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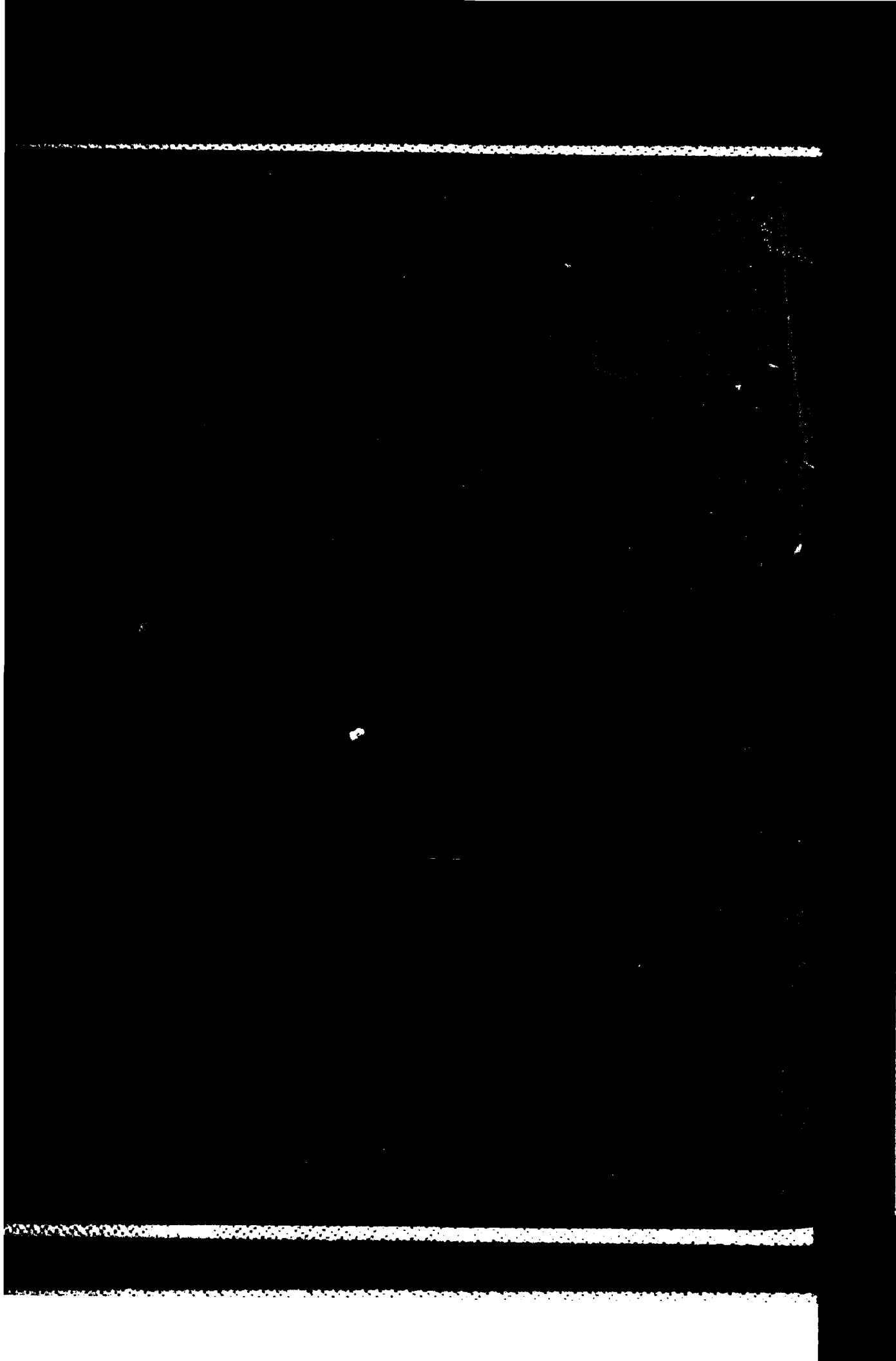NATIONAL BUREAU OF STANDARDS-1963-A

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MEAN-SQUARE OPTIMAL, FULL-ORDER COMPENSATION OF STRUCTURAL SYSTEMS WITH UNCERTAIN PARAMETERS

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ABSTRACT

The minimum information approach to active control of structural systems seeks inherently robust designs by use of mean-square optimization conjoined with a stochastic system model which presumes as little as possible regarding *a priori* information on modal parameter statistics. This report extends earlier results for the regulator problem to the case of full-order dynamic compensation with nonsingular observation noise. Optimality conditions along with sufficient conditions for existence and uniqueness of solutions and for closed-loop stochastic stability are presented. Results concerning asymptotic properties for large uncertainty levels are also given. Numerical results for various simple examples indicate improved robustness properties over standard LQG designs and suggest the possibility that, under the minimum information stochastic approach, the burden of design computation may be reduced to that associated with the relatively well known or "coherent" modes.

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MEAN-SQUARE OPTIMAL, FULL-ORDER COMPENSATION OF STRUCTURAL SYSTEMS WITH UNCERTAIN PARAMETERS

1. INTRODUCTION

Many techniques advanced for the active control of highly flexible structural systems implicitly assume the existence of a large order "verification" model in which the values of structural parameters are precisely known. Where the critically important issue of parameter uncertainties (arising from inherent limitations in the analysis of high order structural modes) is specifically addressed within a computationally tractable approach, the formulation entails either (a) an essentially geometric (in the linear algebraic sense) approach to sensitivity reduction,[*] (b) ad hoc combination of inherently robust controls with more standard LQ design techniques[**] and/or (c) the *a posteriori* verification of robustness by use of methods entirely extrinsic to the design process.[†]

In contrast, it may be argued that although verification models are precise, they are necessarily false; that the structure must be regarded as a stochastically parametered mechanical system (for which limited *a priori* statistical data is unavailable); and that design optimization must proceed from a measure of performance defined over the entire parameter

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[*] For example, the work of Sesak [1,2] as reinterpreted by Coradetti [3] actually involves static decoupling of sensitivity derivatives and/or modal coordinates for model order reduction.
[**] e.g., the rate-output feedback of Balas [4] or low-authority control of Aubrun [5] would be charged with the control of high-order, poorly-known modes while providing enhanced stabilization of a "high authority" LQ control designed for a reduced order model of the low frequency modes.
[†] Using, for example, general multivariable measures of robustness along the lines of Ref. [6].

statistical ensemble. Very recently, such a stochastic design approach was outlined and specifically elaborated for the special case of full state feedback regulation of structural systems with *a priori* uncertainties in the modal frequencies [7,8].

Entailing, as it does, a fundamental revision of the traditional, deterministic approach to dynamical modelling, the earlier developments [7-9] must be reviewed here. To retain the spirit of linear-quadratic optimization, we choose as the performance measure the average of a quadratic functional of state and controls over the parameter ensemble. Secondly, as a complete empirical specification of parameter statistics is never provided in practice, the design approach must accept the kind of severely limited statistical data that is actually available. To avoid ad hoc assumptions, the full probability assignment required for determination of the mean-square optimal control must be consistent with the data on hand but maximally unpresumptive with regard to unavailable data. This is accomplished by resort to a maximum entropy principle (Jaynes' principle).

Furthermore, to achieve particular simplicity and design conservatism, we acknowledge as "available" the minimum possible *a priori* statistical data needed to induce a complete probability assignment while preserving fidelity of the overall model at high levels of uncertainty (or for high order modes).

Thus, the approach is "minimum information" in two respects--first, because we acknowledge as available the bare minimum of *a priori* data and, secondly, because we induce a full probability model from this acknowledged data by use of a minimum information (i.e., maximum entropy) principle.

The resulting probabilistic description induced by this essential data constitutes our fundamental system model and presumes as little parameter information as possible.

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With uncertainties only in the open loop frequencies, we
identified the minimum data set by examining the phenomenology
of frequency uncertainties as reflected in the mean (the par-
ameter ensemble averaged) response, the covariance matrix and
the "expected cost" matrix (i.e., the covariance of the co-state).
The principal effects of modal frequency uncertainty are the
introduction of a spurious damping (the "decorrelation damping")
into the mean response and the suppression of cross-correlation
among distinct modes. Essential to the proper modelling of such
qualitative features are the decorrelation damping time con-
stants, termed the "modal decorrelation times". These are gen-
erally inversely proportional to modal frequency standard devia-
tions and constitute fundamental, albeit unconventional, measures
of frequency uncertainty.

Acknowledgement of only the mean (or nominal) values of
modal frequencies and the modal decorrelation times as available
data induces a white parameter statistical model which reduces
the optimization problem to solution of a modified Riccati
equation (the "stochastic Riccati equation") for the expected
cost matrix. Under mild restrictions, this possesses a unique
positive semi-definite solution which guarantees closed-loop
stochastic stability. Thus the proposed approach reduces the
need for design iteration to achieve robust stability.

The most significant aspect of the stochastic Riccati
equation is the character of its steady-state solutions for
large uncertainties [9]. If the uncertainties in all open-loop
frequencies increase without bound (i.e., all decorrelation
times approach zero) the expected cost reduces to a diagonal
matrix whose elements are independent of modal frequency statis-
tics and are given by simple analytical expressions. This
asymptotic solution gives rise to a velocity feedback control law

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which is stable regardless of the values of modal frequencies or damping ratios. Thus a greater degree of robustness is obtained than was originally sought.

In the more typical case in which the frequencies of low order structural modes are relatively well known while modelling accuracy deteriorates for the high-order modes, the stochastic Riccati equation automatically produces a velocity feedback control (of the asymptotic form) for the high-order, poorly-known modes. At the same time, for low-order modes having small uncertainties, the control closely resembles the deterministic plant solution. In other words, a "high authority", essentially deterministic control for well-known modes and a "low authority" velocity feedback control for relatively uncertain modes naturally emerge as limiting regimes of a global control law which is guaranteed to be stable over the parameter ensemble.

This general behavior has immediate consequences for the computational effort required for high order systems. In brief, the computational burden required for determination of the control gain is mainly associated with the relatively few well-known ("coherent") modes. Provided that the dimension of the coherent system is moderate, the stochastic Riccati equation is amenable to numerical solution of acceptable accuracy for systems of arbitrary order.

