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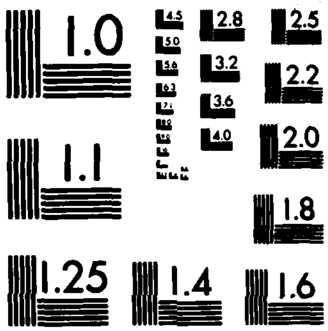
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THE EFFECT OF SURFACE TENSION ON THE SHAPE OF THE KIRCHHOFF JET

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THE EFFECT OF SURFACE TENSION ON THE SHAPE OF THE KIRCHHOFF JET

Jean-Marc Vanden-Broeck*

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ABSTRACT

The effect of surface tension on the shape of a two-dimensional jet emerging from an orifice is considered. It is shown that the slope of the surface profile of the jet is not continuous at the separation points. Both velocity and curvature are infinite at these points. The problem is solved numerically by series truncation. Jet profiles are presented for various values of the surface tension. In addition, perturbation solutions for both small and large values of the surface tension are derived.

AMS (MOS) Subject Classification: 76B10

Key Words: Jet, surface tension

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SIGNIFICANCE AND EXPLANATION

The classical Kirchhoff[1] solution for the shape of a two-dimensional jet emerging from an orifice yields infinite curvature at the separation points. It is shown that this singularity is not removed by including surface tension in the boundary condition. On the contrary surface tension makes the problem more singular by introducing a discontinuity in the slope. This result is in agreement with Vanden-Broeck's[4,5] findings.

The problem is solved numerically by collocation for arbitrary values of the surface tension. In addition perturbation solutions for both small and large values of the surface tension are presented.

The responsibility for the wording and views expressed in this descriptive summary lies with MRC, and not with the author of this report.

THE EFFECT OF SURFACE TENSION ON THE SHAPE OF THE KIRCHHOFF JET

Jean-Marc Vanden-Broeck*

1. INTRODUCTION.

The classical Kirchhoff[1] solution for the shape of a two-dimensional jet emerging from an orifice yields infinite curvature at the separation points. A similar singularity arises in cavitating flow problems such as those considered by Ackerberg[2], Cumberbatch and Norbury[3] and Vanden-Broeck[4,5].

Vanden-Broeck[4] considered the influence of surface tension on the cavitating flow past a flat plate. He provided numerical and analytical evidence that the slope of the free surface is discontinuous at the separation points. Thus the inclusion of surface tension did not remove the infinite curvature at the separation points. On the contrary it made the problem more singular by introducing a discontinuity in slope. These results were generalized by Vanden-Broeck[5] for the cavitating flow past a curved obstacle.

In the present paper we investigate the effect of surface tension on the Kirchhoff[1] jet. A numerical scheme based on series truncation is presented to solve the problem for arbitrary values of the surface tension. The numerical results are qualitatively similar to those obtained by Vanden-Broeck[4,5]. The slope is not continuous at the separation points. Both velocity and curvature are infinite at these points. In addition perturbation solutions for both small and large values of the surface tension are presented.

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The problem is formulated in Section 2. The numerical results are presented in Section 3. The perturbation calculations are derived in Section 4 and 5.

2. FORMULATION.

We consider a two-dimensional jet emerging from an orifice (see Figure 1). The orifice is assumed to be hole in a plane wall of small thickness. The effects of gravity, viscosity and compressibility are neglected. Far downstream from the orifice the speed in the interior of the jet is equal to a constant U and the two streamlines bounding the jet are straight and parallel. We denote by L the thickness of the jet far downstream from the orifice.

We define dimensionless variables by choosing L as the unit length and U as the unit velocity. We introduce the potential function ϕ and the stream function ψ. Without loss of generality we choose $\phi = 0$ at the separation points and $\psi = 0$ on the streamline IJ. It follows from the choice of the dimensionless variables that $\psi = 1$ on the streamline IJ'.

We denote the complex velocity by $u - iv$ and we define the function $\tau - i\theta$ by the relation

$$u - iv = e^{\tau - i\theta} \qquad (1)$$

We shall seek $\tau - i\theta$ as an analytic function of $f = \phi + i\psi$ in the half plane $\psi < 0$. The complex potential plane is sketched in Figure 2.

