ADA130501

MRC Technical Summary Report # 2529

GRANULAR JUMPS DOWN AN INCLINED PLANE

M. C. Shen

Mathematics Research Center

University of Wisconsin—Madison

610 Walnut Street

Madison, Wisconsin 53706

June 1983

(Received May 19, 1983)

DTIC FILE COPY

DTIC
ELECTE
JUL 2 1 1983
S
E
D

Approved for public release
Distribution unlimited

83 07 20 039

UNIVERSITY OF WISCONSIN - MADISON
MATHEMATICS RESEARCH CENTER

GRANULAR JUMPS DOWN AN INCLINED PLANE

M. C. SHEN[*]

Technical Summary Report #2529

June 1983

ABSTRACT

This document studies

~~We study~~ granular jumps generated in the gravity flow of granular

materials down an inclined plane. The equations governing the granular flow

are reduced to a sequence of boundary value problems of linear ordinary

differential equations by means of an asymptotic method. Their solutions are

used to determine the Burgers equation, which possesses a progressive wave

solution to describe a smooth granular jump. The wave speed and a criterion

for the stability of the granular jump are also obtained.

AMS (MOS) Subject Classifications: 73N99, 76A99

Key Words: Asymptotic method, granular jump, gravity flow, Burgers equation,

stability

Work Unit Number 2 - Physical Mathematics

---

[*]

Department of Mathematics and Mathematics Research Center, University of
Wisconsin, Madison, WI 53706.

SIGNIFICANCE AND EXPLANATION

Problems of flow of granular materials have attracted much attention recently because of their importance in industrial applications and geophysical situations. It is observed in laboratory tests that a granular jump may appear in the gravity flow of a granular material down an inclined plane. The granular jump may be considered as a wave front moving with constant velocity on the free surface of the granular material and connecting one uniform depth to another. The main contribution of this report is to develop an asymptotic method, which can describe the speed, wave profile and stability of the granular jump.

Accession For	
NTIS GRA&I	X
DTIC TAB	☐
Unannounced	☐
Justification	
By	
Distribution/	
Availability Codes	
Dist	Avail and/or Special
A	

GRANULAR JUMPS DOWN AN INCLINED PLANE

M. C. Shen[*]

1. Introduction

In recent years there has been much interest in the study of the gravity
flow of granular materials. Needless to say, granular flow problems occur
frequently in industry as well as in nature such as handling of grains and
sands, snow avalanches and rock falls. It is reported in laboratory tests
(Savage, 1979) that smooth granular jumps may be observed when granular
materials are moving down an inclined chute. As seen from loose snow
avalanche or earth slide in nature, such a granular jump also seems to
appear. On the other hand, since Goodman and Cowin (1972) proposed their
continuum theory for flow of granular materials, only a few analytical results
based upon their equations have been found, for example, steady state
solutions by Goodman and Cowin (1971), and solutions of the linearized
equations by Nunziato and Walsh (1977). The main purpose of this report,
therefore, is to develop an asymptotic method for the study of the time-
dependent granular flow under gravity down an inclined plane, which provides a
means to describe the development of a smooth granular jump. Some results
concerning granular jumps have been obtained by Morrison and Richmond (1976),
and Savage (1979). However, their approach is based upon algebraic
considerations of conservation laws for hydraulic jumps.

[*]
 Department of Mathematics and the Mathematics Research Center, University of
Wisconsin, Madison, WI 53706.

In this report we adopt the equations formulated by Goodman and Cowin (1972) for the so-called cohesionless Coulomb granular material, and our approach used here is essentially an extension of the method we developed in a previous report (Shen, 1983) for the compressible viscous flow down an inclined channel. We briefly explain the method as follows. Assume that the granular flow down an inclined plane is two dimensional. We choose a coordinate system moving with the wave front of a granular jump so that the horizontal axis is parallel to the plane. A small parameter ε as the ratio of the vertical length scale H to the horizontal length scale L is introduced. We assume that the vertical velocity component is much smaller than the horizontal velocity component, and consider the granular flow for large time. Then we use ε to stretch various variables non-dimenionalized by appropriate units so that these assumptions are met. The solution of the governing equations is expanded in an asymptotic series in integral powers of ε, and the equations for the successive approximations determine the wave speed and an evolution equation for the free surface, which is the well-known Burgers equation. The signs of some coefficients in the Burgers equation determine whether the equation is ill-posed and provide a criterion for the stability of the flow.

The governing equations used here involve four material constants; two of them are related to the equilibrium stress and the other two appear in the dynamic stress. We first develop a general method in which all the material constants are assumed to be finite. However, the calculations of the wave speed and the coefficients in the Burgers equations are rather prohibitive. To simplify the derivations, we shall consider a special case, that is, we choose one of the material constants to be large in comparison with the others. In this case the equilibrium solution reduces to the one for the

incompressible viscous flow down an inclined plane, and the wave speed and the coefficients in the Burgers equation can be explicitly expressed in terms of the remaining constants. We also note that many other constitutive equations have been proposed for granular materials lately (Cowin and Satake, 1978; Savage, 1979; Nunziato, Passman and Thomas, 1980; Savage and Jeffrey, 1981; Sayed and Savage, 1983). Most of them are generalizations of the original formulation due to Goodman and Cowin (1972). The basic ideas developed here could also apply to these more general cases.

