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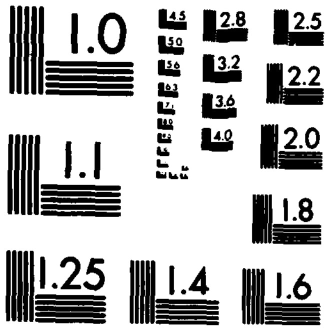

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FINGERS IN A HELE-SHAW CELL WITH SURFACE TENSION

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ABSTRACT

McLean and Saffman's [1] model for the fingering in a Hele-Shaw cell is solved numerically. The results suggest that a countably infinite number of solutions exist for non-zero surface tension. This set of solutions contains the solution previously obtained by McLean and Saffman [1].

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SIGNIFICANCE AND EXPLANATION

Two dimensional flow in a porous medium may be studied by using an analogue proposed by Hele-Shaw. The analogue is based on the fact that the mean velocity in a two dimensional porous medium and the velocity of the flow between two parallel plates satisfy the same equations.

In the present paper we consider the steady two dimensional flow produced by a finger advancing between the two plates. The analogous porous medium flow occurs in oil recovery. This problem was first considered by Saffman and Taylor [2]. They obtained an exact solution for zero surface tension. McLean and Saffman [1] generalized the result of Saffman and Taylor [2] by including the effect of surface tension at the interface. They solved the problem numerically and obtained one family of solutions. In the present paper we solve the problem by a different numerical scheme. Our results suggest the existence of a countable number of solutions for non-zero surface tension. This infinite set of solutions contains the solution previously obtained by McLean and Saffman [1].

The responsibility for the wording and views expressed in this descriptive summary lies with MRC, and not with the author of this report.

FINGERS IN A HELE-SHAW CELL WITH SURFACE TENSION

Jean-Marc Vanden-Broeck*

In a recent paper McLean and Saffman [1] modelled the fingering in a Hele-Shaw Cell by a two dimensional potential flow with surface tension included at the interface (see Figure 1). They introduced the notation that U is the velocity of the finger, $2a$ the lateral width of the channel, b the transverse thickness, T the surface tension and μ the velocity. In addition they denoted the ratio of the width of the finger to the width of the channel by the parameter λ.

Taking a as the unit length and $(1 - \lambda)U$ as the unit velocity they derived the following nonlinear integro-differential equation for the unknown shape of the free surface

$$\log q(S) = -\frac{S}{\pi}\int_0^1 \frac{\theta(S')}{S'(S'-S)}\,dS' \tag{1}$$

$$\kappa\, q\, S \frac{d}{dS}\left(q\, S \frac{d\theta}{dS}\right) - q = -\cos\theta \tag{2}$$

$$\theta(0) = 0 \qquad q(0) = 1 \tag{3}$$

$$\theta(1) = -\frac{\pi}{2} \qquad q(1) = 0 \ . \tag{4}$$

Here

$$\theta = \hat{\theta} - \pi \tag{5}$$

$$q = (1-\lambda)\hat{q} \tag{6}$$

$$\kappa = \frac{Tb^2\pi^2}{12\mu\ Ua^2(1-\lambda)^2} \ . \tag{7}$$

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The integral in (1) is of Cauchy principal value form. The variables $\hat{q}$ and $\hat{\theta}$ in (5) and (6) are defined in terms of the dimensionless complex velocity $\hat{u} - i\hat{v}$ by the relation

$$\hat{u} - i\hat{v} = \hat{q}e^{-i\hat{\theta}} \quad . \tag{8}$$

The flow configuration is illustrated in Figure 1. The $\hat{x}$-axis is parallel to the walls of the channel and is the axis of symmetry of the finger. The points A and B in Figure 1 correspond to $s = 0$ and $s = 1$ respectively.

After a solution for θ and q is obtained, the shape of the finger is given by

$$\hat{x}(S) + i\hat{y}(S) = -\frac{1-\lambda}{\pi} \int_S^1 \frac{e^{i\theta}}{Sq}\, dS \quad . \tag{9}$$

For $\kappa = 0$, Saffman and Taylor [2] obtained the following closed form solution

$$q = \left[\frac{(1-S)(1-\lambda)^2}{(1-\lambda)^2 + S(2\lambda-1)}\right]^{1/2} \tag{10}$$

$$\theta = \cos^{-1} q \quad .$$

The solution (10) leaves the parameter λ undetermined.

McLean and Saffman [1] discretized the system (1) - (4) with $\kappa \neq 0$ and solved the resultant algebraic equations by Newton's method. They obtained one family of solutions. Romero [3] repeated the calculations with a different numerical scheme and different initial guesses. He obtained two other families of solutions.

