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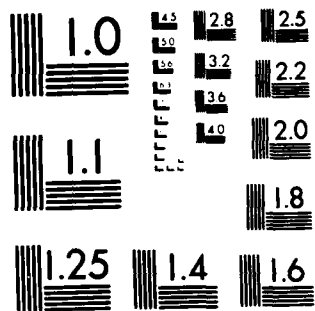
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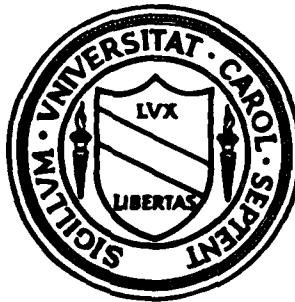
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WEAK AND STRONG LAW RESULTS FOR A FUNCTION OF THE SPACINGS

William P. McCormick

TECHNICAL REPORT #30

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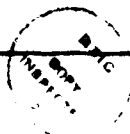
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max min  $S_{j,n}$ . Exact and weak limit results are obtained for the  $M_{\ell,n}$ .  
 $1 \leq i \leq n - \ell$  is  $j \leq i + \ell$ . Further we show that with probability one

$$\lim_{n \rightarrow \infty} \frac{(\ell+1)nM_{\ell,n}}{\log n} = 1.$$

This extends results of Cheng.

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# WEAK AND STRONG LAW RESULTS FOR A FUNCTION OF THE SPACINGS

William P. McCormick

## Abstract

Let  $\{U_n, n \geq 1\}$  be i.i.d. uniform on  $(0,1)$  random variables and define  $S_{i,n} = U_{i,n-1} - U_{i-1,n-1}$ ,  $i=1, \dots, n$  where the  $U_{i,n-1}$  are the order statistics from a sample of size  $n-1$  and  $U_{0,n-1} = 0$  and  $U_{n,n-1} = 1$ . The  $S_{i,n}$  are called the spacings divided by  $U_1, \dots, U_{n-1}$ . For a fixed integer  $\ell$ , set  $M_{\ell,n} = \max_{1 \leq i \leq n-\ell} \min_{i \leq j \leq i+\ell} S_{j,n}$ . Exact and weak limit results are obtained for the  $M_{\ell,n}$ . Further we show that with probability one

$$\lim_{n \rightarrow \infty} \frac{(\ell+1)nM_{\ell,n}}{\log n} = 1.$$

This extends results of Cheng.

Keywords: Order statistic, spacings, limiting distribution, strong law.

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# 1. Exact distribution.

Let  $X_1, X_2, \dots, X_n$  be i.i.d. having continuous distribution  $F$ . Let  $\ell$  be a fixed integer and define a random variable  $Y_{\ell, n} = \max_{1 \leq i \leq n-\ell} \min_{i \leq j \leq i+\ell} X_j$ . In this paper we will determine the exact and limiting distribution of  $Y_{\ell, n}$ . In section 3 these results are then applied to obtain weak and strong laws for spacings generalizing previous work of Cheng [5]. Further these results may be of independent interest and we mention [4] in which a similar analysis has been carried out.

Let  $X_{1,n} \leq X_{2,n} \leq \dots \leq X_{n,n}$  denote the order statistics and define the random index  $R_n$  by  $Y_{\ell, n} = X_{R_n, n}$ . If  $r_1, r_2, \dots, r_n$  denote the ranks of  $X_1, X_2, \dots, X_n$ , it is clear that

$$(1.1) \quad R_n = \max_{1 \leq i \leq n-\ell} \min_{i \leq j \leq i+\ell} r_j.$$

Observe that  $R_n$  is independent of  $X_{1,n}, \dots, X_{n,n}$  and  $R_n$  has the distribution of the permutation statistic defined by the right hand side of (1.1) with all permutations equally likely. In the following we take  $R_n$  as defined on the space of permutations of  $1, 2, \dots, n$ . Then  $Y_{\ell, n} \stackrel{d}{=} X_{R_n, n}$ , that is, we have equality in distribution.

We introduce some convenient terminology. Define an  $r$ -component of a permutation as a collection of consecutive entries each of which is greater than  $r$  and the collection is maximal with respect to this property. The size of an  $r$ -component is defined to be the number of elements in the component. Further let  $\beta_{j_1, \dots, j_\ell}^{r, n} = \beta_{j_1, \dots, j_\ell}^{r, n}$  equal the number of permutations on  $n$  elements with exactly  $j_k$   $r$ -components of size  $k$  and no  $r$ -component of size greater than  $\ell$ .

