||| 1.0

4.5
5.0
5.6

||| 2.8 ||| 2.5

||| 3.2 ||| 2.2

||| 3.6

||| 1.1

||| 4.0 ||| 2.0

||| 1.8

||| 1.25 ||| 1.4 ||| 1.6

MICROCOPY RESOLUTION TEST CHART
NATIONAL BUREAU OF STANDARDS-1963-A

RADC-TR-83-26
In-House Report
February 1983

ON THE ODD-SYMMETRY OF MINIMUM PHASE-ONLY PERTURBATIONS

Robert A. Shore

DTIC
ELECTE
JUL 2 0 1983

A

ROME AIR DEVELOPMENT CENTER
Air Force Systems Command
Griffiss Air Force Base, NY 13441

83 05 20 032

This report has been reviewed by the RADC Public Affairs Office (PA) and is releasable to the National Technical Information Service (NTIS). At NTIS it will be releasable to the general public, including foreign nations.

RADC-TR-83-26 has been reviewed and is approved for publication.

APPROVED: *Philipp Blacksmith*

PHILIPP BLACKSMITH
Chief, EM Techniques Branch
Electromagnetic Sciences Division

APPROVED: *Allan C. Schell*

ALLAN C. SCHELL
Chief, Electromagnetic Sciences Division

FOR THE COMMANDER: *John P. Huss*

JOHN P. HUSS
Acting Chief, Plans Office

If your address has changed or if you wish to be removed from the RADC mailing list, or if the addressee is no longer employed by your organization, please notify RADC (EECS) Hanscom AFB MA 01731. This will assist us in maintaining a current mailing list.

Do not return copies of this report unless contractual obligations or notices on a specific document requires that it be returned.

REPORT DOCUMENTATION PAGE		READ INSTRUCTIONS BEFORE COMPLETING FORM
1. REPORT NUMBER RADC-TR-83-26	2. GOVT ACCESSION NO. AD-A130 514	3. RECIPIENT'S CATALOG NUMBER
4. TITLE (and Subtitle) ON THE ODD-SYMMETRY OF MINIMUM PHASE-ONLY PERTURBATIONS		5. TYPE OF REPORT & PERIOD COVERED Inhouse
		6. PERFORMING ORG. REPORT NUMBER
7. AUTHOR(s) Robert A. Shore		8. CONTRACT OR GRANT NUMBER(s)
9. PERFORMING ORGANIZATION NAME AND ADDRESS Rome Air Development Center (RADC/EEC) Hanscom AFB Massachusetts 01731		10. PROGRAM ELEMENT, PROJECT, TASK AREA & WORK UNIT NUMBERS 61102F 230J304
11. CONTROLLING OFFICE NAME AND ADDRESS Rome Air Development Center (RADC/EEC) Hanscom AFB Massachusetts 01731		12. REPORT DATE February 1983
		13. NUMBER OF PAGES 24
14. MONITORING AGENCY NAME & ADDRESS(if different from Controlling Office)		15. SECURITY CLASS. (of this report) Unclassified
		15a. DECLASSIFICATION DOWNGRADING SCHEDULE N/A

16. DISTRIBUTION STATEMENT (of this Report)

Approved for public release; distribution unlimited.

17. DISTRIBUTION STATEMENT (of the abstract entered in Block 20, if different from Report)

18. SUPPLEMENTARY NOTES

RADC Project Engineer - Robert A. Shore, EEC

19. KEY WORDS (Continue on reverse side if necessary and identify by block number)
Antenna
Linear arrays
Nulling
Phase-only control

20. ABSTRACT (Continue on reverse side if necessary and identify by block number)
It is proved that the minimum phase-only weight perturbations required to impose a single null in a real, linear array, antenna pattern are odd-symmetric with respect to a phase reference at the center of the array. Hence the perturbed pattern is likewise real. Additional evidence is given for supporting the hypothesis that the minimum phase-only weight perturbations are odd-symmetric for any number of imposed nulls.

DD FORM 1473 EDITION OF 1 NOV 65 IS OBSOLETE

Preface

The author wishes to thank Dr. Hans Steyskal for many helpful discussions.

