $\lambda\nu$-plane consists of a butterfly and touching dual cusps (see Figure 1 and [9]). The dual-cusp point corresponds to x_0.

The global definition (4.6) of the vector field is possible only in the finite-dimensional case. However, we do not need a determinant to define an orientation of a local vector field in a neighborhood of each $x \in M_0$. In other words, it may be expected that augmented equations of the form (4.3) always define 1-distributions on certain relatively open submanifolds M_0 of M. The numerical methods mentioned earlier are then designed to compute specific integral manifolds of such 1-distributions. The connection between augmented equations and certain 1-distributions is taken up in detail in the next section. Once augmented equations are associated with 1-distributions, they can then be used to classify an arbitrary point of M and to provide an alternate formulation of Theorem 3.2. This is the topic of Section 6.

Our example shows that the singular behavior at the point $x_0 = (\frac{1}{2}, 0, \frac{1}{2}, 0, \frac{1}{8})^T$ arises only because we considered a particular parameter combination and the corresponding 1-distribution. Otherwise, the point is regular for F. Moreover, instead of fixing $\nu = 0$, $\gamma = \frac{1}{8}$ and letting λ vary as we did in (4.3),

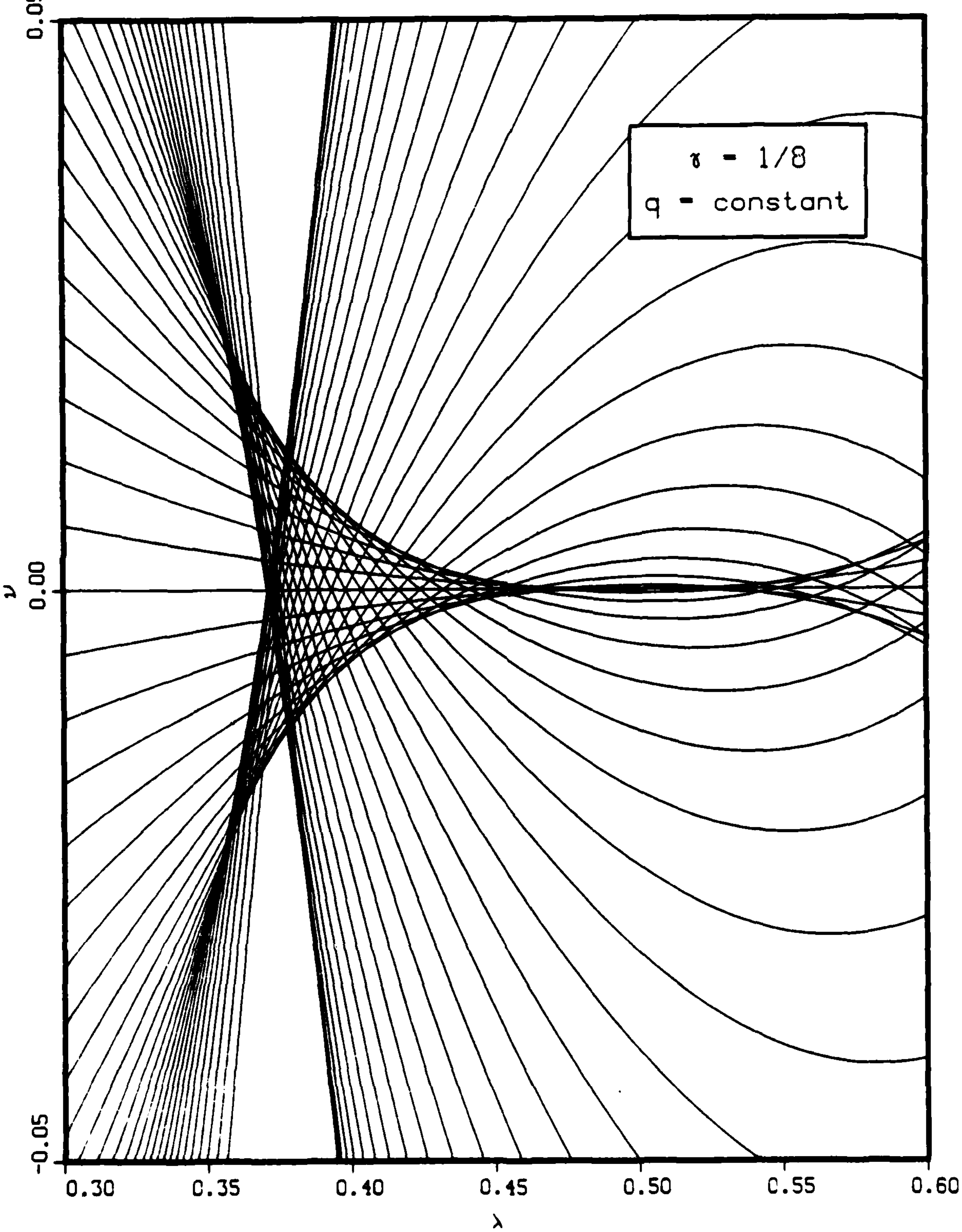


Figure 1

we may fix $\nu = \frac{1}{2}$, $\gamma = \frac{1}{8}$ and allow ν to vary. Then we are led to the augmented equation

$$G(x) \equiv \begin{bmatrix} F(x) \\ \lambda \\ \gamma \end{bmatrix} = \begin{bmatrix} 0 \\ \frac{1}{2} \\ \frac{1}{8} \end{bmatrix}. \tag{4.7}$$

For this new G, $DG(x_0)$ has rank 4 so that x_0 now becomes a regular point of G. This opens up the possibility of considering numerical methods which avoid the singularity entirely by working with other more suitable 1-distributions such as that defined by the augmented equation (4.7).

5. 1-Distributions and Augmented Equations

In the last section, we discussed the idea behind continuation methods for analyzing solution manifolds of parametrized equations and how parameter combinations lead to 1-distributions and one-dimensional integral manifolds. In this section, we consider the relationship between such integral manifolds and certain augmented equations.

Throughout this and the next section, let M denote the m-dimensional C^r-manifold (2.1) of the equation (2.2) and assume that $X = Z \oplus \Lambda$ is a parameter splitting of X. A combination of the natural parameters with one degree of freedom corresponds to the choice of a one-dimensional subspace $S \subset \Lambda$ which defines the remaining degree of freedom. We call S a reduced-parameter space and any splitting $X = W \oplus S$ with $Z \subset W$ a reduced-parameter splitting. For example, in (4.3) the reduced-parameter space is $S = \{(0,0,\lambda, 0,0)^T\}$, whereas in (4.7) $S = \{(0,0,0,\nu,0)^T\}$.

