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Nonperiodic Fluctuations Induced By Stationary Surface Waviness on a Semi-Infinite Plate

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An analytical/numerical study of flow over an aerodynamic model or vehicle with surface waviness illustrates how surface waviness can generate several families of waves. The classical Kelvin-Helmholtz solution for irrotational flow over a wavy wall is extended to include the effects of the leading edge of a semi-infinite plate. The solution, obtained by conformal mapping and integral transforms, shows that a monochromatic surface waviness can generate a spectrum of standing waves with a continuum of x-wavenumbers, as well as the Kelvin-Helmholtz solution. Downstream of the leading edge, each of those standing waves decays exponentially

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in the streamwise direction and oscillates sinusoidally in the direction normal to the plate. These standing waves satisfy the usual boundary conditions on a flat plate, although they are initiated by the combined effects of the waviness of the wall and the leading edge. The spectrum depends upon the phase of the surface waviness with respect to the leading edge. Far downstream of the leading edge, only the Kelvin-Helmholtz solution remains. Upstream of the leading edge, the flow field can be represented as a superposition of exponentially-growing standing waves.

The theory for flow over a stationary wavy wall is applicable to the case of flow over a traveling wavy wall if the velocity is properly nondimensionalized and the phase of the surface waviness is replaced by the product by frequency and time.

PREFACE

This report is one in a series of AEDC Technical Reports which document the solutions of the Orr-Sommerfeld and Rayleigh equations and how these fluctuations are generated. The reports in this series are:

(1) Rotational and Irrotational Freestream Disturbances Interacting Inviscidly with a Semi-Infinite Plate (AEDC-TR-83-3)

(2) Exponentially-Varying, Standing Waves in Parallel-Flow Boundary Layers (AEDC-TR-83-4)

(3) Waves Which Travel Upstream in Boundary Layers (AEDC-TR-83-7)

(4) Spatially-Decaying Arrays of Rectangular Vortices Interacting with Falkner-Skan Boundary Layers (AEDC-TR-83-8)

(5) The Boundary-Value Problem for Two-Dimensional Fluctuations in Boundary Layers (AEDC-TR-83-9)

(6) Nonperiodic Fluctuations Induced by Stationary Surface Waviness on a Semi-Infinite Plate (this report) (AEDC-TR-83-10)

Reports (5) and (6) are concerned with unsteady freestream disturbances and with stationary surface waviness, respectively. However, the two studies are closely related mathematically. If the solutions are properly expressed, then the solution for (6) is imbedded within the solution for (5) when the frequency vanishes and when the surface waviness is in-phase on both sides of the plate. Hence, an analogy exists between these physically quite-different cases.

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1.0 INTRODUCTION AND LITERATURE SURVEY

1.1 Introduction and motivation for this study

Aerodynamic surfaces are wavy in various degrees as a result of machining and fabrication of the surface. The deformation of the structure due to aerodynamic loading and body forces can also cause wrinkling and local buckling of the skin. Subscaled models built for aerodynamic testing are particularly subject to adverse affects of surface waviness, even when the aerodynamic surfaces are manufactured with very small tolerances, because the waviness can be large nondimensionally. Unsteady surface waves may arise through the use of active or passive compliant surfaces, panel flutter, a layer of water flowing along the surface having a wavy air/water interface, etc. While the theory we develop in this study can be easily adapted to the case of traveling waves, the problem we consider has stationary waviness.

The surface waviness can affect laminar and turbulent skin friction, transition to turbulent flow, and the structure of the boundary layers. If the pressure along the surface is not the same at equal heights of surface displacement, $h(x)$, then the waviness will cause pressure drag. The waviness (on an otherwise flat plate) can lead to local separations after only a few wavelengths. At the very high Reynolds number flows of aerospace applications, Görtler instabilities are possible in the regions of streamline concavity.

While it is useful to analytically assume that the wavy wall extends to infinity upstream and downstream, the wavy region may end at leading and trailing edges, or the wavy wall may connect with (or flare into) smooth walls. While it is useful to numerically study flows which are spatially periodic, the inflow and outflow boundary conditions will not be the same if edges or flaring regions are "nearby". The nonperiodicity of the inflow/outflow boundary conditions will influence the flowfield, even though the surface displacement may be periodic. Even the periodic inflow/outflow conditions can be altered by edges of the wavy regions.

The case of inviscid, irrotational flow over a small-amplitude sinusoidal surface was analyzed by Kelvin and Helmholtz (Ref. 1). In a coordinate system where the surface waves are steady, and introducing a phase shift θ which is useful later, the classical solutions for the longitudinal and normal velocity fluctuations are

$$u(x,y)=u_0\sin(\alpha x-\theta)\exp(-\alpha y)$$

$$v(x,y)=v_0\cos(\alpha x-\theta)\exp(-\alpha y) \qquad (1.1a,b)$$

However, generally if leading or trailing edges are present, the Kelvin-Helmholtz solutions are not mathematically complete; the flow cannot be represented as a superposition of the Kelvin-Helmholtz solutions. Other waveforms may be required in the analysis. Even for the steady case of an inviscid uniform mean flow, a mathematically complete set includes a pair of steady, exponentially-varying standing waves of form

$$u(x,y)=\cos(\beta y)\exp(\pm\beta x)$$

$$v(x,y)=\sin(\beta y)\exp(\pm\beta x) \qquad (1.2a,b)$$

where β is a real number. These waves grow or decay exponentially in the streamwise direction and satisfy impermeability along a flat plate. While there also is the possibility of another family of waves representing vortical fluctuations in the freestream, and some other variations to (1.2a,b), the above forms are sufficient for this discussion. We are particularly interested in the decaying form of these standing waves and how they are linked to the Kelvin-Helmholtz solutions through the presence of the leading edge. The issue of mathematical completeness is further discussed in Section 1.6.

Solutions (1.2a,b) are simplified versions of waves of form

$$u(x,y)=f(y)\exp(\pm\beta x-i\omega t)$$

$$v(x,y)=\phi(y)\exp(\pm\beta x-i\omega t) \qquad (1.3a,b)$$

found in viscous, parallel-flow boundary layers. These waves have been calculated by Rogler and Tsugé (Ref. 2), and Rogler (Refs. 3a,b) as solutions of the Orr-Sommerfeld equation for Falkner-Skan boundary layers.

A motivation for this study is to better understand the role and properties of these additional solutions. Even for cases where superpositions of the Kelvin-Helmholtz solutions are possible (e.g. the case of a semi-infinite flat plate joined to a semi-infinite sinusoidal wall) it may be useful to represent the flowfield in terms of mathematically complete sets in the two regions, and join them smoothly at the junction. An objective of this study is to obtain the flowfield over a semi-infinite sinusoidal wall. While we consider the wavy surface to be stationary, with appropriate nondimensionalizations and phase shifts, the solution for the flowfield is also applicable for flow over a semi-infinite plate with traveling surface waviness. The theory with traveling surface waviness is documented in Ref.22.

1.2 An introduction to the influences of leading and trailing edges of wavy surfaces and flaring sections between wavy and smooth surfaces

Several investigators have been aware of (a) the influence of the phase angle θ when a leading edge is present, (b) the length of the flat section which joins to the wavy region, and (c) geometrical details of the flaring section. There are several mechanisms by which the leading/trailing edges or flaring sections can influence the flowfield. For simplicity, we will confine this discussion to first-order boundary layer effects, where these factors can influence the freestream velocity at the boundary layer edge, $U_\infty(x)$.

(1) In the doubly-infinite Kelvin-Helmholtz solution, the phase θ of the surface waviness can be eliminated by a simple coordinate shift. However, the calculations by Dryden (Refs. 4,6, based on Ref. 5) for the boundary layer development along a plate show that the separation point depends on the phase θ of the sinusoidally imposed freestream velocity

$$U_\infty(x) = U_\infty + \varepsilon\sin(\alpha x-\theta) \tag{1.4}$$

(We have freely changed the definition and sign of the phase angle used by other investigators to be consistent with the present work.) With a leading edge present, the phase angle of the waviness becomes a relevant parameter because the phase influences the initial condition, the imposed pressure gradient associated with the velocity (1.4), and the edge boundary condition in the boundary layer equations. Dryden's calculations (Ref. 4) systematically considered these effects.

(2) The flow around the leading edge can substantially influence the initial conditions, the pressure gradient, and the edge boundary conditions. This flow depends on the phase of the waviness. Based on the theory presented herein, the freestream velocity takes the form

$$U_\infty(x) = U_\infty + \varepsilon\sin(\alpha x-\theta) + \varepsilon u^{(s)}(x,\theta) \tag{1.5}$$

- $\varepsilon u^{(s)}(x,\theta)$ — Secondary flow caused by the leading and trailing edges, flaring sections, etc.
- $\varepsilon\sin(\alpha x-\theta)$ — Kelvin-Helmholtz solution evaluated along y = 0
- U_∞ — uniform mean flow which would exist if the plate were flat

This expression is valid only along y=0 where a sinusoidal wall is present. The secondary flow is also proportional to the amplitude of the surface waviness. The secondary flow is <u>not periodic in x</u>. We shall show that it is composed of a superposition of decaying waves of form (1.2a,b). While $u^{(s)}$ would vanish far-downstream of the leading edge, the influence in the boundary layer could persist downstream because of the development of the boundary layer in the first half-wavelength or so of the waviness. A similar behavior is possible with a flat section connected to a wavy section. In elliptic problems, the (possibly different) phases of the waviness on the top side of the plate and the bottom side of the plate would be relevant. In the present work, we assume that the phases are the same.

(3) If <u>both leading and trailing edges are present</u>, then the waviness can influence the circulation about the plate. The secondary flow will now depend on the plate length, L. This secondary flow in eqn.(1.5) could remain significant for many wavelengths from the leading/trailing edges. A similar effect could arise with a <u>finite-length of surface waviness</u> along a doubly-infinite plate. The feature shared by both of these cases is that the end regions of the wavy surface can establish high and low pressures, and the difference between those pressures divided by the length of the wavy section is a pressure gradient which could have a relatively long range influence. In elliptic cases, another feature of relevance is whether the model support or attachment of the wavy surface to the wind tunnel wall will influence the circulation about the plate.

(4) The geometrical details of any (non-sinusoidal) <u>flaring sections</u> which connect the wavy surface to adjacent non-wavy surfaces could be treated as

still another superposed flowfield. This flow would be excited by the wall boundary condition which is the difference between the sinusoidal wall and the flaring region.

Analytical solutions have been obtained (Refs. 7-9) for the interaction of various freestream disturbances with semi-infinite plates. Within the framework of inviscid linearized airfoil theory, the closely related solution for flow over stationary surface waviness on a semi-infinite plate can be developed. For this reason, a semi-infinite wavy wall is used in this analysis. The closed-form solution which results can be rearranged to illustrate the origin of exponentially-varying standing waves. In this report, we will not directly consider either the combined effects of leading and trailing edges, or the effects of nonsinusoidal flaring regions.