Note

$$(1.2) \quad \#(R_n \leq r) = \sum_{\underline{j} \in I_{n-r}} \beta_{j_1, \dots, j_\ell}^{r, n}$$

where  $\#$  denotes cardinality and  $I_{n-r} = \{\underline{j} = (j_1, \dots, j_\ell) : \sum_{t=1}^{\ell} t j_t = n - r \text{ and } j_t \text{ is a nonnegative integer}\}.$

$\beta_{j_1, \dots, j_\ell}$  may be evaluated by the following elementary counting argument. First select out of  $\sum_1^\ell j_t$  places  $j_1$  places for 1-components,  $j_2$  places for 2-components, ...,  $j_\ell$  places for  $\ell$ -components. This is done in  $(\sum_1^\ell j_t)! / [\prod_1^\ell j_t!]^{-1}$  ways. Next arrange the numbers  $r+1, \dots, n$  in one of  $(n-r)!$  ways and the numbers  $1, \dots, r$  in one of  $r!$  ways. Finally choose  $\sum_1^\ell j_t$  spaces among the  $r+1$  spaces separating the numbers  $1, \dots, r$ . This is done in  $\binom{r+1}{\sum_1^\ell j_t}$  ways.

Notice that a permutation counted by  $\beta_{j_1, \dots, j_\ell}$  can be constructed as follows. Designate the spaces chosen in the last step as being a 1-component, 2-component, ..., or  $\ell$ -component according to the selection made in step 1. In these spaces make the appropriately sized component by placing the numbers  $r+1, \dots, n$  according to the order given them by their permutation. Between these components place the numbers  $1, \dots, r$  in the order given them by their permutation. This construction is also reversible. Hence

$$(1.3) \quad \beta_{j_1, \dots, j_\ell} = \frac{(j_1 + j_2 + \dots + j_\ell)!}{j_1! j_2! \dots j_\ell!} (n-r)! r! \binom{r+1}{j_1 + \dots + j_\ell}.$$

Therefore by (1.2) and (1.3) we have proved the following.

Theorem 1.1

Let  $X_1, X_2, \dots, X_n$  be i.i.d. with continuous distribution  $F$ . Then

$$P\{Y_{\ell, n} \leq x\} = \sum_{r=1}^n P\{X_{r, n} \leq x\} P\{R_n = r\}$$

where

$$(1.4) \quad P\{R_n \leq r\} = \frac{(n-r)! r! (r+1)!}{n!} \sum_{\underline{j} \in I_{n-r}} \frac{1}{\prod_1^\ell j_t! (r+1 - \sum_1^\ell j_t)!}, I_{n-r} = \{\underline{j}: \sum_1^\ell j_t = n-r\}.$$

Remark: Note Theorem 1.1 remains true if the  $X_i$  are assumed only to be exchangeable.

The distribution of  $Y_{\ell, n}$  can be obtained in another way which yields a simpler expression than that given in Theorem 1.1. For  $A \subset \{1, 2, \dots, n\}$  let  $M(A) = \max\{X_i, i \in A\}$  and  $W(A) = \min\{X_i, i \in A\}$  with the convention  $W(\emptyset) = \infty$ . Let  $E_{k, \ell}$

equal the class of all  $k$  element subsets of  $\{1, 2, \dots, n\}$  which do not contain an interval of length greater than  $\ell$ . Then

$$(1.5) \quad P\{Y_{\ell, n} \leq x\} = \sum_{k=0}^n \sum_{A \in E_{k, \ell}} P\{W(A) > x, M(A^c) \leq x\} = \sum_{k=0}^n \#(E_{k, \ell}) (1 - F(x))^k F^{n-k}(x).$$

To evaluate  $\#(E_{k, \ell})$  we partition  $E_{k, \ell}$  into the following sets. Let  $\beta_{j_1, \dots, j_\ell}$  be the class of all  $k$  element subsets of  $\{1, \dots, n\}$  containing  $j_i$  intervals of length  $i$  and no interval of length greater than  $\ell$ . Then

$$(1.6) \quad \#(E_{k, \ell}) = \sum_{\underline{j} \in I_k} \#(\beta_{j_1, \dots, j_\ell}).$$