On the surface of the jet the Bernoulli equation and the pressure jump due to surface tension yield

$$\frac{1}{2} q^2 - \frac{T}{\rho} K = \frac{1}{2} U^2 \qquad (2)$$

Here q is the flow speed, K the curvature of the jet surface, T the surface tension and ρ the density. In dimensionless variables this becomes

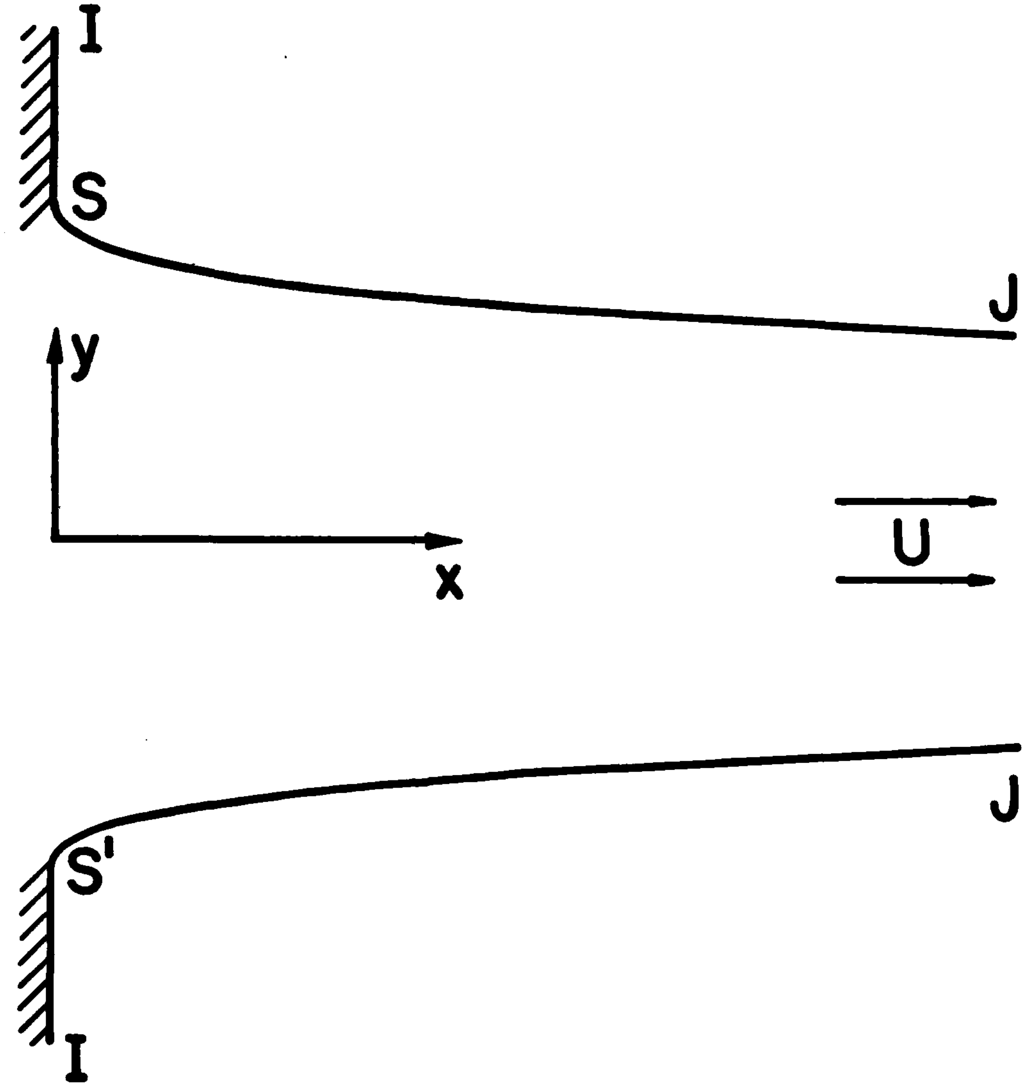

Figure 1. Sketch of the flow and the coordinates.