In Sections 1.3-1.9, we outline eight approaches to the problem of studying flow over wavy walls. While the survey is not exhaustive, additional references in any category can be found through the papers and reports which we cite. We did not thoroughly survey the related topics of (1) flows over traveling wavy walls, (2) geophysical studies of the excitation of water waves by the wind, (3) airflow over a flat plate with a film of fluid with a wavy surface, (4) Görtler instabilities in the concave streamlines over a wavy surface, (5) influence of waviness on the Tollmien-Schlichting instability, (6) the triple-deck analyses of flows over a small bump in a boundary layer, (7) a layer of liquid flowing over a wavy surface, such as the flow of a river over an undulating riverbed, (8) heat transfer in corrugated pipes, (9) low Reynolds number lubrication problems with wavy surfaces, (10) flow in blood vessels and pipes with wavy walls, (11) flows along flat eels and other animals with undulating surfaces, and (12) boundary layer development with surface roughness.

1.3 Irrotational flows past small-amplitude and large-amplitude wavy walls

Irrotational incompressible flow over a small-amplitude sinusoidal wall was analyzed by Kelvin and Helmholtz (Ref. 1). The streamlines of this flow are plotted in Figure 1. This solution was extended by Ackeret (Refs. 10-13) to the cases of compressible subsonic and supersonic flow. For the subsonic cases, the pressure is symmetric on the windward and leeward sides of the wave, and there is no pressure drag. For the supersonic case, wave drag is generated because of the asymmetric variation in pressure.

Transonic flow over a wavy wall was analyzed by C. Kaplan (Ref. 14) and Hosokawa (Ref. 15) who included the nonlinear transonic term in their analyses. Both assumed the flow to be isentropic and periodic, and thus tacitly neglect the increases in entropy across the shocks. Moore and Gibson (Ref. 16) and Vincenti (Refs. 17,18) analyzed inviscid compressible flow over a sinusoidal wall of a gas in vibrational or chemical non-equilibrium. They focused on the effects of finite-rate chemical/vibrational adjustments, and neglected the effects of any (nonlinear) transonic term in their consideration of subsonic, transonic, and supersonic flows. Ryhming (Ref. 19) discussed the combined problem of transonic effects and finite-rates of adjustment. For the (equilibrium) transonic cases, the mixed supersonic/subsonic flows cause pressure drag. For the non-equilibrium cases, the finite rates of adjustment will cause pressure drag even in the subsonic cases.

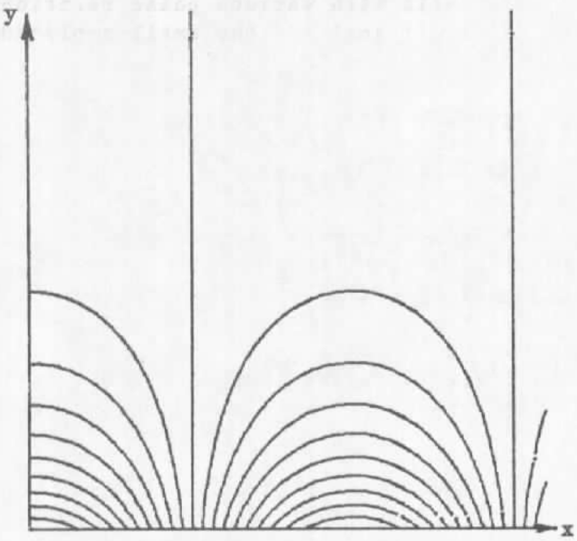


Figure 1. Streamlines for the periodic Kelvin-Helmholtz solution, $\psi^{(kh)}$, for incompressible, irrotational flow over a wavy wall. The phase of the waviness in this figure is the same as in Figures 2b, 5b, and 6f.

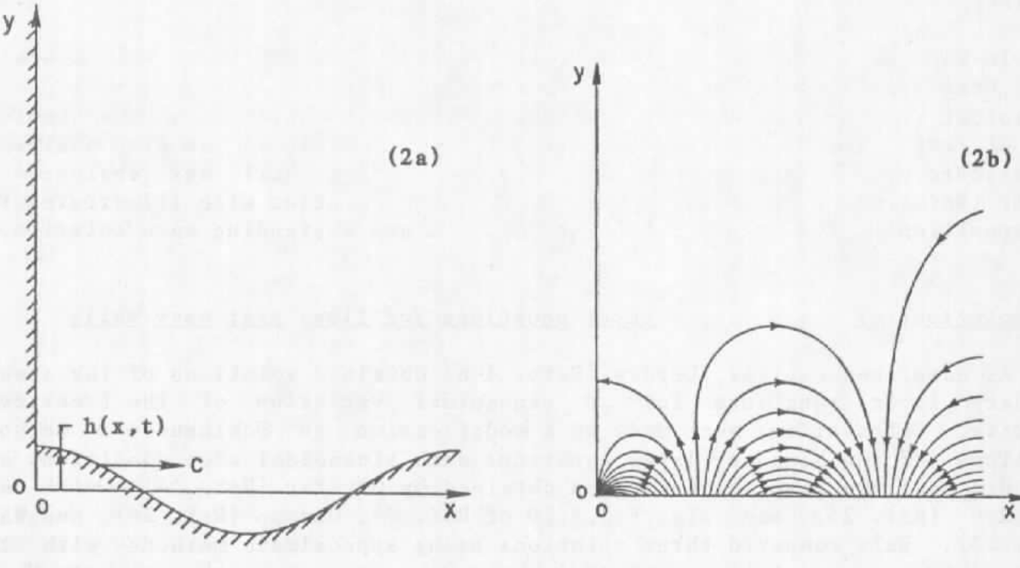


Figure 2. Geometry and streamlines for nonperiodic flow in a corner induced by a traveling wavy wall (Ref. 23).

Horlock (Ref. 20) analyzed incompressible flow in an unsteady wind tunnel with small-amplitude wavy walls with various phase relationships between the two wavy walls. Rogler (Ref. 21) analyzed the small-amplitude cases of wall amplitudes varying as

$$h(x,t) = h_0 \exp[i\alpha(x-ct)] \text{ with } \alpha \text{ complex} \tag{1.6a}$$

and

$$h(x,t) = (h_0+h_1x+h_2t)\exp[i\alpha(x-U_\infty t)] \tag{1.6b}$$

The latter "galloping" wall shape is a surface wave which induces disturbances that propagate at the freestream speed, $c=U_\infty$. The sinusoidal wall shape (1.6a) with α real cannot induce disturbances propagating at the freestream speed.

Flow in a wind tunnel with large-amplitude, non-sinusoidal wavy walls was analyzed in Ref. 21 also. For the cases analyzed, velocity disturbances of the form

$$u,v = [f(y),\phi(y)]\exp[i\alpha(x-ct)] \tag{1.7a}$$

with one x-wavenumber, α, were induced by large wall displacements which were expressed as the Fourier series

$$h(x,t) = \sum_{n=0}^{\infty} [h_n^{(1)} \sin\alpha_n(x-ct) + h_n^{(2)} \cos\alpha_n(x-ct)] \tag{1.7b}$$

with the series of x-wavenumbers, $\alpha_n=n\alpha$, $n=0,1,2,\cdots$. The velocities were expressed analytically, and the wall shapes and Fourier coefficients were obtained numerically.

In a companion study to the present report, Rogler (Ref. 22) analyzed flow over a semi-infinite plate with traveling surface waves, including both sinusoidal and the more general "galloping" waves of eqn.(1.6b). The combined case of freestream disturbances and traveling wall waviness was also analyzed. Nonperiodic flow in a corner induced by a sinusoidal wall was analyzed by Rogler (Ref. 23) as shown in Figures 2a,b. That solution also illustrates the superposition of the Kelvin-Helmholtz solution and a standing wave solution.

1.4 Solutions of the boundary layer equations for flows past wavy walls

As described earlier, Dryden (Refs. 4-6) obtained solutions of the steady boundary layer equations for an sinusoidal variation of the freestream velocity. Calculations were made by a modification to Pohlhausen's method. Solutions of the boundary layer equations with sinusoidal edge conditions and periodic pressure gradient have been obtained by Görtler (Ref. 24), Quick and Schröder (Ref. 25; see also Fig.A,10 of Ref. 6), Geropp (Ref. 26), and Walz (Ref. 27). Walz compared three solutions using approximate methods with the finite-difference solutions of Quick and Schroder, where the velocity $U_\infty(x)$ was designated as (1) a uniform flow, (2) the sum of a uniform flow and a sinusoidal perturbation, and (3) the sum of a uniform flow plus a nonsinusoidal perturbation in an intermediate sector to represent the effects of a flaring section. Results without a starting length were also compared.

Fannelöp and Flügge-Lotz (Ref. 28) solved the compressible boundary layer equations for subsonic and supersonic flow over a wavy wall. They used an implicit finite-difference solution and addressed the problem of a flat plate joined to a wavy wall. The effects of boundary layer displacement were incorporated by assuming the sinusoidal edge conditions to be those corresponding to the combined effects of surface waviness and displacement thickness. The boundary layer flow was found, the local displacement thickness was calculated, and the wall shape was found indirectly. Walz (Ref. 27) also compares two approximate methods at supersonic Mach numbers with and without heat transfer. Interestingly, Walz (page 242) plots the amplitude ε of the sinusoidal velocity which leads to <u>laminar separation within the first wavelength</u>. This amplitude varies from 2.13% at $M_\infty=0$ to about 0.5% at $M_\infty=4.0$, but also depends on heat transfer. Presumably, based on Dryden's calculations, this amplitude would also depend on the phase Θ of the waviness. The smallness of these amplitudes emphasizes the sensitivity of the boundary layer to waviness, at least when the mean pressure gradient vanishes, $dP_\infty/dx=0$.

Rogers (Ref. 29) developed an approximate method of accounting for boundary layer effects on pressure drag produced by a compressible flow over a wavy wall.

Numerical solutions for supersonic flows over single and multiple sinusoidal protuberances were obtained by Polak, Werle, Vatsa, and Bertke (Ref. 30). Local separations and reattachments were calculated using boundary layer equations where the interaction of the boundary layer with the isentropic supersonic inviscid flow was modeled by linking the pressure gradient parameter to the local inclination of the total displacement body. This displacement body included the effects of the wavy surface and the boundary layer displacement thickness.

In these references, the pressure at the boundary layer edge was impressed on the layer, in contrast to the studies of Section 1.5 where the pressure is not required to be constant across the boundary layer.

1.5 <u>Incompressible and compressible parallel-flow boundary layers with small-amplitude wavy walls</u>

Effects of the mean boundary layer along the wavy wall have been studied analytically and numerically by assuming a parallel-flow boundary layer and seeking solutions of linear ordinary-differential equation(s) subject to linearized boundary conditions. These studies assume that the disturbance flow is of the form

$$u(x,y) = f(y)\exp(i\alpha x)$$

$$v(x,y) = \phi(y)\exp(i\alpha x) \qquad (1.8a,b)$$

and similarly for the other disturbance variables in a coordinate system moving with the surface wave. This solution form has the same x-wavenumber as the surface waviness, and is compatible with the linearized boundary conditions along y=0. However, it excludes steady waves of form (1.3a,b) which may be present.