The above qualitative features provide strong motivation for extension of the formulation to linear, dynamic compensation. Here we consider a structural system as described by a finite number of its "all-elastic" normal modes with uncertainties in the open-loop frequencies. We remove, however, the restriction of previous work to full-state feedback and consider dynamic output feedback compensation. Chapter 2 sets forth the minimum information stochastic modelling approach and derives the

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stochastic Lyapunov equation which must be appended to the mean-square optimization problem as a constraint. Various properties of this equation are related to closed-loop stochastic stability and the existence of steady state, constant gain controls. In Chapter 3 we specialize to full-order dynamic compensation, and derive explicit stationary conditions. Existence and uniqueness of solutions to these conditions as well as their asymptotic properties for large levels of modal frequency uncertainty are explored. Chapter 4 concludes by a presentation of computational procedures and various numerical examples. From these theoretical developments and supporting numerical results, it will be seen that the desirable features of the stochastic design approach as applied to full-state feedback regulation are indeed retained in this less idealized setting.

2. THE MINIMUM INFORMATION FORMULATION OF THE OPTIMIZATION PROBLEM

2.1 Problem Statement

As in the previous work [7], we consider the control of a linear elastic structure subject to small deformations and no rigid body degrees of freedom. Modifications needed to include rigid body modes are straightforward and need not be treated in this initial development. Retaining n normal mode coordinates in the system model, the state-space form of the equations of motion may be written:

$$\dot{x} = (\overline{A} + a(t))x + Bu_1 + w_1$$

$$\eta = Cx + \tilde{w}_2$$ (1)

$$\overline{A} \in R^{2n \times 2n}, \ B \in R^{2n \times \ell}, \ u_1 \in R^{\ell}, \ \eta \in R^{P}$$

where x is the vector of modal coordinates and velocities with its odd indexed elements representing modal displacements and the adjacent even indexed elements giving the corresponding modal velocities. w_1 is a white disturbance noise with intensity $V_1 \geq 0$, u_1 the control input and B the input map:

$$
B \ = \ \begin{bmatrix} 0 & 0 & \cdots & 0 \\ b_{21} & b_{22} & \cdots & b_{2\ell} \\ 0 & 0 & \cdots & 0 \\ b_{41} & b_{42} & \cdots & b_{4\ell} \\ \vdots & \vdots & & \vdots \end{bmatrix} \qquad (2)
$$

where the non-zero elements are proportional to the normal mode shapes at ℓ actuator locations. η is the vector of sensor outputs with output map C and observation noise, $\tilde{w}_2$. We assume that $\tilde{w}_2$ is independent of w_1 and has a nonsingular intensity matrix v_2.

$\overline{A}$ is the nominal or mean value of the system map:

$$
\overline{A} \ \triangleq \ \begin{array}{c} \text{block-diag} \\ k=1,\ldots,n \end{array} \begin{bmatrix} 0 & 1 \\ -\overline{\omega}_k^2 & -2\eta_k\overline{\omega}_k \end{bmatrix} \qquad (3)
$$

where the $\overline{\omega}_k$; $k=1,\ldots,n$ are the nominal design values of the modal frequencies and the η_k are the modal damping ratios representing inherent structural damping. It is assumed throughout that:

$$
0 \ < \ \eta_k \ \ll 1 \ ; \quad k = 1,\ldots,n \qquad (4)
$$

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Finally, a(t) is the random portion (assumed zero mean) of the system map representing possible statistical variation of the modal frequencies.

As a preliminary step, it is convenient to express the above relations in the eigen-basis of $\overline{A}$. In view of the assumption of small damping we may simplify this process by introducing the resonant approximation for $\overline{A}$:

$$
\overline{A} = \begin{array}{c} \text{block-diag} \\ k=1,\ldots,n \end{array} \begin{bmatrix} -\eta_k \overline{\omega}_k & 1 \\ -\overline{\omega}_k^2 & -\eta_k \overline{\omega}_k \end{bmatrix} \tag{5}
$$

so-called because the difference between damped and undamped natural frequencies is neglected. With this replacement, the eigenvector matrix of $\overline{A}$ is:

$$
\Phi = \begin{array}{c} \text{block-diag} \\ k=1,\ldots,n \end{array} \begin{bmatrix} 1 & 1 \\ i\overline{\omega}_k & -i\overline{\omega}_k \end{bmatrix} \tag{6}
$$

Then, defining:

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$$\overline{\mu} \;\triangleq\; \Phi^{-1}\,\overline{A}\,\Phi$$

$$= \text{diag}\;\{\overline{\omega}_1(i-\eta_1),\overline{\omega}_1(-i-\eta_1),\ldots,\overline{\omega}_n(i-\eta_n),\overline{\omega}_n(-i-\eta_n)\} \qquad \text{a.}$$

$$\nu(t) \;\triangleq\; \Phi^{-1}\,a\,\Phi \qquad\qquad\qquad \text{b.}$$

$$\beta \;\triangleq\; \Phi^{-1}\,B \qquad\qquad\qquad\qquad \text{c.} \left.\right\} \;(7)$$

$$\tilde{w}_1 \;\triangleq\; \Phi^{-1}\,w_1 \qquad\qquad\qquad\quad \text{d.}$$

$$v_1 \;\triangleq\; \Phi^{-1}\,V_1\,\Phi^{-1H} \qquad\qquad\quad \text{e.}$$

$$\gamma \;\triangleq\; C\,\Phi \qquad\qquad\qquad\qquad\quad \text{f.}$$

the equations of motion written in terms of the state-space
modal coordinate vector:

$$\xi \;\triangleq\; \Phi^{-1}\,x \qquad\qquad\qquad\qquad\qquad (8)$$

assume the form:

$$\dot{\xi} \;=\; \mu\xi + \beta\,u_1 + \tilde{w}_1 \qquad\qquad\qquad \text{a.}$$

$$\eta \;=\; \gamma\xi + \tilde{w}_2 \qquad\qquad\qquad\qquad \text{b.} \left.\right\} \;(9)$$

$$\mu \;\triangleq\; \overline{\mu} + \nu(t) \qquad\qquad\qquad\qquad \text{c.}$$

Furthermore, we suppose that *a priori* uncertainty exists
only in the open-loop frequencies so that:

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$$\nu(t) = \operatorname*{diag}_{k=1,\ldots,2n} \{i\operatorname{Im}(\overline{\mu}_k)\,\delta_k(t)\} \qquad (10)$$

where $\operatorname{Im}(\ldots)$ denotes the imaginary part. The $\delta_k(t)$, $k=1,\ldots,2n$, are assumed real valued, zero-mean, stationary random processes in time and statistically independent of $\tilde{w}_1$ and $\tilde{w}_2$.

Obviously, a general treatment would require that all system maps be random. However, the above restriction offers an appropriately simple point of departure and permits relatively easy interpretation. Moreover, as will ultimately be seen, the very special problem considered here still exhibits important features.

With the above restrictions, suppose the control to be provided by interconnection of (9) with a fixed order dynamic compensator. Specifically, the controlled system equations take the form:

$$\dot{\xi} = \mu\xi + \beta u_1 + \tilde{w}_1 \quad ; \quad \xi \in C^{2n} \qquad \text{a.}$$

$$\dot{q} = \alpha q + u_2 \qquad ; \quad q \in C^{N_q} \qquad \text{b.} \qquad \Big\} \quad (11)$$

where

$$u_1 \triangleq -\kappa q$$

$$u_2 \triangleq f\eta \qquad\qquad\qquad\qquad \Big\} \quad (12)$$

$$\eta = \gamma\xi + \tilde{w}_2$$

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$$\text{Intensity} \quad \begin{pmatrix} \tilde{w}_1 \\ \tilde{w}_2 \end{pmatrix} = \begin{bmatrix} v_1 & 0 \\ 0 & v_2 \end{bmatrix} \quad ; \quad v_1 \geq 0, \ v_2 > 0 \qquad (13)$$

and where q is the compensator state and κ, f are the control gains.

As a measure of performance of system (11), we take the quadratic functional:

$$\bar{J} = E\left[\int_{t_o}^{t_1} dt \ [\xi^H r_1 \xi + u_1^H R_2 u_1] \right] \qquad \text{a.}$$

$$R_2 > 0 \ , \ \epsilon \ R^{\ell x \ell} \qquad \text{b.}$$

$$r_1 \stackrel{\Delta}{=} \phi^H R_1 \ \phi \qquad \text{c.}$$

$$R_1 \geq 0 \ , \ \epsilon \ R^{2n x 2n} \qquad \text{d.}$$

$$\left. \rule{0pt}{60pt} \right\} \quad (14)$$

where the averaging operation includes the parameter ensemble (i.e., in this case the ensemble of the $\delta_k(t)$). Note that terminal state weighting is ignored in (14) since its inclusion would entail only superficial modification of the following results.

The problem is to determine κ and f in (12) (with appropriate choice of α) to minimize $\bar{J}$. It is expeditious, at this point to recast this problem in terms of the augmented state:

$$X \stackrel{\Delta}{=} \begin{pmatrix} \xi \\ q \end{pmatrix} \quad \epsilon \ C^{2n+N}q \qquad (15)$$

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in which case (11) through (14) become:

$$\min_{\kappa,f,\alpha}: \quad \bar{J} = E[\int_{t_o}^{t_1} dt \; X^H \; R'X] \qquad (16)$$

$$\dot{X} = [\bar{A}' + \nu' \; -F'] \; X + W(t) \qquad (17)$$

where

$$W(t) \triangleq \begin{pmatrix} \tilde{w}_1 \\ f\tilde{w}_2 \end{pmatrix} \qquad (18)$$

and

$$\bar{A}' \triangleq \begin{bmatrix} \bar{\mu} & 0 \\ 0 & \alpha \end{bmatrix} \qquad \qquad \text{a.}$$

$$\nu' \triangleq \begin{bmatrix} \nu(t) & 0 \\ 0 & 0 \end{bmatrix} \qquad \qquad \text{b.}$$

$$F' \triangleq \begin{bmatrix} 0 & \beta\kappa \\ -f\gamma & 0 \end{bmatrix} \qquad \qquad \text{c.} \qquad \left. \begin{array}{c} \\ \\ \\ \\ \\ \end{array} \right\} (19)$$

$$V' \triangleq \begin{bmatrix} v_1 & 0 \\ 0 & fv_2 f^H \end{bmatrix} \qquad \qquad \text{d.}$$

$$R' \triangleq \begin{bmatrix} r_1 & 0 \\ 0 & \kappa^H R_2 \kappa \end{bmatrix} \qquad \qquad \text{e.}$$

Finally, after straightforward manipulation, we can restate the optimization problem in terms of the augmented co-state matrix as follows:

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$$\min_{\kappa, f, \alpha} \quad \bar{J} = \text{tr} \int_{t_0}^{t_1} dt\, \bar{P}'(t)\, V' \qquad (20)$$

$$\bar{P}' = E[P'] \; \epsilon \; C^{(2n+N_q) \times (2n+N_q)} \qquad \text{a.} \;\Bigg\}$$

$$-\dot{P}' = [\bar{A}' + \nu' - F']^H P' + P'[\bar{A}' + \nu' - F'] + R' \qquad \text{b.} \;\Bigg\} \quad (21)$$

$$P'(t_1) = 0 \qquad \text{c.} \;\Bigg\}$$

where the averaging in (21.a) now extends only over the parameter ensemble. The quantity $\bar{P}'$ (which we term the _expected cost matrix_) is the covariance of the augmented co-state and gives a direct measure of mean-square performance by virtue of (20).