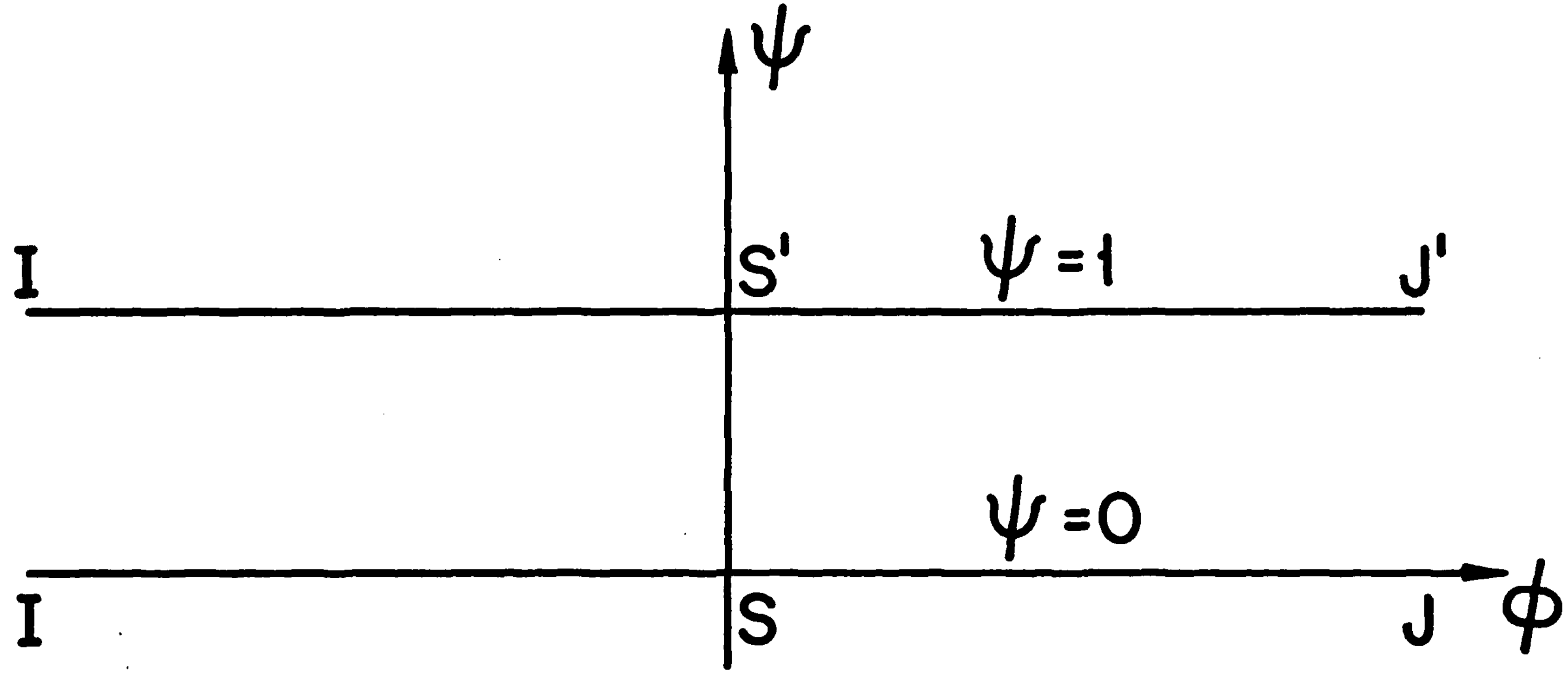

Figure 2. The image of the flow in the plane of the complex potential $f = \phi + i\psi$.

$$e^{\tau}\frac{\partial\theta}{\partial\phi} = \frac{\alpha}{2}(e^{2\tau} - 1), \quad \psi = 0 \quad \phi > 0 \tag{3}$$

$$e^{\tau}\frac{\partial\theta}{\partial\phi} = -\frac{\alpha}{2}(e^{2\tau} - 1), \quad \psi = 1 \quad \phi > 0 \tag{4}$$

Here α is the Weber number defined by

$$\alpha = \frac{\rho L U^2}{T} \tag{5}$$

The kinematic condition in IS and IS' yield

$$\theta = 0, \quad \psi = 0 \quad \phi < 0 \tag{6}$$

$$\theta = 0, \quad \psi = 1 \quad \phi < 0 \tag{7}$$

This completes the formulation of determining the function $\tau - i\theta$. For each α, $\tau - i\theta$ must be analytic is the half plane $\psi < 0$ and satisfy the boundary conditions (3), (4), (5) and (6).

3. NUMERICAL RESULTS

Following Birkhoff and Zarantonello[6] we define the new variable t by the relation

$$e^{-\pi(\phi+i\psi)} = -\frac{1}{2}\left(t + \frac{1}{t}\right) \tag{8}$$

This transformation maps the flow domain onto the unit circle in the complex t-plane so that the walls go onto the real diameter, and the free surfaces onto the circumference (see Figure 3).