$\#(\beta_{j_1, \dots, j_\ell})$  is obtained by the following counting argument. Consider  $n-k$  blocks into which integers will be put and the  $n-k+1$  spaces between the blocks. Among these  $n-k+1$  spaces choose  $j_1$  to be designated as a single element space,  $j_2$  for a two element space, ...,  $j_\ell$  for an  $\ell$ -element space. Then a  $k$  element subset of  $\{1, \dots, n\}$  belonging to  $\beta_{j_1, \dots, j_\ell}$  is obtained by writing the numbers 1 to  $n$  in their natural order putting one integer in each of the  $n-k$  blocks and  $j_i$  consecutive integers in a space designated as an  $i$ -element space. The  $k$ -element set is then obtained by choosing the numbers put into the spaces. Hence

$$(1.7) \quad \#(\beta_{j_1, \dots, j_\ell}) = \frac{(n-k+1)!}{\prod_1^\ell j_t! (n-k+1 - \sum_1^\ell j_t)!}.$$

### Theorem 1.2

Under the assumptions of Theorem 1.1 we have

$$(1.8) \quad P\{Y_{\ell, n} \leq x\} = \sum_{k=0}^n \sum_{\underline{j} \in I_k} \frac{(n-k+1)! (1-F(x))^k F^{n-k}(x)}{\prod_1^\ell j_t! (n-k+1 - \sum_1^\ell j_t)!}.$$

If the  $X_i$  are assumed only to be exchangeable and  $F^{(k)}(x) = P\{W(A) > x, M(A^c) \leq x\}$  where  $A \subset \{1, \dots, n\}$  is any  $k$  element subset then

$$(1.9) \quad P\{Y_{\ell,n} \leq x\} = \sum_{k=0}^n \sum_{j \in I_k} \frac{(n-k+1)! F^{(k)}(x)}{\prod_1^{\ell} j_t! (n-k+1-\sum_1^{\ell} j_t)!}.$$

Proof: (1.8) is immediate from (1.5), (1.6), and (1.7) while (1.9) follows for the same reasons except that in (1.5) the expression  $(1-F(x))^k F^{n-k}(x)$  is replaced by  $F^{(k)}(x)$ .

## 2. Limiting distribution

In this section we derive the asymptotic behavior of  $Y_{\ell,n}$ . Preliminary to this work we obtain an asymptotic result for the permutation statistic  $R_n$ . In our analysis we rely on a method for obtaining the asymptotic behavior of sums with positive terms. A description of this tool may be found in the expository paper [3].

### Lemma 2.1

Let  $R_n$  be the permutation statistic defined in (1.1) and having distribution given in (1.4). Then

$$(2.1) \quad \lim_{n \rightarrow \infty} P\left\{\frac{n-R_n}{n^{\ell/\ell+1}} \leq x\right\} = \begin{cases} 0, & x < 0 \\ 1 - e^{-x^{\ell+1}}, & x \geq 0 \end{cases}.$$

Proof: In order to obtain the asymptotic behavior of the sum in (1.4) we first locate the maximum summand and introduce a change of variables so that the largest term occurs at the zero point.

Observe that if  $j_1^*, \dots, j_{\ell}^*$  are defined as the solution to the equations

$$(2.2) \quad \begin{aligned} j_1^* + 2j_2^* + \dots + \ell j_{\ell}^* &= n - r \\ (j_1^*)^t &= j_t^* (r - j_1^* - \dots - j_{\ell}^*)^{t-1}, \quad t=2,3,\dots,\ell \end{aligned}$$

then the maximum summand in (1.4) occurs in a suitable neighborhood of  $(j_1^*, \dots, j_{\ell}^*)$ . Let  $h_i = (j_i^*)^{1/2}$ ,  $i=2, \dots, \ell$  and  $j_i = j_i^* + x_i h_i$ ,  $i=2, \dots, \ell$  where  $x_i$  is a fractional index with step size  $(h_i)^{-1}$ . Making the change of variables in (1.4) and restricting attention to a neighborhood of the maximum summand, we consider

$$(2.3) \quad \frac{(n-r)!r!(r+1)!}{n!} \sum^* \left[ \prod_{t=2}^{\ell} (j_t^* + x_t h_t)! (n-r - \sum_{t=2}^{\ell} t j_t^* - \sum_{t=2}^{\ell} t x_t h_t)! \right]^{-1} \\ \cdot \left[ (r+1 - \sum_{t=2}^{\ell} (t-1) j_t^* + \sum_{t=2}^{\ell} (t-1) x_t h_t)! \right]^{-1}$$

where the summation is over the  $x_i$  such that  $x_i$  has step size  $h_i^{-1}$  and  $\max_{2 \leq i \leq \ell} |x_i| \leq A$  where  $A$  is a fixed positive constant. Then with  $\max_{2 \leq i \leq \ell} |x_i| \leq A$ , Stirling's formula and (2.2) we have

$$j_t! = (j_1^*)^{t j_t} (r - j_1^* - \dots - j_{\ell}^*)^{-(t-1) j_t} h_t \sqrt{2\pi} \\ (1 + O(\frac{1}{j_t^*})) \exp \{ j_t \left[ \ell n(1 + \frac{x_t}{h_t}) - 1 \right] \} \quad , \quad t=2, \dots, \ell.$$