Miles (Ref. 31) solved the inviscid Rayleigh equation for flow over a traveling wave in an analysis focused on predicting the energy transfer between a parallel-flow (air) boundary layer and gravity waves in water. The rate at which energy was tranferred was proportional to the profile curvature at the critical point.

Benjamin (Ref. 32) analytically studied flow over a stationary wavy wall, along with other cases of traveling surface waves. With the critical point at the wall, viscosity was retained in a thin layer adjacent to the wall where the mean velocity was assumed to vary linearly. Miles (Ref. 33) also included viscosity in his analysis.

Numerical solutions of the incompressible, viscous Orr-Sommerfeld equation with a parallel-flow Blasius boundary layer and with a wavy wall were obtained by Lessen and Gangwani (Ref. 34) and Aldoss and Reshotko (Ref. 35). Boundary-value problems related to Refs. 32, 34, and 35 are described in Section 1.6 below.

Inger and Williams (Ref. 36) solved the compressible, parallel-flow equations for flow over a stationary sinusoidal wall. The Prandtl number was assumed to be unity. The inviscid, nonconducting solutions were matched to solutions in the sublayer which accounted for viscosity and heat conduction. McClure (Ref. 37) also obtained solutions for compressible disturbances in pseudo-laminar boundary layers with turbulent mean profiles. He developed some justification for studying turbulent boundary layers over wavy walls by using the turbulent mean profiles in the linearized, parallel-flow equations, and by using the ordinary viscosity coefficient rather than some combination of the ordinary and eddy coefficients. Numerical solutions of the 6th-order, viscous, heat-conducting equations for flow over stationary wavy walls were obtained by Lekoudis, Nayfeh, and Saric (Ref. 38). Mean boundary layers representative of laminar layers and turbulent layers were used in their calculations.

The reader is referred to Section 1.10 for related parallel-flow studies of the interaction of steady freestream disturbances with a boundary layer along a flat plate.

1.6 Boundary-value problems with wavy walls

The subject of this section is the spatial evolution of disturbances in parallel-flow boundary layers along wavy walls. The studies surveyed in Section 1.5 assumed solutions of the form (1.8a,b) which have the same x-wavenumber as the surface waviness. As discussed in Section 1.1, however, these solutions are not mathematically complete. Any other solution which satisfies the same equation, except with homogeneous boundary conditions along $y = 0$, can be superposed. There are other steady solutions, as well as other unsteady solutions.

None of the boundary-value problems studied by Tsugé and Rogler (Refs. 39a,39b,40) and Aldoss and Reshotko (Ref. 41) include all of the flat-plate solutions and the wavy-wall solution as well. Based on the linearity of the equation and boundary conditions, however, we can state that a complete set includes

(1) a finite set of eigenmodes, including the
Tollmien-Schlichting stability wave
(2) decaying standing waves
(3) growing standing waves
(4) downstream propagating continuous spectrum
(5) upstream propagating continuous spectrum
(6) wavy wall solution corresponding to
Refs. 32, 34 and 35

Refs. 39a, 39b, 40, and 41 treated the problem by Laplace transform methods, and formally decomposed the initial conditions along the y-axis into the various waveforms. Aldoss and Reshotko (Ref. 41) demonstrated that the effect of the small-amplitude wavy wall was to introduce two additional mathematical poles in the Laplace plane. These poles represent the stationary wavy-wall solutions of the Orr-Sommerfeld equation calculated by Benjamin (Ref. 32), Lessen and Gangwani (Ref. 34) and Aldoss and Reshotko (Ref. 35).

A difficulty with these boundary-value problems is that the waviness will influence the boundary conditions on the steady fluctuations specified along the y-axis. Additional information is required to specify those conditions in an elliptic problem.

The present work addresses the issue of proper boundary condition when a leading edge is present. While each of the solutions (1-6) are independent and superposable, we shall show that there is a linking together between the steady waves (2 and 6) if a leading edge is present. Instead of the quarter-plane problem $(0 \leq x \leq \infty, 0 \leq y \leq \infty)$, of Refs. 39-41, we consider a whole-plane problem $(-\infty < x < +\infty,\ -\infty < y < +\infty)$. It may be said that a <u>semi-infinite wavy wall "excites" several families of waves with interrelated amplitudes, phases, and wavenumbers</u>. This process is closely related to the excitation of standing waves by freestream disturbances, as analyzed by Rogler and Reshotko (Ref. 9).

1.7 Flow over a nonsinusoidal bump treated as a superposition of linear solutions for flows over sinusoidal walls

If the equations and boundary conditions for flow past a sinusoidal wall are linear, then a superposition of the solutions is possible, of course. Periodic wall shapes can be represented as Fourier series, and nonperiodic wall shapes can be represented as Fourier integrals. For example, Benjamin (Ref. 32) treated the case of flow over a bump as a superposition of his analytical solutions of the Orr-Sommerfeld equation for flow past a sinusoidal wall.

Generally, the flow over finite-length and semi-infinite length plates with wavy surfaces cannot be represented as merely the superposition of the solutions obtained with a plate that extends from $-\infty$ to $+\infty$. As noted earlier, the reason is that these wavy solutions do not form a complete mathematical set, and other waveforms arise which need to be included. The present analysis describes some of these additional waves and obtains their amplitudes and phases. We note that there are special cases where the solutions for a doubly-infinite plate can be used to construct the solution for a semi-infinite or finite-length plate. An example is the symmetric case of inviscid, irrotational flow over a plate with waviness on the top side which is 180° out of phase with the waviness on the bottom side.

1.8 Numerical solutions of partial differential equations for flows past wavy walls

Markatos (Ref. 42) obtained numerical solutions of a set of elliptic partial differential equations with constant density and with periodic boundary conditions on the upstream and downstream boundaries of the computational domain

$$v(x+p,y) = v(x,y), \; u(x+p)=u(x,y) \qquad (1.9a)$$

In a curvilinear, orthogonal coordinate system, solutions are obtained by a finite difference method. Flows were calculated with a nonsinusoidal surface waviness of the form

$$h(x) = -(2\pi)^{-1/2}\exp(-x^2/2)[C_0 - C_1x + C_2(x^2-1)]$$

This surface represents water waves in a coordinate system moving at the wave speed. Markatos considered both laminar and turbulent cases, and examined the temperature and concentration fields as well as the flowfields in an exceptionally thorough study. The mean flows were planar, and a two-equation turbulence model was used, where two additional differential equations were solved for the kinetic energy of turbulence and its rate of dissipation.

Using Fourier and Chebyshev series, Balasubramian and Orszag (Ref. 43) calculated steady, planar flows over sinusoidal and non-sinusoidal periodic walls with periodic inflow and outflow boundary conditions. The walls were stationary, and both laminar and turbulent flows were studied. A one-equation eddy viscosity model was used in the simulations of turbulent flow. Conformal mapping was used to transform the region above the plate into a rectangular strip, then this strip was mapped into a finite rectangular domain.

For low Reynolds number flows, Chin (Refs. 44,45) developed and applied conformal mapping techniques to flows over wavy walls. Although we are not directly concerned with lubrication problems, we cite these references because the mapping techniques are useful.

1.9 Experiments with flows past wavy walls

Stanton, Marshall and Houghton (Ref. 46) experimentally measured flow over a wave train where the amplitude and wavelength grew in the downstream direction. Motzfeld (Ref. 47) investigated flow over a rigid wavy surface composed of three wavelengths.

A wavy wall with waves propagating either downstream or upstream was used by Kendall (Ref. 48) to study their influence on the structure of a turbulent boundary layer developing along that wall. Kachanov, Kozlov, Kotjolkin, Levchenko, and Rudnitsky (Ref. 49) experimentally measured the laminar velocity profiles over stationary waviness and compared the results with Görtler's solutions of the boundary layer equations (Ref. 24). As noted earlier, McClure (Ref. 37) and Inger and Williams (Ref. 36) experimentally and analytically studied the compressible turbulent flows above small-amplitude stationary wavy walls. Bertram, et al (Ref. 50) investigated hypersonic flows past wavy surfaces.

A wavy-wall wind tunnel was built at Cambridge University and used for studies with unsteady airfoils and cascades (Refs. 51–54). In that facility over a length of 2.7m, a system of cams and springs deformed flexible metal sheets to produce a sinusoidal wall shape of wavelength 1.8m over a central region of 1.2m. The remaining 1.5m were used for the two end regions where the amplitudes diminished to zero so that the wavy wall would flare smoothly into the stationary walls of the tunnel.

Holmes (Ref. 51) found that significant discrepancies existed between the theory and the experimental data when the tunnel operated in the out-of-phase mode where purely longitudinal disturbances were introduced along the centerline.

We believe that these discrepancies between theory and experiment are related to the presence of standing waves we find in the present analysis and arise because the sinusoidal wavy walls do not extend forever upstream and downstream. In addition to the traveling waves analyzed by Horlock (Ref. 20) and Rogler (Ref. 21), one would expect to find a superposition of unsteady exponentially-decaying and growing standing waves of the form (1.3a,b). A neutral standing wave of form $u(t)=u_0 \exp(-i\omega t)$, $v=0$ could also arise for certain wall geometries and phase relations between the walls in some sections of the tunnel.

1.10 Boundary layer over a flat plate with a steady sinusoidal exterior boundary condition

While wavy walls introduce disturbances at the wall and the freestream is ordinarily assumed to be quiescent in analytical studies, a related problem is the introduction of disturbances from the freestream and the plate is assumed to be flat. If the x-wavenumbers and phase speeds of disturbances in the two cases are the same, then many features are quite similar. They have the same critical height y_c where $\bar{U}(y_c)=c$, and they can have similar viscous sublayers at the wall. Based on first-order boundary layer theory, the edge conditions and the pressure gradients may be the same or similar. If the disturbances are weak and linearized equations and boundary conditions are adequate, then the two problems may share certain superposable parts of their solutions. If the disturbances are stationary (c=0), then the critical layer and viscous sublayer merge into a single layer at the wall.

The problem of a parallel-flow boundary layer along a flat plate with periodic fluctuations at the boundary layer edge is the steady limit ($\omega = 0$) of the problem analyzed by Rogler (Ref. 55). The boundary layer was represented as a parallel-flow Blasius layer, and the Orr-Sommerfeld equation was solved subject to the exterior behavior

$$\phi(y) = Ae^{+\alpha y} + Be^{+\gamma y} + Ce^{-\alpha y} + De^{-\gamma y}$$

and subject to the usual (flat) wall boundary conditions of impermeability and no-slip at y=0. The constants A and B were specified, and the constants C and D were found numerically. This problem is related to the parallel-flow treatment of incompressible flow over a wavy wall (Refs. 32,34,35).

Lighthill (Ref. 56) analyzed the inviscid, nonconducting case of a stationary, weak disturbance impinging on a parallel-flow boundary layer along a flat plate with a supersonic freestream. This is related to the cases of compressible flow over a stationary wavy wall (Refs. 36-38).