We introduce the function $\Omega(t)$ by the relation

$$\tau - i\theta = \ln t - \frac{2\beta}{\pi}\ln(1 - t^2) + \Omega(t) \tag{9}$$

Here β is the value of θ at $\phi = \psi = 0$. The function $\Omega(t)$ is bounded and continuous in the unit circle $|t| < 1$, and analytic in the interior. The conditions (6) and (7) show that $\Omega(t)$ can be expressed in the form of a Taylor expansion in even powers of t. Hence

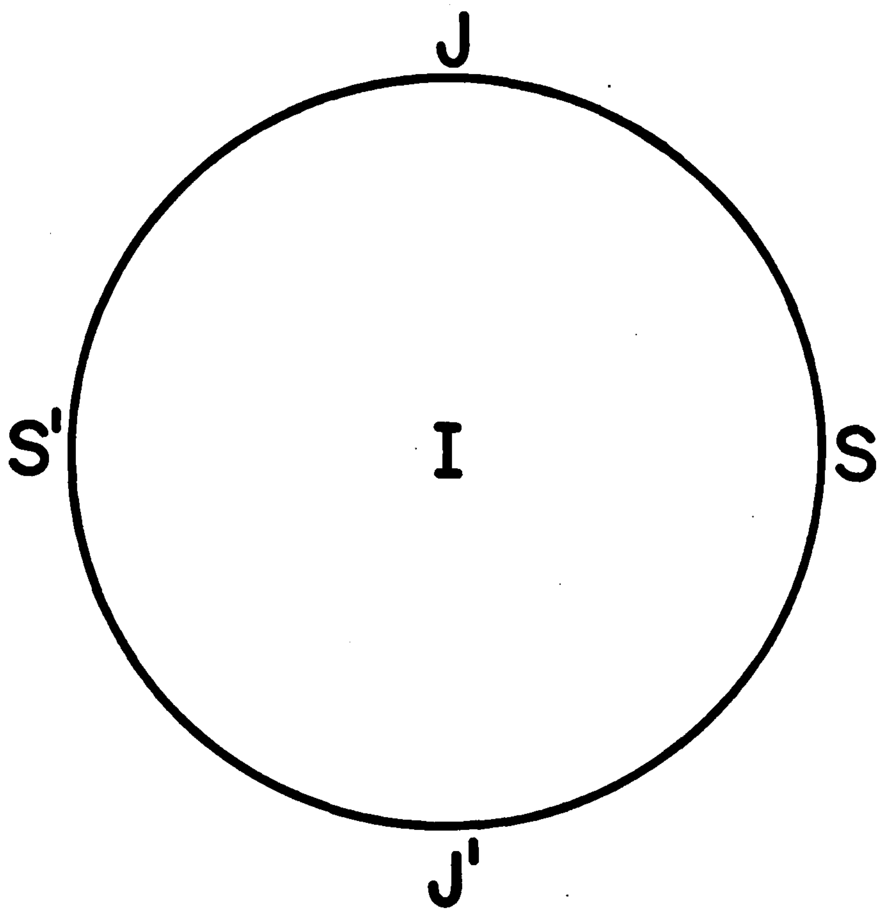

Figure 3. The t-plane.

$$\tau - i\theta = \ell n\, t - \frac{2\beta}{\pi}\, \ell n(1 - t^2) + \sum_{n=0}^{\infty} U_{n+1} t^{2n} \tag{10}$$

The function (10) satisfy (5) and (6). The coefficients U_n and the constant β have to be determined to satisfy (3). The condition (4) will then be automatically satisfied by symmetry.

We use the notation $t = |t|e^{i\sigma}$ so that points on SJ are given by $t = e^{i\sigma}$, $0 < \sigma < \frac{\pi}{2}$. Using (8) we rewrite (3) in the form

$$\pi \frac{\cos \sigma}{\sin \sigma} e^{\tilde{\tau}} \frac{d\tilde{\theta}}{d\sigma} = \frac{\alpha}{2} (e^{2\tilde{\tau}} - 1) \tag{11}$$

Here $\tilde{\tau}(\sigma)$ and $\tilde{\theta}(\sigma)$ denote the values of τ and θ on the free surface SJ.