3

Contents

Illustrations

5

On the Odd-Symmetry of Minimum Phase-Only Perturbations

I. INTRODUCTION

The increasing importance of phased arrays has been accompanied by considerable interest in phase-only control of array element weights for adaptive nulling[1-5] and for null synthesis.[6-13] Unlike for combined phase and amplitude control, the problem of imposing nulls in a linear array pattern with phase-only weight control is nonlinear and cannot be solved exactly analytically. Approximations and/or numerical techniques must be employed to calculate the required phase perturbations.

A problem of some practical interest in null synthesis is that of imposing nulls in a given array antenna pattern at a prescribed set of locations in such a way as to minimize the effect on the given pattern at other locations. If the given pattern is real and the weight perturbations are of the element phases only, it is intuitive* that the perturbed pattern should likewise be real and hence that the phase perturbations be odd-symmetric with respect to a phase reference taken at the center of

---

(Received for publication 10 February 1983)

*In their basic paper on adaptive nulling with phase-shifters, Baird and Rassweiler[1] make an implicit assumption based on intuition that the optimal phase-only weights for suppressing noise from a set of specified directions are odd-symmetric. The coefficients γ_k in their Eq. (8) are implicitly assumed to be real.

(Due to the large number of References cited above, they will not be listed here. See References, page 15.)

the array. If numerical techniques such as nonlinear programming are used to compute the desired phase perturbations, the odd-symmetry property can be used to reduce the number of independent unknown phases by a factor of a half. Although the odd-symmetry property of the minimum phase-only perturbations may be intuitive, nevertheless a proof of this property would be desirable.

The principal object of this report is to prove the odd-symmetry property of the minimum phase-only weight perturbations required to impose a single null in a real array antenna pattern. The proof unfortunately does not appear to lend itself to generalization to the case of more than one null. However, we present additional evidence to support the conjecture that the minimum phase-only perturbations are in general odd-symmetric, and it is hoped that this paper will serve as a stepping stone to the finding of a general proof.

2. ANALYSIS

Consider a linear array of equispaced isotropic elements (see Figure 1). The spacing between the elements is d and the phase reference center is taken to be at the center of the array. Let a_n, $n = 1, 2, \ldots, N$, be the amplitude of the nth array element, and assume symmetry of the amplitudes with respect to the reference center; that is,

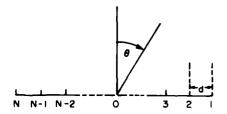$$a_n = a_{N-n+1}, \quad n = 1, 2, \ldots, N. \tag{1}$$

Figure 1. Geometry of Array

Then the array field pattern, $p_0(u)$, is

$$p_0(u) = \sum_{n=1}^{N} a_n e^{jd_n u}$$

where

$$d_n = \frac{N-1}{2} - (n-1), \quad n = 1, 2, \ldots, N$$

8

and

$$u = kd \sin \theta$$

with

$$k = \frac{2\pi}{\lambda}$$

and θ the angle measured from broadside to the array. The $\{d_n\}$ are odd-symmetric with respect to the phase reference center; that is,

$$d_n = -d_{N-n+1}, \quad n = 1, 2, \ldots, N . \tag{2}$$

The pattern $p_0(u)$ is real because of the symmetry of the $\{a_n\}$ and the $\{d_n\}$.

Now suppose it is desired to impose a null in the pattern at the location $u = u_1$ with phase-only perturbations $e^{j\phi_n}$ of the array weights. That is, we would like to have

$$\sum_{n=1}^{N} a_n e^{j\phi_n} e^{j d_n u_1} = 0 . \tag{3}$$

At the same time we wish to minimize the weight perturbations in a least squares sense. Since the perturbation of the nth weight is $a_n(e^{j\phi_n} - 1)$, we wish to minimize the quantity

$$\sum_{n=1}^{N} a_n^2 \left| e^{j\phi_n} - 1 \right|^2 = 4 \sum_{n=1}^{N} a_n^2 \sin^2\left(\frac{\phi_n}{2}\right) . \tag{4}$$