Lighthill (Ref. 57) also considered the above case except he included the effects of viscosity in a sublayer near the wall. The low Mach number mean flow near the wall was represented as an incompressible constant property flow with disturbances governed by the Orr-Sommerfeld equation with a linear mean profile and phase speed zero. Neglecting terms of the order of α^2, the solution indicated that "the effect of the sublayer on the flow outside it is exactly as if the inner part $0<y<0.78L$ of it were replaced by a solid wall and the remainder were not subject to viscosity", where the viscous length is $L=[\nu_w/i\alpha U_y(0)]^{1/3}$ and ν_w is the wall value of the kinematic viscosity. The case where the α^2 terms were included in the analysis was also considered.

Other analyses with freestream disturbances interacting with semi-infinite plates which are closely related mathematically to the present work are the interaction of an array of square vortices with a semi-infinite plate (Ref. 7), seven other disturbance forms interacting with plates (Refs. 8), and the representation of those interactions as the superposition of traveling and standing waves (Ref. 9).

1.11 Plan of this study

The solution for incompressible, irrotational flow over a semi-infinite sinusoidal wall is obtained by conformal mapping and integral transforms in Section 2.0. In Section 3.0, this solution downstream of the leading edge is represented as the sum of the periodic Kelvin-Helmholtz solution and a secondary flow which arises because of the leading edge.

The normal velocity along a line x=constant is found above the plate, and the velocity associated with the Kelvin-Helmholtz solution is subtracted out. The remaining velocity is Fourier analyzed in Section 4.0 and represented as a Fourier integral of waves of form $\exp(-\beta x)\sin\beta y$. Upstream of the leading edge, the flow is represented as a superposition of growing waves of form $\exp(+\beta x)\cos\beta y$ in Section 5.0 The study is summarized in Section 6.0.

2.0 FORMULATION AND SOLUTION

By conformal mapping, we will obtain the flowfield with stationary surface waviness on a semi-infinite plate. The method of solution is similar to the procedure used to analyze the interaction of freestream disturbances with a semi-infinite plate (Refs. 7-9). We are seeking a solution which satisfies Laplace's equation, $\nabla^2\psi^{(i)} = 0$, where $\psi^{(i)}$ is the streamfunction associated with the flow induced by the surface waviness. For sign convention, the streamfunction is related to the longitudinal and normal velocities by $-\psi_y^{(i)} = u^{(i)}$ and $\psi_x^{(i)} = v^{(i)}$ respectively.

The geometry and notation for the semi-infinite plate with a wavy surface is shown in Figure 3. The displacement of the wall is the shifted sine wave

$$h(x,\Theta) = h_0 \sin(\alpha x - \Theta) \quad (2.1)$$

where h_0 is the amplitude, Θ is the phase angle of the waviness, and α is the x-wavenumber.

The linearized relation between the surface displacement and the normal velocity along the x-axis is

$$v^{(i)}(x \geq 0, 0) = U_\infty \partial h/\partial x \quad (2.2)$$

$$= U_\infty \alpha h_0 \cos(\alpha x - \Theta) = \psi_x^{(i)} (\text{for } y=0, x \geq 0) \quad (2.3a)$$

The streamfunction is

$$\psi^{(i)}(x \geq 0, 0) = U_\infty h_0 \sin(\alpha x - \Theta) + \psi_0 \quad (2.3b)$$

where the streamfunction ψ_0, which is a constant or function of time, can be neglected with this semi-infinite plate. These boundary conditions are valid only downstream of the leading edge.

Hereafter, the velocities are nondimensionalized against $\alpha U_\infty h_0$, the streamfunction is nondimensionalized against $\pi U_\infty h_0$, and the x and y coordinates have been nondimensionalized against the half-wavelength $\lambda/2$ of the sinusoidal surface waviness.

For the case of the waviness being in-phase on opposite sides of the plate, the streamfunction will be symmetric about the x-axis. Hence, the homogeneous Neumann boundary condition is applicable upstream of the plate

$$\psi_y^{(i)} = 0 \quad (\text{for } y=0,\ x<0) \quad (2.4)$$

The solution is assumed to vanish far-upstream from the plate and at great distances laterally from the plate. Far-downstream of the leading edge, one would expect that the Kelvin-Helmholtz solution would be recovered. Hence, we require that the solution be bounded as $x \to +\infty$. These exterior boundary conditions, along with the Dirichlet and Neumann conditions along the x-axis and Laplace's equation yield a well-posed elliptic mathematical system. The streamfunction will be symmetric about the x-axis, so we can confine attention to either half-plane which is above or below the plate.

We conformally map the half-plane above the x-axis onto the quarter-plane as shown in Figure 4 by the mapping, $\rho = z^{1/2}$, where $z = x+iy$ and $\rho = \xi+i\eta$. The conformal mapping alleviates the problem of split boundary conditions along the x-axis, with a Neumann boundary condition for $x<0$ and a Dirichlet boundary condition for $x>0$ (Eqns. 2.3b,2.4). Under this mapping, the resultant mathematical system is defined in Figure 4.

The solution is obtained by a cosine integral transform in the ξ direction. The details are presented in Ref. 7. The solution for the streamfunction of the flow induced by the semi-infinite plate with a sinusoidal wall is

$$\begin{aligned}\psi^{(i)} = \{&\cos\theta\ \mathrm{Real}[-\cos\pi z S_2(-\pi z)-\sin\pi z C_2(-\pi z)\\ &+\cos\pi z C_2(-\pi z)-\sin\pi z S_2(-\pi z)+\sin\pi z]\\ &-\sin\theta\ \mathrm{Real}[-\cos\pi z S_2(-\pi z)-\sin\pi z C_2(-\pi z)\\ &-\cos\pi z C_2(-\pi z)+\sin\pi z S_2(-\pi z)+\cos\pi z]\}/\pi\end{aligned} \tag{2.5}$$

where S_2 and C_2 are the Fresnel integrals

$$S_2(z) = (2\pi)^{-1/2}\int_0^z t^{-1/2}\sin(t)\ dt \tag{2.6a}$$

$$C_2(z) = (2\pi)^{-1/2}\int_0^z t^{-1/2}\cos(t)\ dt \tag{2.6b}$$

The Fresnel integrals are evaluated numerically, either as asymptotic series valid for large argument or series expansions valid for small argument (Ref. 58).

The velocities corresponding to solution (2.5) are

$$u^{(i)} = -\psi_y^{(i)} = -iF(z,\theta) \tag{2.7a}$$

$$v^{(i)} = \psi_x^{(i)} = F(z,\theta) \tag{2.7b}$$

where F is the complex function

$$\begin{aligned}F = &\sin(\pi z-\theta)[S_2(-\pi z)-C_2(-\pi z)]\\ &-\cos(\pi z-\theta)[S_2(-\pi z)+C_2(-\pi z)-1]\\ &-(\sin\theta+\cos\theta)/[\pi(-2z)^{1/2}])\end{aligned} \tag{2.7c}$$

For large $|z|$, computational experience has shown that it is better to recast the solution as

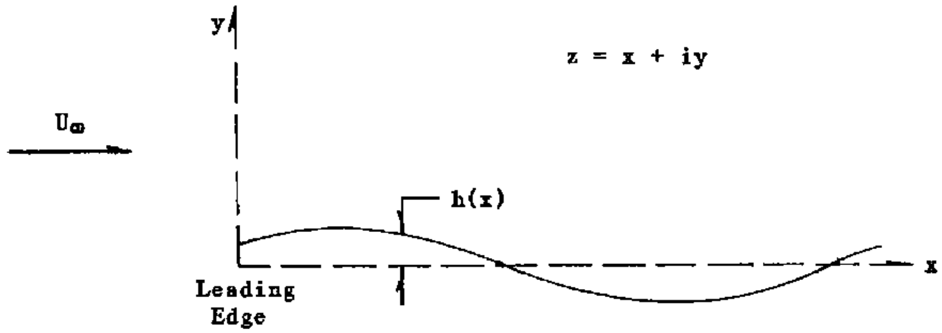


Figure 3. Geometry and notation for flow past a wavy, semi-infinite plate.

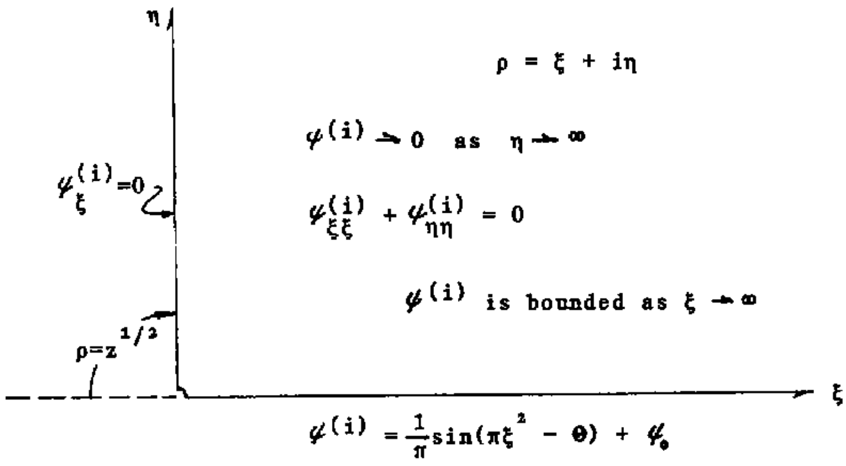


Figure 4. Conformal mapping of the upper half-plane onto the quarter-plane.

$$F = \sin\theta[f - g - 1/[\pi(-2z)^{1/2}]] + \cos\theta[f + g - 1/[\pi(-2z)^{1/2}]] \quad (2.7d)$$

where f(z) and g(z) are related to the Fresnel integrals by

$$f(z) = [\tfrac{1}{2} - S(z)]\cos(\pi z^2/2) - [\tfrac{1}{2} - C(z)]\sin(\pi z^2/2)$$

$$g(z) = [\tfrac{1}{2} - C(z)]\cos(\pi z^2/2) + [\tfrac{1}{2} - S(z)]\sin(\pi z^2/2)$$

$$S(z) = S_2(\pi z^2/2); \; C(z) = C_2(\pi z^2/2)$$

The asymptotic series for f and g (Ref. 58, page 302)

$$\pi z f(z) \sim 1 + \sum_{m=1}^{\infty} (-1)^m \, [1 \cdot 3 \cdots (4m-1)/(\pi z^2)^{2m}]$$

$$\pi z g(z) \sim \sum_{m=0}^{\infty} (-1)^m \, [1 \cdot 3 \cdots (4m+1)/(\pi z^2)^{2m+1}]$$

converge rapidly. The solution form (2.7d) avoids the problems of calculating the small differences between large numbers. While the direct evaluation of solution (2.7b) is limited to about $|z|=1.8$ when 32 bit arithmetic is used, and is limited to about $|z|=5.2$ when 64 bit arithmetic is used, solution (2.7d) has been evaluated at $|z|=400$, which is 200 wavelengths from the leading edge.