Similarly

$$j_1! = (j_1^*)^{j_1} \sqrt{2\pi j_1^*} (1 + O(\frac{1}{j_1^*})) \exp \{ j_1 \left[ \ell n(1 - \frac{1}{j_1^*} \sum_{t=2}^{\ell} t x_t h_t) - 1 \right] \}.$$

Therefore using (2.2) and  $\ell n(1+x) = x - \frac{x^2}{2} + O(x^3)$  as  $x \rightarrow 0$  it can be checked that

$$\prod_{t=1}^{\ell} j_t! = (j_1^*)^m \exp \{ \frac{\ell}{2} \sum_{t=2}^{\ell} x_t^2 - \sum_{t=1}^{\ell} j_t^* + O(\frac{1}{h_{\ell}} + \frac{j_2^*}{j_1^*}) \} \\ \prod_{t=2}^{\ell} \left[ (r - \sum_{t=1}^{\ell} j_t^*)^{-(t-1) j_t} h_t \sqrt{2\pi} \sqrt{2\pi j_1^*} (1 + O(\frac{1}{j_t^*})) \right]$$

where  $m = n - r$ .

Therefore we find the expression at (2.3) equals

$$(2.4) \quad \exp \{ m \ell n(\frac{m}{j_1^*}) + 2r \ell n(1 + \frac{\sum_{t=1}^{\ell} j_t^*}{r - \sum_{t=1}^{\ell} j_t^*}) + n \ell n(1 - \frac{m + \sum_{t=1}^{\ell} j_t^*}{n}) \} (1 + O(\frac{1}{h_{\ell}} + \frac{j_2^*}{j_1^*})) \\ \frac{1}{\prod_{t=2}^{\ell} \sqrt{2\pi} h_t} \sum^* e^{-\frac{1}{2} \sum_{t=2}^{\ell} x_t^2}.$$

Since the  $x_i$  have step size  $h_i^{-1}$  we have

$$\frac{1}{\prod_{t=2}^{\ell} \sqrt{2\pi} h_t} \sum^* e^{-\frac{1}{2} \sum_{t=2}^{\ell} x_t^2} \rightarrow \left[ \int_{-A}^A \frac{1}{\sqrt{2\pi}} e^{-\frac{x^2}{2}} dx \right]^{\ell-1} \quad \text{as } n \rightarrow \infty$$

so that it suffices to consider

$$(2.5) \quad \exp\left\{m\ln\left(\frac{m}{j_1^*}\right) + 2r\ln\left(1 + \frac{\sum_1^\ell j_t^*}{r - \sum_1^\ell j_t^*}\right) + n\ln\left(1 - \frac{m + \sum_1^\ell j_t^*}{n}\right)\right\}.$$

Let  $x = \frac{j_1^*}{r - \sum_1^\ell j_t^*}$  and  $\varepsilon = \frac{m}{r - \sum_1^\ell j_t^*}$ . Then by (2.2) we have

$$(2.6) \quad \sum_1^\ell tx^t = \varepsilon.$$

For  $f(x) = \sum_{k=0}^K a_k x^k + R(x)$  with  $R(x) = O(x^{K+1})$  as  $x \rightarrow 0$ , let  $\lceil f(x) \rceil_r = \sum_{k=0}^r a_k x^k$ ,  $0 \leq r \leq K$ . Using (2.6) we observe

$$(2.7) \quad \frac{m}{j_1^*} = \frac{\varepsilon}{x} = \sum_{t=0}^{\ell+1} (t+1)x^t - (\ell+1)x^\ell - (\ell+2)x^{\ell+1} = \lceil (1-x)^{-2} \rceil_{\ell+1} - (\ell+1)x^\ell - (\ell+2)x^{\ell+1}.$$

$$(2.8) \quad \left(1 + \frac{\sum_1^\ell j_t^*}{r - \sum_1^\ell j_t^*}\right)^2 = \left(1 + \sum_1^\ell x^t\right)^2 = \lceil (1-x)^{-2} \rceil_{\ell+1} - 2x^{\ell+1} + O(x^{\ell+2}).$$

$$(2.9) \quad 1 - \frac{m + \sum_1^\ell j_t^*}{n} = \lceil 1 + \varepsilon + \sum_1^\ell x^t \rceil^{-1} = \left(\sum_{t=0}^\ell (t+1)x^t\right)^{-1} = \left(\lceil (1-x)^{-2} \rceil_{\ell+1} - (\ell+2)x^{\ell+1}\right)^{-1}.$$