We claim that the desired phase perturbations are then odd-symmetric with respect to the phase reference center; that is,

$$\phi_n = -\phi_{N-n+1}, \quad n = 1, 2, \ldots, N \tag{5}$$

so that the resulting field pattern remains real. The procedure we will use to prove Eq. (5) is first to assume that a set of phase perturbations has been found to satisfy Eq. (3); then to express these phase perturbations as a sum of two sets of perturbations, one with even-symmetry and the other with odd-symmetry; and finally, to show that a set of odd-symmetric phase perturbations can be constructed that satisfies Eq. (3) and yields a lower value of the quantity

9

$$\sum_{n=1}^{N} a_n^2 \sin^2\left(\frac{\phi_n}{2}\right)$$

than the starting set of phase perturbations with an even-symmetric component.

Accordingly, we begin with a set of phase perturbations, ϕ_n, $n = 1, 2, \ldots, N$, that satisfy Eq. (3), and let

$$\phi_n = \phi_{ne} + \phi_{no}, \quad n = 1, 2, \ldots, N \tag{6}$$

where*

$$\phi_{ne} = \frac{1}{2}(\phi_n + \phi_{N-n+1})$$

$$n = 1, 2, \ldots, N$$

$$\phi_{no} = \frac{1}{2}(\phi_n - \phi_{N-n+1}).$$

Then

$$\phi_{N-n+1, e} = \phi_{ne}$$

$$n = 1, 2, \ldots, N \tag{7}$$

$$\phi_{N-n+1, o} = -\phi_{no}$$

so that the $\{\phi_{ne}\}$ are the even-symmetric part and $\{\phi_{no}\}$ the odd-symmetric part of the phase perturbations $\{\phi_n\}$. Substituting Eq. (6) in Eq. (3),

$$\sum_{n=1}^{N} a_n e^{j(\phi_{ne}+\phi_{no})} e^{jd_n u_1} = 0$$

from which it is straightforward to obtain, utilizing the symmetries of Eqs. (1), (2), and (7), the equation

$$\sum_{n=1}^{N} a_n \cos\phi_{ne} \cos(d_n u_1 + \phi_{no}) + j\sum_{n=1}^{N} a_n \sin\phi_{ne} \cos(d_n u_1 + \phi_{no}) = 0 . \tag{8}$$

We now note that a purely odd-symmetric solution $\{\phi'_{no}\}$ of Eq. (3) can be constructed by defining ϕ'_{no} to be the solution of the equation

---

If the number of elements in the array is odd, the phase of the center element is, of course, zero since this element is the phase reference for the array.

$$\cos \phi_{ne} \cos(d_n u_1 + \phi_{no}) = \cos(d_n u_1 + \phi'_{no}), \quad n = 1, 2, \dots, N \tag{9}$$

with the smallest absolute value. (It is clear that a solution of Eq. (9) always exists since the left hand side of Eq. (9) has magnitude less than or equal to one.) For the set of phase perturbations so constructed, the terms of the real part of Eq. (8) remain unchanged and so their sum continues to equal zero, while the imaginary part of Eq. (8) is zero because all the $\{\phi'_{ne}\}$ are identically equal to zero. To show that the set of odd-symmetric phase perturbations obtained by solving Eq. (9) gives a smaller value of the quantity

$$\sum_{n=1}^{N} a_n^2 \sin^2 \left(\frac{\phi_n}{2} \right)$$

than the starting set of phase perturbations with both an even- and odd-symmetric component, we first note that in view of Eq. (7) it is sufficient to show, taking the elements of the array by symmetrically located pairs, that

$$\sin^2 \left(\frac{\phi_{ne} + \phi_{no}}{2} \right) + \sin^2 \left(\frac{\phi_{ne} - \phi_{no}}{2} \right) \geq 2 \sin \left(\frac{\phi'_{no}}{2} \right). \tag{10}$$

Using the trigonometric identity

$$\sin^2 \left(\frac{\theta}{2} \right) = \frac{1}{2} (1 - \cos \theta)$$

and the formulas for the cosine of a sum and difference of two angles, it is simple to show that the inequality, (10), is equivalent to the inequality

$$\cos \phi_{ne} \cos \phi_{no} \leq \cos \phi'_{no}. \tag{11}$$

In Appendix A we show that the solution ϕ'_{no} of Eq. (9) with the smallest absolute value satisfies the inequality, (11), (strict inequality holds unless $d_n u_1$ is an integral multiple of π) and thus complete the proof that the minimum phase-only perturbations that result in an imposed null at a prescribed location in a real pattern have odd-symmetry with respect to the reference center of the array.