In this paper we present a systematic numerical procedure to compute all the possible solutions of the system (1) - (4). We show that for each $\kappa \neq 0$ there exists a countably infinite number of solutions. This set of solutions contain the families of solutions previously obtained by McLean and Saffman [1] and Romero [3]. Our results show that the degeneracy of the Saffman-Taylor [2] solutions is not completely removed for $\kappa \neq 0$.

In order to find all the solutions of (1) - (4), we define a modified problem which has solutions for all values of λ and κ. This modified problem is obtained by replacing (4) by

$$q(1) = 0 \quad . \tag{11}$$

Therefore $\theta(1)$ becomes a free parameter to be found as part of the solution.

We will solve the modified problem defined by (1) - (3) and (11) and obtain solutions for all values of λ and κ. Then we will obtain the solutions of the original problem by selecting among the solutions of the modified problem those for which $\theta(1) = -\pi/2$.

Following McLean and Saffman [1] we introduce the change of variable

$$s^{\tau} = 1 - \zeta^{\gamma} \quad . \tag{12}$$

Here τ is the smallest root of

$$\frac{1}{\tau^2} \cot \pi\tau = \kappa \quad . \tag{13}$$

With (12), θ is twice differentiable with respect to ζ at both end points. McLean and Saffman [1] chose $\gamma = 2$ in (12). In order to solve the modified problem we choose $\gamma = 4$.

To solve the system (1) - (3) and (11) we introduce the N mesh points

$$\zeta_I = (I-1)/N \quad I = 1,\ldots,N \quad . \tag{14}$$

We also define the quantities

$$\theta_I = \theta\left[\left(1 - \zeta_I^{\gamma}\right)^{1/\tau}\right] \; I = 1,\ldots,N \quad . \tag{15}$$

We discretize the system (1) - (3) and (11) by following the procedure outlined by McLean and Saffman [1]. Thus we obtain $N - 1$ nonlinear algebraic equations for the $N - 1$ unknowns θ_I, $I = 2,\ldots,N$. For given values of λ and κ this system is solved by Newton's method.

In Figure 2 we present numerical values of $\theta(1)$ versus λ for $\kappa = 0.273$. As λ approaches zero, $\theta(1) \to -\pi$ and the finger collapses on the negative $\hat{x}$-axis. As λ approaches one, $\theta(1)$ oscillates infinitely often around $-\pi/2$.

Figure 2 shows that there exists a countably infinite number of values of λ for which $\theta(1) = -\pi/2$. The solutions corresponding to these values of λ are the solutions of the original problem.

The three smallest values of λ for which $\theta(1) = -\frac{\pi}{2}$ are 0.54, 0.61 and 0.67. The corresponding profiles are shown in Figure 3. The value $\lambda = 0.54$ corresponds to the solution obtained by McLean and Saffman [1].

Similar results were found for other values of κ. As κ decreases, the amplitudes and the wavelengths of the oscillations in Figure 2 decrease. In Figure 4 we present the values of three smallest values of λ as functions of κ. The three curves approach $\lambda = \frac{1}{2}$ as κ tends to zero. The lowest of the three curves is the family of solutions previously obtained by McLean and Saffman [1]. The two other curves correspond to the families of solutions obtained by Romero [3]. The values of λ presented by Romero agree with our numerical results within 4% for $\kappa > 0.2$. However, Romero did not obtain reliable results for small values of κ.

Acknowledgement

I wish to thank Professor P. G. Saffman for bringing to my attention the work of Romero.