Figures 5a,b show the disturbance streamline pattern for the flow induced by the semi-infinite wavy wall with two phase angles. These figures, and the others, do not include the uniform mean flow. The induced flow depends on the phase, θ, of the surface wave. The wavy surfaces on opposite sides of the plate are in phase. One can clearly see the upstream influence of the waviness, as expected since Laplace's equation is elliptic.

A singularity occurs at the leading edge for the velocities, with the validity of the theory consistent with linearized airfoil theory. The region where the velocities are large can be made as small as desired by making the wall waviness smaller. Hence, this is the proper first-order solution as long as the flow is unseparated at the leading edge. Since we are interested in the practical case of flows past bodies with streamlined leading edges, the assumption of unseparated flow implicitly contained in the Laplace's equation appropriately models that feature without unduly complicating the analysis with details of the nose geometry and plate thickness.

Far-downstream of the leading edge, it appears in Figures 5a,b that the Kelvin-Helmholtz solution is recovered. However, the Kelvin-Helmholtz solution is one part of the solution for <u>all</u> $x \geq 0^+$. We shall demonstrate this feature in the next section by rearranging the solution.

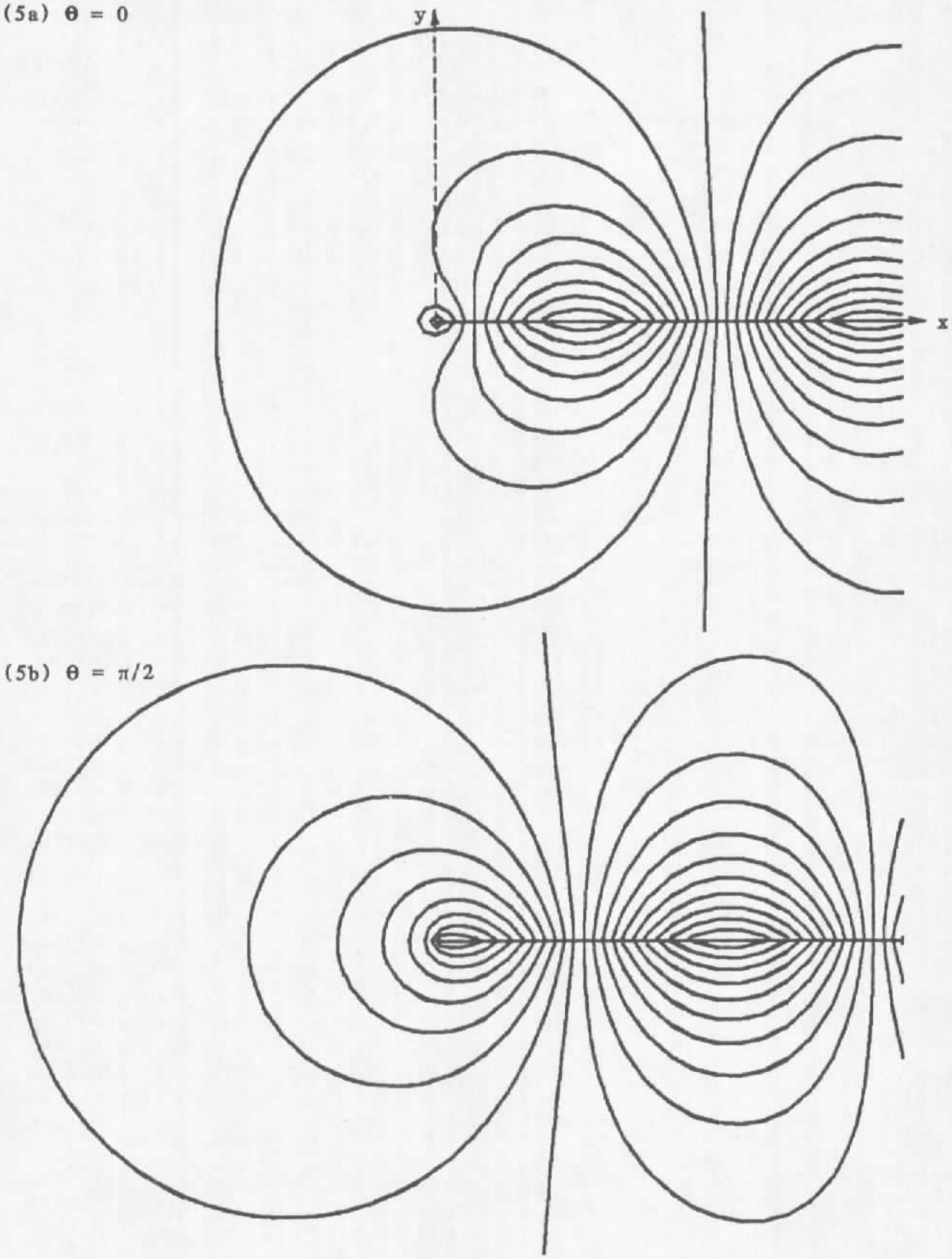


Figure 5. Streamlines for the flow induced by surface waviness on a semi-infinite plate for two phase angles of the surface waviness.

3.0 A SEPARATION OF THE SOLUTION DOWNSTREAM OF THE LEADING EDGE INTO TWO CLASSES OF WAVES

Recent analyses (Refs. 39a,39b,40,41) which have identified the basic solutions of the Orr-Sommerfeld equation offer guidance on how to decompose solution (2.5,2.7a,b) into its basic Fourier-Laplace constituents. Rather than resort to a formal boundary-value problem in the quarter-plane $x \geq 0, y \geq 0$ for this inviscid problem with a uniform mean flow, we will list the simplified versions of the waves discussed in Section 1.6. The eigenmodes and the upstream-propagating continuous spectrum do not appear if the mean flow is uniform and viscous effects are neglected. For the region <u>downstream</u> of the leading edge, the Rayleigh equation with a uniform mean flow is

$$(1 - \frac{\omega}{k})(D^2 - k^2)\phi(y) = 0 \tag{3.1}$$

where the normal velocity and its amplitude are related by $v(x,y,t)=\phi(y)\exp(ikx-i\omega t)$. The x-wavenumber can be complex. Solutions are assumed to be bounded as $y \to \infty$. The boundary condition along $y=0$ depends on the wavenumber and frequency, and we shall note the appropriate condition for each wave. The formal boundary-value problems properly sort out these solutions and conditions through the use of Fourier and Laplace transforms, but for brevity, a more direct statement will be made here for the subject problem.

The Rayleigh equation (3.1) only admits solutions of four forms which are bounded as $y \to \infty$:

(1) The (steady) classical Kelvin-Helmholtz fluctuating flow with phase speed zero. Only this solution has a nonhomogeneous boundary condition along the x-axis, and the wavenumber k is the same wavenumber as the sinusoidal wall.

(2) Irrotational, exponentially-decaying standing waves, of the form

$$v = \sin(\beta y)\exp(-\beta x)$$

where β is real and the wavenumber is pure imaginary, $k=i\beta$. This solution satisfies the usual impermeability condition along a flat plate. At this point, we do not know the value of β. Later, we shall show that there is a spectrum of these waves.

(3) Irrotational, exponentially-growing standing waves of the form

$$v = \sin(\beta y)\exp(+\beta x)$$

These growing waves must be excluded <u>downstream</u> of the leading edge of this semi-infinite plate, unless some influence originating from positive infinity is to be included. Growing waves of form $v(x,y)=\cos(\beta y)\exp(+\beta x)$ do exist upstream of the plate, however, as discussed in Section 5.0.

(4) Vortical fluctuations convecting downstream, which can be ruled out here since no vortical freestream disturbances have been introduced in this problem, and none can diffuse from the wall in this inviscid analysis. These fluctuations would satisfy impermeability on a flat plate.

Hence, for $x>0$, we write solution (2.5) as the sum of two parts

$$\psi^{(i)} = \psi^{(kh)} + \psi^{(s)}$$

where

$$\psi^{(kh)} = \frac{1}{\pi}\sin(\pi x-\theta)e^{-\pi y} \quad \text{(Kelvin-Helmholtz solution)} \tag{3.2a}$$

$$\psi^{(s)} = \psi^{(i)} - \psi^{(kh)} \quad \text{(Secondary flow)} \tag{3.2b}$$

The secondary flow represents the influence of the leading edge of the plate. This secondary flowfield satisfies the impermeability condition <u>for a flat plate</u>.

The streamfunction $\psi^{(i)}$ has been evaluated downstream of the leading edge, and the Kelvin-Helmholtz solution has been subtracted out. The streamlines of the secondary flow which remains are plotted in Figures 6a-h for eight different phase angles. Note in Figures 6d-6g that the flow reverses itself near the wall, and that the size of the cell along the x-axis depends on the phase angle of the surface waviness. The longitudinal velocity along the x-axis is the function $u^{(s)}(x,\theta)$ which appeared in eqn. 1.5.

The normal velocity associated with this second flow, $v^{(s)} = \psi_x^{(s)}$ along the line, $x_1 = 0.5$, is plotted in Figure 7 for two phase angles. These velocity profiles were calculated along the dashed lines drawn in Figures 6a,f. The maximum y-value is much larger than for the other figures so that the decay can be illustrated. Note that this second flow satisfies impermeability ($v^{(s)}=0$ at $y=0$) in contrast to the nonzero value for the Kelvin-Helmholtz solution.