Suppose now that $\Delta\colon M_0 \to TM$ is a 1-distribution of class C^{r-1} on some open set $M_0 \subset M$ for which $\Pi\Delta_x = S$ for all $x \in M_0$. As in Section 3, Π denotes the projection of X onto Λ along Z. Clearly, the condition $\Pi\Delta_x = S$ is equivalent to $\Delta_x \subset Z \oplus S$. We call a 1-distribution with this property a 1-distribution with respect to S.

1-distributions with respect to S can be discussed conveniently in the context of certain augmented forms of the equation (2.2). For any given reduced-parameter space S, there exist linear operators $L\colon \Lambda \to \mathbb{R}^{m-1}$ with $\ker L = S$. With any choice of such L, we define the augmented mapping

$$(5.1) \qquad G\colon E \subset X \to Y \times \mathbb{R}^{m-1}, \quad G(x) = (F(x), L\Pi x), \quad x \in E,$$

and with it the augmented equation

$$G(x) = (y_0, L\Pi a_0), \tag{5.2}$$

where $a_0 \in X$ is as yet arbitrary. For example, in (4.3) and (4.7) $L: \Lambda \to \mathbb{R}^2$ is defined by

$$L = \begin{bmatrix} 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix} \quad \text{and} \quad L = \begin{bmatrix} 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix},$$

respectively.

A number of useful facts about (5.1) are collected together in the following technical lemma.

Lemma 5.1: Let $x_0 \in M$. Then:

(i) $DG(x_0)$ is a Fredholm operator of index 1.

(ii) $DF(x_0)|Z$, and hence $DG(x_0)|W$, is a Fredholm operator of index 0.

(iii) $\ker DG(x_0) = T_{x_0}M \cap (Z \oplus S)$.

(iv) $Z_0 \equiv Z \cap T_{x_0}M = W \cap \ker DG(x_0)$.

(v) $S \cap \Pi T_{x_0}M = \{0\}$ if and only if $Z_0 = \ker DG(x_0)$.

Proof: Note that

$$DG(x_0) = (DF(x_0), L\Pi) = (DF(x_0), 0) + (0, L\Pi),$$

where $(DF(x_0),0): X \to Y \times \mathbb{R}^{m-1}$ is a Fredholm operator with index $m - (m-1) = 1$ and $(0, L\Pi): X \to Y \times \mathbb{R}^{m-1}$ is a compact operator. This implies that (i) holds (see [10; p. 114]). The injection $j: Z \to X$, $jz = z$, $z \in Z$, is a Fredholm operator of index $0 - m = -m$. Therefore, $DF(x_0)|Z = DF(x_0)j$ is a Fredholm operator with index $m - m = 0$ ([10; p. 111]) which proves (ii). Now (iii) follows directly from the definition of G and the fact that $u \in \ker DG(x_0)$ if and only if $u \in \ker DF(x_0) = T_{x_0}M$ and $\Pi u \in \ker L = S$. Furthermore, (iv) is a consequence of (iii) and $Z \subset W$. Finally, to obtain (v), we note that

$$
\begin{aligned}
S \cap \Lambda_0 = \{0\} &\iff \Pi(T_{x_0}M \cap (Z \oplus S)) = \{0\} \\
&\iff \Pi \ker DG(x_0) = \{0\} \quad \text{by (iii)} \\
&\iff \ker DG(x_0) \subset Z \\
&\iff \ker DG(x_0) = Z_0 \quad \text{by (iv)}.
\end{aligned}
$$

This completes the proof.

Note that none of the assertions of the lemma depends upon the particular choice of L in the definition of G. A consequence of part (iii) is the following connection between 1-distributions with respect to S and augmented equations.

<u>Theorem 5.2</u>: Let $X = W \oplus S$ be a reduced-parameter splitting and $\Delta: M_0 \to TM$ a 1-distribution with respect to S of class C^{r-1} on a relatively open set $M_0 \subset M$. Moreover, let G be an augmented function (5.1). Then $\Delta_x \subset \ker DG(x)$ for any $x \in M_0$ and hence any integral manifold N of Δ on M_0 is a solution manifold of the augmented problem (5.2), where a_0 is a given point of N.

So far, we began with a 1-distribution with respect to S and then related it to an augmented equation. In practice, the starting point is usually an augmented equation which is then used to characterize and determine a 1-distribution with respect to S. This was the procedure we followed in the example of the previous section. How this can be accomplished in general is our next goal.

As before, let $X = W \oplus S$ be a reduced-parameter splitting of X and G an augmented function (5.1). We denote by $E_0 \subset E$ the open set of all regular points of G and set $M_0 = M \cap E_0$.

<u>Theorem 5.3</u>: The mapping $\Delta: M_0 \to TM$, $\Delta_x = \ker DG(x)$, $x \in M_0$, is a 1-distribution with respect to S of class C^{r-1} on M_0. Moreover, if $r \geq 2$, then for any given $a_0 \in M_0$ the regular-solution set of the augmented equation (5.2) is an integral manifold of Δ on M_0.

<u>Proof</u>: By Lemma 5.1(i), $DG(x)$ is, for any $x \in M_0$, a Fredholm operator of index 1 and since $DG(x)$ is surjective we must have $\dim DG(x) = 1$. Consequently, the mapping Δ is well-defined and by Lemma 5.1(iii) we have $\Delta_x = T_xM \cap (Z \oplus S)$ for $x \in M_0$. Therefore, it remains to verify only the second property of a 1-distribution. By the theory of Fredholm operators [2], [10], for $x_0 \in M_0$, there exist a vector $u_0 \in X$ and a continuous linear functional $\phi_0 \in X^*$ such that $\Delta_{x_0} = \ker DG(x_0) = \mathrm{span}\{u_0\}$, $\phi_0(u_0) = 1$, and $X = \ker \phi_0 \oplus \Delta_{x_0}$. The mapping

$$H: E \times X \to Y \times \mathbb{R}, \quad H(x,u) = (DG(x)u, 1-\phi_0(u)), \quad (x,u) \in E \times X,$$

is of class C^{r-1} and satisfies $H(x_0,u_0) = 0$. As in the proof of part (i) of Lemma 5.1, the partial derivative

$$D_uH(x_0,u_0)\cdot = (DG(x_0)\cdot,0) + (0,-\phi_0(\cdot))$$

is the sum of a Fredholm operator of index 0 and a compact operator and hence is itself a Fredholm operator of index 0. If $D_uH(x_0,u_0)v = 0$, then $v \in \ker \phi_0 \cap \Delta_{x_0} = \{0\}$; that is, $D_uH(x_0,u_0)$ is an isomorphism. By the implicit function theorem, there exist a nieghborhood U of x_0 in X and a C^{r-1}-mapping $\xi: U \to X$ such that

$$H(x,\xi(x)) = (DG(x)\xi(x),1-\phi_0(\xi(x))) = (0,0), \quad x \in U.$$

Therefore, $\xi: M_0 \cap U \to TM$ is a local vector field and $\Delta_x = \mathrm{span}\{\xi(x)\}$ for $x \in M_0 \cap U$. This completes the proof that Δ is a 1-distribution with respect to S of class C^{r-1}. The second part follows from the standard existence and uniqueness theory for flows since we assumed $r \geq 2$ (for example, see [8]).