We solve the problem approximately by truncating the infinite series in (10) after N terms. We find the N coefficients U_n and the constant β by collocation. Substituting $t = e^{i\sigma}$ into (10) and taking the real and imaginary parts we obtain

$$\tilde{\tau}(\sigma) = -\frac{\alpha}{\pi} \log(2 \sin \sigma) + \sum_{n=0}^{N-1} U_{n+1} \cos 2n\sigma \tag{12}$$

$$\tilde{\theta}(\sigma) = \sigma - \frac{2\beta}{\pi} \sigma + \beta + \sum_{n=0}^{N-1} U_{n+1} \sin 2n\sigma \tag{13}$$

We now introduce the $N + 1$ mesh points

$$\sigma_I = \frac{\pi}{2(N + 1)} \left(I - \frac{1}{2}\right), \quad I = 1, \ldots, N + 1 \tag{14}$$

Using (12) and (13) we obtain $[\tilde{\tau}(\sigma)]_{\sigma=\sigma_I}$ and $[\frac{d\tilde{\theta}}{d\sigma}]_{\sigma=\sigma_I}$ in terms of the coefficients U_n and the constant β. Substituting these expressions into (11) at the points σ_I we obtain $N + 1$ nonlinear algebraic equations for the $N+1$ unknowns U_n, $n = 1,\ldots,N$ and β. We solve this system by Newton's method.

Once this system is solved for a given value of α, the shape of the jet is obtained by numerically integrating the exact relations

$$\frac{d\tilde{x}}{d\sigma} = \frac{1}{\pi} tg\sigma e^{-\tau} \cos\theta \tag{15}$$

$$\frac{d\tilde{y}}{d\sigma} = \frac{1}{\pi} tg\sigma e^{-\tau} \sin\theta \tag{16}$$

Here $\tilde{x}(\sigma)$ and $\tilde{y}(\sigma)$ are the values of x and y on the free surface SJ. Relations (15) and (16) follow directly from (1) and (8).

Some of the numerical results are shown in Table 1. These values are correct to the number of decimal places shown. As n increases, the coefficients U_n decrease rapidly. Typical profiles are shown in Figure 4.

Our numerical results (see Table 1) indicate that $\beta \neq 0$ for all values of $\alpha \neq \infty$. Therefore the slope of the surface profile of the jet is discontinuous at the separation points for all values of $\alpha \neq \infty$. Both velocity and curvature are infinite at these points.

In Figure 5 we present a graph of β versus α. As α varies between 0 and ∞, β varies continuously from $\frac{\pi}{2}$ to 0.

For $\alpha = \infty$, the condition (11) reduces to the free-streamline condition $\tilde{\tau} = 0$. The solution is then

$$\beta = 0, \quad U_n = 0, \quad n = 1,2,3,\ldots . \tag{17}$$

Substituting (17) into (10) we obtain the following exact solution

$$\tau_\infty - i\theta_\infty = \ln t, \quad \beta = 0 \tag{18}$$

This is the classical Kirchhoff[1]'s solution. Here the subscript ∞ denotes $\alpha = \infty$.

As $\alpha \to 0$, the profile of the jet approaches two horizontal straight lines. The solution is then

	$\alpha=\infty$	$\alpha=20$	$\alpha=5$	$\alpha=0$
β	0	0.488	0.944	1.571
C	0.61	0.70	0.87	1.0
U_1	0	0.2025	0.4860	0.693
U_2	0	-0.531×10^{-1}	0.268×10^{-1}	0
U_3	0	-0.339×10^{-1}	-0.315×10^{-1}	0
U_4	0	-0.667×10^{-2}	-0.047×10^{-2}	0
U_5	0	-0.797×10^{-2}	-0.625×10^{-2}	0
U_6	0	-0.205×10^{-2}	-0.036×10^{-2}	0
U_7	0	-0.323×10^{-2}	-0.237×10^{-2}	0
U_8	0	-0.086×10^{-2}	-0.020×10^{-2}	0
U_9	0	-0.17×10^{-2}	-0.119×10^{-2}	0
U_{10}	0	-0.04×10^{-2}	-0.012×10^{-2}	0
U_{11}	0	-0.09×10^{-2}	-0.069×10^{-2}	0
U_{12}	0	-0.02×10^{-2}	-0.008×10^{-2}	0
U_{13}	0	-0.06×10^{-2}	-0.045×10^{-2}	0
U_{14}	0	-0.01×10^{-2}	-0.005×10^{-2}	0
.	.	.	.	.
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U_{30}	0	-9×10^{-6}	-4×10^{-6}	0

Table 1. Values of β, C and U_n for various values of α.