4.0 THE SPECTRUM OF STANDING WAVES

It is often useful to express the solution in terms of its Fourier-Laplace constituents, each which can be calculated and the waves then superimposed. The Kelvin-Helmholtz solution (3.2a) is already in the form of a neutrally oscillating wave. The normal velocity for the secondary flow will be decomposed by taking the half-range sine transform

$$v^{(s)}(x_1,\beta,\theta) = (2/\pi)^{1/2}\int_0^\infty v^{(s)}(x_1,y,\theta)\sin\beta y \, dy \tag{4.1a}$$

with inverse

$$v^{(s)}(x_1,y,\theta) = (2/\pi)^{1/2}\int_0^\infty v^{(s)}(x_1,\beta,\theta)\sin\beta y \, d\beta \tag{4.1b}$$

Since each of the waves decay exponentially in the x-direction, the spectrum at position x is related to the spectrum at position x_1 by the relation

$$v^{(s)}(\beta,x,\theta) = v^{(s)}(\beta,x_1,\theta)\exp[-\beta(x-x_1)]$$

for $x,x_1>0$. Hence, for other x-positions, the velocity can be expressed in terms of the spectrum at $x=x_1$ by the relation

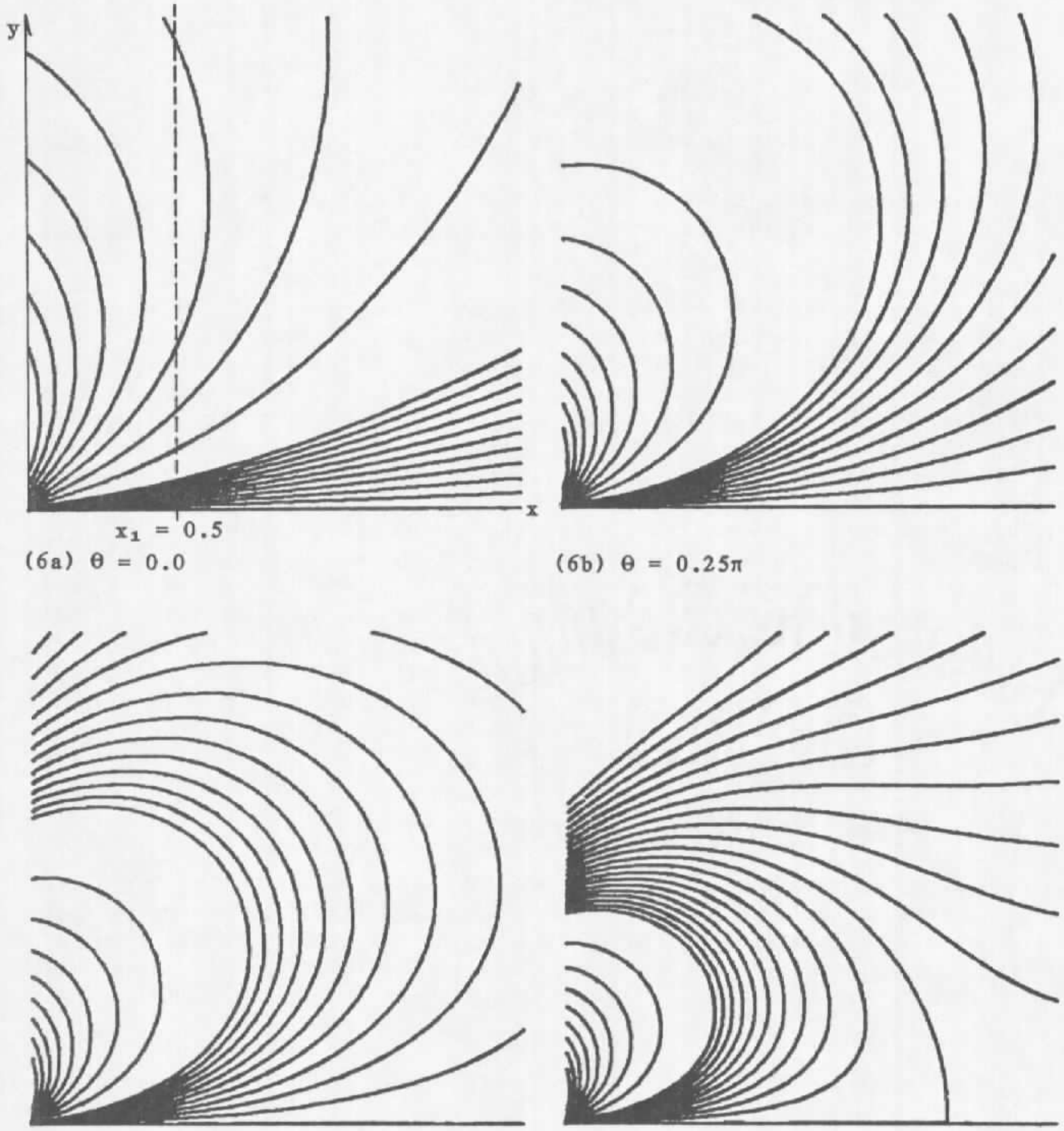


Figure 6. Streamlines for the standing wave pattern, $\psi^{(s)}$, for various phase angles of the surface waviness. This flow represents the alteration to the Kelvin-Helmholtz solution produced by the leading edge. It is composed of a superposition of the basic waves shown in Figure 8. Other phase angles are shown in Figures 6e-h. The incremental value of the streamfunction has two values, with one value ten times smaller than the other to better illustrate the flow.

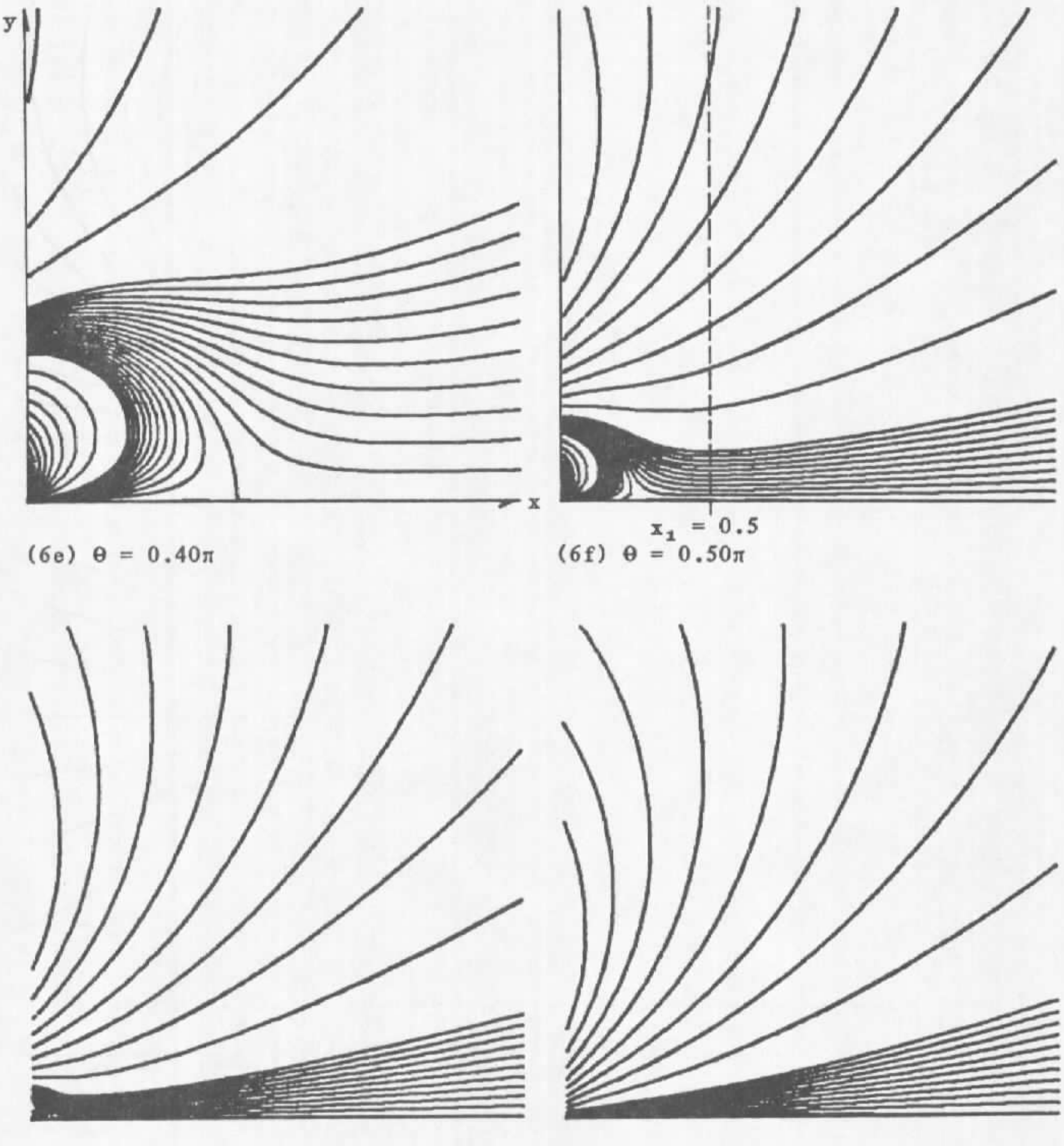


Figure 6. Concluded.

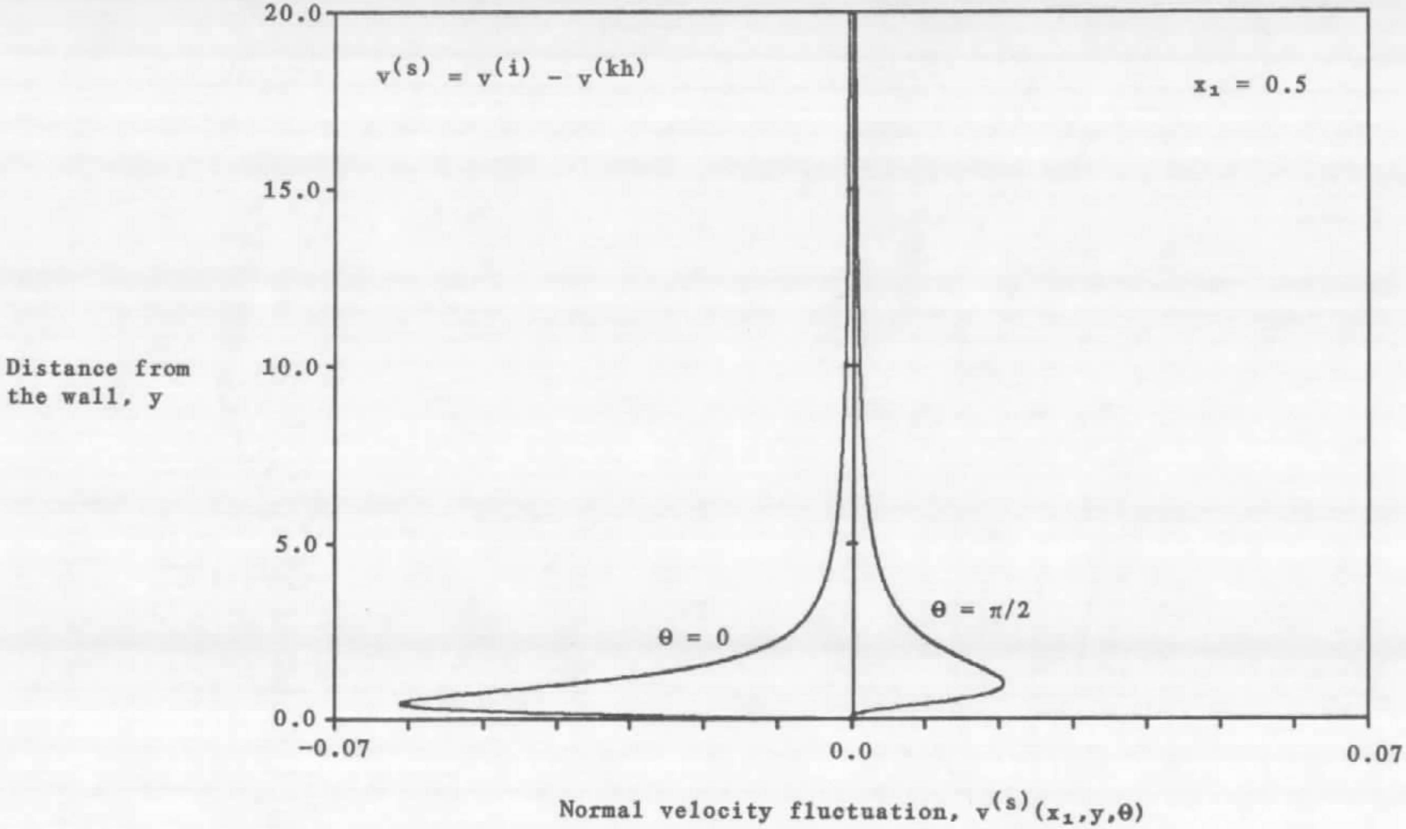


Figure 7. The normal velocity fluctuation for the superposition of standing waves for two phase angles. These are the velocity fluctuations along the dashed lines drawn in Figures 6a and 6f. The maximum ordinate is much larger for this figure than for the other figures to illustrate the slow decay.

$$v^{(s)}(x,y,\theta) =$$

$$(2/\pi)^{1/2}\int_0^{\infty} v^{(s)}(\beta,x_1,\theta)e^{-\beta(x-x_1)}\sin\beta y \, d\beta \tag{4.1c}$$

The streamlines for exponentially-decaying standing waves are plotted in Figs. 8a-c with three values of the x-wavenumber, $k = i\beta$. The three values of k are approximately the values corresponding to the maximum Fourier amplitudes in Figure 9.