From the assumptions made concerning $W(t)$, the disturbance noise is a differential process (with unbounded variation) and care is needed in the precise specification of the meaning of (17). Such a specification should also allow the parameter noise, ν', to be a differential process. To answer this need, we shall take (17) to mean the truth with probability one of the equality:

$$\int_a^b h(t)\, [-dX(t) + [\bar{A}' - F']X(t)dt + d\nu'(t)X + dW(t)] = 0 \qquad (22.a)$$

for all $t_0 \le a < b \le t_1$ and any continuous matrix function, $h(t)$; where the above stochastic integrals are interpreted according to Stratonovich [10,11] and Wong and Zakai [12]:

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$$\int_a^b h(t) \, d\nu'(t) \, X(t) =$$

$$\text{l.i.m.} \sum_i h(\tilde{t}_i) [\nu'(t_i) - \nu'(t_{i-1})][\tfrac{1}{2}(X(t_k) + X(t_{i-1}))] \qquad (22.b)$$
$$\varepsilon \to 0$$

where $\{t_k\}$ is a partition of the interval $[a,b]$ and

$$\left. \begin{aligned} \varepsilon &\overset{\Delta}{=} \max_i (t_i - t_{i-1}) \\[2ex] \tilde{t}_i &\in [t_{i-1}, t_i] \end{aligned} \right\} \qquad (23)$$

Interpretation (22) is the appropriate one for our application since, in writing (17) we have in mind a mechanical system whose parameters are perturbed by a noise of <u>finite</u> total power. Adopting the more familiar Itô differential for (17) would ignore this fact. However, in the case in which ν' is white, (22) yields results corresponding to a bandpass parameter noise in the limit as the passband approaches infinity.

Thus, with (22), ν' may be treated as a process of bounded variation (almost everywhere) and we may state various formal results for the system response in terms of the transition matrix of $(\overline{A}' - F' + \nu')$ as follows:

Theorem 1

Suppose that $\nu'(t)$ is a stationary zero mean random matrix process. Define an increment in the nonstationary process W' by:

$$W'(t_1, t_2) \overset{\Delta}{=} W'(t_2) - W'(t_1) \overset{\Delta}{=} \int_{t_1}^{t_2} d\tau \, \nu'(\tau) \; ; \; t_2 \geq t_1 \qquad (24)$$

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and assume that $W'(t_1, t_2)$ possesses joint moments of all orders for all finite $t_2 - t_1$. Further, suppose that $F'(t)$ is bounded and continuous. Then under interpretation (22) and in a probability-one sense:

A.   The transition matrix, $\phi'(t, \tau)$, for system (17) is given by:

$$\phi'(t, \tau) = \sum_{k=0}^{\infty} \phi'_k(t, \tau) \quad ; \quad t \geq \tau \qquad \text{a.}$$

where

$$\phi'_o(t, \tau) = \exp[\overline{A}'(t-\tau) + W'(\tau, t)] \qquad \text{b.}$$

$$\phi'_k(t, \tau) = (-1)^k \int_\tau^t d\tau_1 \int_\tau^{\tau_1} d\tau_2 \ldots \int_\tau^{\tau_{k-1}} d\tau_k \qquad (25)$$
$$(k>0)$$

$$\times [\phi'_o(t, \tau_1) \, F'(\tau_1) \phi'_o(\tau_1, \tau_2) \, F'(\tau_2) \ldots$$

$$\ldots \phi'_o(\tau_{k-1}, \tau_k) \, F'(\tau_k) \phi'_o(\tau_k, \tau)] \qquad \text{c.}$$

and where the integrals are the usual Riemann-Stieltjes sums and extend over the left semi-closed intervals.

B. $\phi'(t, \tau)$ is almost everywhere continuous and its first and second moments are continuous and differentiable in both arguments.

C.   Eqs. (21.b,c) possess the unique, positive semi-definite solution:

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$$P'(t) = \int_t^{t_1} d\tau\ \psi(t,\tau) \quad ; \quad t \in [t_o, t_1] \tag{26}$$

where, for $\tau \geq t$, $\delta \leq \tau - t$:

$$\psi(t,\tau) = \phi'^H(t+\delta,t)\ \psi(t+\delta,\tau)\ \phi'(t+\delta,t) \qquad \text{a.}$$

$$\psi(\tau,\tau) = R'(\tau) \qquad \text{b.} \tag{27}$$

D. $\overline{\psi}(t,\tau) \triangleq E[\psi(t,\tau)]$; $t \in [t_o, t_1]$ is continuous and differentiable in t and τ.

The above are entirely analogous to the results of Theorem 1 of [7] and the proof may be omitted here.

In the case in which ν' is a differential process, we have an alternative formulation. In such a case, by virtue of (22), system (17) possesses the Itô differential:

$$d\ X'(t) = (\overline{A}' - F'(t) + \tfrac{1}{2}I')\ X(t)\ dt$$

$$+\ d\nu'(t)\ X(t) + dW(t) \tag{28}$$

$$I' = \lim_{\Delta \to 0} \frac{1}{\Delta}\ E[W'^2(t, t+\Delta)]$$

where $\tfrac{1}{2}I'$ is the so-called Stratonovich correction.

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2.2 The Minimum Information Model - Derivation of the Stochastic Lyapunov Equation

As may be seen from Theorem 1, a complete specification of the statistical structure of open-loop frequency deviations permits explicit determination of $\overline{P}'$. However, a complete probability model of the parameters based upon empirical determinations can never be provided in practice and we are faced with limited available data on parameter statistics. To induce a complete probability model uniquely from the available data we define the desired probability assignment as the one which, under the constraints imposed by available data, is maximally noncommittal with regard to unavailable data, i.e., maximizes the entropy of the underlying processes.

More specifically, we may introduce measures of information reposed in the statistics of relative deviations of modal frequencies in the following way. First, note that only increments in the non-stationary processes $\delta_k(0,t)$; defined by

$$\delta_k(t_1,t_2) \triangleq \delta_k(0,t_2) - \delta_k(0,t_1) = \int_{t_1}^{t_2} d\delta_k(\tau)$$

$$t_2 \geq t_1 \quad ; \quad k = 1,\ldots,2n$$

(29)

actually enter into (26) and (27) and thus need to be considered. Define $\{t_m^{(k)}\}$ as an arbitrary division of the real line with $m = 0,1,\ldots,N_k$; N_k finite and $k \in N$ where N comprises a set of distinct integers in the range 1 to 2n. For notational convenience let $\{t_{N_k}^{(k \in N)}\}$ be the totality of such divisions. Then a measure of information contained in the finite set of increments:

$$\delta_k(t_o^{(k)}, t_1^{(k)}), \delta_k(t_1^{(k)}, t_2^{(k)}) \ldots, \delta_k(t_{N_k-1}^{(k)}, t_{N_k}^{(k)}) \quad ; \quad k \; \varepsilon \; N \qquad (30)$$

is given by the relative entropy of these increments.

$$H[\delta; \{t_{N_k}^{(k \varepsilon N)}\}] \triangleq -\int d\sigma \; P[\delta; \{t_{N_k}^{(k \varepsilon N)}\}] \; \ln P[\delta; \{t_{N_k}^{(k \varepsilon N)}\}] \qquad (31)$$

where $d\sigma$ is the volume element in the sample space of (30) and $P[\delta; \{t_{N_k}^{(k \varepsilon N)}\}]$ is the joint probability density of the increments (30). Although it is possible to define a measure of entropy for the stochastic system as a whole, the measures (31) defined directly on the $\delta_k(t_1, t_2)$ will suffice for present purposes.

Now in practice, we may suppose that specific numerical values may be assigned to various statistics of the $\delta_k(t_1, t_2)$ (defined as functionals of $P[\delta; \{t_{N_k}^{(k \varepsilon N)}\}]$) based on empirical determinations. This constitutes the "available data" of the problem. To avoid ad hoc assumptions on the probability distributions of frequency uncertainties, we choose $P[\delta; \{t_{N_k}^{(k \varepsilon N)}\}]$ to maximize a measure of our ignorance of the increments (30) in the light of the available data. In other words, given constraints on $P[\delta; \{t_{N_k}^{(k \varepsilon N)}\}]$ implied by the available data, we determine $P[\delta; \{t_{N_k}^{(k \varepsilon N)}\}]$ to maximize $H[\delta; \{t_{N_k}^{(k \varepsilon N)}\}]$ of (31) for all choices of N, N_k and $\{t_m^{(k)} ; m \; \varepsilon \; [0, N_k], \; k \; \varepsilon \; N\}$.

The idea of employing a statistical model which is maximally unpresumptive with regard to parameter data can be carried still further. We may choose to acknowledge as available a data set which is essential to the proper modelling of open-loop statistical response and is just sufficient to induce a well-defined probability assignment via a maximum entropy principle.

The discussion of section (3) of [13] established various quali-
tative features of the open-loop system response (in particular
the mean, covariance and expected cost), and introduced the con-
cepts of "decorrelation damping", "coherence limit" and "incoher-
ent range". It was concluded that, at the very least, any
approximating probability model of the $\delta_k(0,t); k=1,\ldots,2n$ should
preserve the time scales of decorrelation damping, provide a cor-
rect estimate of the coherence limit and satisfy the bound given
by (19.b) of [13] for the cross-correlations of high order modes.
This is possible only if the "modal decorrelation times", T_k:

$$
\left.
\begin{aligned}
T_k &\triangleq \left(|\text{Im}(\bar{\mu}_k)| I_k \right)^{-1} \\[1em]
&\triangleq \int_0^\infty dt \left| E\left[\exp i\text{Im}(\bar{\mu}_k) \int_0^t d\tau\, \delta_k(\tau) \right] \right|^2 \\[1em]
k &= 1,\ldots,2n \\[1em]
T_{2m} &= T_{2m-1} \quad ; \quad m = 1,\ldots,n
\end{aligned}
\right\} \tag{32}
$$

are admitted as fundamental data. In essence, numerical values
assigned to the T_k establish the scales of frequency deviations
relative to the remaining time scales of the problem.