Therefore by (2.7), (2.8), and (2.9) we see that (2.5) can be written

$$\begin{aligned} & \exp\{m\ln(\lceil (1-x)^{-2} \rceil_{\ell+1} - (\ell+1)x^\ell - (\ell+2)x^{\ell+1}) \\ & + r\ln(\lceil (1-x)^{-2} \rceil_{\ell+1} - 2x^{\ell+1} + O(x^{\ell+2})) \\ & - n\ln(\lceil 1-x \rceil_{\ell+1} - (\ell+2)x^{\ell+1})\} \\ & = \exp\left\{- (\ell+1)m\left(\frac{j_1^*}{r - \sum_1^\ell j_t^*}\right)^\ell - 2r\left(\frac{j_1^*}{r - \sum_1^\ell j_t^*}\right)^{\ell+1} \right. \\ & \quad + n(\ell+2)\left(\frac{j_1^*}{r - \sum_1^\ell j_t^*}\right)^{(\ell+1)} + mO\left(\left(\frac{j_1^*}{r - \sum_1^\ell j_t^*}\right)^{(\ell+1)}\right) \\ & \quad \left. + (1 - \frac{m + \sum_1^\ell j_t^*}{n})\left(\frac{j_1^*}{r - \sum_1^\ell j_t^*}\right)^{(\ell+2)}\right\}. \end{aligned} \quad (2.10)$$

By taking  $m = xn^{\frac{\ell}{\ell+1}}$  we have that  $m \sim j_1^*$ ,  $r \sim n$ , as  $n \rightarrow \infty$ . Hence (2.10) is asymptotic to

$$(2.11) \quad \exp\{-x^{\ell+1} + o(n^{-\frac{1}{\ell+1}})\}$$

Finally outside the neighborhood of the maximum summand we have that

$$(2.12) \quad \begin{aligned} & \frac{(n-r)!r!(r+1)!}{n!} \sum^{**} \left[ \prod_{t=2}^{\ell} (j_t^* + x_t h_t)! (n-r - \sum_{t=2}^{\ell} t j_t^* - \sum_{t=2}^{\ell} t x_t h_t)! \right]^{-1} \\ & \cdot \left[ (r+1 - \sum_{t=2}^{\ell} (t-1) j_t^* + \sum_{t=2}^{\ell} (t-1) x_t h_t)! \right]^{-1} \\ & = e^{-x^{\ell+1}} \left[ 1 - \int_{-A}^A \frac{1}{\sqrt{2\pi}} e^{-\frac{x^2}{2}} dx \right]^{(\ell-1)} O(1) \text{ as } n \rightarrow \infty \end{aligned}$$

where  $\sum^{**}$  denotes the summation over the  $x_i$  such that  $x_i$  has step  $h_i^{-1}$  and  $\max_{2 \leq i \leq \ell} |x_i| > A$ . Hence Lemma 2.1 holds by (2.11) and (2.12).

Lemma 2.1 and results of Balkema and de Haan [1] and [2] allow us information on the asymptotic behavior of  $Y_{\ell,n}$  even when classical extreme value theory does not apply. However, we must allow random normalization in the following weak limit results. Since the proofs of Theorem 2.1 and Theorem 2.2 are the same as the proofs of Proposition 6 and Proposition 8 of [4], we give only statements of the Theorems.

#### Theorem 2.1

Let  $X_1, X_2, \dots$  be an i.i.d. sequence with distribution  $F$  satisfying

$$(i) \quad F(x) < 1, \quad -\infty < x < \infty$$

$$(ii) \quad \frac{F(ru) - F(u)}{\frac{\ell+2}{2}} \rightarrow \lambda \log r, \quad 0 < r < \infty$$

$$(1 - F(u))^{\frac{2}{\ell+2}}$$

as  $u \rightarrow \infty$  for some  $\lambda > 0$ . Let  $Z_n = \frac{X_{R_{n,n}}}{f(\frac{R_n}{n})}$  where  $f$  denotes the right continuous inverse of  $F$ . Then

inverse of  $F$ . Then

$$\lim_{n \rightarrow \infty} P\{Z_n \leq z\} = \int_0^\infty \Phi(\lambda w^{\frac{\ell+1}{2}} \ln z) (\ell+1) w^\ell e^{-w^{\ell+1}} dw, \quad 0 < z < \infty$$

$$\text{where } \Phi(x) = \int_{-\infty}^x \frac{1}{\sqrt{2\pi}} e^{-\frac{u^2}{2}} du.$$

Remark: It can be checked that the hypothesis of Theorem 2.1 is satisfied for  $F$  of the form  $F(x) = 1 - c(\ln x)^{-(2/\ell)}$  for  $x$  large. By classical results in extreme value theory, such distributions do not belong to the domain of attraction of any extreme value distribution.