3. DISCUSSION

The proof of the odd-symmetry property of the minimum phase-only perturbations we have given above is based on the assumption that there is only one null to

be imposed in the antenna pattern. If there is more than one imposed null, the procedure of the proof does not appear to be capable of generalization. Suppose, for example, that nulls are to be imposed at two locations, u_i, $i = 1, 2$. Then, corresponding to Eq. (8), we now have the pair of equations

$$\sum_{n=1}^{N} a_n \cos \varrho_{ne} \cos(d_n u_i + \varrho_{no}) + j \sum_{n=1}^{N} a_n \sin \varrho_{ne} \cos(d_n u_i + \varrho_{no}) = 0$$
$$i = 1, 2$$

and there is no way in general that ϱ'_{no} can be found to simultaneously satisfy the pair of equations

$$\cos \varrho_{ne} \cos (d_n u_i + \varrho_{no}) = \cos(d_n u_i + \varrho'_{no}), \quad i = 1, 2$$

as required by the proof [see Eq. (9)]. It is true that we can find a different set of odd-symmetric phase perturbations $\varrho'_{no, i}$, $i = 1, 2$ to satisfy each of the equations individually, but the nonlinearity of the phase-only nulling equation, Eq. (3), then does not allow any simple way of combining the two sets of odd-symmetric phase-only perturbations, one for each null, to give one set of phase perturbations that will result in nulls at both of the locations simultaneously.

Although the proof we have given cannot be directly generalized to the case of more than one imposed null, additional evidence can be cited to support the conjecture that the minimum phase-only perturbations required to impose nulls in a real pattern at a prescribed set of locations have odd-symmetry. For one thing, Shore and Steyskal[9] have shown that if the assumption of small phase perturbations is made and used to linearize the phase-only nulling equation

$$\sum_{n=1}^{N} a_n e^{j\varphi_n} e^{j d_n u_i} = 0, \quad i = 1, 2, \ldots, M,$$

the resulting set of equations can be solved exactly analytically subject to the constraint that the weight perturbations be minimized in a generalized least squares sense. The phases thus obtained are odd-symmetric with respect to the reference center located at the center of the array.

Additional evidence to support the odd-symmetry hypothesis comes from the use of nonlinear programming techniques[14] to calculate the minimum phase-only

---

14. Shore, R. (to be published, 1983), Phase-Only Nulling as a Nonlinear Programming Problem, RADC-TR-83-37.

perturbations required to impose nulls in a pattern at a prescribed set of locations. It is found that even if no symmetry requirement is placed on the solution so that there are N unknown phases for an array of N elements, N even, (N - 1 if N is odd) the solution obtained has odd-symmetry and is identical to that obtained if the assumption of odd-symmetry is made at the outset to reduce the number of unknown phases by a factor of one half. A sample output is given in Appendix B.

In further support, in Appendix C it is shown, using the Lagrange multiplier method, that the minimum phase perturbations satisfy the equation

$$\sum_{n=1}^{N} a_n \sin \phi_n = 0$$

a property consistent with the odd-symmetry hypothesis in view of the even-symmetry of the $[a_n]$ assumed throughout [see Eq. (1)].

Thus it appears likely, although as yet unproven in general, that the odd-symmetry hypothesis is true, and it is hoped that this report will stimulate further effort to find a general proof.

References

1. Baird, C., and Rassweiler, G. (1976) Adaptive sidelobe nulling using digitally controlled phase-shifters, IEEE Trans. Antennas Propag. AP-24:638-649.