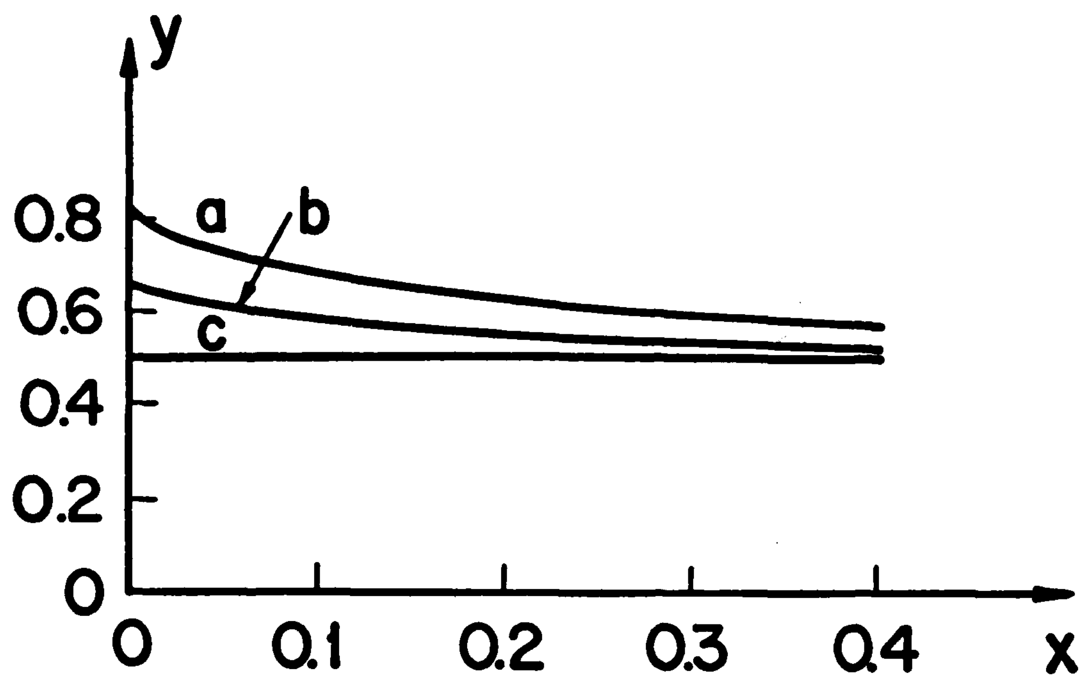

Figure 4. Computed profiles for various values of α. The curves (a), (b) and (c) correspond respectively to $\alpha = \infty$, 10, 0.