We formulate the problem for the general case in Section 2. The wave speed and the Burgers equation are derived in Section 3. In Section 4, we consider the special case. In Section 5, we define the criterion of stability, derive an expression for the wave profile of a granular jump and present some problems for further study.

2. Formulation

We consider a cohesionless Coulomb granular material flowing down a rough inclined plane so that we may impose the no-slip condition there. The flow is assumed to be two dimensional and a coordinate system moving at a speed σ^* in the x^*-direction is chosen so that the x^*-axis is parallel to the plane and the y^*-axis is positive upward (Figure 1). The governing equations are the following:

$$\nu^*_{t^*} + \nabla^* \cdot (\nu^* \bar{q}^*) = 0 \ , \tag{1}$$

$$\gamma \nu^* (\bar{q}^*_{t^*} + \bar{q}^* \cdot \nabla^* \bar{q}^*) = \nabla^* \cdot \bar{T}^* + \gamma \nu^* \bar{g}^* \ , \tag{2}$$

subject to the boundary conditions:

At the free surface $\xi^* = y^* - \eta^*(x^*, t^*) = 0$,

$$\bar{T}^* \cdot \bar{n}^* = 0 \ , \tag{3}$$

$$\bar{h}^* \cdot \bar{n}^* = 0 \ , \tag{4}$$

$$\eta^*_{t^*} + u^* \eta^*_{x^*} - v^* = 0 \ ; \tag{5}$$

at the rough rigid plane,

$$\bar{q}^* = -\sigma^* \ , \quad \nu^* = \nu^*_p \ . \tag{6}$$

Here $\nabla^* = (\partial/\partial x^*, \partial/\partial y^*) = (\partial/\partial x^*_1, \partial/\partial x^*_2)$, ν^* is the volume distribution function, γ is the mass density assumed to be constant, $\bar{q}^* = (u^*, v^*) = (u^*_1, u^*_2)$ is the velocity of the flow, $\bar{T}^*$ is the stress tensor defined by

$$T^*_{ij} = [-\beta^*(\nu^*)^2 + \alpha^*(\nabla^* \nu^*)^2 + 2\alpha^* \nu^* (\nabla^*)^2 \nu^*] \delta_{ij} - 2\alpha^* \nu^*_{x^*_i} \nu^*_{x^*_j} +$$

$$\lambda^* \nabla^* \cdot \bar{q}^* \delta_{ij} + 2\mu^* \sigma^*_{ij} \ ,$$

$$\sigma^*_{ij} = (u^*_{ix^*_j} + u^*_{jx^*_i})/2 \ ,$$

$\bar{g} = (g \sin \theta, -g \cos \theta)$ is the constant gravitational acceleration, θ is the angle of inclination of the plane to the horizontal assumed to be greater than the angle of internal friction but less than $\pi/2$, $\bar{h}^*$ is the equilibrated stress vector given by $\bar{h}^* = 2\alpha^* \nabla^* \nu^*$, and $\bar{n}^*$ is the vector
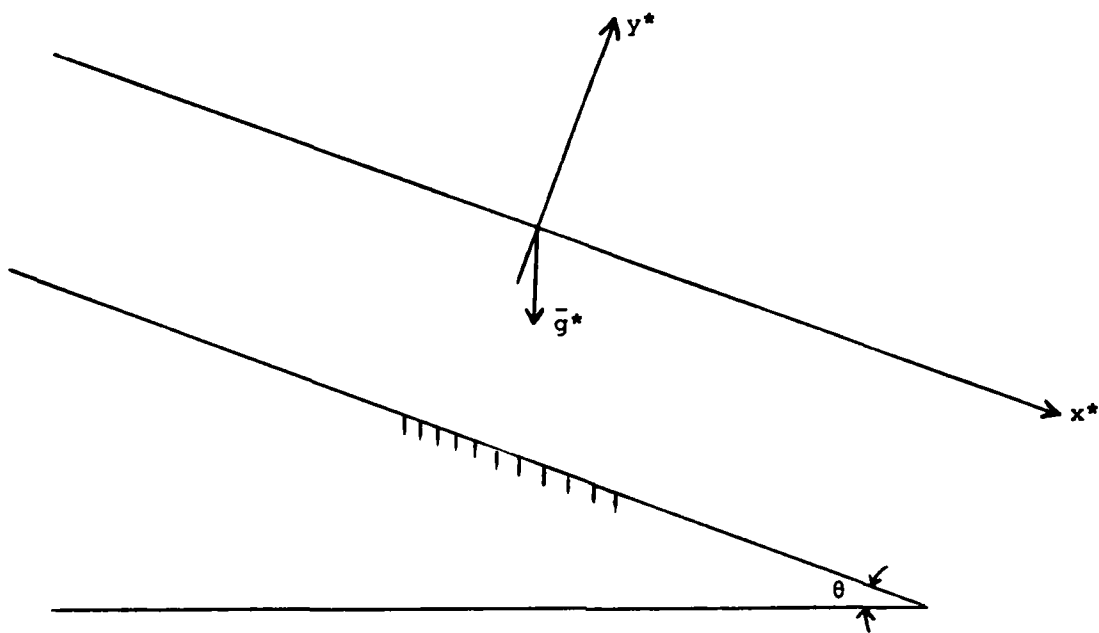
Figure 1. Coordinate system for the granular flow

$(\eta^*_{x^*},-1)$ normal to the free surface. For simplicity, we assume that α^*,
β^*, μ^* and ν^*_p are positive constants, and λ^* is also constant but
satisfies $3\lambda^* + 2\mu^* > 0$.