2. Mendelovicz, E., and Oestreich, E.T. (1979) Phase-only adaptive nulling with discrete values, IEEE AP-S International Symposium, 1979 International Symposium Digest, Antennas and Propagation, Vol. I:193-198.

3. Hockham, G.A., et al (1980) Null-steering techniques for application to large array antennas, Conference Proceedings, Military Microwaves 80, October 1980, pp. 623-628.

4. Ananasso, F. (1981) Nulling performance of null-steering arrays with digital phase-only weights, Electron. Lett. 17:255-257.

5. Ananasso, F. (1981) Null-steering uses digital weighting, Microwave System News, 11:78-94.

6. Davies, D.E.N. (1967) Independent angular steering of each zero of the directional pattern for a linear array, IEEE Trans. Antennas Propag. AP-15:296-298.

7. Cheng, D.K. (1971) Optimization techniques for antenna arrays, Proc. IEEE, 59:1664-1674.

8. Guo, Y.C., and Smith, M.S. (1981) Phase weighting for linear antenna arrays, Electron. Lett. 17:121-122.

9. Shore, R., and Stevskal, H. (1982) Nulling in Linear Array Patterns With Minimization of Weight Perturbations, RADC-TR-82-32, AD A118695.

10. Shore, R. (1982) An Iterative Phase-Only Nulling Method, RADC-TR-82-49, AD A116949.

11. Shore, R. (1982) Phase-Only Nulling as a Least Squares Approximation to Complex Weight Nulling, RADC-TR-82-129, AD A118722.

12. Shore, R. (1982) A unified treatment of nulling in linear array patterns with minimized weight perturbations, IEEE AP-S International Symposium, 1982 APS Symposium Digest, Antennas and Propagation, Vol. II:703-706.

13. Steyskal, H. (1982) Simple method for pattern nulling by phase only, IEEE AP-S International Symposium, 1982 APS Symposium Digest. Antennas and Propagation, Vol. II:707-710.

14. Shore, R. (to be published, 1983), Phase-Only Nulling as a Nonlinear Programming Problem, RADC-TR-83-37.

15. Pierre, D.A., and Lowe, M.J. (1975) Mathematical Programming Via Augmented Lagrangians: An Introduction With Computer Programs: Addison-Wesley Publishing Co., Reading, Massachusetts.

16. Fletcher, R. (1981) Practical Methods of Optimization. Volume 2. Constrained Optimization. John Wiley and Sons, New York.

Appendix A

Proof of a Trigonometric Inequality

In Appendix A we prove that the solution u to the equation

$$\cos x \cos (z + y) = \cos (z + u) \tag{A1}$$

with the smallest absolute value satisfies the inequality

$$\cos x \cos y \leq \cos u \tag{A2}$$

with strict inequality holding for all values of z not an integral multiple of π. Let x and y be arbitrary angles and consider the behavior of u as a function of z for z in the interval $[-\pi, \pi]$. Defining $\cos^{-1}(\theta)$ to lie in the interval $[0, \pi]$, we see that there are two solutions of Eq. (A1)

$$u_1 (z) = \cos^{-1} [\cos x \cos (z + y)] - z \tag{A3a}$$

and

$$u_2 (z) = -\cos^{-1} [\cos x \cos (z + y)] - z . \tag{A3b}$$

17

Since

$$u_2 (-z) = u_1 (\pi - z)$$

(A4)

$$u_1 (-z) = u_2 (\pi - z)$$

it suffices to consider the behavior of $u_1 (z)$ and $u_2 (z)$ for z in the interval $[0, \pi]$. As z increases from 0 to π, cos x cos $(z + y)$ varies from cos x cos y to cos x cos $(\pi + y)$ = -cos x cos y and correspondingly, $\cos^{-1}$ [cos x cos $(z + y)$] varies from $\cos^{-1}$ (cos x cos y) to $\cos^{-1}$ (-cos x cos y) = π - $\cos^{-1}$ (cos x cos y). Hence $u_1 (z)$ varies from $\cos^{-1}$ (cos x cos y) to $-\cos^{-1}$ (cos x cos y) while $u_2 (z)$ varies from $-\cos^{-1}$ (cos x cos y) to $\cos^{-1}$ (cos x cos y) -2π. Furthermore, since

$$\frac{d}{dx} \cos^{-1} (x) = - \frac{1}{\sqrt{1 - x^2}} \quad ,$$

differentiating Eqs. (A3a, A3b),

$$u'_{\genfrac{}{}{0pt}{}{1}{2}} (z) = \pm \frac{\cos x \sin (z + y)}{\sqrt{1 - \cos^2 x \cos^2 (z + y)}} - 1 .$$