Thus, we propose to acknowledge only the $T_k (k=1,\ldots,2n)$ as
the "available" data. It remains to determine the probability
assignment which maximizes the entropy (31) for all N, N_k,
$\{t_m^{(k)} : m \in [0,N_k], k \in N\}$ subject to the constraints implied by
(32). We term the resulting probability model the "maximum
entropy probability assignment induced by the data" (32).

Here we repeat the answer to this problem given in Theorem
3 of [7]:

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Theorem 2

Assuming that the processes (29) possess finite variances for all $t_1, t_2 \in (0, \infty)$ and stationary increments, the maximum entropy probability assignment induced by (32) is the one under which the $\delta_k(0,t)$; $k=1,\ldots,2n$ are independent Wiener-Levy processes with intensities $I_k / \bar{\omega}_{N(k)}$:

$$E[\delta_k^2 (0,t)] = I_k |t| / |\mathrm{Im}\bar{\mu}_k|$$

$$I_{2m} = I_{2m-1} \quad ; \quad m = 1, \ldots, n$$

(33)

It must be noted that if the available data encompasses more than specification of the decorrelation times, the resulting maximum values of the information measures (31) are smaller than the values corresponding to the above probability assignment. On the other hand, if the available data omits some of the decorrelation times, a maximum entropy probability assignment for which (29) possess finite variances for finite t does not exist. This arises because (31) monotonically increases with the noise intensities so that, loosely speaking, the maximum entropy model involves white noise of unbounded intensity. Thus, in this sense, the decorrelation times constitute the minimum data required to induce any "reasonable" maximum entropy probability assignment.

Most importantly, the $\delta_k(t)$ are modelled as white noise so that posterior learning is impossible and the stochastic control problem is nondual. In consequence, the white parameter uncertainty model provides a worst case situation from the point of view of parameter identification. Indeed the model may be used

to determine performance degradation due to parameter uncertainty and to assess the need for identification and adaptive algorithms.

With this maximally unpresumptive statistical model we are in a position to determine a single closed equation for $\bar{P}'$ and restate the optimization problem as follows:

Theorem 3

Under the maximum entropy statistical model induced by the decorrelation times as given in Theorem 2 and $F'(t)$ bounded and continuous in $t \in [t_o, t_1]$, the variational problem of (20) and (21) becomes:

$$\min_{\kappa, f, \alpha}: \quad \bar{J} = \int_{t_o}^{t_1} dt \ tr[\bar{P}'V'] \tag{34}$$

where $\bar{P}'$ is the unique, hermitian, positive semi-definite solution of:

$$-\dot{\bar{P}}' = [\bar{A}' - F' - \tfrac{1}{2}I']^H \bar{P}' + \bar{P}'[\bar{A}' - F' - \tfrac{1}{2}I'] + R' + I'\{\bar{P}'\}$$

$$\tag{35}$$

$$\bar{P}'(t_1) = 0$$

where

$$I' \triangleq \begin{bmatrix} I & 0 \\ 0 & 0 \end{bmatrix} \qquad\qquad a. \Big\}$$

$$\tag{36}$$

$$I \triangleq diag \ [\bar{\omega}_1 I_1, \ \bar{\omega}_1 I_1, \dots, \bar{\omega}_n I_n, \ \bar{\omega}_n I_n] \qquad b.$$

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and where, for any square matrix, M:

$$\{M\} \overset{\Delta}{=} \underset{k}{\text{diag}} [M_{kk}] \tag{37}$$

Proof

With the statistical model of Theorem 3, and the definition (22) of stochastic integrals, the results of Theorem 1 may be used. In particular, (18) yields:

$$\psi(t,\tau) = \phi'^{H}(t+\delta,t) \; \psi(t+\delta,\tau) \; \phi'(t+\delta,t) \qquad \text{a.} \left.\begin{array}{c} \\ \\ \\ \\ \end{array}\right\} \tag{38}$$

$$\psi(t+\delta,\tau) = \phi'^{H}(\tau,t+\delta) \; R'(\tau) \; \phi'(\tau,t+\delta) \qquad \text{b.}$$

where $\delta > 0$. Equation (25) shows that $\phi'(t+\delta,t)$ depends upon $\delta_k(t_1,t_2)$ only for $t_1,t_2 \; \varepsilon (t,t+\delta]$, while $\phi'(\tau,t+\delta)$ depends upon $\delta_k(t_1,t_2)$ only for $t_1,t_2 \; \varepsilon [\tau,t+\delta)$. Since these intervals are disjoint and the increments of $\delta_k(0,t)$ are independent, the ensemble average of (37.a) becomes:

$$\overline{\psi}(t,\tau) = E[\phi'^{H}(t+\delta t) \; \overline{\psi}(t+\delta,\tau) \; \phi'(t+\delta,t)] \tag{39}$$

where

$$\overline{\psi}(t,\tau) \overset{\Delta}{=} E[\psi(t,\tau)]$$

Now examine (25). Keeping in mind that 1) the ϕ_o's appearing in the integrals of (25.c) are each dependent on increments of the $\delta_k(0,t)$ over mutually disjoint intervals, 2) that partial sums of (25.a) are almost everywhere convergent, and 3) the $\delta_k(0,t)$ are Gaussian with variances (33), it is seen that the contribution of $\sum_{k=2}^{\infty} \phi_k'(t+\delta,t)$ to $\phi'(t+\delta,t)$ produces terms of order δ^2 on the right side of (39).

Equation (39) may thus be written:

$$\overline{\psi}(t,\tau) = E[(\phi_o'(t+\delta,t) - \int_t^{t+\delta} d\tau_1 \phi_o'(t+\delta,\tau_1) F'(\tau_1) \phi_o'(\tau_1,t))^H$$

$$\times \ \overline{\psi}(t+\delta,\tau)(\phi_o'(t+\delta,t) - \int_t^{t+\delta} d\tau_1 \phi_o'(t+\delta,\tau_1) F'(\tau_1) \phi_o'(\tau_1,t))]$$

$$+ O(\delta^2)$$

Similarly, using the expression (25.b) for the ϕ_o's appearing above, we have:

$$\overline{\psi}(t,\tau) = E[\Lambda^H \overline{\psi}(t+\delta,\tau) \Lambda] + O(\delta^2)$$

$$\Lambda \triangleq I + \overline{A}'\delta + W'(t,t+\delta) + \tfrac{1}{2}W'^2(t,t+\delta) - \int_t^{t+\delta} d\tau_1 F'(\tau_1)$$

After expanding out, rearranging and dividing by δ:

$$- \frac{1}{\delta} \left[\overline{\psi}(t+\delta,\tau) - \overline{\psi}(t,\tau) \right]$$

$$= \left(\overline{A}' - F'(t) + \frac{1}{2\delta} E[W'^2(t,t+\delta)] \right)^H \overline{\psi}(t+\delta,\tau)$$

$$+ \overline{\psi}(t+\delta,\tau) \left(\overline{A}' - F'(t) + \frac{1}{2\delta} E[W'^2(t,t+\delta)] \right)$$

$$+ \frac{1}{\delta} E[W'^H(t,t+\delta) \ \overline{\psi}(t+\delta,\tau) \ W'(t,t+\delta)] + O(\delta)$$

Next use Theorem 3 to evaluate the above averages, then pass to the limit $\delta \downarrow 0$. Recalling that $\frac{\partial}{\partial t} \overline{\psi}(t,\tau)$ exists by Theorem 1.D, we obtain:

$$- \frac{\partial}{\partial t} \overline{\psi}(t,\tau) = \left(\overline{A}' - F'(t) - \tfrac{1}{2} I' \right)^H \overline{\psi}(t,\tau)$$

(40.a)

$$+ \overline{\psi}(t,\tau) \left(\overline{A}' - F'(t) - \tfrac{1}{2} I' \right) + I' \{ \overline{\psi}(t,\tau) \}$$

with I given as in (36). Also, (27.b) yields directly:

$$\overline{\psi}(\tau,\tau) = R'(\tau)$$

(40.b)

Finally, integration of all terms in (40) over $\tau \in [t,t_1]$ and use of (26) gives (35).

The linearity of this equation guarantees the uniqueness of the solution, and the positive-semidefiniteness of $P'(t)$ noted in Theorem 1.C implies the same property for $\overline{F}'(t)$. $\quad \Box$

Under the maximum entropy statistical model we thus obtain
a modified Lyapunov equation for $\bar{P}'$ which must be appended to
the variational problem as a constraint. Clearly as the decor-
relation times approach infinity, the matrix I' approaches zero
and (35) reduces to the familiar Lyapunov equation for a deter-
ministic plant. Note that (35) could have been shown by pro-
ceeding directly from the Itô differential, (28). However, the
method of the above proof illustrates the simplicity and unity
afforded by the formalism of Theorem 1.

2.3 Stochastic Stability and the Steady State Case

Because of the relative ease with which constant gain con-
trols may be implemented, we hence forth consider only the steady
state case and suppose that κ, f and α are time-independent. As
a preliminary step, we first consider stochastic stability and
introduce "equivalent coefficient matrices" in the sense of
Kleinman [14]. In the following, the equivalent coefficient
matrix of $\bar{P}'$ in (35), for example, will be denoted by:

$$\Delta_{\bar{P}'}[(\bar{A}' - F' - \tfrac{1}{2}I')^H \bar{P}' + \bar{P}'(\bar{A}' - F' - \tfrac{1}{2}I') + I'\{\bar{P}'\}]$$

or more simply by $\Delta_{\bar{P}'}$ whenever clarity permits. By definition,
the limit of $\bar{P}'$ as t_1 approaches infinity exists if and only if
$\Delta_{\bar{P}'}$ is exponentially stable. When this is so, we term κ, f and
α __admissible__ controls. Note that the equivalent coefficient
matrix, $\Delta_{\bar{Q}}$ of the covariance of X (which is adjoint to $\bar{P}'$) is
simply $\Delta_{\bar{P}'}^H$. Thus admissible controls imply second mean
stability (for a discussion of this and related concepts of
stochastic stability see [15]).

More precisely, we have the result (see Ref. [16] or [17]).

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.