The Fourier spectrum of the velocity plotted in Figure 7 is shown in Figure 9. The spectrum illustrates that the velocity $v^{(s)}$ is composed of a superposition of steady waves, each of form

$$v^{(s)}(x,y,\beta,\theta) = v^{(s)}(\beta,\theta)e^{-\beta x}\sin\beta y \tag{4.2}$$

This flow is also an irrotational flow. Unlike the classical Kelvin-Helmholtz solution, however, this solution decays exponentially downstream of the leading edge, and satisfies the impermeability condition at the surface of a flat plate. Although this wave oscillates neutrally as $y \to \infty$, we have shown that such waves can originate from the wall. The superposition of these waves, however, yields a flow which vanishes far away from the plate.

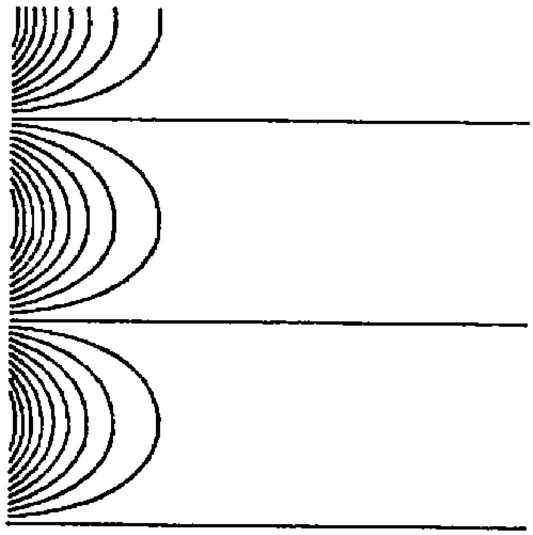
5.0 SOLUTION UPSTREAM OF THE LEADING EDGE REPRESENTED AS A SUPERPOSITION OF GROWING STANDING WAVES

Solution (2.5) for the streamfunction $\psi^{(i)}$ and solutions (2.7a,b,c) for the velocities are valid both upstream and downstream of the leading edge. However, the representation $v^{(i)} = v^{(kh)} + v^{(s)}$, with $v^{(s)}$ expressed in equation (4.1c) as a superposition of decaying standing waves, is valid only downstream of the leading edge. What is the corresponding representation upstream of the leading edge in the half-plane $x<0$, $-\infty<y<\infty$?

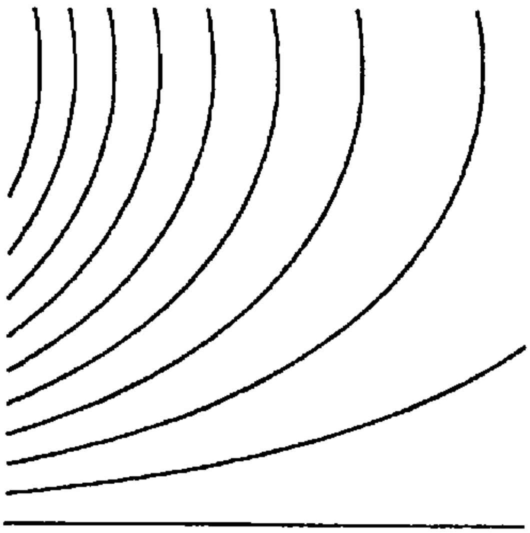
Upstream of the leading edge, there is no constraint that the solution of the Rayleigh equation (3.1) satisfy the impermeability boundary condition. Hence, the Kelvin-Helmholtz solution does not appear in the upstream half-plane. Hence the possible solutions for $v^{(s)}$ are growing and decaying standing waves and vortical fluctuations of the forms

Standing waves	Rectangular arrays of vortices convecting downstream	
(a) $\exp(+\beta x)\cos\beta y$	(e) $\sin\gamma(x-t)\sin\beta y$	
(b) $\exp(+\beta x)\sin\beta y$	(f) $\sin\gamma(x-t)\cos\beta y$	
(c) $\exp(-\beta x)\cos\beta y$	(g) $\cos\gamma(x-t)\sin\beta y$	
(d) $\exp(-\beta x)\sin\beta y$	(h) $\cos\gamma(x-t)\cos\beta y$	(5.1)

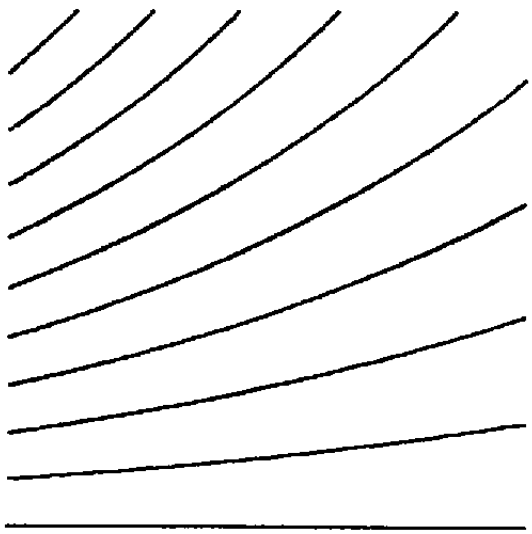
Of these possibilities, the form (b) can be eliminated by the homogeneous Neumann condition. Forms (c,d) blow up as $x \to -\infty$ and can be eliminated. The last four solutions (e-h) represent vortical freestream disturbances convected downstream, and they do not appear in the present problem. Only the waveform (a) remains. Hence, the normal velocity upstream of the leading edge can be represented as the cosine integral transform



(8a) α = 5.0i



(8b) α = 1.1i



(8c) α = 0.5i

Figure 8. Streamlines for three examples of the basic Fourier-Laplace constituents of the flow illustrated in Figures 6a-h. These are stationary waves which decay like sin(βy)exp(-βx). The flow in Figures 6a-h represent a superposition of these waves, with spectrums plotted in Figure 9. These three values of the x-wavenumber correspond approximately to the local maximums of the Fourier amplitudes in Figure 9.

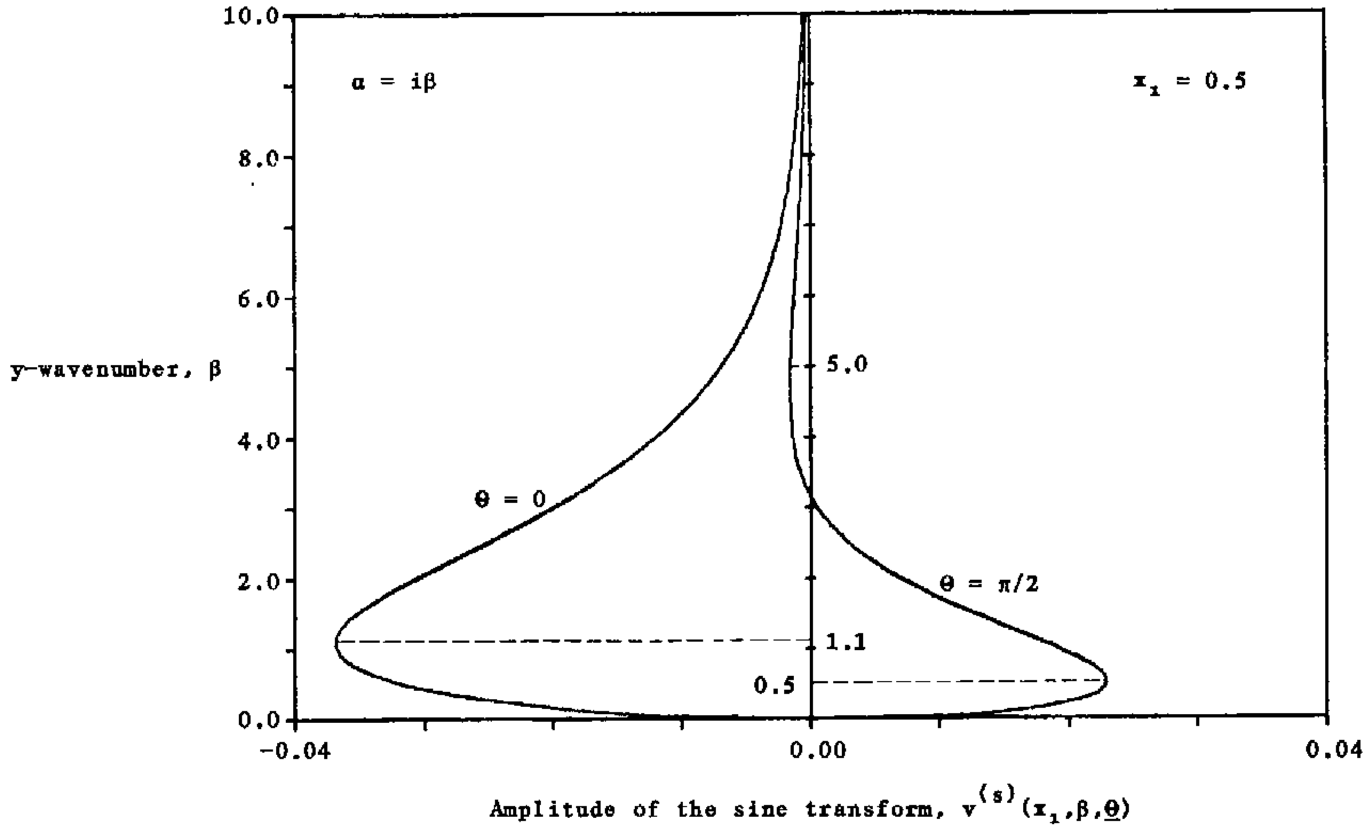


Figure 9. The spectrums of exponentially-decaying standing waves for two phase angles of the surface waviness.

$$v^{(s)}(x_1,\beta,\theta) = (2/\pi)^{1/2}\int_0^\infty v^{(s)}(x_1,y,\theta)\cos\beta y \; dy \qquad (5.2a)$$

with inverse

$$v^{(s)}(x_1,y,\theta) = (2/\pi)^{1/2}\int_0^\infty v^{(s)}(x_1,\beta,\theta)\cos\beta y \; d\beta \qquad (5.2b)$$

Of course for an even function in y, a cosine transform is adequate. The fact that they are growing standing waves appears in the next step. These are the Fourier amplitudes and velocity along some line $x=x_1<0$. Since each of the waves grows exponentially in the streamwise direction, the amplitude at x is related to the amplitude at x_1 by

$$v(x,\beta,\theta)=v(x_1,\beta,\theta)\exp[+\beta(x-x_1)]$$

Hence, the velocity at the position x,y is

$$v^{(s)}(x,y,\theta) = $$
$$= (2/\pi)^{1/2}\int_0^\infty v^{(s)}(x_1,\beta,\theta)e^{+\beta(x-x_1)}\cos\beta y \; d\beta \qquad (5.3)$$

for x, $x_1<0$. There is no "wavy-wall" solution for $x<0$. Hence, $v^{(i)}=v^{(s)}$ for $x<0$. The cosine integral expresses the solution as a superposition of exponentially-growing, symmetric, standing waves.