### Theorem 2.2

Let  $F$  be a distribution function with upper endpoint  $y_1$  which has a strictly positive density  $F'(y) = e^{u(y)}$  for  $y$  in a left neighborhood of  $y_1$ . Suppose  $F$  satisfies one of the following conditions:

- (i)  $1 - F(y') \sim 1 - F(y)$  for  $y \rightarrow y_1$  implies  $F'(y') \sim F'(y)$ ,
- (ii)  $\frac{F''(y)(1-F(y))}{(F'(y))^2}$  is bounded in a left neighborhood of  $y_1$ ,
- (iii)  $F'$  varies regularly in  $y_1^-$  with exponent  $p \neq -1$ ,
- (iv)  $\limsup_{y \rightarrow y_1^-} \frac{u''(y)}{(u'(y))^2} < 1$  or  $\liminf_{y \rightarrow y_1^-} \frac{u''(y)}{(u'(y))^2} > 1$ .

Then if  $f$  denotes the right continuous inverse of  $F$ , we have

$$\lim_{n \rightarrow \infty} P\left\{ \frac{Y_{\ell,n} - f(\frac{R_n}{n})}{f(\frac{R_n}{n} + \frac{R_n(n-R_n)}{n^3}) - f(\frac{R_n}{n})} \leq x \right\} = \Phi(x), \quad -\infty < x < \infty.$$

Remark: The conditions of Theorem 2.2 are satisfied for the normal, Laplace, Cauchy, beta, gamma distributions and all limit distributions for extreme order statistics  $X_{k,n}$  with  $k$  fixed or  $n-k$  fixed.

Let  $V_i = \min_{i \leq j \leq i+\ell} X_j$ ,  $i=1, \dots, n-\ell$ . Then  $Y_{\ell,n} = \max_{1 \leq i \leq n-\ell} V_i$  and the  $V_i$  form an  $\ell$ -dependent stationary sequence, that is  $V_i$  and  $V_j$  are independent if  $|i-j| > \ell$ .

Extreme value theory for dependent stationary sequences has received considerable attention. A basic result which relates the asymptotic behavior of the maximum of a stationary sequence to the asymptotic behavior of the maximum of the associated i.i.d. sequence is the following:

Let  $\{\xi_n, n \geq 1\}$  be a stationary sequence and let  $F_{i_1, \dots, i_n}(x) = P\{\xi_{i_1} \leq x, \dots, \xi_{i_n} \leq x\}$ . Condition  $D(u_n)$  is said to hold if for any integers  $1 \leq i_1 < \dots < i_p < j_1 < \dots < j_q \leq n$  with  $j_1 - i_p \geq \ell$  we have

$$|F_{i_1 \dots i_p j_1 \dots j_q}(u_n) - F_{i_1 \dots i_p}(u_n) F_{j_1 \dots j_q}(u_n)| \leq \alpha_{n, \ell}$$

where  $\alpha_{n, \ell} = o(1)$  for some sequence  $\ell_n = o(n)$ . Condition  $D'(u_n)$  is said to hold for the stationary sequence  $\{\xi_n, n \geq 1\}$  if

$$\lim_{n \rightarrow \infty} \frac{1}{n} \sum_{j=2}^{\lfloor n/k \rfloor} P\{\xi_1 > u_n, \xi_j > u_n\} \rightarrow 0 \text{ as } k \rightarrow \infty.$$

In [7] it is shown (Theorem 3.5.2) that if  $M_n = \max_{1 \leq i \leq n} \xi_i$  and  $\hat{M}_n = \max_{1 \leq i \leq n} \hat{\xi}_i$  where the  $\hat{\xi}_i$  are i.i.d. with the same underlying distribution as the  $\xi_i$  then if for every  $x$  conditions  $D(u_n)$  and  $D'(u_n)$  hold where  $u_n = x/a_n + b_n$ ,  $a_n > 0$ , we have

$$\lim_{n \rightarrow \infty} P\{M_n \leq \frac{x}{a_n} + b_n\} = G(x), \quad -\infty < x < \infty$$

if and only if

$$\lim_{n \rightarrow \infty} P\{\hat{M}_n \leq \frac{x}{a_n} + b_n\} = G(x), \quad -\infty < x < \infty.$$