Theorem 5.3 guarantees that the regular-solution set of an augmented equation can be used to determine an integral manifold. But an augmented function (5.1) is defined on all of E; that is, an augmented problem can also be considered at nonregular points of the augmented function. We consider this topic in the next section.

6. Characterization of Points Using Augmented Functions; Coordinate Splittings

We continue the development begun in the previous section and consider how an augmented function can be used to characterize the nature of an arbitrary point on the regular-solution manifold M of equation (2.2) with respect to a given parameter splitting $X = Z \oplus \Lambda$ and reduced-parameter space S. Theorem 5.3 applies only on the submanifold M_0 of M consisting of the regular points of the augmented function. Nevertheless, any augmented function is defined everywhere on M and, as noted earlier, that observation enables us to use an augmented function to characterize different types of points on M.

A general examination of the possible types of points on M is not our intent. Instead, we consider three commonly occurring cases which arise frequently in practice. The following definition is based upon fairly standard terminology.

Definition 6.1: Let $X = W \oplus S$ be a reduced-parameter splitting and G any corresponding augmented function (5.1). A point $x_0 \in M$ is a

(i) nonsingular point of G if $\dim \ker DG(x_0) = 1$ and $W \cap \ker DG(x_0) = \{0\}$;

(ii) limit point of G if $\dim \ker DG(x_0) = 1$ and $\ker DG(x_0) \subset W$; and

(iii) simple critical point of G if $\dim \ker DG(x_0) = 2$ and $\dim \{W \cap \ker DG(x_0)\} = 1$.

Note that this definition depends only on W and S and is otherwise independent of the choice of G. In fact, the definition involves only W and $\ker DG(x_0)$ and, in view of Lemma 5.1(iii), $\ker DG(x_0)$ depends only on S and not on the particular choice of the linear operator $L: \Lambda \to \mathbb{R}^{m-1}$. Also, we remark that nonsingular points and limit points of G are regular points of G.

For the three cases, we may characterize completely the nature of a point $x_0 \in M$ in terms of the subspaces $Z_0 = Z \cap T_{x_0}M$, $\Lambda_0 = \Pi T_{x_0}M$, and S. In the following discussion, we assume the setting of Theorem 3.2, (5.1), and Definition 6.1. We also assume that $r \geq 2$ in assumption (A) (in order to apply Theorem 5.3).

We begin with the case $Z_0 = \{0\}$. Then from Lemma 5.1(ii) and (iv), it follows that x_0 is a nonsingular point of G. By Theorem 5.3 (with $a_0 = x_0$) and Theorem 3.2 (applied to G), $X = W \oplus S$ is a coordinate splitting for the regular-solution submanifold of the augmented problem

$$G(x) = (y_0, L\Pi x_0) \tag{6.1}$$

at x_0. Moreover, by Lemma 5.1(iv) and Theorem 3.2 (applied to F), $X = Z \oplus \Lambda$ is a coordinate splitting for the manifold M at x_0.

Next we assume that $\dim Z_0 = 1$ and $S \cap \Lambda_0 = \{0\}$. From Lemma 5.1(v), it follows that x_0 is a limit point of G, and Theorems 5.3 and 3.2 imply that $X = [(W \ominus Z_0) \oplus S] \oplus Z_0$ is a coordinate splitting for the regular-solution submanifold of the augmented problem (6.1) at x_0. (Note that the use of $Z_0 = \ker DG(x_0)$ as the coordinate space is the basis for the Lyapunov-Schmidt procedure [12].) To extend this coordinate splitting for the submanifold to a coordinate splitting for the full manifold M, we combine Lemma 3.1(i) with Lemma 5.1(v) to conclude that $S \oplus \Lambda_0 = \Lambda$ and then use Theorem 3.2. This shows that $X = [(Z \ominus Z_0) \oplus S] \oplus [Z_0 \oplus \Lambda_0]$ is a coordinate splitting for M at x_0.

Finally, let $\dim Z_0 = 1$ and $S \cap \Lambda_0 \neq \{0\}$ (that is, $S \cap \Lambda_0 = S$). Again from Lemma 5.1, it follows that x_0 is a simple critical point of G. Although $DG(x_0)$ remains a Fredholm operator of index 1, its null space is now two-dimensional and x_0 is not a regular point of G. Nevertheless, x_0 is still a regular point of F and the manifold M does have coordinate splittings at x_0. In particular, by Theorem 3.2 once more, $X = [(Z \ominus Z_0) \oplus (\Lambda \ominus \Lambda_0)] \oplus [Z_0 \oplus \Lambda_0]$ is a coordinate splitting for M at x_0. It is important to observe that now S is not an acceptable choice for the complementary subspace $\Lambda \ominus \Lambda_0$.

We summarize the above results on coordinate splittings in our next theorem.

<u>Theorem 6.2</u>: Assume the conditions and notation of Theorem 3.2, (5.1), and Definition 6.1.

(i) If $Z_0 = \{0\}$, then x_0 is a nonsingular point of the augmented function G; $X = W \oplus S$ is a coordinate splitting for the regular-solution submanifold of the augmented problem (6.1) at x_0, and $X = Z \oplus \Lambda$ is a coordinate splitting for the regular-solution manifold M of (2.2) at x_0.

(ii) If $\dim Z_0 = 1$ and $S \cap \Lambda_0 = \{0\}$, then x_0 is a limit point of the augmented function G; $X = [(W \ominus Z_0) \oplus S] \oplus Z_0$ is a coordinate splitting for the regular-solution submanifold of the augmented problem (6.1) at x_0, and $X = [(Z \ominus Z_0) \oplus S] \ominus [Z_0 \oplus \Lambda_0]$ is a coordinate splitting for the regular-solution manifold M of (2.2) at x_0.

(iii) If $\dim Z_0 = 1$ and $S \cap \Lambda_0 \neq \{0\}$, then x_0 is a simple critical point of the augmented function G; $X = [(Z \ominus Z_0) \oplus (\Lambda \ominus \Lambda_0)] \oplus [Z_0 \oplus \Lambda_0]$ is a coordinate splitting for the regular-solution manifold M of (2.2) at x_0. Here, S is not an acceptable choice for $\Lambda \ominus \Lambda_0$.