We introduce the following nondimensional variables:

$$\nu = \nu^*/\nu^*_p, \quad (u,v) = (u^*,\varepsilon v^*)(gH)^{-1/2} \quad,$$

$$t = \varepsilon^{-2}t^*(H/g)^{-1/2}, \quad (x,y) = (\varepsilon^{-1}x^*,y^*)H^{-1} \quad,$$

$$\sigma = \sigma^*(gH)^{-1}, \quad \beta = \beta^*(\gamma\nu^*_p gH)^{-1} \quad,$$

$$\alpha = \alpha^*(\gamma\Delta gH^3)^{-1}, \quad (\lambda,\mu) = (\lambda^*,\mu^*)(\gamma\nu^*_p(gH)^{1/2}H)^{-1}$$

$$\varepsilon = H/L \quad.$$

In terms of them, (1) to (6) become

$$\varepsilon\nu_t + (\nu u)_x + (\nu v)_y = 0 \quad, \tag{7}$$

$$\varepsilon\nu(\varepsilon^2 u_t + uu_x + vu_y) = -2\beta\varepsilon\nu\nu_x + 2\alpha\nu\varepsilon(\varepsilon^2\nu_{xx} + \nu_{yy})_x$$
$$+ (\lambda + \mu)\varepsilon^2(u_x + v_y)_x + \mu(\varepsilon^2 u_{xx} + u_{yy}) + \nu \sin\theta \quad, \tag{8}$$

$$\varepsilon^2\nu(\varepsilon v_t + uv_x + vv_y) = -2\beta\nu\nu_y + 2\alpha\nu(\varepsilon^2\nu_{xx} + \nu_{yy})_y$$
$$+ (\lambda + \mu)\varepsilon(u_x + v_y)_y + \mu\varepsilon(\varepsilon^2 v_{xx} + v_{yy}) - \nu \cos\theta \quad; \tag{9}$$

at $y = \eta(x,t)$,

$$[-\beta\nu^2 + \alpha(\varepsilon^2\nu_x^2 + \nu_y^2) + 2\alpha\nu(\varepsilon^2\nu_{xx} + \nu_{yy}) - 2\alpha\varepsilon^2(\nu_x)^2$$
$$+ \lambda\varepsilon(u_x + v_y) + 2\mu\varepsilon u_x]\varepsilon\eta_x + 2\alpha\varepsilon\nu_x\nu_y - \mu(u_y + \varepsilon^2 v_x) = 0 \quad, \tag{10}$$

$$[-2\alpha\varepsilon\nu_x\nu_y + \mu(u_y + \varepsilon^2 v_x)]\varepsilon\eta_x + \beta\nu^2 - \alpha(\varepsilon^2\nu_x^2 + \nu_y^2)$$
$$-2\alpha\nu(\varepsilon^2\nu_{xx} + \nu_{yy}) + 2\alpha(\nu_y)^2 - \lambda\varepsilon(u_x + v_y) - 2\mu\varepsilon v_y = 0 \quad, \tag{11}$$

$$\varepsilon^2\eta_x\nu_x - \nu_y = 0 \quad, \tag{12}$$

-6-

$$\varepsilon\eta_t + u\eta_x - v = 0 \quad ; \tag{13}$$

at $y = -1$,

$$u = -\sigma, \ v = 0, \ \nu = 1 \quad . \tag{14}$$

3. Wave speed and the Burgers equation

Assume that u, v, ν and η possess an asymptotic expansion of the form

$$\phi = \phi_0 + \epsilon\phi_1 + \epsilon^2\phi_2 + \cdots \quad . \tag{15}$$

Without loss of generality, we may assume $\sigma = \sigma_0 + \epsilon\sigma_1$. Substitution of (15) in (7) to (14) yields a sequence of equations and boundary conditions for the successive approximations. The equations for the zeroth approximation are

$$\mu\, u_{0yy} = -\nu_0 \sin\theta \quad , \tag{16}$$

$$2\,\alpha\, \nu_{0yyy} - 2\beta\nu_{0y} = \cos\theta \quad ; \tag{17}$$

at $y = 0$,

$$u_{0y} = 0 \quad , \tag{18}$$

$$2\,\alpha\, \nu_{0yy} - \beta\nu_0 = 0 \quad , \tag{19}$$

$$\nu_{oy} = 0 \quad ; \tag{20}$$

at $y = -1$,

$$u_0 = -\sigma_0, \; \nu_0 = 1 \tag{21}$$

where u_0 is assumed to be a function of y only and $v_0 = \eta_0 = 0$. The solutions of (16) to (21) for u_0 and ν_0 can be easily found, and a discussion of these solutions is given in Goodman and Cowin (1971). Following them, we define

