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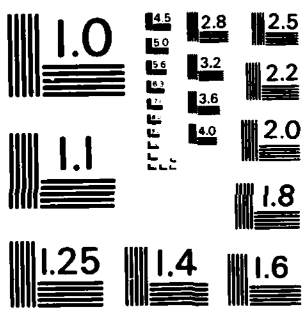
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FOREIGN TECHNOLOGY DIVISION

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by

Karl Nickel

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SOLUTION OF A MINIMUM PROBLEM IN AIRCRAFT WING THEORY

Karl Nickel in Tübingen

ABSTRACT

The third fundamental problem of Prandtl's theory of the supporting line is extended to the case of a finite number of linear supplementary conditions, and solved. Three examples demonstrate cases which occur in practice, with such supplementary conditions.

1. Formulation of the problem

In wing theory I of the "Four Essays"[1], L. Prandtl states the following minimum problem as the "Third Fundamental Problem" (op. cit., p. 28):

"Given is the total lift and the wing span, also ρ and V; to be found is that distribution of the lift over the wing span for which the drab becomes a minimum".

With the aid of the given formulas (op. cit., p. 27), the mathematical formulation of this problem reads as follows in a coordinate system in which the wing extends from -1 to +1:

Let

$$w(x) = \frac{1}{4\pi} \oint_{-1}^{1} \frac{d\Gamma(y)}{dy} \frac{dy}{x-y} \;{}^{2)} \qquad (1)$$

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1) Prandtl, L., and Betz., A. Vier Abhandlungen zur Hydrodynamik und Aerodynamik. Neudruck aus den Verhandlungen des III. Internationalen Mathematiker-Kongresses zu Heidelberg und aus den Nachrichten der Gesellschaft der Wissenschaften zu Goettingen. ("Four Essays on Hydrodynamics and Aerodynamics. Reprint from the Proceedings of the Third International Congress of Mathematicians at Heidelberg and from the Communications of the Society of Sciences at Goettingen"). Goettingen, 1927.

2) The sign $\oint$ is to be understood as the Cauchy principal value of the integral.

and $\Gamma(-1)=\Gamma(+1)=0$ [where $\Gamma(x)$ is the local lift density and $w(x)$ the induced downwash velocity at position x on the wing]. Let the function $\Gamma(x)$ on the interval $\langle -1, +1 \rangle$ be determined such that

$$\int_{-1}^{1} \Gamma(x)\, w(x)\, dx \qquad (2)$$

becomes a minimum under the supplementary condition

$$\int_{-1}^{1} \Gamma(x)\, dx = A \qquad (3)$$

(A = the total lift, arbitrarily prescribed).

This formulation of the problem was extended by M. Munk[3] to the case of arbitrarily distributed and directed lift, and solved in general.

One can also generalize the above minimum problem in another way, by prescribing, instead of the supplementary condition (3), other--and if necessary several--supplementary conditions. Three examples will illustrate this more exactly:

a) A flying wing is to be made into a (shallow) curve by a (small) aileron deflection. What form must be taken by the lift distribution which is added to the symmetric distribution by the aileron deflection in order that the increase in the induced drag stays as small as possible? If one knows this distribution, one can approximate it with a suitable aieleron shape, in order to prevent superfluous losses when deflecting the ailerons. The question here therefore reads: The lift distribution is sought which makes the induced drag (2) a minimum, when the roll torque

$$\int_{-1}^{1} \Gamma(x)\, x\, dx \qquad (4)$$

is to have a prescribed value. The use of equation (1) is in this case exactly correct only in the first instant of aileron deflection, so long as there is no yawing or rolling motion, because only then the departing vortex lines are rectilinear. At any rate, for small 73

[3] Munk, M. Isoperimetrische Aufgaben aus der Theorie des Fluges. Inaugural-Dissertation. ("Isoperimetric Problems from the Theory of Flight. Inaugural Dissertation"). Goettingen, 1919.

yaw and roll velocities one can still use the solution of the formulated problem as an approximation.