Thus, from a knowledge of the subspaces Z_0, Λ_0, and S alone, we are able to characterize the nature of the point x_0 with respect to a combination of parameters. In Theorem 6.2(iii), it is not possible to identify the type of simple critical point because that would require an examination of second and higher derivatives. However, it is interesting to note that the coordinate splitting for the manifold M is independent of the type of simple critical point at x_0; in other words, it is not necessary to know the type of simple critical point when choosing a coordinate splitting for M.

To illustrate this theorem, we return to the problem of the buckling of a spring system mentioned in Section 4. With $x_0 = (\frac{1}{2}, 0, \frac{1}{2}, 0, \frac{1}{8})^T$, we calculated Z_0 and Λ_0 in (4.4). Therefore, according to Theorem 6.2, since $\dim Z_0 = 1$, x_0 is either a limit point or a simple critical point of any augmented function G. The augmented function G of (4.3) is associated with the reduced-parameter space $S = \{(0,0,\lambda,0,0)^T\}$ for which $S \cap \Lambda_0 \neq \{0\}$. By Theorem 6.2(iii), x_0 must be a simple critical point of that augmented function and (4.5) is a coordinate splitting for the regular-solution manifold of (4.1) at x_0. On the other hand, the augmented function G of (4.7) is associated with the reduced-parameter space $S = \{(0,0,0,\nu,0)^T\}$ for which $S \cap \Lambda_0 = \{0\}$. By Theorem 6.2(ii), x_0 is a limit point of that augmented function; $X = \{(p,0,\lambda,\nu,\gamma)^T\} \oplus \{(0,q,0,0,0)^T\}$ is a coordinate splitting for the regular-solution submanifold of the augmented problem (4.7) at x_0, and (4.5) again serves as a coordinate splitting for the regular-solution manifold of (4.1) at x_0.

7. Finite-Dimensional Approximations and Error Estimates

We consider now the fundamental problem of determining suitable approximations to the regular-solution manifold M of the equation (2.2) and of obtaining corresponding estimates for the error. As it turns out, the discussion in [4] applies essentially unchanged to the case of an m-dimensional manifold. Therefore, we merely introduce the notation and state the principal results in our current, more general setting and refer to [4] for the proofs which are easily extended to this case.

As always we assume that the information in (A) is given. To formulate an appropriate approximate problem for (2.2), we remark that in applications it is the state variables and not the natural parameters which are discretized. With that in mind, suppose that a parameter splitting

$$X = Z \oplus \Lambda \qquad (\dim \Lambda = m) \tag{7.1}$$

of X is available and that Z and Y are related through an operator Q as follows:

$$\begin{aligned} &Q \in L(X,Y), \quad \ker Q = \Lambda, \\ &Q|Z \text{ is an isomorphism of } Z \text{ onto } Y. \end{aligned} \tag{7.2}$$

A discretization of the problem (2.2) is specified by a collection $\{P_h: h > 0\}$ of finite-rank projections $P_h \in L(Y)$ converging strongly to the identity on Y; that is,

$$\lim_{h\to 0} P_h y = y, \quad y \in Y. \tag{7.3}$$

The projections P_h and the operator Q determine finite-dimensional subspaces

$$(7.4) \qquad Y_h = P_h Y, \quad Z_h = (Q|Z)^{-1} Y_h, \quad X_h = Z_h \oplus \Lambda,$$

and out interest is directed towards the regular-solution manifold

$$(7.5) \qquad M_h = \{x \in E_h: \ x \text{ regular}, \ F_h(x) = y_{oh}\}$$

of the equation

$$(7.6) \qquad F_h(x) = y_{oh}, \quad y_{oh} = P_h y_o,$$

where

$$(7.7) \qquad F_h: E_h \subset X_h \to Y_h, \quad F_h(x) = P_h F(x), \quad x \in E_h = E \cap X_h.$$

We refer to the information contained in (7.3)-(7.7) as a basic discretization of the problem (2.2).

It is easier to compare a discretized problem (7.6) with the original problem (2.2) if both are formulated on the same spaces. A convenient way of doing this is to extend the discretized mapping F_h to all of $E \subset X$ by defining

$$(7.8) \qquad \bar{F}_h: E \subset X \to Y, \quad \bar{F}_h(x) = (I-P_h)Qx + P_h(F(x)-y_o), \quad x \in E,$$

where I denotes the identity on Y.

Some important properties of $\bar{F}_h$ and its connection to a discretized problem (7.6) are summarized in the next lemma.

<u>Lemma 7.1</u>: $\bar{F}_h$ is a C^r-mapping with the following properties:

(i) $\bar{F}_h(x) = 0$ for $x \in E$ if and only if $x \in E_h$ and $F_h(x) = y_{oh}$.

(ii) $D\bar{F}_h(x)X_h \subset Y_h$, $x \in E$.

(iii) $\ker D\bar{F}_h(x) \subset X_h$, $x \in E$.

(iv) $D\bar{F}_h(x) \in L(X,Y)$ is a Fredholm operator of index m for $x \in E$.

(v) $P_h DF(x)X_h = Y_h$ for some $x \in E$ implies that x is a regular point of $\bar{F}$.

To compare the regular-solution manifolds M of (2.1) and M_h of (7.5), let x_o be any point of M. Moreover, let $X = V \oplus T$ *be a coordinate* splitting for M at x_o and $A: Y \to V$ an isomorphism of Y onto V. The relationship between the coordinate splitting and the above basic discretization is expressed in the following <u>stability condition</u>:

(S) $$||D\bar{F}_h(x_o)Ay|| \geq \delta||y||, \quad y \in Y, \quad h > 0 \text{ sufficiently small},$$

where δ is a positive constant independent of y and h. A detailed discussion of the implications of this stability condition, including equivalent formulations, is deferred to the next section.

The stability condition (S) combined with generalized versions of the inverse and implicit function theorems enables us to obtain the main result regarding existence of approximate solutions and error estimates.

<u>Theorem 7.2</u>: (1) Let $y_o \in F(E)$, $X = Z \oplus \Lambda$ a parameter splitting of X, $X = V \oplus T$ a coordinate splitting for M at a given point $x_o \in M$, and

$A: Y \to V$ an isomorphism. Suppose that an operator (7.2) and a basic discretization are chosen so that the stability condition (S) is satisfied. Then for all sufficiently small $h > 0$, there exist $x_{oh} \in M_h$ such that $\lim_{h \to 0} x_{oh} = x_o$.