$$\ell = (\beta/\alpha)^{1/2} \quad \text{and} \quad M = \cos\theta/(2\beta) \quad , \tag{22}$$

and obtain

$$\nu_0 = M\ell^{-1} \sinh \ell y$$
$$+ (1 - M + M\ell^{-1} \sinh \ell)(1 + \cosh \ell)^{-1}(1 + \cosh \ell y) - My \quad , \tag{23}$$

$$u_0 = \phi_0 - \sigma_0$$
$$= \mu^{-1} \sin\theta\{-M\ell^{-3}(\sinh \ell + \sinh \ell y) + (1-M+M\ell^{-1} \sinh \ell)x$$
$$(1 + \cosh \ell)^{-1}[(1-y^2)/2 + \ell^{-2}(\cosh \ell - \cosh \ell y)] + (M/6)(1+y^3)$$
$$+ M\ell^{-2}(1+y)\} - \sigma_0. \tag{24}$$

For small ℓ, we may expand v_0 and ϕ_0 in integral powers of ℓ and obtain

$$v_0 \simeq 1 + \ell^2 [(M/6)y^3 + y^2/4 + (M/6 - 1/4)] + \cdots \cdots , \qquad (25)$$

$$\phi_0 \simeq \mu^{-1} \sin \theta \{ (1-y^2)/2 + \ell^2 [(1-y^2)(M/12 - 1/8) + (1-y^4)/48$$

$$\qquad (26)$$

$$- M(1+y^5)/120] + \cdots \cdots \} \quad .$$

We see from (25) and (26) that as $\ell \to 0$, v_0, u_0 tend to the solutions for an incompressible viscous flow fown an inclined plane (Landau and Lifshitz, 1959). This result motivates our choice of α as a large parameter in the next section so that expressions for the wave speed and the coefficients in the Burgers equation can be greatly simplified. On the other hand, if $\ell \to \infty$, v_0 tends to a discontinuous solution when $M \neq 1$ and becomes zero at $y = 0$ (Goodman and Cowin, 1971). Therefore, we shall not study this case in the sequel.

The equations for the first approximation are

$$(v_1 u_0 + v_0 u_1)_x + (v_0 v_1)_y = 0 \quad , \qquad (27)$$

$$\mu \, u_{1yy} = -v_1 \sin \theta \quad , \qquad (28)$$

$$\alpha \, v_{1yyy} - \beta \, v_{1y} = 0 \quad ; \qquad (29)$$

at $y = 0$,

$$u_{1y} + u_{0yy} \eta_1 = 0 \quad , \qquad (30)$$

$$\alpha (v_0 v_{1yy} + v_{0yy} v_1) - \beta v_0 v_1 = -\alpha \, v_0 v_{0yyy} \eta_1 \quad , \qquad (31)$$

$$v_{1y} + v_{0yy} \eta_1 = 0 \quad ; \qquad (32)$$

$$u_0 \eta_{1x} - v_1 = 0 \qquad (33)$$

at $y = -1$,

$$u_1 = -\sigma_1, \; v_1 = 0, \; v_1 = 0 \quad . \qquad (34)$$

As observed from (28) to (32) and (34), we may express v_1 and u_1 as

$$v_1 = \psi_1 \eta_1, \; u_1 = \phi_1 \eta_1 - \sigma_1 \quad , \qquad (35)$$

at $y = 0$,

$$(-\beta v_0^2 + 2\alpha v_0 v_{0yy})\eta_{1x} - \mu(u_{2y} + u_{1y}\eta_1 + u_{0yy}\eta_2 + u_{0yyy}\eta_1^2/2) = 0$$

$$\beta(2v_0 v_{0yy}\eta_1^2/2 + v_1^2 + 2v_0 v_{1y}\eta_1 + 2v_0 v_2) + 2[(v_{0yy})^2\eta_1^2 + (v_{1y}$$

$$+ 2v_{0yy}v_{1y}\eta_1] - 2\alpha[v_0(v_{0yyy}\eta_2 + v_{0yyyy}\eta_1^2/2 + v_{1yyy}\eta_1 + v_2$$

$$+ v_{0yy}(v_{0yy}\eta_1^2/2 + v_{1y}\eta_1 + v_2 + v_1 v_{1yy}) + v_1(v_{0yyy}\eta_1 + v_{1y}$$

$$\lambda(u_{1x} + v_{1y}) - 2\mu v_{1y} = 0 \quad ,$$

$$v_{2y} + v_{1y}\eta_1 + v_{0yy}\eta_1^2/2 = 0 \quad ,$$

$$\eta_{1t} + u_0 \eta_{2x} + u_1 \eta_{1x} + u_{0y}\eta_1\eta_{1x} - v_2 - v_{1y}\eta_1 = 0 \quad ;$$