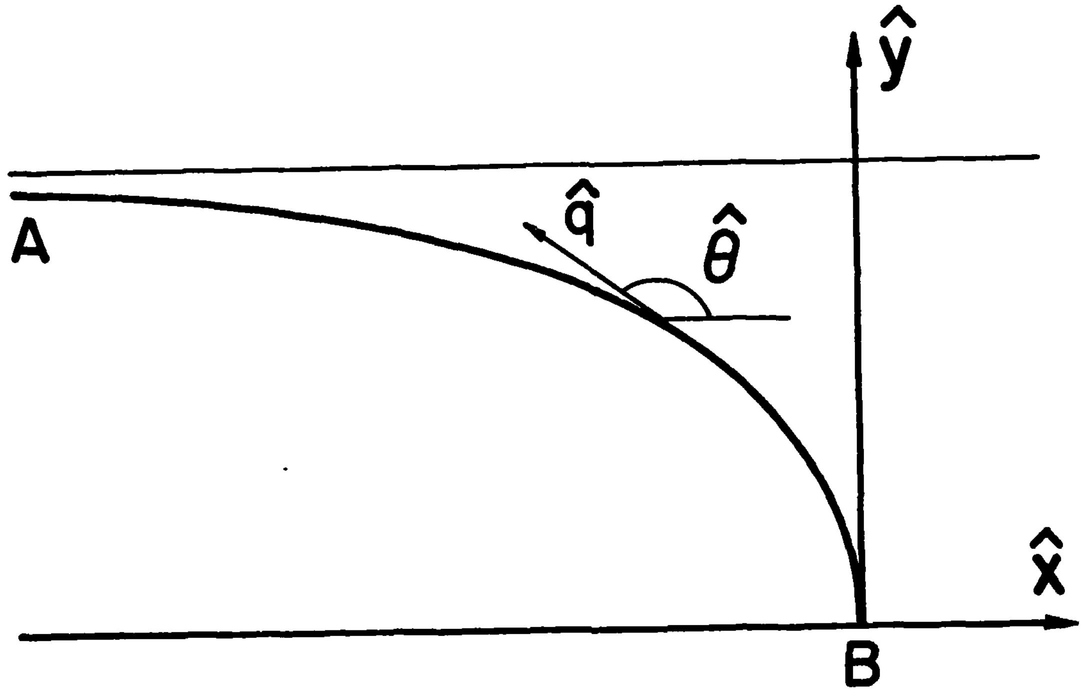

Figure 1: Sketch of the flow and the coordinates

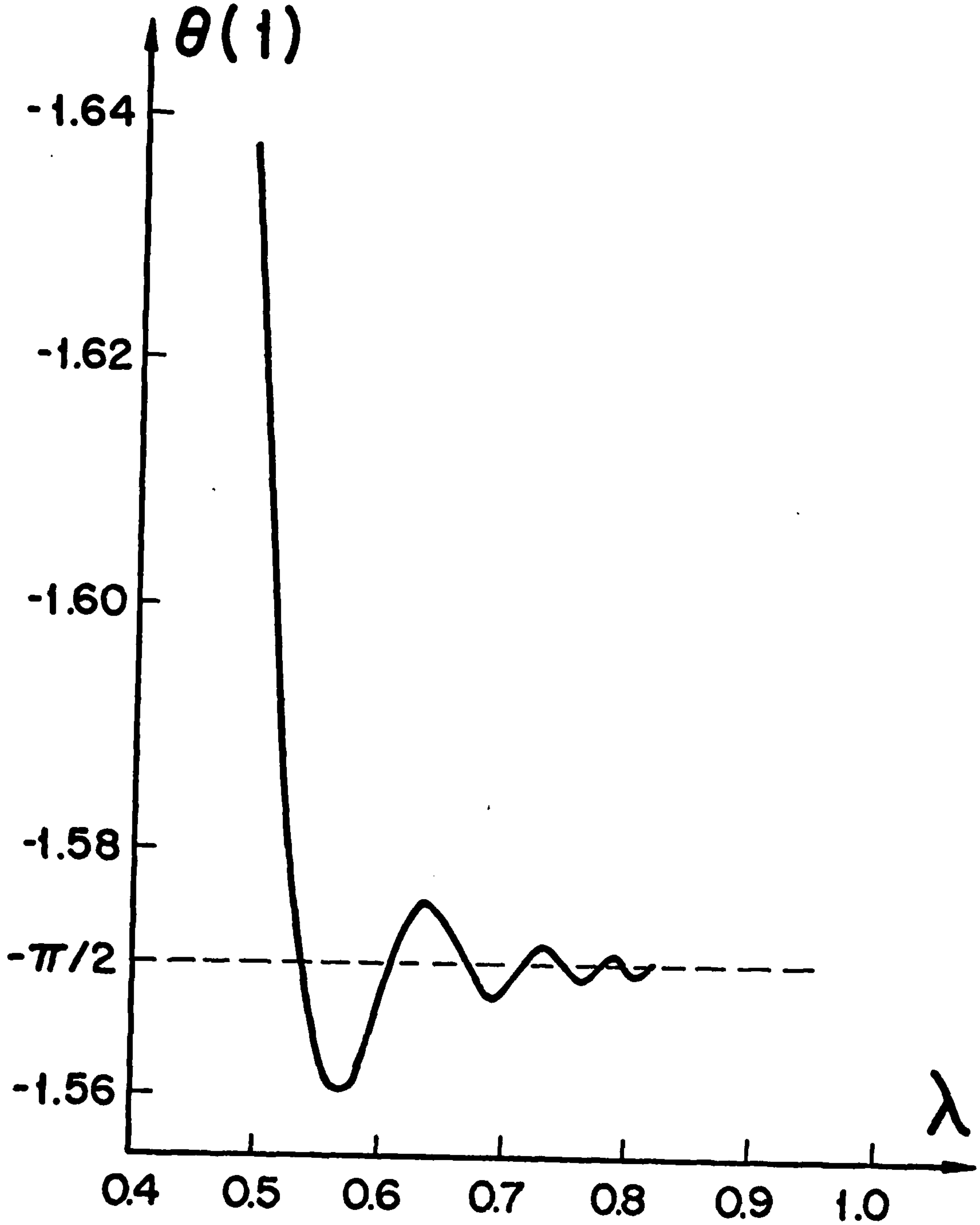

Figure 2: Values of $\theta(1)$ versus λ for $\kappa = 0.273$

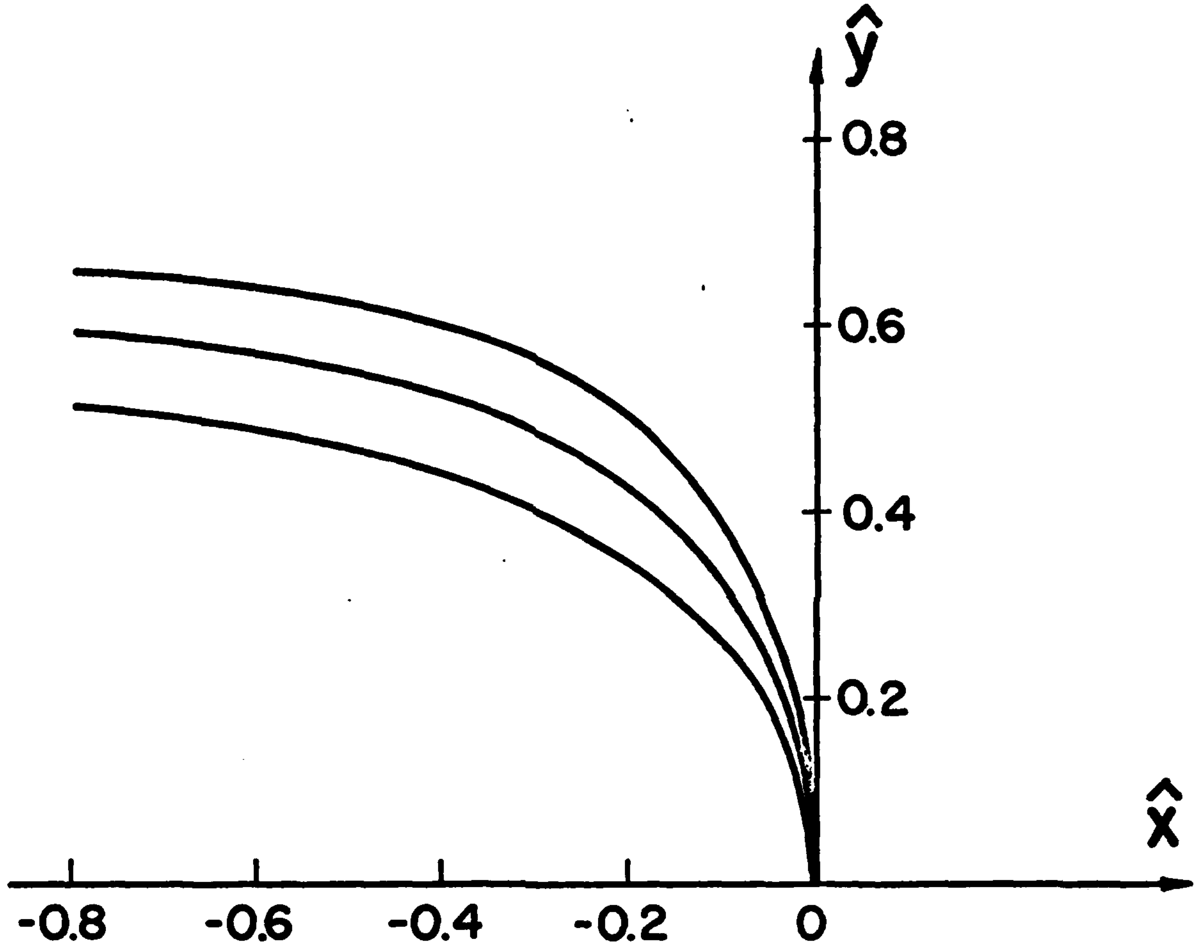

Figure 3: Profiles of the finger for $\kappa = 0.273$.
The values of λ are 0.54, 0.61 and 0.67