Lemma 1

Consider:

$$-\dot{\Lambda} = A^H \Lambda + \Lambda A + I'\{\Lambda\} + S \quad , \quad t \, \varepsilon [t_o, t_1]$$

$$\Lambda(t_1) = \Lambda_1 \tag{41}$$

with $S \geq 0$ ($S \leq 0$). If $\Delta_\Lambda [A^H \Lambda + \Lambda A + I'\{\Lambda\}]$ is asymptotically stable,

$$\lim_{t_1 \uparrow \infty} \Lambda(t) = \Lambda_\infty$$

where Λ_∞ exists as the unique positive semi-definite (negative semi-definite) solution to:

$$0 = A^H \Lambda_\infty + \Lambda_\infty A + I'\{\Lambda_\infty\} + S \tag{42}$$

In consequence, under admissible controls, the steady state performance:

$$\overline{J}_s \triangleq \lim_{|t_1 - t_o| \uparrow \infty} \overline{J}/|t_1 - t_o|$$

exists and assumes the form:

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$$\overline{J}_s = \text{tr}[\overline{P}'V']$$ (43)

where $\overline{P}'$ is a positive semi-definite solution of:

$$0 = [\overline{A}' - F' - \tfrac{1}{2}I']^H \overline{P}' + \overline{P}'[\overline{A}' - F' - \tfrac{1}{2}I']$$
$$+ R' + I'\{\overline{P}'\}$$ (44)

We now consider (43) and (44). Preparatory to determination of the optimal control gains we note the following results:

Lemma 2

Let $\delta[\dots]$ denote the first variation of $[\dots]$ consequent upon variations in κ, f and α subject to the admissibility condition and (44). Then the stationary condition:

$$\delta[\overline{J}_s] = 0$$ (45)

determines a minimum of $\overline{J}_s$.

Proof

The condition for extremalization of $\overline{J}_s$ requires that $\delta[\overline{P}']$ vanish. Consequently, the second variation of $\overline{P}'$ may be computed from (44) as:

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$$0 = [\bar{A}_O' - F_O' - \tfrac{1}{2}I']^H \delta^2[\bar{P}'] + \delta^2[\bar{P}'][\bar{A}_O' - F_O' - \tfrac{1}{2}I']$$

$$+ I'\{\delta^2[\bar{P}']\} + \begin{bmatrix} 0 & 0 \\ 0 & \delta[\kappa]^H R_2 \delta[\kappa] \end{bmatrix}$$

where $\bar{A}_O'$ and F_O' denote $\bar{A}'$ and F' evaluated with α, κ and f as determined in accordance with (45). Since $\delta[\kappa]^H R_2 \delta[\kappa] \geq 0$ and $\Delta_\delta^2[\bar{P}']$ is asymptotically stable, $\delta^2[\bar{P}']$ is positive semi-definite by Lemma 1. Since $V' \geq 0$, the second variation of $\bar{J}_s$ is non-negative. □

In the following, we shall concentrate on the derivation of the stationary conditions, i.e., the conditions imposed on α, κ and f by the requirements:

$$\delta[\bar{J}_s] = 0 \qquad \qquad \text{a.}$$

$$\bar{J}_s \triangleq \text{tr}[\bar{P}'V'] \qquad \qquad \text{b.} \qquad \qquad (46)$$

$$0 = [\bar{A}' - F' - \tfrac{1}{2}I']^H \bar{P}' + \bar{P}'[\bar{A}' - F' - \tfrac{1}{2}I']$$

$$\qquad \qquad \text{c.}$$

$$+ R' + I'\{\bar{P}'\}$$

Subsequently, at least for the case of full order compensation, ($N_q = 2n$), the conditions under which $\Delta_{\bar{P}'}$ is asymptotically stable will be established.

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3. THE MEAN-SQUARE OPTIMAL FULL-ORDER COMPENSATOR

3.1 Derivation of the Stationary Conditions

Here and in the remainder of this report we consider the case of full-order dynamic compensation for which $N_q = 2n$ in (46). Leaving aside the question of admissibility for the moment, let us now derive the stationary condition (46.a).

To handle the constraint imposed by (46.c) most expeditiously, we introduce the hermitian multiplier matrix, $Q' \varepsilon C^{4n \times 4n}$ and form the Hamiltonian:

$$H \triangleq tr[\overline{P}'V' + ([\overline{A}' -F' -\tfrac{1}{2}I']^H \overline{P}' + \overline{P}'[\overline{A}' -F' -\tfrac{1}{2}I']$$

$$+ R' + I'\{\overline{P}'\})Q'] \tag{47}$$

Then (46.a) reduces to the requirement that the first variation of H consequent upon unrestricted variation in α, κ, f and $\overline{P}'$ vanish. Partitioning $\overline{P}'$ thus:

$$\overline{P}' \triangleq \begin{bmatrix} P_\xi & P_{\xi q} \\ P_{\xi q}^H & P_q \end{bmatrix} \tag{48}$$

with similar notation for Q', we obtain the following specific conditions:

$$\frac{\partial H}{\partial \overline{P}'} = [\overline{A}'-F'-\tfrac{1}{2}I']Q' + Q'[\overline{A}'-F'-\tfrac{1}{2}I']^H + I'\{Q'\} + V' \qquad \text{a.}$$
$$= 0$$

$$\frac{\partial H}{\partial \kappa} = 2[-\beta^H(P_\xi Q_{\xi q} + P_{\xi q}Q_q) + R_2\kappa Q_q] = 0 \qquad \text{b.}$$

$$\frac{\partial H}{\partial f} = 2](P_{\xi q}^H Q_\xi + P_q Q_{\xi q}^H)\gamma^H + P_q f v_2] = 0 \qquad \text{c.}$$

$$\frac{\partial H}{\partial \alpha} = 2[P_{\xi q}^H Q_{\xi q} + P_q Q_q] = 0 \qquad \text{d.}$$

$$(49)$$

From (49.a), Q' can now be identified as the covariance of the augmented state. Furthermore, it is easily checked that by virtue of (46.c) and (49.a,b,c), (49.d) is an identity. This manifests the well-known result that the stationary conditions furnish no determination of the compensator dynamic matrix, α. Indeed it is fortunate that we may choose α so as to simplify (46.c) and (49) very greatly.

In preparation for the main result, define the state transformation:

$$\begin{pmatrix} \xi \\ q \end{pmatrix} = T \begin{pmatrix} \tilde{\xi} \\ \tilde{q} \end{pmatrix}$$

$$T \triangleq \begin{bmatrix} I_{2n} & 0 \\ I_{2n} & -I_{2n} \end{bmatrix}$$

$$(50)$$

so that $\tilde{q}$ is the observation error if we view the compensator as a full-order observer. Noting that

$$\bar{P}' = T^T \tilde{P} T$$

$$Q' = T \tilde{Q} T^T$$

$$\left.\right\} \quad (51)$$

partition $\tilde{P}$, and $\tilde{Q}$ in accordance with (48):

$$\tilde{P} = \begin{bmatrix} \tilde{P}_\xi & \tilde{P}_{\xi q} \\ \tilde{P}^H_{\xi q} & \tilde{P}_q \end{bmatrix}$$

Finally, with the notation:

$$\bar{\mu}_m \triangleq \bar{\mu} - \tfrac{1}{2}I \quad (52)$$

we have:

Theorem 4

With the above definitions and the choice:

$$\alpha = \bar{\mu}_m - \beta\kappa - f\gamma \quad (53)$$

the specifications:

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$$\tilde{P}_{\xi q} = 0 \qquad \text{a.}$$

$$\tilde{Q}_q = \tilde{Q}_{\xi q}^H \qquad \text{b.}$$
$$\left.\right\} \quad (54)$$

and

$$\kappa = R_2^{-1} \beta^H \tilde{P}_\xi \qquad \text{a.}$$

$$f = \tilde{Q}_q \gamma^H v_2^{-1} \qquad \text{b.}$$
$$\left.\right\} \quad (55)$$

$$0 = (\bar{\mu}_m - \beta\kappa)^H \tilde{P}_\xi + \tilde{P}_\xi(\bar{\mu}_m - \beta\kappa) + r_1 + \kappa^H R_2 \kappa + I\{\tilde{P}_\xi + \tilde{P}_q\}$$

$$0 = (\bar{\mu}_m - \beta\kappa)\tilde{Q}_\xi + \tilde{Q}_\xi(\bar{\mu}_m - \beta\kappa)^H + \beta\kappa\tilde{Q}_q + \tilde{Q}_q\kappa^H \beta^H$$

$$+ I\{\tilde{Q}_\xi\} + v_1$$
$$\left.\right\} \quad (56)$$

$$0 = (\bar{\mu}_m - f\gamma)^H \tilde{P}_q + \tilde{P}_q(\bar{\mu}_m - f\gamma) + \kappa^H R_2 \kappa$$

$$0 = (\bar{\mu}_m - f\gamma)\tilde{Q}_q + \tilde{Q}_q(\bar{\mu}_m - f\gamma)^H + I\{\tilde{Q}_\xi\} + v_1 + fv_2f^H$$
$$\left.\right\} \quad (57)$$

identically satisfy the stationary conditions (46.c) and (49).
The proof is given in Appendix 1.

With the choice (53), system (17) assumes the form:

$$\dot{\xi} = \mu\xi - \beta\kappa q + \tilde{w}_1 \qquad \text{a.}$$

$$\dot{q} = \bar{\mu}_m q - \beta\kappa q + f\gamma(\xi - q) + f\tilde{w}_2 \qquad \text{b.}$$
$$\left.\right\} \quad (58)$$

so that the compensator structure is as shown in Figure 1. Furthermore, referring to (28), the mean value of ξ is seen to satisfy:

$$\dot{\overline{\xi}} \;=\; \overline{\mu}_m \, \overline{\xi} \, -\beta\kappa\overline{q}$$

Thus, it is clear from (58.b) that with α chosen as in (53), the compensator is a full-order observer of the mean state.