(2) In addition, assume that $r \geq 2$ and let $x: J \subset T \to M$ be a C^r-function representing the manifold M locally near x_o as given by Theorem 2.2. Then there exist a compact ball $J_o \subset J$, $0 \in J_o$, and C^r-functions $x_h: J_o \to M_h$ representing the approximate manifolds M_h locally near x_{oh} such that

$$(7.9) \qquad ||x(t)-x_h(t)|| \leq C||(I-P_h)Qx(t)||, \quad t \in J_o,$$

where C *is independent of* h *and* t.

Actually, we can say more. It turns out that, for sufficiently small h, $X = V \oplus T$ is also a coordinate splitting of X for each M_h at x_{oh}. Moreover, if $x: J \to M$ is written as $x(t) = x_o + t + A\eta(t)$, then each $x_h: J_o \to M_h$ has the form $x_h(t) = x_o + t + A\eta_h(t)$. The proof of Theorem 7.2 follows verbatim the corresponding proof in [4].

8. Discrete Convergence and Stability Conditions

A central requirement of the theory in the previous section is the condition (S). We formulated (S) as a stability condition similar to those arising typically in convergence studies of discretization methods. In certain practical situations, (S) is readily verified. For instance, in [4] we used decompositions $F(x) = N(x) + G(x)$ with $Q = DN(x_0)$ and compact $DG(x_0)$, in which case the stability condition (S) holds for any coordinate splitting.

Nevertheless, the need remains for other necessary and sufficient conditions for the validity of (S) and for a closer analysis of the interplay between the quantities F, P_h, Q, and A entering into (S). This is the topic of the present section.

We follow Vainikko [13] in the following formulations of various types of discrete convergence. A sequence $\langle x_n \rangle$ of elements in a Banach space X is called discretely compact (d-compact for short) if every subsequence of $\langle x_n \rangle$ has a convergent subsequence.

Let B_h $(h > 0)$ and B be bounded linear operators from a Banach space X to a Banach space Y such that $B_h x \to Bx$ for all $x \in X$.

The convergence $B_h \to B$ is regular, denoted by $B_h \overset{r}{\to} B$, if

(8.1) $\quad \langle x_{h_n} : h_n \to 0 \rangle$ bounded, $\langle B_{h_n} x_{h_n} \rangle$ d-compact $\Rightarrow$ $\langle x_{h_n} \rangle$ d-compact.

The convergence $B_h \to B$ is stable, denoted by $B_h \overset{s}{\to} B$, if

(8.2) $\quad B_h^{-1}$ exist and are uniformly bounded for sufficiently small h.

The convergence $B_h \to B$ is compact, denoted by $B_h \overset{c}{\to} B$, if

(8.3) $\quad \langle x_{h_n} : h_n \to 0 \rangle$ bounded $\Rightarrow$ $\langle B_{h_n} x_{h_n} \rangle$ d-compact.

Looking back at our condition (S), we note that

$$D\bar{F}_h(x_o)Ay = (I-P_h)QAy + P_h DF(x_o)Ay \to DF(x_o)Ay, \quad y \in Y,$$

and that (S) is equivalent with stable convergence - that is, with

$$D\bar{F}_h(x_o)A \xrightarrow{s} DF(x_o)A. \tag{8.4}$$

A less obvious equivalence is the content of the following result.

Proposition 8.1: The convergence condition (8.4) and hence the condition (S) holds if and only if $D\bar{F}_h(x_o)A \xrightarrow{r} DF(x_o)A$.

The proof follows directly from a theorem of Vainikko [13; p. 655] and the observation that rge $DF(x_o)A = Y$, ker $DF(x_o)A = \{0\}$, and each $D\bar{F}_h(x_o)A$ is a Fredholm operator of index 0.

Both regular and stable convergence involve a complex interplay among F, P_h, Q, and A. The following sufficient condition for (S) is helpful here.

Proposition 8.2: If $P_h(-Q+DF(x_o))A \xrightarrow{s} (-Q+DF(x_o))A$, then $D\bar{F}_h(x_o)A \xrightarrow{r} DF(x_o)A$ and hence (S) holds.

Proof: From $D\bar{F}_h(x_o)A = QA + P_h(-Q+DF(x_o))A$, it follows that QA is a Fredholm operator of index 0. If QA is an isomorphism, then we have trivially $QA \xrightarrow{s} QA$ and rge $QA = Y$, and the result follows immediately [13; p. 654]. If QA is not an isomorphism, then by the Fredholm nature of QA and with Y_o = ker QA, Y_3 = rge QA there exist closed subspaces Y_1 and Y_2 of Y such that $Y = Y_o \oplus Y_1 = Y_2 \oplus Y_3$, $\dim Y_o = \dim Y_2 < \infty$ and QA is an isomorphism of Y_1 onto Y_3. Set $V_0 = AY_0 = V \cap \Lambda$ and $V_1 = AY_1$; then $V = V_0 \oplus V_1$. Let $\bar{V}_0 = X \ominus (V_1 \oplus \Lambda)$ and $\bar{V} = \bar{V}_0 \oplus V_1$. Since $\dim \bar{V}_0 = \dim V_0$, there exists an isomorphism B of V_0 onto $\bar{V}_0$ and with the projection P of Y onto Y_0 along Y_1 we may define the isomorphism

$$\bar{A}: Y \to \bar{V}, \quad \bar{A}y = A(I-P)y + BAPy, \quad y \in Y.$$

Then $Q\bar{A}$ is an isomorphism of Y and

$$D\bar{F}_h(x_0)A = Q\bar{A} + Q(A-BA)P + P_h(-Q+DF(x_0))A.$$

We now have trivially $Q\bar{A} \overset{S}{\to} Q\bar{A}$ and $\operatorname{rge} Q\bar{A} = Y$, and thus

$$Q(A-BA) + P_h(-Q+DF(x_0))A \overset{C}{\to} Q(A-BA)P + (-Q+DF(x_0))A.$$

The result follows again as above [13; p. 654].