But

$$\frac{\cos^2 x \sin^2 (z + y)}{1 - \cos^2 x \cos^2 (z + y)} = \frac{\cos^2 x - \cos^2 x \cos^2 (z + y)}{1 - \cos^2 x \cos^2 (z + y)} \le 1$$

with equality if and only if $\cos^2 x = 1$. Hence if $x \ne N\pi$, N = 0, ± 1, both $u_1'(z)$ and $u_2'(z)$ are negative over the entire range of z and so $u_1 (z)$ and $u_2 (z)$ are monotonically decreasing functions of z. It follows that as z increases from 0 to π, $u_1 (z)$ decreases monotonically from $\cos^{-1}$ (cos x cos y) to $-\cos^{-1}$ (cos x cos y) while $u_2 (z)$ decreases monotonically from $-\cos^{-1}$ (cos x cos y) to $\cos^{-1}$ (cos x cos y) -2π, (see Figure A1). Hence, in the interval $0 \le z \le \pi$, u = $u_1 (z)$ is the solution of Eq. (A1) with the smallest absolute value and takes on its largest absolute values at the endpoints of this interval. Since at the endpoints cos u = cos x cos y, it follows that for all values of z in between, cos x cos y < cos u and so we have demonstrated inequality (A2). If x = 0
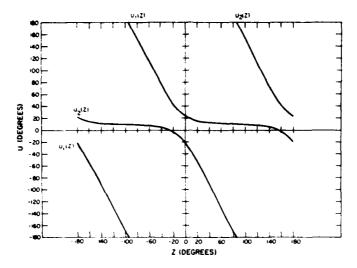
18

Figure A1. Plot of u_1 (z) = $\pm$ cos^{-1} [cos x cos (z + y)] - z,
$x = 20°$, $y = 10°$

$$u_1' (z) = \begin{cases} 0, & z + y \text{ in 1st and 2nd quadrants} \\ \\ -2, & z + y \text{ in 3rd and 4th quadrants} \end{cases}$$

$$u_2' (z) = \begin{cases} -2, & z + y \text{ in 1st and 2nd quadrants} \\ \\ 0, & z + y \text{ in 3rd and 4th quadrants} \end{cases}$$

while if $x = \pm \pi$,

$$u_1' (z) = \begin{cases} 0, & z + y \text{ in 3rd and 4th quadrants} \\ \\ -2, & z + y \text{ in 1st and 2nd quadrants} \end{cases}$$

$$u_2' (z) = \begin{cases} -2, & z + y \text{ in 3rd and 4th quadrants} \\ \\ 0, & z + y \text{ in 1st and 2nd quadrants} \end{cases}$$

and the inequality (A2) remains true as before, the only difference being that equality holds over an interval of z instead of at just the endpoints of the interval $0 \leq z \leq \pi$. In the interval $-\pi \leq z \leq 0$, we choose $u = u_2$ (z) and use Eq. (A4) and the above results for $0 \leq z \leq \pi$ to demonstrate inequality (A2), (see Figure A1). Thus the proof is complete.