at $y = -1$,

$$u_2 = v_2 = v_2 = 0 \quad .$$

From (17), (19), (24), (35), (36) and (42), we may rewrite (
and (53) as follows:

$$L_1(u_2) = \mu u_{2yy} = -v_2 \sin\theta + [v_0(\phi_0 - \sigma_0)\phi_1 + 2\beta v_0\psi_1 - 2\alpha v$$

$$+ v_0 f_1 \phi_{0y}]\eta_{1x} \quad ,$$

$$L_2(v_2) = 2\alpha v_0 v_{2yyy} - \alpha\beta v_0 v_{2y} = -[(\lambda+\mu)(\phi_{1y} + f_{1yy}) + \mu f_{1yy}]$$

at $y = 0$,

$$R_1(u_2) = \mu u_{2y} = (-\beta v_0^2 + 2\alpha v_0 v_{0yy})\eta_{1x} - \mu(\phi_{1y} + \phi_{0yyy}/2)\eta_1^2 - \mu\phi_{0yy}$$

$$B_2(v_2) = 2\alpha v_0 v_{2yy} - \beta v_0 v_2 = [\lambda(\phi_1 + f_{1y}) - 2\mu f_{1y}]\eta_{1x}$$

$$+ \{\beta(v_0 v_{0yy} + \psi_1^2 + 2v_0\psi_{1y}) - 2\alpha[(v_{0yy})^2/2 - (\psi_{1y})^2/2 + v_0\gamma$$

$$+ \psi_{1yyy}) + v_{0yy}\psi_{1y}\psi_{1yy} + \psi_1(v_{0yyy} + \psi_{1yy})]\}\eta_1^2$$

$$- 2\alpha v_0 v_{0yyy}\eta_2 \quad .$$

-11-

$$B_3(v_2) = v_{2y} = -(\psi_{1y} + v_{0yy}/2)n_1^2 \quad , \tag{58}$$

at $y = -1$,

$$u_2 = v_2 = 0 \quad . \tag{59}$$

It is observed from (54) to (59) that u_2 and v_2 can be expressed in the following forms

$$u_2 = \phi_2 n_{1x} + \phi_3 n_1^2 + \phi_4 n_2 \quad , \tag{60}$$

$$v_2 = \psi_2 n_{1x} + \psi_3 n_1^2 + \psi_4 n_2 \quad . \tag{61}$$

Here ψ_2 to ψ_4 and ϕ_2 to ϕ_4 are functions of y and satisfy

$$L_2(\psi_i) = F_i \quad , \quad i = 2,3,4 \quad , \tag{62}$$

$$B_2(\psi_i) = G_i \quad , \quad B_3(\psi_i) = H_i \quad , \quad \text{at } y = 0 \quad , \tag{63}$$

$$\psi_i = 0 \quad \text{at } y = -1 \quad ; \tag{64}$$

$$L_1(\phi_j) = I_j \quad , \quad j = 2,3,4 \quad , \tag{65}$$

$$B_1(\phi_j) = J_j \quad \text{at } y = 0 \quad , \tag{66}$$

$$\phi_j = 0 \quad \text{at } y = -1 \quad , \tag{67}$$

where

$$F_2 = -(\lambda+\mu)(\phi_{1y} + f_{1yy}) - \mu f_{1yy} \quad ,$$

$$F_i = 0 \quad , \quad i = 3,4 \quad ,$$

$$G_2 = \lambda(\phi_1 + f_{1y}) - 2\mu f_{1y} \quad ,$$

$$G_3 = \beta(v_0 v_{0yy} + \psi_1^2 + 2v_0\psi_{1y}) - 2\alpha[(v_{0yy})^2/2 - (\psi_{1y})^2/2 + v_0(v_{0yyyy}/2$$
$$+ \psi_{1yyy}) + v_{0yy}\psi_{1y}\psi_{1yy} + \psi_1(v_{0yyy} + \psi_{1yy}) \quad ,$$

$$G_4 = -2\alpha v_0 v_{0yyy} \quad ,$$

$$H_3 = -(\psi_{1y} + v_{0yy}/2), \quad H_i = 0, \quad i = 3,4 \quad ,$$

$$I_2 = -\psi_2 \sin\theta + v_0(\phi_0 - \sigma_0)\phi_1 + 2\beta v_0\psi_1 - 2\alpha v_0\psi_{1yy} +$$
$$v_0 f_1\phi_{0y}, \quad I_3 = -\psi_3 \sin\theta, \quad I_4 = -\psi_4 \sin\theta \quad ,$$

$$J_1 = -\beta v_0^2 + 2\alpha v_0 v_{oyy}, \quad J_2 = 0, \quad J_3 = -\mu(\phi_{1y} + \phi_{0yyy}/2) \quad ,$$

$$J_4 = -\mu \phi_{0yy} \quad .$$

In principle, we can solve (62) to (67) for ψ_i and ϕ_j, and in turn u_2 and v_2 can be determined from (60), (61). Now we are in a position to determine the Burgers equation. We integrate (46) with respect to y from $y = -1$ to $y = 0$ and make use of (18), (33) to (35), (52) to (53), (60) and (61), to obtain