A second problem, which leads to the same (mathematical) problem, and in which these difficulties do not arise, was kindly communicated to me by Dr. Prandtl. This is the question of the lift distribution with minimum drag, for a wing with an eccentrically applied load, which creates a roll torque (4).

b) The solution of the third fundamental problem stated by Prandtl reads (ref. 1, page 32) in the formulation given above:

$$\Gamma(x) = \frac{2A}{\pi}\sqrt{1-x^2},$$

that is, the lift is distributed over the wing span in the form of a half ellipse (see the solid line in Figure 1). If one now modifies $\Gamma(x)$ somewhat, in the manner shown by the dashed line, letting A in (3) keep its value, the induced drag (in the vicinity of its minimum) will change only slightly. The spar-bending torque at the wing root

$$\int_0^1 \Gamma(x)\, x\, dx,$$

which can be replaced by

$$\int_{-1}^{1} \Gamma(x)\, |x|\, dx \qquad (5)$$

for symmetric lift distributions (apart from a factor of 1/2), will however become smaller.

For a free-flying wing, this means the following: in consequence of the reduced stress, the wing can be built lighter in weight. Because, however, in constant horizontal flight the lift and the weight are equal, this means that the total lift A in (3) also becomes smaller.

If $\Gamma(x)$ is replaced by $\lambda \cdot \Gamma(x)$, the lift (3) and spar-bending torque (5) change in proportion to λ, while the induced drag is proportional to λ^2. One would expect, therefore, somewhat more favorable drag conditions with a lift distribution which has somewhat

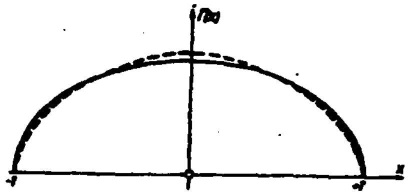

Figure 1

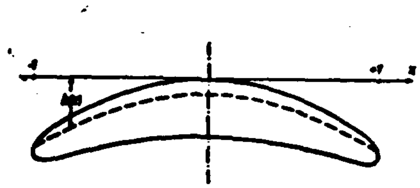

Figure 2

smaller values at the wing tips, than with the elliptical lift distribution. At any rate, it must be noted that the dependence of the flying weight on the spar bending torque is very small for the usual wing construction methods, so that the effect described above is small.

This consideration leads to the following formulation of the problem: The induced drag (2) is to be made a minimum under the two supplementary conditions, that the lift (3) and the spar bending moment (5) possess prescribed values.

c) If one considers a wing which is curved in the direction of flight (see Figure 2), the longitudinal torque would also have to be held constant, in addition to the total lift, in solving the third fundamental problem of wing theory[4)] for this case. One can now require that the constancy of the longitudinal torque should result exclusively from lifting forces on the wing (longitudinal stabilization through wing warping, such as in bird-flight). Let h(x) be the distance of the central line of pressure (dashed in Figure 2) from the x-axis. Then this requirement means that

$$\int_{-1} \Gamma(x)\, h(x)\, dx = M \qquad (6)$$

4) It is true (see reference 1, page 25) that the idea of the supporting line can no longer be used in this case; however, as M. Munk has shown (reference 3, page 21), in considering the drag it suffices to consider only the two-dimensional problem of a wing which is not curved in the direction of flight. $\Gamma(x)$ here is thus the projection of the lift onto a plane normal to the direction of flight.

(longitudinal torque M prescribed) and the problem above reads: The lift distribution $\Gamma(x)$ is sought, for which (2) becomes a minimum under the two supplementary conditions (3) and (6). It might be remarked here that in the general case of an arbitrarily shaped wing the central line of pressure will not be the line of the for- 74
ward neutral point in the two-dimensional problem (T/4 line). The task of determining this line for an arbitrarily prescribed wing poses great difficulties which, to my knowledge, have not yet been solved. As an approximation, one can nevertheless replace the central line of pressure by the T/4 line, so long as the wing does not deviate too much from a straight wing.