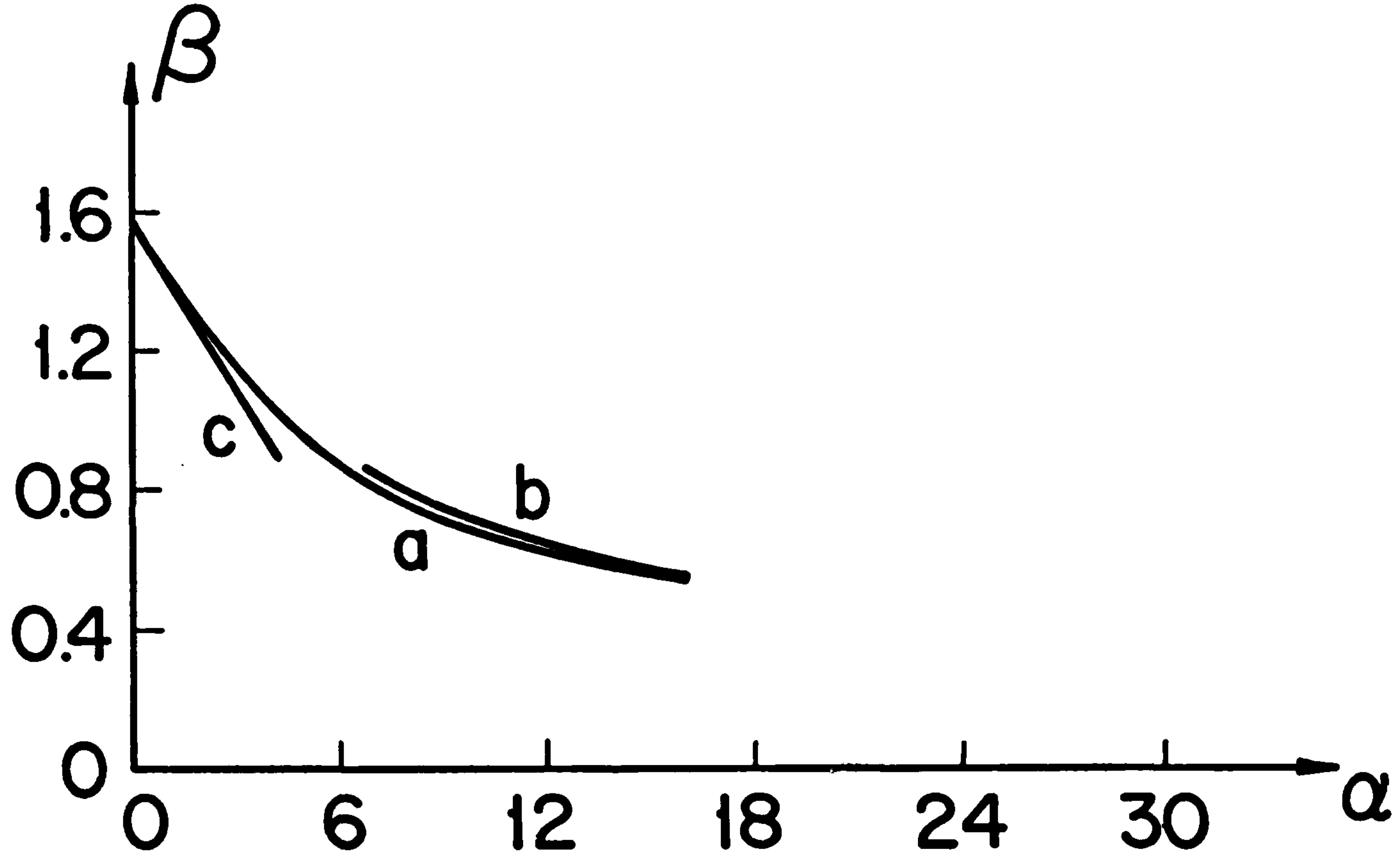

Figure 5. The slope β of the free surface at the separation points as a function of the Weber number α as given by the numerical scheme (curve a), the formula (22) (curve b) and the formula (27) (curve c).

$$\beta = \frac{\pi}{2}, \quad U_1 = \ell n\ 2$$
$$U_n = 0, \quad n = 2,3,\ldots \tag{19}$$

Substituting (19) into (10) we obtain the following exact solution

$$\tau_0 - i\theta_0 = \ell n \frac{2t}{1 - t^2} \tag{20}$$

Here the subscript 0 denotes $\alpha = 0$.

The contraction ratio C of the jet is defined as the ratio of the width of the jet far away from the orifice to the width of the orifice.

For $\alpha = \infty$ we obtain from (18)

$$C_\infty = \frac{\pi}{\pi + 2} \sim 0.61$$

For $\alpha = 0$ we obtain from (20)

$$C_0 = 1 .$$

In Figure 6 we present numerical values of C versus α. As α decreases from infinity, the contraction ratio increases monotonically from C_∞ to C_0.

4. PERTURBATION SOLUTION FOR α LARGE

For $\alpha = \infty$, the solution is given by (1.8). Using (8) and (18) we obtain after some algebra

$$\frac{\partial\theta_\infty}{\partial\phi} = -\left(\frac{2}{\pi}\phi\right)^{-1/2} \quad \text{as} \quad \phi \to 0, \quad \psi = 0 \tag{21}$$

Therefore the Kirchhoff[1] solution yields infinite curvature at the separation points.

A perturbation solution for α large can be derived by using the method of matched asymptotic expansions. The details of the calculation follow closely the work of Vanden-Broeck[4,5]. Therefore they will not be repeated here.