It is also evident that under (53), Eqs. (55), (56) and (57) comprise the only essential stationary conditions. Introducing further notation:

$$
\left.
\begin{aligned}
P &\triangleq \tilde{P}_\xi \\[1.5em]
\hat{P} &\triangleq \tilde{P}_q \\[1.5em]
Q &\triangleq \tilde{Q}_q \\[1.5em]
\hat{Q} &\triangleq \tilde{Q}_\xi - \tilde{Q}_q
\end{aligned}
\right\} \quad (59)
$$

$$
\left.
\begin{aligned}
\sigma &\triangleq \beta R_2^{-1} \beta^H \\[1.5em]
\overline{\sigma} &\triangleq \gamma^H v_2^{-1} \gamma
\end{aligned}
\right\} \quad (60)
$$

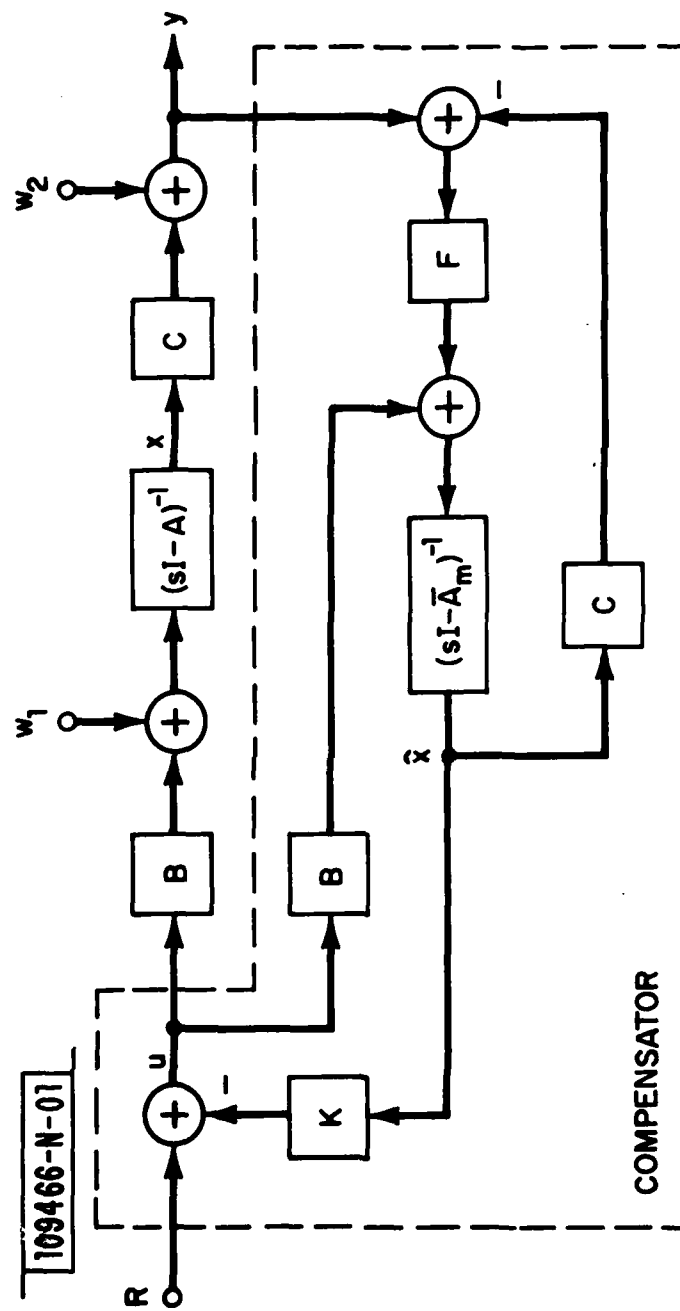slight rearrangement of (55) through (57) yields the following stationary conditions:

Fig. 1 Full-order compensator structure

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Corollary 1

With the full-order compensator form given in (58.b), the gains:

$$\kappa = R_2^{-1} \beta^H P$$

$$f = Q\gamma^H v_2^{-1}$$

(61)

where P, Q, $\hat{P}$, $\hat{Q} \geq 0$ satisfy:

$$0 = \bar{\mu}_m^H P + P \bar{\mu}_m + I\{P + \hat{P}\} + r_1 - P \sigma P \qquad \text{a.}$$

$$0 = \bar{\mu}_m Q + Q \bar{\mu}_m^H + I\{Q + \hat{Q}\} + v_1 - Q \bar{\sigma} Q \qquad \text{b.}$$

(62)

$$0 = (\bar{\mu}_m - Q \bar{\sigma})^H \hat{P} + \hat{P} (\bar{\mu}_m - Q \bar{\sigma}) + P \sigma P \qquad \text{a.}$$

$$0 = (\bar{\mu}_m - \sigma P) \hat{Q} + \hat{Q} (\bar{\mu}_m - \sigma P)^H + Q \bar{\sigma} Q \qquad \text{b.}$$

(63)

yield an extremum for the steady-state performance, $\bar{J}_s$.

The particularly simple form of (62) and (63) is to be noted. Observe that in the deterministic plant case ($I = 0$), (62) form the only restrictions and reduce to the familiar uncoupled regulator and observer Riccati equations. Also in the absence of the terms $I\{\hat{P}\}$ and $I\{\hat{Q}\}$, (62) are of the form of the stochastic Riccati equation arising from the regulator problem treated earlier. In general, however, (62.a) and (62.b)

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are coupled through $I\{\hat{P}\}$ and $I\{\hat{Q}\}$ so that the effects of modal frequency uncertainties preclude the separation principle. Also the coupling terms effectively augment the diagonal elements of r_1 and v_1, thereby demanding greater control authority and stabilization. Moreover, from (63), $\hat{P}$ and $\hat{Q}$ tend to increase with the controller input $\kappa^H R_2 \kappa$ and its dual $f^H v_2 f$.

We may say that $\hat{P}$ and $\hat{Q}$ represent error "leaking through" the regulator to the observer and vice-versa by virtue of parameter uncertainty.

To summarize, we consolidate the above results and simplify the requisite conditions for admissibility of the controls:

Theorem 5

Suppose that P, Q, $\hat{P}$, $\hat{Q}$ exist as positive semi-definite solutions to (62) and (63). Then with constant gains given by (61):

A. $(\overline{\mu}_m - f\gamma)$, $(\overline{\mu} - f\gamma)$, $(\overline{\mu}_m - \beta\kappa)$ and

$(\overline{\mu} - \beta\kappa)$ are asymptotically stable.

B.   The stochastic system (58) is second mean exponentially stable and almost surely exponentially stable, i.e., the control is admissible.

C.   Eq. (61) yield a minimum of the steady state performance, $\mathcal{J}_s$.

The proof is given in Appendix 2.

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At this point, it is well to recapitulate the above developments, indicating where the derivation is subject to significant generalization. First, we chose to treat the problem of mean-square optimal dynamic compensation of structural systems with uncertainties in the open loop frequencies within the dual formulation, involving the augmented co-state or expected cost matrix. This gives rise to the statement embodied in Eqs. (20) and (21) which is applicable to any statistical model of parameter uncertainties. Next we imposed the minimum information statistical model of frequency uncertainties to obtain the variational problem of Eq. (46) which involves a variational constraint imposed by the stochastic Lyapunov equation for the expected cost. Treatment of less restricted types of parameter uncertainty would proceed analogously to obtain an appropriately generalized stochastic Lyapunov equation.

Assuming admissibility of the controls, and specializing to the steady state, full-order case, we derived the optimality conditions associated with the variational problem of Eqs. (46). Choice of the compensator as a full-order observer for the mean state is found to permit great simplification of these conditions resulting in Eqs. (61) through (63). Clearly, the case of time-varying controls may be handled in a similar manner. Moreover, we may anticipate that assuming $q \in C^N q$, $N_q < 2n$ and proceeding as in this section, the optimality conditions for reduced-order dynamic compansation may be derived. Once again, judicious choice of the compensator dynamic matrix, α, will result in drastic simplification and yield appropriately modified versions of Eqs. (61), (62) and (63).

Finally, Theorem 5 shows that existence of positive semi-definite solutions to the optimality conditions, Eqs. (62) and (63), guarantees the minimum property and second mean and almost

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sure exponential stability for the closed loop system. Thus the
new approach to full-order compensation holds the promise of
ensuring robust stability under a design-conservative statisti-
cal model of parameter uncertainties. The issues of existence
and uniqueness are immediately addressed in the following section.

3.2 Existence and Uniqueness of Solutions to the Optimality Conditions

The results of this section represent an extension of the
earlier work on full state feedback regulation to the more gen-
eral problem posed by Eqs. (62) and (63). As in the classic
work of Wonham [17,18], or the developments of Merriam [16], the
basic technique is to establish the existence of bounded mono-
tone sequences of positive semi-definite hermitian maps. In
this connection the following result is essential [19]:

Lemma 3

Every sequence, $\{X_i\}$, of hermitian positive semi-definite
matrices bounded below (above, resp.) with $(X_{i+1} - X_i)$ negative
semi-definite (positive semi-definite, resp.) for each i con-
verges to a positive semi-definite limit.

In addition, we shall need a preliminary lemma on the
familiar Lyapunov equation [16,20].

Lemma 4

Consider:

$$A^H P + P A + S = 0 \tag{64}$$

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A.   If $S \geq 0$ (≤ 0, resp.) and A is stable, (64) has a
unique solution P, and $P \geq 0$ (≤ 0, resp.). If, in
addition $(S^{\frac{1}{2}}, A)$ is reconstructible, $P > 0$ (< 0, resp.).

B.   Suppose $P \geq 0$ (≤ 0, resp.), $S \geq 0$ (≤ 0, resp.) satisfy
(64) and $(S^{\frac{1}{2}}, A)$ is detectable. Then A is stable.

With regard to stochastic Lyapunov equations of the form:

$$0 = A^H P + P A + I\{P\} + S \qquad (65)$$

we can state [14]:

Lemma 5

If $\Delta_p[A^H P + P A + I\{P\}]$ is asymptotically stable and $S \geq 0$
(≤ 0, resp.) then the unique solution of (65) is positive (nega-
tive, resp.) semi-definite.

Finally, we must recapitulate earlier results for the
stochastic Riccati equation of the regulator problem [7]:

Lemma 6

Consider:

$$0 = A_m^H P + P A_m + I\{P\} + r - P \sigma P$$

$$A_m \overset{\Delta}{=} A - \frac{1}{2} I \qquad (66)$$

where $r \geq 0$ and I and σ are as defined by (36.b) and (60.a). Under the conditions:

$$(A, \sigma^{\frac{1}{2}}) \quad \text{stabilizable}$$

$$(r^{\frac{1}{2}}, A) \quad \text{detectable}$$

(66) possesses a unique positive semi-definite solution P, and $(A - \sigma P)$, $(A - \frac{1}{2}I - \sigma P)$ and $\Delta_P [A_m^H P + P A_m + I\{P\}]$ are asymptotically stable. Moreover, suppose that P_1 and P_2 are the solutions of (66) with $r = r_1$ and $r = r_2$, respectively. Then $r_1 \geq r_2$ implies $P_1 \geq P_2$.

The dualization of Lemmas 4, 5 and 6 is immediate and need not be considered explicitly.