On the basis of these examples, it is now natural to extend the minimum problem formulated above in such a way that finitely many supplementary conditions, linear in $\Gamma(x)$, are prescribed. One seeks therefore to determine $\Gamma(x)$ in the interval $\langle -1, +1 \rangle$ such that

$$\int_{-1}^{1} \Gamma(x)\, w(x)\, dx \qquad (2)$$

becomes a minimum, under the N supplementary conditions

$$\int_{-1}^{1} \Gamma(x)\, h_n(x)\, dx = A_n \qquad (n = 1, \ldots, N) \qquad (7)$$

$[\Gamma(-1) = \Gamma(+1) = 0;$ $h_n(x)$ and A_n arbitrarily prescribed, except that for $h_m(x) = 0$ naturally only $A_m = 0$ is allowed]. For $N = 1;\ h_1(x) \equiv 1,\ A_1 = A$ the problem formulated by Prandtl results as a special case; likewise the examples a), b) and c) can be obtained through analogous specialization.

2. Transformation of the problem

For the mathematical treatment of the problem, let the following transformations be undertaken:

One sets

$$x = \cos s, \qquad y = \cos t,$$
$$w(x)\sqrt{1 - x^2} = f(s), \qquad \Gamma(x) = Z(s)$$

and uses the notation $h_n(x)\sqrt{1-x^2} = h_n(s)$ $(n = 1, \ldots, N)$. Then[5] from (1), (2) (neglecting a factor of 1/4) and (7):

$$f(s) = \frac{1}{\pi}\oint_0^\pi \frac{dZ(t)}{dt}\,\frac{\sin s\,dt}{\cos t - \cos s} \quad \ldots\ldots\ldots\ldots\ldots \text{(1a)},$$

$$= \int_0^\pi Z(s)f(s)\,ds \quad \ldots\ldots\ldots\ldots\ldots\ldots \text{(2a)},$$

$$\int_0^\pi Z(s)\,h_n(s)\,ds = A_n \qquad (n = 1, \ldots, N) \quad \ldots\ldots \text{(7a)}.$$

For known f(s), (1a) is a Fredholm integral equation of the first kind for dZ(t)/dt. As is well known, its solution[6] is

$$\frac{dZ(s)}{ds} = \frac{1}{\pi}\oint_0^\pi f(t)\,\frac{\sin t\,dt}{\cos s - \cos t}.$$

from which follows by integration

$$Z(s) = \frac{1}{\pi}\int_0^\pi f(t)\log\frac{\sin\dfrac{s+t}{2}}{\sin\left|\dfrac{s-t}{2}\right|}\,dt \qquad (8)$$

if one writes for brevity

$$\frac{1}{\pi}\log\frac{\sin\dfrac{s+t}{2}}{\sin\left|\dfrac{s-t}{2}\right|} = S(s,t),$$

and inserts (8) in (2a) and (7a), one obtains 75

$$\int_0^\pi\int_0^\pi f(s)f(t)\,S(s,t)\,ds\,dt \qquad (2b)$$

and

$$\int_0^\pi\int_0^\pi f(s)\,h_n(t)\,S(s,t)\,ds\,dt = A_n \qquad (7b)$$

Since the functions Z(s) and f(s) are unambiguously related to

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5) The conditions under which the integrals and infinite series which occur exist, and under which the transformations used (changing the order of integration, exchanging integrations and summations) are allowed, are examined in a comprehensive work, which is to appear shortly under the title Solution of a Particular Minimum Problem in the Mathematische Zeitschrift. In it a second class of supplementary conditions will be added to those already considered. Note added in proof: This has meanwhile appeared in the Mathematische Zeitschrift, vol. 53 (1950), pp. 21-52.