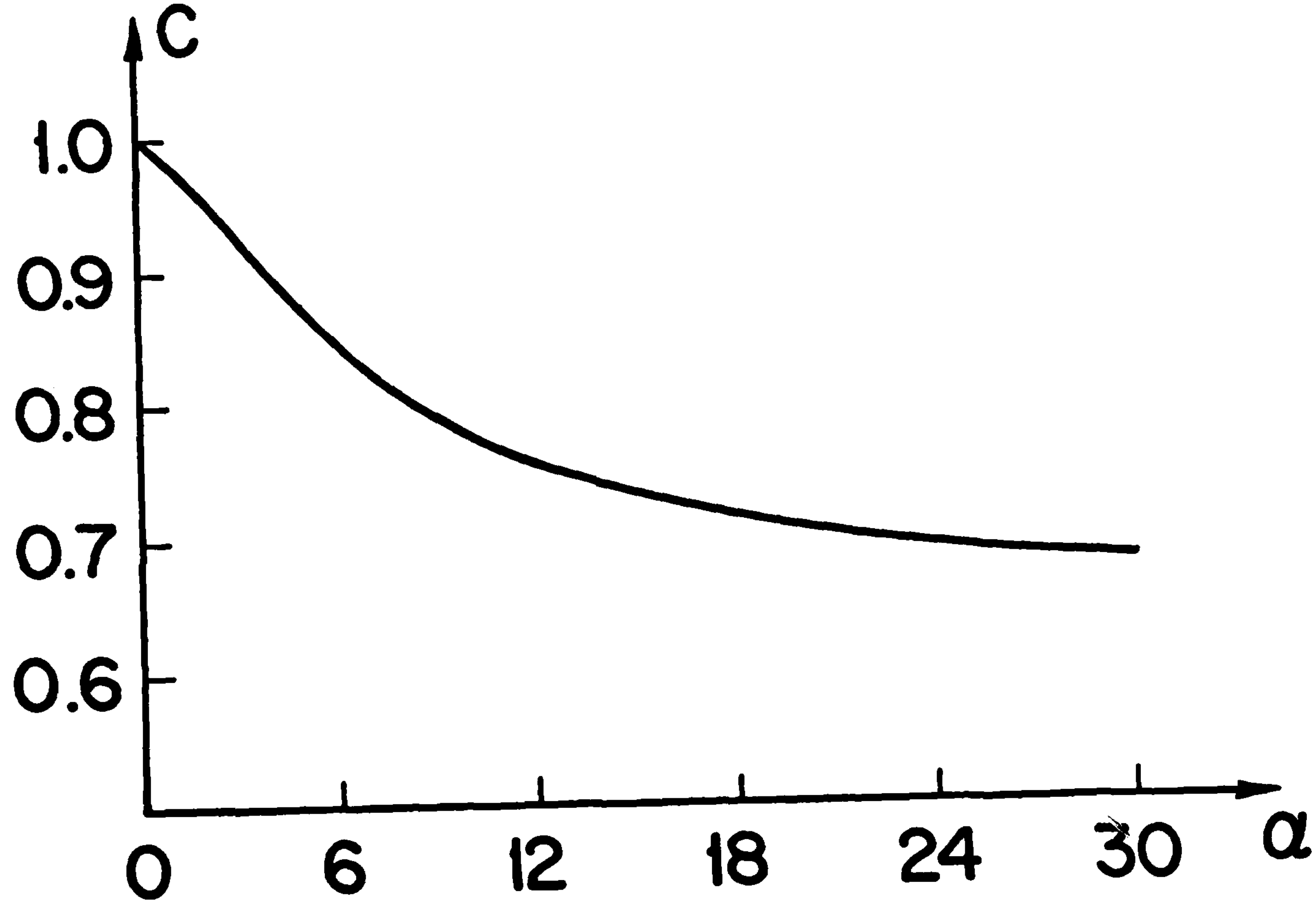

Figure 6. The contraction ratio C as a function of the Weber number α.

The asymptotic results for α large confirm that the slope is not continuous at the separation points. In particular the asymptotic expression for β is

$$\beta \sim \pi(2\alpha)^{-1/2} \quad \text{as} \quad \alpha \to \infty \tag{22}$$

Relation (22) is shown in Figure 5. The numerical results are in good agreement with (22) for α large. For $\alpha = 20$ the value of β predicted by (22) agrees with the numerical results within two percent.

5. PERTURBATION SOLUTION FOR α SMALL

For $\alpha = 0$, the solution is given by (20). We seek a solution for α small in the form of an expansion in powers of α. Thus we write

$$\tau = \tau_0 + \alpha\tau_1 + 0(\alpha^2) \tag{23}$$

$$\theta = \theta_0 + \alpha\theta_1 + 0(\alpha^2) \tag{24}$$

Here τ_0 and θ_0 are defined by (20). Substituting (23) and (24) into (11) we have

$$\frac{d\tilde{\theta}_1}{d\sigma} = \frac{\alpha}{2\pi} \cos \sigma \tag{25}$$

Integrating (25) and using the condition $\tilde{\theta}_1\left(\frac{\pi}{2}\right) = \frac{\pi}{2}$ we obtain

$$\tilde{\theta}_1 = \frac{\sin \sigma - 1}{2\pi} \alpha + \frac{\pi}{2} \tag{26}$$

Relation (26) and the definition of β imply

$$\beta = \frac{\pi}{2} - \frac{\alpha}{2\pi} \tag{27}$$

Relation (27) is shown in Figure 5. For $\alpha = 1$ the value of β predicted by (15) agrees with the numerical results within two percent.

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20. ABSTRACT (Continue on reverse side if necessary and identify by block number)

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