To simplify this initial development, attention is henceforth restricted to the case:

$$(\overline{\mu}, \sigma^{\frac{1}{2}}) \quad , \quad (\overline{\mu}, v_1^{\frac{1}{2}}) \quad \text{controllable}$$

$$(r_1^{\frac{1}{2}}, \overline{\mu}) \quad , \quad (\overline{\sigma}^{\frac{1}{2}}, \overline{\mu}) \quad \text{reconstructible}$$

although, as subsequent numerical results suggest, it is likely that these conditions can be weakened considerably.

Within the above restrictions, consider the sequences P_i, Q_i, $\hat{P}_i$ and $\hat{Q}_i$ defined by:

$$0 = \bar{\mu}_m^H P_i + P_i \bar{\mu}_m + r_1 + I\{P_i\} - P_i \sigma P_i + I\{\hat{P}_i\} \qquad a.$$

$$0 = \bar{\mu}_m Q_i + Q_i \bar{\mu}_m^H + v_1 + I\{Q_i\} - Q_i \bar{\sigma} Q_i + I\{\hat{Q}_i\} \qquad b. \qquad (67)$$

$$0 = \hat{P}_{i+1}(\bar{\mu}_m - Q_i\bar{\sigma}) + (\bar{\mu}_m - Q_i\bar{\sigma})^H \hat{P}_{i+1} + P_i \sigma P_i \qquad a.$$

$$0 = \hat{Q}_{i+1}(\bar{\mu}_m - \sigma P_i)^H + (\bar{\mu}_m - \sigma P_i) \hat{Q}_{i+1} + Q_i \bar{\sigma} Q_i \qquad b. \qquad (68)$$

for all $i \geq 0$. The following results on the boundedness of these sequences is essential.

Theorem 6

Under the conditions stated above, there exist bounded, positive definite P^u, P^ℓ, $\hat{P}^u$, $\hat{P}^\ell$, Q^u, Q^ℓ, $\hat{Q}^u$, $\hat{Q}^\ell$ with

$$P^u \geq P^\ell, \ \hat{P}^u \geq \hat{P}^\ell, \ Q^u \geq Q^\ell, \ \hat{Q}^u \geq \hat{Q}^\ell$$

such that:

$$\hat{P}_o = \hat{P}^\ell \ , \ \hat{Q}_o = \hat{Q}^u \qquad (69)$$

in conjunction with (67) and (68) implies:

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$$P^{\ell} < P_i \le P^u, \quad Q^{\ell} < Q_i \le Q^u$$

$$(\overline{\mu}_m - \sigma P_i) \; , \; (\overline{\mu}_m - Q_i \, \overline{\sigma}) \quad \text{stable}$$

and for i and

$$\hat{P}^{\ell} < \hat{P}_i \le \hat{P}^u, \quad \hat{Q}^{\ell} < \hat{Q}_i \le \hat{Q}^u$$

for $i \ge 1$.

The proof is contained in Appendix 3.

This prepares the way for the main conclusion:

Theorem 7

With $\overline{\mu}_m$, I, r_1, v_1 σ and $\overline{\sigma}$ as defined previously and the conditions:

$$(\overline{\mu}, \; \sigma^{\frac{1}{2}}) \quad , \quad (\overline{\mu}, \; v_1^{\frac{1}{2}}) \quad \text{controllable}$$

$$(r_1^{\frac{1}{2}}, \; \overline{\mu}) \quad , \quad (\overline{\sigma}^{\frac{1}{2}}, \; \overline{\mu}) \quad \text{reconstructible}$$

Eqs. (62) and (63) possess unique positive definite solutions for P, $\hat{P}$, Q and $\hat{Q}$. Moreover, $(\overline{\mu}_m - \sigma P)$ and $(\overline{\mu}_m - Q\overline{\sigma})$ are asymptotically stable.

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Proof

First, let us stipulate that $\hat{P}_o$ and $\hat{Q}_o$ be chosen in accordance with (69) and Theorem 6. Then, for all $i \geq 0$, P_i, $\hat{P}_i$, Q_i and $\hat{Q}_i$ as defined by (67) and (68) are positive definite and $(\overline{\mu}_m - \sigma P_i)$ and $(\overline{\mu}_m - Q_i \overline{\sigma})$ are asymptotically stable.

Defining:

$$z_{Pi} \triangleq \hat{P}_{i+1} - \hat{P}_i$$

$$z_{Qi} \triangleq \hat{Q}_{i+1} - \hat{Q}_i \tag{70}$$

and

$$\Sigma_{Pi} \triangleq P_{i+1} - P_i$$

$$\Sigma_{Qi} \triangleq Q_{i+1} - Q_i \tag{71}$$

manipulation of (67) and (68) yields

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$$0 = Z_{P_i}(\overline{\mu}_m - Q_i\,\overline{\sigma}) + (\overline{\mu}_m - Q_i\,\overline{\sigma})^H Z_{P_i} + \Sigma_{P_{i-1}}\sigma\,\Sigma_{P_{i-1}}$$
$$- [\hat{P}_i\Sigma_{Q_{i-1}}\overline{\sigma} + \overline{\sigma}\,\Sigma_{Q_{i-1}}\hat{P}_i] + P_{i-1}\sigma\,\Sigma_{P_{i-1}} + \Sigma_{P_{i-1}}\sigma\,P_{i-1} \tag{72.a}$$

$$0 = Z_{Q_i}(\overline{\mu}_m - \sigma P_i)^H + (\overline{\mu}_m - \sigma P_i)\,Z_{Q_i} - \Sigma_{Q_{i-1}}\overline{\sigma}\,\Sigma_{Q_{i-1}}$$
$$- [\hat{Q}_i\Sigma_{P_{i-1}}\sigma + \sigma\,\Sigma_{P_{i-1}}\hat{Q}_i] + \Sigma_{Q_{i-1}}\overline{\sigma}\,Q_i + Q_i\overline{\sigma}\,\Sigma_{Q_{i-1}} \tag{72.b}$$

and

$$0 = (\overline{\mu}_m - \sigma P_i)^H \Sigma_{P_i} + \Sigma_{P_i}(\overline{\mu}_m - \sigma P_i)$$
$$+ I\{\Sigma_{P_i} + Z_{P_i}\} - \Sigma_{P_i}\sigma\,\Sigma_{P_i} \qquad \text{a.}$$

$$0 = (\overline{\mu}_m - Q_i\,\overline{\sigma})\,\Sigma_{Q_i} + \Sigma_{Q_i}(\overline{\mu}_m - Q_i\,\overline{\sigma})^H$$
$$+ I\{\Sigma_{Q_i} + Z_{Q_i}\} - \Sigma_{Q_i}\overline{\sigma}\,\Sigma_{Q_i} \qquad \text{b.}$$

$$\tag{73}$$

Now suppose $Z_{P_{i-1}} \geq 0$ and $Z_{Q_{i-1}} \leq 0$. Considering (73.a), it is clear that since $\overline{\mu}_m - \sigma P_i$ is stable, $(\overline{\mu}_m - \sigma P_{i-1}, \sigma^{\frac{1}{2}})$ and $((I\{Z_{P_{i-1}}\})^{\frac{1}{2}}, \overline{\mu}_m - \sigma P_{i-1})$ are stabilizable and detectable, respectively. Then, by Lemma 6, $\Sigma_{P_{i-1}}$ exists as the unique positive semi-definite solution of (73.a). Similarly, from (73.b), Lemma 6, and the assumption $Z_{Q_{i-1}} \leq 0$; $\Sigma_{Q_{i-1}}$ is negative semi-definite.

Since $-Q_i \leq \Sigma_{Q_{i-1}} \leq 0$ and $\bar{\sigma} \geq 0$, use of (68.a) and Lemma 4 shows that $[\hat{P}_i \Sigma_{Q_{i-1}} \bar{\sigma} + \bar{\sigma} \Sigma_{Q_{i-1}} \hat{P}_i]$ is negative semi-definite. Similarly $[P_{i-1}\sigma \Sigma_{P_{i-1}} + \Sigma_{P_{i-1}} \sigma P_{i-1}] \geq 0$ so that the second line of (72.a) is positive semi-definite. Consequently, since $(\bar{\mu}_m - Q_i\bar{\sigma})$ is stable, Z_{P_i} is the unique positive semi-definite solution of (72.a) by virtue of Lemma 4.A. Analogous reasoning on (72.b) shows that $Z_{Q_i} \leq 0$.

Thus, on the assumption that $Z_{P_{i-1}} \geq 0$ and $Z_{Q_{i-1}} \leq 0$, we have shown that $\Sigma_{P_{i-1}} \geq 0$, $\Sigma_{Q_{i-1}} \leq 0$, $Z_{P_{i-1}} \geq 0$ and $Z_{Q_{i-1}} \leq 0$. Induction on i shows:

$$\Sigma_{P_i} \quad , \quad Z_{P_i} \geq 0$$

$$\Sigma_{Q_i} \quad , \quad Z_{Q_i} \leq 0$$

for all i provided that $\hat{P}_1 - \hat{P}_o \geq 0$ and $\hat{Q}_i - \hat{Q}_o \leq 0$. Under the choice (69) this must be so, for by Theorem 6, $\hat{P}_1 > \hat{P}^\ell = \hat{P}_o$ and $\hat{Q}_1 \leq \hat{Q}^u = \hat{Q}_o$.

Thus (67), (68) and (69) define P_i and $\hat{P}_i$ as positive definite nondecreasing sequences and Q_i and $\hat{Q}_i$ as positive definite non-increasing sequences. Moreover, by Theorem 6, these sequences are bounded both from above and from below. Therefore Lemma 3 implies that the sequences P_i, $\hat{P}_i$, Q_i and $\hat{Q}_i$ possess positive definite limits, which by virtue of (67) and (68) satisfy (62) and (63).

Furthermore, $(\overline{\mu}_m - \sigma P_i)$ and $(\overline{\mu}_m - Q_i \overline{\sigma})$ are asymptotically stable so that $\lim_{i \uparrow \infty} (\overline{\mu}_m - \sigma P_i)$ and $\lim_{i \uparrow \infty} (\overline{\mu}_m - Q_i \overline{\sigma})$ are also stable. Finally, since $\lim_{i \uparrow \infty} \hat{P}_i$ and $\lim_{i \uparrow \infty} \hat{Q}_i$ exist the results of Lemma 6 immediately apply to (67). Thus buth $\overline{\mu} - \sigma P$ and $\overline{\mu} - Q \overline{\sigma}$ are asymptotically stable.