As suggested, the converse of Proposition 8.2 is not true. For example, let Y be an infinite-dimensional separable Hilbert space and with $X = Y \times \mathbb{R}$ consider the function

$$F: X \to Y, \quad F(x) = \frac{3}{2}y, \quad x = (y,\lambda) \in X.$$

For a coordinate splitting $X = V \oplus T$, take $V = Y \times \{0\}$ and $T = \{0\} \times \mathbb{R}$, and define $A: Y \to V$ by $Ay = (y,0)$ for $y \in Y$. The operator $Q: X \to Y$ is defined as $Qx = y$ for $x = (y,\lambda) \in X$ and, if $\{e_i\}$ is an orthonormal basis for Y, the discretization projections P_h are taken to be the orthogonal projections onto the subspaces spanned by $e_1,\ldots,e_n$ with $\frac{1}{n+1} < h \leq \frac{1}{n}$. Then we have, for any x_0,

$$DF_h(x_o)A = I + \frac{1}{2} P_h \xrightarrow{r} \frac{3}{2} I = DF(x_o)A,$$

but

$$P_h(-Q+DF(x_o))A = \frac{1}{2} P_h \overset{s}{\not\to} \frac{1}{2} I = (-Q+DF(x_o))A.$$

In applications, the choice of Q is at our disposal and a suitable choice of Q may ensure the validity of (S). In our example, the slightly different operator Q defined by $Qx = \frac{3}{2} y$ for $x = (y,\lambda) \in X$ turns out to allow the application of Proposition 8.2.

Compact convergence does not appear to be any easier to verify than either regular or stable convergence, and so Proposition 8.2 mainly has theoretical interest. Our next result is a step towards rectifying that.

<u>Proposition 8.3</u>:

(i) If $(-Q+DF(x_o))A$ is a compact operator, then $P_h(-Q+DF(x_o))A \xrightarrow{c} (-Q+DF(x_o))A$.

(ii) If $P_h(-Q+DF(x_o))A \xrightarrow{c} (-Q+DF(x_o))A$ and if Y is separable, then $(-Q+DF(x_o))A$ is a compact operator.

<u>Proof</u>: Set $C = (-Q+DF(x_o))A$. To prove (i), let $\langle y_{h_n} \rangle$ be a bounded sequence in Y. Since C is compact, any subsequence $\langle Cy_{h_{n_k}} \rangle$ of $\langle Cy_{h_n} \rangle$ has a convergent subsequence $\langle Cy_{h_{n_{k_i}}} \rangle$. If y is the limit, then

$$||P_{h_{n_{k_i}}} Cy_{h_{n_{k_i}}} - y|| \le ||P_{h_{n_{k_i}}} Cy_{h_{n_{k_i}}} - P_{h_{n_{k_i}}} y|| + ||P_{h_{n_{k_i}}} y - y||$$

$$\le \alpha ||Cy_{h_{n_{k_i}}} - y|| + ||P_{h_{n_{k_i}}} y - y|| \to 0,$$

where $||P_h|| \leq \alpha$ for sufficiently small h (by the uniform boundedness principle). Consequently, any subsequence of $\langle P_{h_n} C y_{h_n} \rangle$ has a convergent subsequence and hence $P_h C \overset{c}{\subset} C$. (ii) follows directly from a result of Vainikko [13; p. 654].

Proposition 8.3(ii) remains true for certain nonseparable spaces Y as well. However, most applications involve separable spaces and so there is no real loss in stating the result as we did.

Proposition 8.3 effectively replaces the need for testing the discrete convergence by a simple compactness condition on the operator $(-Q+DF(x_o))A$ and thus represents a significant simplification. The projections P_h are removed from consideration, and the condition appears to involve only A and Q. We may actually go one step further and remove A from the picture. The final condition, expressed in the next proposition, is a condition on Q and Q alone.

<u>Proposition 8.4</u>: $(-Q+DF(x_o))A$ is compact if and only if $-Q+DF(x_o)$ is compact.

<u>Proof</u>: Set $K = -Q+DF(x_o)$. If K is compact, obviously KA is compact. So assume KA is compact. Let $\langle x_n \rangle$ be a bounded sequence in X. Since $X = V \oplus T$, we can write $x_n = v_n + t_n$ and the sequences $\langle v_n \rangle$ and $\langle t_n \rangle$ of components are bounded. Since $\dim T = m$, $\langle t_n \rangle$ has a convergent subsequence $\langle t_{n_k} \rangle$. Set $y_n = A^{-1} v_n$; then $\langle y_{n_k} \rangle$ is a bounded sequence and hence, by the compactness of KA, $\langle KAy_{n_k} \rangle$ has a convergent subsequence $\langle KAy_{n_{k_i}} \rangle$. Now it follows that

$$Kx_{n_{k_i}} = Kv_{n_{k_i}} + Kt_{n_{k_i}} = KAy_{n_{k_i}} + Kt_{n_{k_i}},$$

so that $\langle Kx_{n_{k_i}} \rangle$ converges. Therefore, K is compact.

We summarize all the above implications and equivalences in a theorem.

<u>Theorem 8.5</u>: Let $y_0 \in F(E)$, $X = Z \oplus \Lambda$ a parameter splitting of X, $X = V \oplus T$ a coordinate splitting for M at a given point $x_0 \in M$, and $A: Y \to V$ an isomorphism. Suppose that an operator Q (7.2) and a basic discretization are chosen and consider the following statements:

(i) stability condition (S) is satisfied;

(ii) $D\bar{F}_h(x_0)A \xrightarrow{s} DF(x_0)A$;

(iii) $D\bar{F}_h(x_0)A \xrightarrow{r} DF(x_0)A$;

(iv) $P_h(-Q+DF(x_0))A \xrightarrow{c} (-Q+DF(x_0))A$;

(v) $(-Q+DF(x_0))A$ is a compact operator;

(vi) $-Q+DF(x_0)$ is a compact operator.

Then the following implications hold:

$$(i) \Leftrightarrow (ii) \Leftrightarrow (iii) \Leftarrow (iv) \Leftarrow (v) \Leftrightarrow (vi).$$

If, in addition, Y is separable, (iv) <=> (v).

Thus, condition (vi) becomes an extremely simple condition on the operator Q alone which guarantees the stability condition (S). Since (iii) and (iv) are not equivalent, (vi) is not the only way to the stability condition (S), but it is one of the easiest to apply.