Therefore in view of the above result the asymptotic behavior of the  $\max v_i$  will be exactly the same as if the  $v_i$  were independent once  $D(u_n)$  and  $D'(u_n)$  have been established where  $u_n$  is determined by  $(1-F(u_n))^{\ell+1} = \frac{\tau}{n} + o(\frac{1}{n})$  as  $n \rightarrow \infty$  where  $F$  is the underlying distribution of the  $X_i$ . Such a sequence  $\{u_n, n \geq 1\}$  exists for  $F$  belonging to the domain of attraction of an extreme value distribution. Finally the verification of Conditions  $D(u_n)$  and  $D'(u_n)$  is easy.

In order to establish a strong law result in section 3 we need the following result.

Theorem 2.3

Let  $\{x_n, n \geq 1\}$  be an i.i.d. sequence with underlying distribution  $F$ . Let  $Y_{\ell, n} = \max_{1 \leq i \leq n-\ell} \min_{i \leq j \leq i+\ell} x_j$ . If  $x_n$  is any sequence such that  $n(1-F(x_n))^{\ell+2} = o(1)$  then

$$P\{Y_{\ell, n} \leq x_n\} = (1 + o(n(1-F(x_n))^{\ell-1})) + o(n(1-F(x_n))^{\ell+2}) \exp\{-n(1-F(x_n))^{\ell+1}\}.$$

Proof: Since the technique to obtain the asymptotic behavior of  $Y_{\ell, n}$  is the same as the one used for  $R_n$ , we present a sketch only. In order to obtain the asymptotic behavior for

$$H(x) = \sum_{(\underline{j}, r) \in J} \frac{(r+1)!(1-F(x))^{n-r}(F(x))^r}{\prod j_t! (r+1-\sum_1^{\ell} j_t)!}$$

where  $J = \{(\underline{j}, r) : 0 \leq j_t, r, t=1, 2, \dots, \ell \text{ and } \sum_1^{\ell} t j_t = n-r\}$ , we find that the maximum summand occurs in a neighborhood of the point  $(\underline{j}^*, r^*)$  satisfying

$$(2.13) \quad \begin{aligned} (i) \quad & \sum_1^{\ell} t j_t^* = n - r^* \\ (ii) \quad & (j_1^*)^t = (j_t^*) (r^* - \sum_1^{\ell} j_t^*)^{(t-1)}, \quad t=2, \dots, \ell \\ (iii) \quad & \alpha r^* j_1^* = (r^* - \sum_1^{\ell} j_t^*)^2 \end{aligned}$$

where  $\alpha = F(x)(1-F(x))^{-1}$ .

By the usual calculations we find that

$$(2.14) \quad H(x) = \left(\frac{\alpha}{1+\alpha}\right)^n \left(\frac{r}{\alpha j_1^*}\right)^{n/2} (1 + o(\frac{\alpha}{n}))^{\ell}.$$

Let  $u = \alpha j_1^*/r^*$ . Then by (2.13 ii) we find that  $j_t^*/j_1^* = (u/\alpha)^{(t-1)}$  so that from (2.13 iii)  $u^{-2} = (u^{-2} - \alpha^{-1} \sum_{t=0}^{\ell-1} (u/\alpha)^t)^2$  which implies  $(1 + \alpha^{-1})u - (u/\alpha)^{\ell+1} + o(\alpha^{-(\ell+2)}) = 1$ . Hence

$$(2.15) \quad u = \frac{\alpha}{\alpha+1} + (1+\alpha)^{-(\ell+1)} + o(\alpha^{-(\ell+2)})$$

By (2.14) and (2.15) we find

$$\begin{aligned} H(x) &= (1 + o(n^{-1}\alpha^\ell)) \\ &\exp\{n\ell n(\frac{\alpha}{1+\alpha}) - n\ell n(\frac{\alpha}{1+\alpha} + (1+\alpha)^{-(\ell+1)} + o(\alpha^{-(\ell+2)}))\} \\ &= (1 + o(n^{-1}\alpha^\ell) + o(n\alpha^{-(\ell+2)})) \exp\{-n(1+\alpha)^{-(\ell+1)}\} \end{aligned}$$

proving Theorem 2.3.