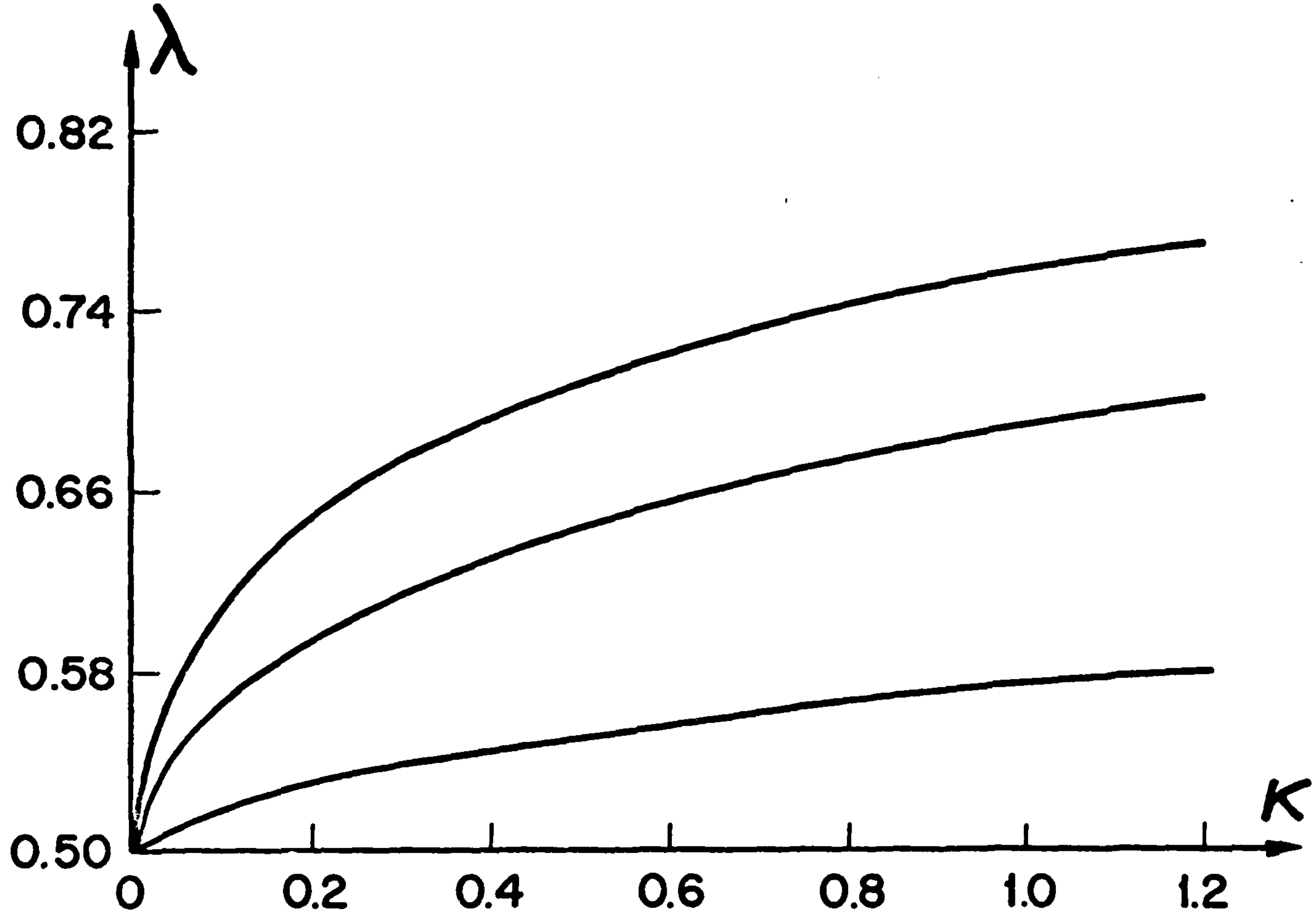

Figure 4: Values of λ versus κ.

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[1] J. W. McLean and P. G. Saffman, J. Fluid Mech. <u>102</u>, 455 (1981).

[2] P. G. Saffman and G. I. Taylor, Proc. Roy. Soc. A <u>245</u>, 312 (1958).

[3] L. Romero, Ph.D. Thesis, California Institute of Technology (1982).

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20. ABSTRACT (Continue on reverse side if necessary and identify by block number)

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