9. Applications

In this final section, we look at two applications. The first example is a mildly nonlinear problem, whereas the second one is much more general in nature.

a. Shallow Arch Subjected to a Load: This problem concerns the deformations of a shallow circular elastic arch which has been used frequently as a test case, and we refer specifically to [7]. In dimensionless form, the total potential energy of the system is given by

$$V = \int_{-\theta_0}^{\theta_0} \{[v' - u + \frac{1}{2}\alpha_1(u')^2]^2 + \alpha_2(u'')^2 - 2\alpha_3 p\}\, d\theta, \tag{9.1}$$

where u and v are (dimensionless) displacements of the arch axis, the primes denote differentiation with respect to the angle θ, $2\theta_0$ is the angle subtended by the circular arch, p is the (dimensionless) load, and $\alpha_1, \alpha_2, \alpha_3$ are (dimensionless) constants. The load $p = p(\theta, u, v, \lambda)$ is permitted to be a sufficiently differentiable function of the variables θ, u, v and of m parameters $\lambda_1, \ldots, \lambda_m$ which we represent as an m-vector λ. For clamped ends, the boundary conditions are

$$u(\pm\theta_0) = u'(\pm\theta_0) = v(\pm\theta_0) = 0. \tag{9.2}$$

The principle of stationary potential energy applied to (9.1) yields the pair of differential equations

$$\begin{aligned} &\alpha_2 u'''' - \alpha_1\alpha_3 p_v u' - [v' - u + \frac{1}{2}\alpha_1(u')^2](\alpha_1 u'' + 1) - \alpha_3 p_u = 0, \\ &v'' - u' + \alpha_1 u'u'' + \alpha_3 p_v = 0. \end{aligned} \tag{9.3}$$

These equations (9.3) together with the boundary conditions (9.2) may be formulated weakly as the problem of finding $(u,v) \in H_0^2[-\theta_0,\theta_0] \times H_0^1[-\theta_0,\theta_0]$ such that

$$\text{(9.4)}\qquad \begin{aligned} &\langle u,\phi\rangle_2 + \langle f(u,v,\lambda),\phi\rangle_0 = 0, \quad \phi \in H_0^2[-\theta_0,\theta_0], \\ &\langle v,\psi\rangle_1 + \langle g(u,v,\lambda),\psi\rangle_0 = 0, \quad \psi \in H_0^1[-\theta_0,\theta_0], \end{aligned}$$

where $f,g\colon H_0^2[-\theta_0,\theta_0] \times H_0^1[-\theta_0,\theta_0] \times \mathbb{R}^m \to L^2[-\theta_0,\theta_0]$ are given by

$$f(u,v,\lambda) = -\frac{1}{\alpha_2}\{\alpha_1\alpha_3 p_v u' + [v' - u + \frac{1}{2}\alpha_1(u')^2](\alpha_1 u'' + 1) + \alpha_3 p_u\},$$

$$g(u,v,\lambda) = u' - \alpha_1 u'u'' - \alpha_3 p_v,$$

and

$$\langle\phi,\psi\rangle_i = \int_{-\theta_0}^{\theta_0} \phi^{(i)}\psi^{(i)}, \quad i = 0,1,2.$$

Suppose that p is such that f and g are C^r-mappings with $r \geq 2$.

We introduce linear operators $k\colon H^{-2}[-\theta_0,\theta_0] \to H_0^2[-\theta_0,\theta_0]$ and $\ell\colon H^{-1}[-\theta_0,\theta_0] \to H_0^1[-\theta_0,\theta_0]$ defined by

$$\langle ku,\phi\rangle_2 = \langle u,\phi\rangle_0, \quad u \in H^{-2}[-\theta_0,\theta_0], \quad \phi \in H_0^2[-\theta_0,\theta_0],$$

$$\langle \ell v,\psi\rangle_1 = \langle v,\psi\rangle_0, \quad v \in H^{-1}[-\theta_0,\theta_0], \quad \psi \in H_0^1[-\theta_0,\theta_0].$$

Then $k\colon L^2[-\theta_0,\theta_0] \to H_0^2[-\theta_0,\theta_0]$ and $\ell\colon L^2[-\theta_0,\theta_0] \to H_0^1[-\theta_0,\theta_0]$ are compact and (9.4) becomes simply

$$u + kf(u,v,\lambda) = 0,$$

(9.5)

$$v + \ell g(u,v,\lambda) = 0.$$

If we now set $Y = H_0^2[-\theta_0,\theta_0] \times H_0^1[-\theta_0,\theta_0]$, $X = Y \times \mathbb{R}^m$, and define $K: H^{-2}[-\theta_0,\theta_0] \times H^{-1}[-\theta_0,\theta_0] \to Y$ by $K(u,v) = (ku,\ell v)$ and $G: X \to L^2[-\theta_0,\theta_0] \times L^2[-\theta_0,\theta_0]$ by $G(u,v,\lambda) = (f(u,v,\lambda),g(u,v,\lambda))$, then (9.5) reduces further to

$$(9.6) \qquad F(w,\lambda) \equiv w + KG(w,\lambda) = 0, \quad w = (u,v) \in Y, \quad \lambda \in \mathbb{R}^m.$$

Thus, the shallow-arch problem has been transformed into a mildly non-linear problem (9.6), and a mildly nonlinear problem can be handled in a straightforward manner. With X and Y as above, there is a natural parameter splitting $X = Z \oplus \Lambda$, where $Z = Y \times \{0\}$ and $\Lambda = \{0\} \times \mathbb{R}^m$. Consequently, a natural choice for the operator $Q \in L(X,Y)$ of (7.2) is the projection

$$Q(w,\lambda) = w, \quad x = (w,\lambda) \in X.$$

In terms of Q, (9.6) can be written as

$$F(x) = Qx + KG(x) = 0, \quad x \in X,$$

and hence $-Q + DF(x_0) = KDG(x_0)$ is compact for any x_0. By Theorem 8.5, the stability condition (S) is satisfied for any coordinate splitting $X = V \oplus T$ and isomophism $A: Y \to V$, and the estimate (7.9) of Theorem 7.2 becomes

$$||w(t)-w_h(t)||_Y + ||\lambda(t)-\lambda_h(t)||_{\mathbb{R}^m} \leq C\ ||(I-P_h)w(t)||_Y, \quad t \in J_0.$$

Setting $P_h w = (P_h^{(1)}u, P_h^{(2)}v)$, we may write this estimate in terms of the original displacements u and v as

$$||u(t)-u_h(t)||_{H_o^2} + ||v(t)-v_h(t)||_{H_o^1} + ||\lambda(t)-\lambda_h(t)||_{\mathbb{R}^m}$$

$$\leq C\,\{||(I-P_h^{(1)})u(t)||_{H_o^2} + ||(I-P_h^{(2)})v(t)||_{H_o^1}\}, \quad t \in J_o.$$

b. A Nonlinear Dirichlet Problem: Let Ω be a bounded domain in $\mathbb{R}^d$ with sufficiently smooth boundary. As usual, we use the multi-index notation $D^\alpha u$ for the partial derivatives of functions u defined on Ω. If $|\alpha| = \sum\alpha_i = \ell$ and there is no danger of confusion, we write also $D^\ell u$ for the generic derivatives $D^\alpha u$ of order $|\alpha| = \ell$.