19

As an example of the odd-symmetry of the phases obtained as a nonlinear pro-
gramming solution of the minimum phase-only weight perturbation problem, we show
results for a 21-element uniform amplitude array with half-wavelength spacing
The locations of the imposed nulls are 12, 15, and 18 degrees. The nonlinear
programming computer code LPNLP[15] was used. When no symmetry requirement
was imposed on the solution, the phases obtained are (in radians)

```
X VALUES,  X(1), . . . , X(N)
   0.568771D+00   0.369075D+00   -0.275962D+00   -0.329874D+00   -0.146163D+00
   0.512906D-01   0.152084D+00    0.695103D-01   -0.890314D-01   -0.103123D+00
   0.103123D+00   0.890314D-01   -0.695103D-01   -0.152084D+00   -0.512906D-01
   0.146163D+00   0.329874D+00    0.275962D+00   -0.369075D+00   -0.568771D+00
```

The first two rows are ϕ_1 through ϕ_{10} while the second two rows are ϕ_{12} through
ϕ_{21}. ($\phi_{11} = 0$ since the 11th element is the phase reference.) The odd-symmetry
of the solution is apparent. If odd-symmetry is imposed as a requirement on the
solution from the outset and the number of unknown phases halved, the output is

---

15. Pierre, D. A., and Lowe, M. J. (1975) Mathematical Programming Via
Augmented Lagrangians: An Introduction With Computer Programs
Addison-Wesley Publishing Co., Reading, Massachusetts.

```
X VALUES,  X(1). ..., X(N)
   0.568771D+00   0.369075D+00   -0.275962D+00   -0.329874D+00   -0.146163D+00
   0.512906D-01   0.152084D+00   0.695103D-01   -0.890314D-01   -0.103123D+00
```

which agrees perfectly with the former solution. The null depths corresponding to
these solutions are < -220 dB.

Appendix C

Proof That the Minimum Phase Perturbations
Satisfy the Equation $\displaystyle\sum_{n=1}^{N} a_n \sin \phi_n = 0$

In Appendix C we use the Lagrange multiplier technique to show that the minimum phase perturbations required to place nulls at the set of prescribed pattern locations u_k, $k = 1, 2, \ldots, M$, have the property that

$$\sum_{n=1}^{N} a_n \sin \phi_n = 0 . \qquad \text{(C1)}$$

From the theory of constrained optimization,[16] a necessary condition for a local minimum of a function $f(\underline{x})$ subject to a set of equality constraints $c_k(\underline{x}) = 0$, $k = 1, 2, \ldots, M$, is that the gradient of f be a linear combination of the gradients of the constraint functions; that is

$$\nabla f(\underline{x}^*) = \sum_{k=1}^{M} \lambda_k \nabla c_k(\underline{x}^*) \qquad \text{(C2)}$$

where $\underline{x}^*$ denotes the value of the vector of variables at the local minimum. The multipliers in this linear combination are referred to as Lagrange multipliers and the superscript * indicates that they are associated with the point $\underline{x}^*$.

---

16. Fletcher, R. (1981) Practical Methods of Optimization. Volume 2. Constrained Optimization. John Wiley and Sons, New York.

Here

$$f(\underline{o}) = \sum_{n=1}^{N} a_n \sin^2\left(\frac{o_n}{2}\right)$$

and the constraints are [see Eq. (3)]

$$\sum_{n=1}^{N} a_n \cos(d_n u_k + o_n) = 0$$

$$k = 1, 2, \ldots, M \qquad (C3)$$

$$\sum_{n=1}^{N} a_n \sin(d_n u_k + \phi_n) = 0 .$$

Taking the gradients of f and the constraint functions, and equating components in Eq. (C2) yields

$$a_n \sin \phi_n = \frac{1}{4}\left[-\sum_{k=1}^{M} \lambda_k a_n \sin(d_n u_k + \phi_n) + \sum_{k=1}^{M} \mu_k a_n \cos(d_n u_k + \phi_n)\right] .$$

Summing over n and using Eqs. (C3) then gives Eq. (C1).

24

MISSION
of
Rome Air Development Center

RADC plans and executes research, development, test and selected acquisition programs in support of Command, Control Communications and Intelligence (C^3I) activities. Technical and engineering support within areas of technical competence is provided to ESD Program Offices (POs) and other ESD elements. The principal technical mission areas are communications, electromagnetic guidance and control, surveillance of ground and aerospace objects, intelligence data collection and handling, information system technology, ionospheric propagation, solid state sciences, microwave physics and electronic reliability, maintainability and compatibility.

END

FILMED

8-83

DTIC