6) See for example Schroeder, K. An Integral Equation of the First Kind in Wing Theory. Sitzungsberichte der Preussischen Akademie der Wissenschaften XXX (1938).

each other through (1a) and (8), one can also consider f(s) as the function to be sought. With this the problem to be solved reads: f(s) is to be determined in the interval $\langle 0, \pi \rangle$, such that (2b) becomes a minimum under the supplementary conditions (7b).

For further simplification of notation, let the following further abbreviation be introduced:

$$(f, g) = \int_0^\pi \int_0^\pi f(s) \cancel{g(t) S(s, t)}\, ds\, dt \tag{9}$$

With the representation[7)]

$$S(s, t) = \frac{2}{\pi} \sum_{n=1}^{\infty} \frac{1}{n} \sin ns \sin nt$$

valid for $0 \leq s, t \leq \pi,\ s \neq t$, there results from (9):

$$\begin{aligned}(f, g) &= \int_0^\pi \int_0^\pi f(s)\, g(t)\, S(s, t)\, ds\, dt \\ &= \frac{2}{\pi} \sum_{n=1}^{\infty} \frac{1}{n} \left(\int_0^\pi f(s) \sin ns\, ds \right) \left(\int_0^\pi g(t) \sin nt\, dt \right).\end{aligned}$$

$$(f, g) = \frac{\pi}{2} \sum_{n=1}^{\infty} \frac{1}{n} a_n b_n \tag{10}$$

with the Fourier sine coefficients

$$a_n = \frac{2}{\pi} \int_0^\pi f(s) \sin ns\, ds, \quad b_n = \frac{2}{\pi} \int_0^\pi g(t) \sin nt\, dt \qquad (n = 1, 2, \ldots).$$

According to (9) and (10), and because S(s,t) = S(t,s), (f,g) has the following properties: (a = a real number)

$$\left.\begin{array}{ll} \text{a)} & (af, g) = a(f, g) \qquad (a = \text{reelle Zahl}) \\ \text{b)} & (f, g) = (g, f) \\ \text{c)} & (f + g, h) = (f, h) + (g, h) \\ \text{d)} & (f, f) > 0 \quad \text{für} \quad f(s) \not\equiv 0 \end{array}\right\} \tag{11}$$

One therefore has the task of finding a function f(s) in the interval $\langle 0, \pi \rangle$ such that (f,f) becomes a minimum under the N

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7) See for example Hamel, G. Integralgleichungen ("Integral Equations"). Berlin: J. Springer, 1937 ; or Jaeckel, K. Determination of a Series Representation for the Kernel in r in Elliptical Coordinates. Z. angew. Math. Mech., vol. 30 (1950), p. 186 formula (16).

$$(f, f) = \mathrm{Min.}$$

supplementary conditions

$$(f, h_n) = A_n \qquad (n = 1, \ldots, N) \qquad (12)$$

Thereby (f,g) is defined by (9) and has the properties (11).

3. Solution of the problem

If the functions $h_n(s)$ are linearly dependent, then under the supplementary conditions (12) either they are supernumerary or they are contradictory and, therefore, cannot be satisfied at all. Therefore, without loss of generality, one may assume the functions $h_n(s)$ to be linearly independent.

Assumption: Let the functions $h_n(s)$ be linearly independent, and let the real numbers $A_n (n = 1, \ldots, N)$ be arbitrarily chosen.

Proposition: Then there is exactly one solution f'(s) to the above minimum problem and this solution has the form

$$f'(s) = \sum_{n=1}^{N} a_n h_n(s)$$

with unambiguously determined real numbers a_n.