$$\int_{-1}^{0} \psi_1 \, dy \, \eta_{1t} + \int_{-1}^{0} [(\psi_2 \eta_{1x} + \psi_3 \eta_1^2 + \psi_4 \eta_2)(\phi_0 - \sigma_0) + \psi_1(\phi_1 \eta_1 - \sigma_1)\eta_1$$

$$+ v_0(\phi_2 \eta_{1x} + \phi_3 \eta_1^2 + \phi_4 \eta_2)]_x \, dy + \psi_1(0)[\phi_0(0) - \sigma_0]\eta_1 \eta_{1x} \qquad (68)$$

$$+ \eta_{1t} + (\phi_0 - \sigma_0)\eta_{2x} + (\phi_1 \eta_1 - \sigma_1)\eta_{1x} + -f_{1y}(0)\eta_1 \eta_{1x} = 0 \quad .$$

Note that the coefficient of η_2 vanishes because of (45). By rearranging the terms, we finally obtain the Burgers equation

$$m_0 \eta_{1t} + m_1 \eta_{1x} + m_2 \eta_1 \eta_{1x} = m_3 \eta_{1xx} \quad , \qquad (69)$$

Which is our main result. Here

$$m_0 = \int_{-1}^{0} \psi_1 dy + 1 \quad , \quad m_1 = -\sigma_1 m_0 \quad ,$$

$$m_2 = 2 \int_{-1}^{0} [\psi_3(\phi_0 - \sigma_0) + \psi_1 \phi_1 + v_0 \phi_3] dy + \psi_1(0)[\phi_0(0) - \sigma_0] + \phi_1(0) - f_{1y}(0) \quad ,$$

$$m_3 = - \int_{-1}^{0} [\psi_2(\phi_0 - \sigma_0) + v_0 \phi_2] dy \quad .$$

(69) can be used to study the development of a granular jump.

4. A special case

In Section 3, we derived the expressions for the wave speed σ_0 and the coefficients for the Burgers equation. However, they depend upon θ, α, β, λ and μ, and are too complicated for applications. Therefore, as motivated by the discussion of ν_0 and ϕ_0 given in the last Section, we assume $\alpha = 1/\varepsilon$ to be a large parameter. Substitution of (15) in (8) to (14) also yields a sequence of equations and bounding conditions. The equations for the zeroth approximation are the following:

$$\mu \, u_{0yy} = -\nu_0 \sin \theta \ , \tag{70}$$

$$\nu_{0yyy} = 0 \ ; \tag{71}$$

at $y = 0$,

$$u_{0y} = 0 \ , \tag{72}$$

$$\nu_{0yy} = 0 \ , \quad \nu_{0y} = 0 \ ; \tag{73}$$

at $y = -1$,

$$u_0 = -\sigma_0, \ \nu_0 = 1 \ . \tag{74}$$

It is easily seen that

$$\nu_0 = 1 \ , \tag{75}$$

$$u_0 = [\sin \theta/(2\mu)](1 - y^2) - \sigma_0 \ . \tag{76}$$

(75) and (76) are solutions for the incompressible viscous flow down an inclined plane.

The equations for the first approximation are

$$(\nu_1 u_0 + \nu_0 u_1)_x + (\nu_0 \nu_1)_y = 0 \ , \tag{77}$$

$$\mu \, u_{1yy} = -2\nu_{1yyx} - \nu_1 \sin \theta \ ; \tag{78}$$

$$2\nu_{1yyy} = \cos \theta \ , \tag{79}$$

at $y = 0$,

$$u_{1y} + u_{0yy}\eta_1 = 0 \ , \tag{80}$$

$$\nu_{1yy} = 0 \ , \tag{81}$$

-14-

$$v_{1y} = 0 \quad , \tag{82}$$

$$u_0 \eta_{1x} - v_1 = 0 \quad ; \tag{83}$$

at $y = -1$,

$$u_1 = -\sigma_1, \ v_1 = \nu_1 = 0 \quad . \tag{84}$$

It is found from (78) to (82), and (84) that

$$v_1 = (\cos \theta/12)(1 + y^3) \tag{85}$$

$$u_1 = [\sin \theta \cos \theta/(2\psi\mu)](9/10 - y^2 - y^5/10) - \sigma_1 + (\sin \theta/\mu)(1+y)\eta_1 \quad . \tag{86}$$

We now integrate (77) with respect to y from $y = -1$ to y and obtain, by (75), (76), (85) and (86),

$$v_1 = -\int_{-1}^{y} u_{1x} dy = -[\sin \theta/(2\mu)](1+y)^2 \eta_{1x} \quad . \tag{87}$$

It follows from (76), (83) and (87), that

$$[\sin \theta/(2\mu) - \sigma_0]\eta_{1x} = -\sin \theta/(2\mu)\eta_{1x} \quad .$$