Let

$$F(u,\lambda) \equiv \Phi(\xi,u,Du,\ldots,D^{2k}u,\lambda) = 0$$

be a given elliptic differential operator on Ω. Here, $\xi \in \Omega$ is the space variable and $\lambda \in \mathbb{R}^m$ a parameter vector, and Φ denotes a sufficiently differentiable function of all its arguments. Given any appropriate function f defined on Ω, we consider the nonlinear Dirichlet problem

$$F(u,\lambda) = f \quad \text{on} \quad \Omega, \tag{9.7}$$

$$D^\alpha u = 0 \quad \text{on} \quad \partial\Omega \quad \text{for} \quad 0 \leq |\alpha| \leq k-1. \tag{9.8}$$

Unlike the previous problem, which was transformed into a mildly nonlinear problem, we analyze (9.7) and (9.8) directly. For a fixed scalar τ, $0 < \tau < 1$, let W be the Banach space of functions in $C^{2k,\tau}(\bar{\Omega})$ which satisfy the boundary conditions (9.8). Then set $X = W \times \mathbb{R}^m$ and $Y = C^{0,\tau}(\bar{\Omega})$. With this, the Dirichlet problem (9.7)-(9.8) becomes simply

$$(9.9) \qquad F(x) = f, \quad x = (u,\lambda) \in X.$$

Again, we have a natural parameter splitting $X = Z \oplus \Lambda$ with $Z = W \times \{0\}$ and $\Lambda = \{0\} \times \mathbb{R}^m$.

For $x_0 = (u_0,\lambda_0)$, the derivative $DF(x_0)$ is given by

$$\begin{aligned} DF(x_0)x &= D_uF(x_0)u + D_\lambda F(x_0)\lambda \\ &= \sum_{|\alpha|\le 2k} \frac{\partial\Phi}{\partial(D^\alpha u)} (\xi,u_0,Du_0,\ldots,D^{2k}u_0,\lambda_0)D^\alpha u \\ &\qquad + \frac{\partial\Phi}{\partial\lambda} (\xi,u_0,Du_0,\ldots,D^{2k}u_0,\lambda_0)\lambda, \quad x = (u,\lambda) \in X. \end{aligned}$$

Now assume that x_0 is a solution of (9.9) where $F(u_0,\lambda_0)$ is strongly elliptic; that is, where the partial derivative $D_uF(x_0)$ is a strongly elliptic linear differential operator. The principal part

$$(9.10) \qquad L_{x_0} = \sum_{|\alpha|=2k} \frac{\partial\Phi}{\partial(D^\alpha u)} (\xi,u_0,Du_0,\ldots,D^{2k}u_0,\lambda_0)D^\alpha, \quad x = (u,\lambda) \in X,$$

of $D_uF(x_0)$ is then an isomorphism of W onto Y. Moreover, the remaining terms comprising $D_uF(x_0)$ are all compact operators. For a proof of these facts, we refer to [1 ; p. 686 ff.]. Hence, it follows that $D_uF(x_0)$ is a Fredholm operator from W to Y with index 0. Let P_W and $P_{\mathbb{R}^m}$ be the projections of X onto W and $\mathbb{R}^m$, respectively. Then P_W is a Fredholm operator of index m and in the decomposition

$$DF(x_o) = D_uF(x_o)P_W + D_\lambda F(x_o)P_{\mathbb{R}^m}$$

the term $D_\lambda F(x_o)P_{\mathbb{R}^m}$ is compact. This shows that $DF(x_o)$ is a Fredholm operator from X to Y of index m. Therefore, if we assume that x_o is a regular solution of (9.9), our theory applies.

The above discussion already suggests a choice for the operator $Q \in L(X,Y)$ of (7.2), namely, $Q = L_{x_o}$. In fact, then $-Q + DF(x_o)$ is compact and by Theorem 8.5 the stability condition (S) is satisfied for any coordinate splitting $X = V \oplus T$ and isomorphism $A: Y \to V$. The estimate (7.9) of Theorem 7.2 now takes the form

$$||u(t)-u_h(t)||_{C^{2k,\tau}} + ||\lambda(t)-\lambda_h(t)||_{\mathbb{R}^m} \leq C\ ||(I-P_h)L_{x_o}u(t)||_{C^{0,\tau}}, \quad t \in J_o,$$

where L_{x_o} is the principal part given in (9.10).

References

[1] S. Agmon, A. Douglis and L. Nirenberg, Estimates near the Boundary for Solutions of Elliptic Partial Differential Equations Satisfying General Boundary Conditions I, Comm. Pure Appl. Math., 12, 1959, 623-727.

[2] Yu. G. Borisovich, V. G. Zvyagin and Yu. I. Sapronov, Nonlinear Fredholm Maps and the Leray-Schauder Theory, Russian Math. Surveys, 32:4, 1977, 1-54.

[3] F. Brezzi, J. Rappaz and P.-A. Raviart, Finite-Dimensional Approximation of Nonlinear Problems.
Part I: Branches of Nonsingular Solutions, Numer. Math., 36, 1980, 1-25.
Part II: Limit Points, Numer. Math., 37, 1981, 1-28.
Part III: Simple Bifurcation Points, Numer. Math., 38, 1981, 1-30.

[4] J. P. Fink and W. C. Rheinboldt, On the Discretization Error of Parametrized Nonlinear Equations, Univ. of Pittsburgh, Inst. for Comp. Math. and Applications, Tech. Report ICMA-82-40, July 1982, to appear in SIAM J. Numer. Anal.

[5] J. P. Fink and W. C. Rheinboldt, On the Error Behavior of the Reduced-Basis Technique for Nonlinear Finite-Element Approximations, Zeit. f. Angew. Math. und Mech.

[6] V. Girault and P.-A. Raviart, An Analysis of Upwind Schemes for the Navier-Stokes Equations, SIAM J. Numer. Anal., 19, 1982, 312-333.

[7] A. D. Kerr and M. T. Soifer, The Linearization of the Prebuckling State and Its Effect on the Determined Instability Loads, Trans. ASME, J. Appl. Mech., 36, 1969, 775-783.

[8] S. Lang, Differential Manifolds, Addison-Wesley, Reading, Massachusetts, 1972.

[9] T. Poston and I. Stewart, Catastrophe Theory and Its Applications, Pitman, London, 1978.

[10] M. Schechter, Principles of Functional Analysis, Academic Press, New York, 1971.

[11] M. Spivak, A Comprehensive Introduction to Differential Geometry, Vol. I, 2nd ed., Publish or Perish, Berkeley, California, 1979.

[12] M. M. Vainberg and V. A. Trenogin, The Methods of Lyapunov and Schmidt in the Theory of Nonlinear Equations and Their Further Development, Russian Math. Surveys, 17, 1962, 1-60.

[13] G. Vainikko, Approximative Methods for Nonlinear Equations (Two Approaches to the Convergence Problem), Nonlinear Analysis, Theory, Methods & Applications, 2, 1978, 647-687.

FILM