6.0 SUMMARY, DISCUSSION AND CONCLUSIONS

Conformal mapping of the irrotational flow induced by surface waviness on a semi-infinite plate has yielded solution (2.7b) for the normal velocity which is valid both upstream and downstream of the leading edge. However, downstream of the leading edge, it is instructive to express this solution in terms of two families of waves:

$$v^{(i)}(x,y,\theta) = \cos(\pi x-\theta)\exp(-\pi y)$$

$$+ (2/\pi)^{1/2}\int_0^\infty v^{(s)}(x_1,\beta,\theta)\sin\beta y \; \exp[-\beta(x-x_1)] \; d\beta \qquad (6.1)$$

(1) The classical Kelvin-Helmholtz flow induced by a wavy wall, with streamlines plotted in Figure 1. This solution satisfies a nonhomogeneous wall boundary condition along the x-axis.

(2) A secondary flow which can be represented as a superposition of waves which decay exponentially in the streamwise direction. This flow accounts for the presence of the leading edge of the semi-infinite plate. This spectrum has resulted from surface waviness which has only one x-wavenumber. These waves and their combined pattern satisfy the wall boundary condition on a flat plate, $v^{(s)}=0$ at $y=0$, although they have been excited by the

combined effects of the wall waviness and the leading edge. The Fourier amplitudes $v^{(s)}(x_1,y,\theta)$ are plotted in Figure 9. Far-downstream of the leading edge, the secondary flow vanishes and the only solution which remains is the classical Kelvin-Helmholtz solution.

Also downstream of the leading edge, it is instructive to write the streamwise velocity along the plate, in dimensional form, as

$$U_\infty(x) = U_\infty + u^{(kh)}(x,0,\theta) + u^{(s)}(x,0,\theta) \tag{6.2}$$

This expression emphasizes that the longitudinal velocity along the plate is composed of the value it would have if the plate were flat, a sinusoidal perturbation associated with the Kelvin-Helmholtz solution, and a secondary flow consisting of a superposition of exponentially-decaying waves.

These simplified forms of the waveforms are based on a uniform mean flow which is inviscid. Table 1 summarizes the Rayleigh solutions (with $\bar{U}=1$) and the Orr-Sommerfeld solutions for the two waveforms described above.

The flowfield <u>upstream</u> of the leading edge can be represented as a superposition of symmetric, growing standing waves

$$v^{(i)} = v^{(s)}(x,y,\theta) =$$

$$=(2/\pi)^{1/2}\int_0^\infty v^{(s)}(x_1,\beta,\theta)\cos\beta y \exp[+\beta(x-x_1)]\ d\beta \tag{6.3}$$

Exponentially-decaying waves of the <u>unsteady</u> form

$$v(x,y,t) = \phi(y)\exp(-\beta x-i\omega t) \tag{6.4}$$

with $\phi(y)=\sin\beta y$ also have been found previously in analyses of vortices colliding with a semi-infinite plate (Ref. 9). Unsteady waves of the more general form (6.4) have also been found as solutions of the Orr-Sommerfeld equation (Ref. 2). This analysis illustrates that steady versions of these standing waves originate in the flow about a wavy semi-infinite plate. Further study is required to describe the effects of viscosity and the mean boundary layer. While these waves oscillate neutrally as $y\to\infty$, this analysis shows that <u>they can originate from the wall</u>.

The <u>viscous</u> form of the <u>steady</u> exponentially-decaying wave, $v = \phi(y)\exp(-\beta x)$, has not been calculated. The <u>unsteady</u> solutions for both decaying and growing standing waves in boundary layers as solutions of the Orr-Sommerfeld equation are documented in Ref. 2. When the frequency vanishes, however, the exterior solution takes on the form of <u>four neutrally oscillating waves</u>

$$\phi(y) = Ae^{i\gamma y} + Be^{-i\gamma y} + Ce^{i\beta y} + De^{-i\beta y} \tag{6.5a}$$

where $\gamma=R^{1/2}(\beta+\beta^2/R_\delta)^{1/2}$, in contrast to the exterior solution

$$\phi(y) = Ae^{-my} + Ce^{i\beta y} + De^{-i\beta y} \tag{6.5b}$$

where $m=(-\beta R_\delta-\beta^2-i\omega R_\delta)^{1/2}$ for the unsteady case. Numerical solutions for very small frequencies suggest that a very slowly decaying, high frequency oscillation appears and survives into the freestream. In the exact steady limit, these oscillations may take the form $\exp(+i\gamma y)$ as noted above. These additional steady, viscous solutions warrant further study.

This study has assumed a semi-infinite plate. We believe that the secondary flow generally will be stronger for finite-length plates, than for semi-infinite plates with wavy walls, at least for certain phase angles and plate lengths. This conjecture warrants detailed study.

Finally, we recognize that there is a relationship between the solutions for the cases of

(1) freestream vorticity disturbances convecting downstream and encountering a plate

(2) irrotational, traveling freestream disturbances propagating at speeds different from U_∞ and encountering a plate

(3) flow past a semi-infinite plate with stationary wall waviness

Table 1
COMPARISON OF SOLUTIONS FOR FLOW PAST WAVY WALLS

(a) Solutions of Rayleigh Equation with a Uniform Mean Flow	(b) Solutions of Orr-Sommerfeld Equation with a Nonuniform Mean Flow
(1a) Kelvin-Helmholtz Solution varying as $v(x,y)=v_0\exp(-\beta y)\sin(\beta x)$ which satisfies nonhomogeneous boundary condition at y=0. (See Figure 1 and Ref. 1)	(1b) Benjamin (Ref. 32), Lessen-Gangwani (Ref. 34), and Aldoss-Reshotko (Ref. 35) solution of form $v(x,y)=\phi(y)\exp(i\alpha x)$ which satisfies nonhomogeneous boundary condition(s) at y=0.
(2a) Steady, Decaying Solution varying as $v(x,y)=v_0\sin(\beta y)\exp(-\beta x)$ which satisfies homogeneous boundary conditions at y=0. (See Figures 8a,b; Figure 6a-h is a superposition of these waves)	(2b) Limiting Steady Case (ω=0) of the Exponentially-Varying Standing Wave varying as $v(x,y,t)=\phi(y)\exp(-\beta x-i\omega t)$ which satisfies homogeneous boundary conditions at y=0. (The unsteady case has been calculated (Ref. 32); the steady case has not been calculated) Stokes solution for an unsteady surging of the freestream is the β=0 limit of the unsteady, standing waves

(the case considered in this study)

(4) flow past a semi-infinite wavy wall with traveling surface waves

The relationships between these flows for seemingly rather different cases of freestream disturbances and wall waviness warrant further study and development.

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NOMENCLATURE

English

$c=\omega/\alpha$	phase speed
$C(z)$	Cosine Fresnel integral, $C(z) = \int_0^z \cos(\pi t^2/2)dt$ $C(z)=C_2(\pi z^2/2)$
$C_2(z)$	Cosine Fresnel integral(2), $C_2(z)=(2\pi)^{-1/2}\int_0^z t^{-1/2}\cos t dt$
$f(z)$	function related to the Fresnel integrals by $f(z) = [\frac{1}{2} - S(z)]\cos(\pi z^2/2) - [\frac{1}{2} - C(z)]\sin(\pi z^2/2)$
$F(z,\theta)$	function defined in eqn.(2.7c)
$g(z)$	function related to the Fresnel integrals by $g(z) = [\frac{1}{2} - C(z)]\cos(\pi z^2/2) + [\frac{1}{2} - S(z)]\sin(\pi z^2/2)$
$h(x)$	displacement of the wavy wall
h_0	amplitude of the sinusoidal wall waviness
h_1, h_2	amplitudes of the galloping surface as defined in eqn.(1.6b)
i	$(-1)^{1/2}$
k	x-wavenumber in the Rayleigh eqn.(3.1)
$S(z)$	Sine Fresnel integral, $S(z) = \int_0^z \sin(\pi t^2/2)dt$ $S(z)=S_2(\pi z^2/2)$
$S_2(z)$	Sine Fresnel integral(2), $S_2(z)=(2\pi)^{-1/2}\int_0^z t^{-1/2}\sin t dt$
t	time
u,v	disturbance velocities in the x and y directions
U_∞	mean x-velocity in the freestream that would exist if the plate were flat
x	coordinate parallel to plate and in streamwise direction
y	coordinate normal to the plate
$z=x+iy$	complex coordinate

Greek and Script

α	x-wavenumber
β	y-wavenumber
∇^2	Laplacian operator
ε	small quantity proportional to the amplitude of the surface waviness
λ	wavelength of the surface waviness
$\psi^{(i)}$	disturbance streamfunction induced by the surface waviness on a semi-infinite plate
$\psi^{(kh)}$	Kelvin-Helmholtz streamfunction for flow past a wavy wall
$\psi^{(s)}$	streamfunction composed of a pattern of standing waves and representing the alteration to $\psi^{(kh)}$ caused by the leading edge
ρ	$\xi+i\eta$, complex coordinate in the conformally mapped plane
θ	phase angle of the surface waviness
$\phi(y)$	complex amplitude of the normal velocity fluctuation
ω	frequency
ξ,η	coordinates in the conformally mapped plane

Superscripts, Subscripts, and Miscellaneous Notation

i	associated with the fluctuating flow induced by surface waviness on a semi-infinite plate
kh	associated with the Kelvin-Helmholtz solution
Real[ ]	real part of [ ]
s	associated with the secondary flow which is generated by the combined effects of the surface waviness and the leading edge

Characteristic quantities used in the nondimensionalizations

$\lambda/2$	characteristic length
$\alpha h_0 U_\infty$	characteristic disturbance velocity
U_∞	characteristic mean velocity
$\alpha h_0 U_\infty/\pi$	characteristic disturbance streamfunction