Proof: According to the orthonormalization procedure of E. 76
Schmidt, one can find functions

$$H_n(s) = \sum_{m=1} c_{mn} h_m(s) \qquad (c_{nn} \neq 0;\ n = 1, \ldots, N)$$

such that

$$(H_m, H_n) = e_{mn} = \begin{cases} 1 & \text{for} \quad m = n \\ 0 & \text{for} \quad m \neq n \end{cases}$$

Inverting these equations gives the representation

$$h_n(s) = \sum_{m=1}^{n} d_{mn} H_m(s) \qquad (d_{nn} \neq 0;\ n = 1, \ldots, N).$$

One now sets

$$f'(s) = \sum_{n=1}^{N} c_n H_n(s)$$

with real coefficients c_n which are as yet undetermined and with the calculation rules (11) obtains

$$(f', h_r) = \sum_{n=1}^{N} c_n (H_n, h_r) = \sum_{n=1}^{N} c_n \sum_{m=1}^{r} d_{mr} (H_n, H_m)$$
$$= \sum_{n=1}^{N} c_n \sum_{m=1}^{r} d_{mr} e_{nm} = \sum_{n=1}^{r} c_n d_{nr} \qquad \text{for} \quad r = 1, \ldots, N.$$

If now $\sum_{n=1}^{r} c_n d_{nr} = A_r \ (r = 1, \ldots, N)$, the c_n are thereby unambiguously determined in a recursive fashion. With the real numbers c_n thus chosen, f' then satisfies the supplementary conditions (12), and further receives the desired form

$$f'(s) = \sum_{m=1}^{N} c_m H_m(s) = \sum_{m=1}^{N} c_m \sum_{n=1}^{m} c_{nm} h_n(s) = \sum_{n=1}^{N} h_n(s) \sum_{m=n}^{N} c_m c_{nm} = \sum_{n=1}^{N} a_n h_n(s).$$

It still remains to show that f' also minimizes (f,f). An arbitrary function f(s) can always be written in the form

f(s) = f'(s) + k(s) (namely with k(s) = f(s) - f'(s)).

Thus

$$(f, h_n) = (f' + k, h_n) = (f', h_n) + (k, h_n) = A_n + (k, h_n) \qquad (n = 1, \ldots, N).$$

f(s), therefore, satisfies the supplementary conditions (12) exactly when $(k, h_n) = 0$ for $n = 1, \ldots, N$.

For all the functions f(s) which satisfy the supplementary conditions (12), the following holds, according to the rules (11):

$$(f,f) = (f' + k, f' + k) = (f', f') + 2(f', k) + (k, k)$$
$$= (f', f') + 2 \sum_{n=1}^{N} a_n (h_n, k) + (k, k)$$
$$= (f', f') + (k, k) \geq (f', f'),$$

whereby the equality sign comes into play only for $k(s) \equiv 0$. The unambiguity of the coefficients a_n follows with (11d) for the linear independence of the functions $h_n(s)$. Thus, the proposition is proven.

If one reverses the transformations introduced above, one finds the following result:

4. Result

Let the functions $h_n(x)$ and the real numbers A_n $(n = 1, \ldots, N)$ be arbitrarily chosen. The function $\Gamma(x)$ in $-1 \leq x \leq +1$ is to be determined such that $\Gamma(-1) = \Gamma(+1) = 0$ ist and such that by setting

$$w(x)=\frac{1}{4\pi}\oint_{-1}^{1}\frac{d\Gamma(y)}{dy}\,\frac{dy}{x-y} \qquad (1)$$

the expression

$$\int_{-1}^{1}\Gamma(x)\,w(x)\,dx \qquad (2)$$

becomes a minimum under the supplementary conditions

$$\int_{-1}^{1}\Gamma(x)\,h_n(x)\,dx=A_n \qquad (n=1,\ldots,N) \qquad (7)$$

If a function $\Gamma(x)$ exists at all, which satisfies the N supplementary conditions (7), then this minimum problem has exactly one solution, which has the form

$$\Gamma(x)=\sum_{n=1}^{N}a_n\int_{-1}^{1}h_n(y)\log\left|\frac{1-xy+\sqrt{(1-x^2)(1-y^2)}}{x-y}\right|dy\,.$$

with real coefficients a_n. If the functions $h_n(x)$ are linearly 77
independent, then the minimum problem always has a solution and the coefficients a_n are unambiguously determined.