Assume that $\eta_{1x} \not\equiv 0$ and we have

$$\sigma_0 = \sin \theta/\mu \quad . \tag{88}$$

Now we proceed to the equations for the second approximation

$$\nu_{1t} + (\nu_2 u_0 + \nu_1 u_1 + \nu_0 u_2)_x + (\nu_1 v_1 + \nu_0 v_2)_y = 0 \quad , \tag{89}$$

$$u_0 u_{1x} + v_1 u_{0y} = 2\nu_{2yyx} + \mu u_{2yy} + \nu_2 \sin \theta \quad , \tag{90}$$

$$- 2\beta\nu_{1y} + 2(\nu_{2yyy} + \nu_1 \nu_{1yyy}) - \nu_1 \cos \theta = 0 \quad ; \tag{91}$$

at $y = 0$,

$$(-\beta+2\nu_{1yy})\eta_{1x} - \mu(u_{2y}+u_{1y}\eta_1+u_{0yy}\eta_2+u_{0yyy}\eta_1^2/2) = 0 \quad , \tag{92}$$

$$2\beta\nu_1 =)\nu_{1y})^2 - 2(\nu_{2yy}-\nu_1\nu_{1yy}+\nu_{1yyy}\eta_1) = 0 \quad , \tag{93}$$

$$\nu_{2y} + \nu_{1yy}\eta_1 = 0 \quad , \tag{94}$$

$$\eta_{1t} + u_0\eta_{2x} + u_1\eta_{1x} + u_{0y}\eta_1\eta_{1x} - v_2 - v_{1y}\eta_1 = 0 \quad ; \tag{95}$$

at $y = -1$,

$$u_2 = v_2 = \nu_2 = 0 \quad . \tag{96}$$

From (85), (91), (93), (94) and (96), we find that

$$\nu_2 = (\beta \cos \theta/24)(-9/10 + y^2 + y^5/10) + (\cos \theta/4)(1-y^2)\eta_1 \quad . \tag{97}$$

Then from (76), (86), (87), (90), (92), (96) and (97) it is obtained that

$$\mu u_2 = [-\beta - \cos\theta/2 - \sin^2\theta/(8\mu^2) - \beta y + y^2\cos\theta/2 + \sin^2\theta(-y^3/6 + y^4/12)]\eta_{1x}$$

$$+ \cos\theta\sin\theta(5-6y^2+y^4)\eta_1/48 - \sin\theta(1+y)\eta_1^2 + \sin\theta(1+y)\eta_2 \qquad (98)$$

$$- \beta\cos\theta\sin\theta(155-189y^2 + 35y^4 + y^7)/10080 \quad.$$

Making use of (74), (76), (83) to (88) and (95) to (98), we integrate (89)
with respect to y from $y = -1$ to $y = 0$ to obtain

$$\eta_{1t} + m_1\eta_{1x} + m_2\eta_1\eta_{1x} = m_3\eta_{1xx} \quad, \qquad (99)$$

where the coefficient of η_2 vanishes because of (88), and

$$m_1 = \cos\theta\sin\theta/(24\mu) - \sigma_1 \quad,$$

$$m_2 = \sin\theta/\mu \quad,$$

$$m_3 = \beta/2 + \cos\theta/3 + 23\sin^2\theta/(240\mu^2) \quad.$$

5. Discussion

It is well known that the Burgers equation as given by (69) or (99) is ill-posed if the coefficients m_0 and m_3 are of opposite signs. Therefore, we may use this result to define the criterion of stability. Since in (69), m_0 and m_3 depend upon θ, α, β, λ and μ, we say a granular jump is stable if m_0 and m_3 are of same sign i.e. $m_3/m_0 = f(\theta,\alpha,\beta,\lambda,\mu) > 0$. For (99), $m_0 = 1$ and $m_3 > 0$. Hence a granular jump is always stable in this sense.

The solution method for the Burgers equation is also well known and we only give a brief discussion of the steady state solutions of (69) and (99) in reference to a moving frame. Assume all the coefficients in (69) are not zero, $m_3/m_0 > 0$ and $\eta_1, \eta_{1x} \to 0$ as $x \to \infty$. It is easily found that

$$\eta_1 = -(m_1/m_2)[1 + \tanh(m_1(x+c)/(2m_3))] \quad .$$

where c is an arbitrary constant and we require $m_1/m_3 < 0$. Since in (69), $m_1 = -m_0\sigma_1$, $m_3/m_0 > 0$ and $m_1/m_3 < 0$ imply $\sigma_1 > 0$. Since $\sigma = \sigma_0 + \epsilon\sigma_1 > \sigma_0$, we say the granular jump moves at a supercritical speed. If $m_0/m_2 > 0$, then $\eta_1 \to 2m_0\sigma_1/m_2 > 0$ as $x \to -\infty$. In this case η_1 decreases smoothly from $2m_0/\sigma_1/m_2$ to zero as x increases. On the other hand, if $m_0/m_2 < 0$, then η_1 increases smoothly from $\omega_0\sigma_1/m_2 < 0$ to zero as x increases. For the special case considered in Section 4, $m_0 = 1$, $m_2 > 0$, $m_3 > 0$. Hence $m_1/m_3 < 0$ implies $\sigma_1 > \cos\theta \sin\theta/(24\mu) > 0$ and the speed of the granular jump is supercritical. Furthermore, $m_2 > 0$ implies that we always have a decreasing granular jump.