### 3. Weak and Strong Laws for Spacings.

In this section we derive weak and strong law results for a particular function of the spacings. Let  $\{U_n, n \geq 1\}$  be i.i.d. uniform on  $(0,1)$  random variables. Let  $U_{1,n} \leq \dots \leq U_{n,n}$  be the order statistics for  $U_1, \dots, U_n$ . Then the random variables  $S_{i,n+1} = U_{i,n} - U_{i-1,n}$ ,  $i=1, \dots, n+1$  are called the spacings divided by  $U_1, \dots, U_n$  where  $U_{0,n} = 0$  and  $U_{n+1,n} = 1$ .

Let  $M_{\ell,n+1} = \max_{1 \leq i \leq n+1-\ell} \min_{i \leq j \leq i+\ell} S_{j,n+1}$ . The quantity  $M_{1,n}$  played a role in the work of Marron [8] and Chow, Geman and Wu [6] in cross-validated kernel density estimation. Statistical properties of  $M_{1,n}$  were analyzed in Cheng [5].

Our first result gives the exact distribution of  $M_{\ell,n}$ .

#### Theorem 3.1

$$P\{M_{\ell,n} \leq x\} = \sum_{k=0}^n \sum_{j \in I_k} \frac{(n-k+1)!}{\prod_{t=1}^{\ell} j_t! (n+1-k-\sum_{t=1}^{\ell} j_t)!} \sum_{t=0}^{n-k} (-1)^t \binom{n-k}{t} \{[1-(k+t)x]_+\}^{n-1}$$

where  $x_+ = x$  if  $x > 0$  or  $= 0$  if  $x \leq 0$ .

Proof: The result follows from Theorem 1.2 and the fact that

$$\begin{aligned} &P\{S_{1,n} > x, \dots, S_{k,n} > x, S_{k+1,n} \leq x, \dots, S_{n,n} \leq x\} \\ &= \sum_{t=0}^{n-k} (-1)^t \binom{n-k}{t} \{[1-(k+t)x]_+\}^{n-1} \end{aligned}$$

which was shown in [5], p. 3.

To obtain an asymptotic result for  $M_{\ell,n}$  the following representation will be useful.

Lemma 3.1 (Pyke) Let  $\{X_n, n \geq 1\}$  be i.i.d. exponential random variable with mean one. Let  $T_n = \sum_{i=1}^n X_i$ . Then

$$(S_{1,n}, S_{2,n}, \dots, S_{n,n}) \stackrel{d}{=} (X_1/T_n, X_2/T_n, \dots, X_n/T_n).$$

Theorem 3.2

$$\lim_{n \rightarrow \infty} P\{M_{\ell,n} \leq \frac{x + \ell n}{n(\ell+1)}\} = \exp(-e^{-x}), \quad -\infty < x < \infty.$$

Proof: Let  $Z_{\ell,n} = \max_{1 \leq i \leq n-\ell} \min_{i \leq j \leq i+\ell} X_j$  where the  $X_i$  are i.i.d.,  $X_i \sim 1 - e^{-x}$ , and let  $T_n = \sum_{i=1}^n X_i$ . Then by Lemma 3.1 we have

$$P\{M_{\ell,n} \leq \frac{x + \ell n}{n(\ell+1)}\} = P\{\frac{Z_{\ell,n}}{T_n} \leq \frac{x + \ell n}{n(\ell+1)}\}.$$

But  $((T_n/n) - 1)\ell n \rightarrow 1$  as  $n \rightarrow \infty$ . See [5], p. 7. Therefore it suffices to consider

$$P\{Z_{\ell,n} \leq (x + \ell n) \frac{1}{\ell+1}\}.$$

Since the variables  $W_i = \min_{i \leq j \leq i+\ell} X_j$  satisfy  $D(u_n)$  and  $D'(u_n)$  with  $u_n$  given by  $P\{W_i > u_n\} = e^{-(\ell+1)u_n} = \frac{e^{-x}}{n}$ , we have that the limiting distribution of  $Z_{\ell,n}$  is the same as if the variables  $W_i$  were independent. Hence the Theorem holds. Alternatively we may apply Theorem 2.3 with  $x_n = (1/(\ell+1))(x + \ell n)$ .

Finally we conclude with a strong law result for the spacings. With Theorem 2.3 established it is easy to check that the method of proof of Theorem 4.8 in [5] carries over with obvious modifications to the present case. Therefore, we only state the result

Theorem 3.3

With probability one

$$\lim_{n \rightarrow \infty} \frac{(\ell+1)nM_{\ell,n}}{\log n} = 1.$$

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