5. Application

In the following table are the associated functions $h(x)$

Function $$\int_{-1}^{1} h(y) \log \left| \frac{1 - xy + \sqrt{(1-x^2)(1-y^2)}}{x-y} \right| dy.$$

$h(x)$	$\int_{-1}^{+1} h(y) \log \left\vert \frac{1 - xy + \sqrt{(1-x^2)(1-y^2)}}{x-y} \right\vert dy$
1	$\pi \sqrt{1-x^2}$
x	$\frac{\pi}{2} x \sqrt{1-x^2}$
x^2	$\frac{\pi}{6} (2x^2+1) \sqrt{1-x^2}$
x^3	$\frac{\pi}{8} x (2x^2+1) \sqrt{1-x^2}$
$\begin{cases} 1 & x>0 \\ -1 & x<0 \end{cases}$	$2x \log \frac{1+\sqrt{1-x^2}}{\vert x\vert}$
$\vert x\vert$	$x^2 \log \frac{1+\sqrt{1-x^2}}{\vert x\vert} + \sqrt{1-x^2}$
$x\vert x\vert$	$\frac{2}{3}\left(x^3 \log \frac{1+\sqrt{1-x^2}}{\vert x\vert} + x\sqrt{1-x^2}\right)$
$x^2\vert x\vert$	$\frac{1}{12}\left(6x^4 \log \frac{1+\sqrt{1-x^2}}{\vert x\vert} + (6x^2+1)\sqrt{1-x^2}\right)$
$\begin{cases} 1 & x_0 < x \leq 1 \\ 0 & -1 \leq x < x_0 \end{cases}$	$(x_0 - x) \log \left\vert \frac{1 - x_0 x + \sqrt{(1-x_0^2)(1-x^2)}}{x_0 - x} \right\vert + \sqrt{1-x^2} \arccos x_0$

With these, one finds the solutions for the first two of the examples stated previously:

a) $$\int_{-1}^{1} \Gamma(x)\, x\, dx = R \qquad \Gamma(x) = \frac{8R}{\pi} x \sqrt{1-x^2}$$

b) $$\left\{ \begin{aligned} \int_{-1}^{1} \Gamma(x)\, dx &= A \\ \int_{-1}^{1} \Gamma(x)\, |x|\, dx &= H \end{aligned} \right. \qquad \begin{aligned} \Gamma(x) = {} & 3\left(\frac{2A}{\pi} - H\right)\sqrt{1-x^2} \\ & + 3\left(3H - \frac{4A}{\pi}\right) x^2 \log \frac{1+\sqrt{1-x^2}}{|x|}. \end{aligned}$$

In Figure 3 is displayed this solution for example a). Figure 4 shows some curves from the family of solutions for example b) for A = constant, H variable (among these is contained for $3\pi H = 4A$ the half-ellipse as a special case).

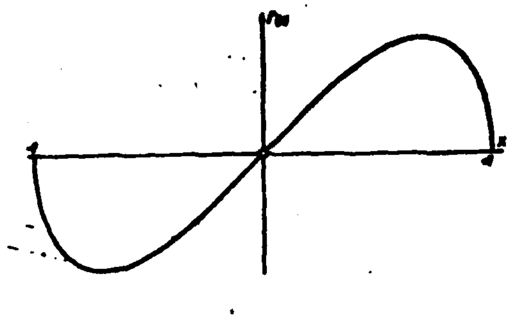

Figure 3

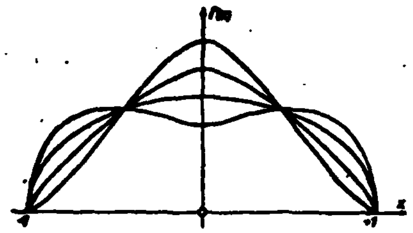

Figure 4

Submitted on January 28, 1950.

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