Finally we consider some problems for further study. If the free surface of the granular material is replaced by a rigid plane, then one boundary condition is lost and we do not have enough boundary conditions to determine the flow. In fact, in the gravity flow of a granular material between two vertical plates (Goodman and Cowin, 1971), a slug flow appears in the central

part of the vertical channel. It should be of interest to extend the approach to the study of the granular flow between two parallel inclined planes. Furthermore, if $m_0/m_3 < 0$, an investigation of the ill-posed problem is certainly of importance. Finally the asymptotic method used here is based upon the existence of a solution given by (15). A justification of the method, needless to say, will be a significant contribution to granular flow problems.

REFERENCES

Cowin, S. C. and Satake, M., (Ed.). Continuum Mechanical and Statistical
Approaches in the Mechanics of Granular Materials, Association for
Science Documents Information, Tokyo, Japan (1978).

Goodman, M. A. and Cowin, S. C., Two Problems in the Gravity Flow of Granular
Materials, J. Fluid Mech. 45, 321-339 (1971).

Goodman, M. A. and Cowin, S. C., A Continuum Theory for Granular Materials,
Arch. Rational Mech. Anal. 44, 249-265 (1972).

Landau, L. D. and Lifshitz, E. M., Fluid Mechanics, Addison-Wealey, Reading,
MA (1960).

Morrison, H. L. and Richmond, O., Application of Spencer's Ideal Soil Model to
Granular Materials Flow, J. Appl. Mech., 43, 49-53 (1976).

Nunziato, J. W., Passman, S. L., and Thomas, J. R., Gravitational Flows of
Granular Materials with Incompressible Grains, J. Rheology, 24,
395-420 (1980).

Nunziato, J. W. and Walsh, E. K., Small-Amplitude Wave Behavior in One-
Dimensional Granular Solids, J. Appl. Mech., 44, 559-564 (1977).

Savage, S. B., Gravity Flow of Cohensionless Granular Materials in Chutes and
Channels, J. Fluid Mech., 92, 53-96 (1979).

Savage, S. B. and Jeffrey, D. J., The Stress Tensor in a Granular Flow at High
Shear Rates, J. Fluid Mech., 110, 255-272 (1981).

Sayed, M. and Savage, S. B., Rapid Gravity Flow of Cohensionless Granular
Materials Down Inclined Chutes, J. Appl. Math. Phys. (ZAMP), 34, 85-99
(1983).

Shen, M. C., Nonlinear Gravity-Capillary Waves on a Compressible Viscous Fluid
with Edge Constraints, Technical Summary Report #2520, Mathematics
Research Center, University of Wisconsin, Madison (1983).

MCS/jvs

REPORT DOCUMENTATION PAGE	READ INSTRUCTIONS BEFORE COMPLETING FORM

1. REPORT NUMBER #2529	2. GOVT ACCESSION NO. AD-A130501	3. RECIPIENT'S CATALOG NUMBER

4. TITLE *(and Subtitle)* Granular Jumps Down an Inclined Plane	5. TYPE OF REPORT & PERIOD COVERED Summary Report - no specific reporting period
	6. PERFORMING ORG. REPORT NUMBER

7. AUTHOR(s) M. C. Shen	8. CONTRACT OR GRANT NUMBER(s) MCS 800-1960 DAAG29-80-C-0041

9. PERFORMING ORGANIZATION NAME AND ADDRESS Mathematics Research Center, University of 610 Walnut Street Wisconsin Madison, Wisconsin 53706	10. PROGRAM ELEMENT, PROJECT, TASK AREA & WORK UNIT NUMBERS Work Unit Number 2 - Physical Mathematics

11. CONTROLLING OFFICE NAME AND ADDRESS See Item 18 below	12. REPORT DATE June 1983
	13. NUMBER OF PAGES 19

14. MONITORING AGENCY NAME & ADDRESS*(if different from Controlling Office)*	15. SECURITY CLASS. *(of this report)* UNCLASSIFIED
	15a. DECLASSIFICATION/DOWNGRADING SCHEDULE

16. DISTRIBUTION STATEMENT *(of this Report)*

Approved for public release; distribution unlimited.

17. DISTRIBUTION STATEMENT *(of the abstract entered in Block 20, if different from Report)*

19. KEY WORDS *(Continue on reverse side if necessary and identify by block number)*

Asymptotic method, granular jump, gravity flow, Burgers equation, stability

20. ABSTRACT *(Continue on reverse side if necessary and identify by block number)*

We study granular jumps generated in the gravity flow of granular materials down an inclined plane. The equations governing the granular flow are reduced to a sequence of boundary value problems of linear ordinary differential equations by means of an asymptotic method. Their solutions are used to determine the Burgers equation, which possesses a progressive wave solution to describe a smooth granular jump. The wave speed and a criterion for the stability of the granular jump are also obtained.

DD ₁ FORM JAN 73 **1473**   EDITION OF 1 NOV 65 IS OBSOLETE

EN

DATE
FILME

8-8

DTIC

ND

DATE

LMED

83

OTIC