To establish uniqueness, suppose that $(P_1, \hat{P}_1, Q_1, \hat{Q}_1)$ and $(P_2, \hat{P}_2, Q_2, \hat{Q}_2)$ are two sets of positive definite solutions of (62) and (63). With $\Sigma_1 \stackrel{\Delta}{=} P_1 - P_2$ and $Z_P \stackrel{\Delta}{=} \hat{P}_1 - \hat{P}_2$, (62.a) may be manipulated to yield:

$$0 = (\overline{\mu}_m - \sigma P_2)^H \Sigma_P + \Sigma_P (\overline{\mu}_m - \sigma P_2) + I\{\Sigma_P\}$$
$$+ I\{Z_P\} - \Sigma_P \sigma \Sigma_P \qquad (74.a)$$

and

$$0 = (\overline{\mu}_m - \sigma P_1)^H \Sigma_P + \Sigma_P (\overline{\mu}_m - \sigma P_1) + I\{\Sigma_P\}$$
$$+ I\{Z_P\} + \Sigma_P \sigma \Sigma_P \qquad (74.b)$$

Employing the same reasoning as used in the Appendix for deriving bounds on $\hat{P}$, it is readily seen that $I\{Z_P\} < \Sigma_P \sigma \Sigma_P$. Thus, since Δ_{Σ_P} in (74.a) is asymptotically stable, Σ_P is negative semi-definite by (74.a) and Lemma 5. Likewise, (74.b) implies $\Sigma_P \geq 0$. Since Σ_P is both positive and negative semi-definite, we conclude that $\Sigma_P = 0$. Similarly, $Q_1 = Q_2$. Finally, $(\hat{P}_1 - \hat{P}_2)$ and $(\hat{Q}_1 - \hat{Q}_2)$ are found to satisfy the homogeneous forms of (63), whence $\hat{P}_1 = \hat{P}_2$ and $\hat{Q}_1 = \hat{Q}_2$. $\square$

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As is clear from the above proof, (67), (68) and (69) give
a computational procedure for the numerical solution of (62) and
(63). When $\hat{P}^{\ell}$ and $\hat{Q}^{u}$ are evaluated in accordance with Eqs.
(A.3.1) through (A.3.4) of the Appendix, monotone convergence is
guaranteed (with P_i, $\hat{P}_i$ non-decreasing and Q_i, $\hat{Q}_i$ non-increasing).
It is interesting to note that if the conditions:

$$\hat{P}_O = \hat{P}^u \quad , \quad \hat{Q}_O = \hat{Q}^{\ell}$$

are taken in place of (69), slight modification of the above
proof again shows convergence of (67) and (68), but with P_i, $\hat{P}_i$
non-increasing and Q_i, $\hat{Q}_i$ non-decreasing.

Theorem 5 is now directly applicable. Thus it is possible
to summarize definite conclusions with regard to stochastic
stability as follows:

Corollary 2

Under the conditions of Theorem 7, let P, Q, $\hat{P}$ and $\hat{Q}$ be the
unique positive definite solutions of (62) and (63). Then with
the constant gains given by (61):

A. $(\overline{\mu}_m - \beta\kappa)$, $(\overline{\mu} - \beta\kappa)$, $(\overline{\mu}_m - f\gamma)$ and $(\overline{\mu} - f\gamma)$ are asymptotically stable.

B. The stochastic system (58) is second mean exponentially stable and almost surely exponentially stable.

C. ــ ١. (61) minimizes the steady state performance index defined by (43).

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Thus, under the minimum information model of modal frequency uncertainties, the compensator design approach embodied in Eqs. (61) through (63) minimizes a measure of performance defined over the entire parameter ensemble and guarantees closed loop stochastic stability.

3.3 Asymptotic Properties for Large Uncertainties

The previous section established the existence and uniqueness of solutions to (62) and (63) under the controlability and reconstructability conditions of Theorem 7 and for all positive I. It is thus natural to inquire what behavior P and Q attain for large uncertainties, i.e., for very small decorrelation times. This case represents the situation in which very great *a priori* uncertainty exists regarding the values of all modal frequencies.

In connection with the stochastic Riccati equation, Theorem 16 of Ref. [17] immediately yields the following:

Lemma 7

Assume the conditions of Theorem 7. Let P and Q be the unique positive definite solutions of (62) given any positive semi-definite $\hat{P}$ and $\hat{Q}$. Introduce a positive scaling parameter, J, into I:

$$I = J \tilde{I} \quad ; \quad J > 0 \ , \ \tilde{I} > 0 \tag{75}$$

Then:

$$\lim_{J\uparrow\infty} P = P*$$

$$\lim_{J\uparrow\infty} Q = Q* \tag{76}$$

where the diagonal matrices $P*$ and $Q*$ are uniquely determined by:

$$0 = \overline{\mu}^H P* + P* \overline{\mu} + I\{\hat{P}\} + \{r_1\} - \{P*\}^2 \{\sigma\}$$

$$0 = \overline{\mu} Q* + Q* \overline{\mu}^H + I\{\hat{Q}\} + \{v_1\} - \{Q*\}^2 \{\overline{\sigma}\} \tag{77}$$

This result readily leads to the main conclusion

Theorem 8

Let P and Q be the unique positive definite solutions of (62) and (63) under the conditions of Theorem 7 and define I as in (75). Then:

$$\lim_{J\uparrow\infty} P = P* \triangleq (-2 \operatorname{Re}\overline{\mu})^{-1} \{r_1\}$$

$$\lim_{J\uparrow\infty} Q = Q* \triangleq (-2 \operatorname{Re}\overline{\mu})^{-1} \{v_1\} \tag{78}$$

Proof

By Theorem 7, $\hat{P}$ and $\hat{Q}$ exist as positive definite solutions of (62) and (63) for all $I \geq 0$ and therefore possess bounded limits $\hat{P}*$ and $\hat{Q}*$, respectively, as J increases without bound. Then (77) yields:

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$$0 = \overline{\mu}^H P* + P* \overline{\mu} + \mathcal{I}\{P*\} + \{r_1\} - P*^2 \{\sigma\}$$

$$0 = \overline{\mu} Q* + Q* \overline{\mu}^H + \mathcal{I}\{Q*\} + \{v_1\} - Q*^2 \{\overline{\sigma}\}$$

(79)

Furthermore, since $\mathcal{I} > 0$, $((P \overline{\sigma} P)^{\frac{1}{2}}, \mathcal{I})$ is reconstructible and $(\mathcal{I}, (Q \overline{\sigma} Q)^{\frac{1}{2}})$ is controllable for all $P, Q > 0$. Then (63.a) gives:

$$\{\hat{P}*\} \triangleq \lim_{J \uparrow \infty} \{\hat{P}\} = \mathcal{I}^{-1} \lim_{J \uparrow \infty} \{P \sigma P\}$$

$$= \mathcal{I}^{-1} \{P*\}^2 \{\sigma\}$$

$(80.a)$

Similarly, from (63.b):

$$\{\hat{Q}*\} \triangleq \lim_{J \uparrow \infty} \{\hat{Q}\} = \mathcal{I}^{-1} \{Q*\}^2 \{\overline{\sigma}\}$$

$(80.b)$

and (78) follows by substitution of (80) into (79). $\square$

Thus, as modal frequency uncertainties increase without bound, the optimality conditions possess very simple asymptotic solutions given by closed analytical expressions. Referring to Theorem 16 of Ref. [7] it must be noted that for the full state feedback regulation problem, diagonalization of the expected cost matrix gives rise to a rate feedback control law which is inherently stable for all values of modal frequencies. Thus

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$$\kappa* \overset{\Delta}{=} \lim_{J\uparrow\infty} \kappa = R_2^{-1}\beta^H P*$$

implies that only the estimated modal valocities are used as control input.

In this case, it is also of interest to investigate more particularly the asymptotic form of the control law for large I. For this purpose, suppose that $\tilde{\mu}$, $\tilde{\beta}$, and $\tilde{\gamma}$ are the modal parameters for a particular realization of the structure as distinct from the nominal or design values, $\overline{\mu}$, β and γ. Then the system equations for the <u>actual</u> structure in connection with the <u>designed</u> compensator are:

$$\dot{\xi} = \tilde{\mu}\,\xi - \tilde{\beta}\,\kappa\,q \qquad\qquad \text{a.} \left.\begin{array}{c} \\ \\ \end{array}\right\} (81)$$

$$\dot{q} = (\overline{\mu}_m - \beta\kappa - f\gamma)\,q + f\,\tilde{\gamma}\,\xi \qquad\qquad \text{b.}$$

Now as $I > 0$ increases, the compensator poles recede into the left half plane owing to the term $-\tfrac{1}{2}I$ in $\overline{\mu}_m$. Thus, for large I we may consider a singular perturbation expansion for q as a functional of ξ. In the first approximation - i.e., to first order in I^{-1}:

$$q \sim 2I^{-1}\,f*\,\tilde{\gamma}\,\xi + O(I^{-2}) \qquad\qquad (82)$$

where $f* \overset{\Delta}{=} \lim_{J\uparrow\infty} f$. Then, to first order, in singular perturbations, the controlled system is given by:

$$\dot{\xi} = \tilde{\mu} \xi - 2 \tilde{\sigma}(P^* \ I^{-1} \ Q^*) \ \tilde{\tilde{\sigma}} \ \xi$$

$$\tilde{\sigma} \triangleq \tilde{\beta} \ R_2^{-1} \beta^H \tag{83}$$

$$\tilde{\tilde{\sigma}} \triangleq \gamma^H \ v_2^{-1} \ \tilde{\gamma}$$

from (82) and (81.a).

To zeroth order, (83) shows that large uncertainties tend to remove the control altogether. However, to first order we have the following property:

Lemma 8

Consider (83) with P* and Q* as given by (78). If $\tilde{\gamma} = \tilde{\beta}^H$, $\gamma = \beta^H$, and $v_2 = vR_2$ (where v is a positive scalar) then (83) is asymptotically stable for all $\tilde{\mu}$ with $\mathrm{Re}\tilde{\mu} < 0$ and all $\tilde{\beta}$, and the control is a rate output feedback law.

Proof

Let $\mathcal{L} \triangleq \xi^H \xi$. Then, from (83):

$$\dot{\mathcal{L}} = 2\xi^H \ [\mathrm{Re}\xi - \Gamma] \ \xi$$

$$\Gamma \triangleq \tilde{\sigma}(P^* \ I^{-1}Q^*) \ \tilde{\tilde{\sigma}} + \tilde{\tilde{\sigma}} \ (P^* \ I^{-1}Q^*) \ \tilde{\sigma} \tag{84}$$

In the case $\tilde{\gamma} = \tilde{\beta}^H$, $\gamma = \beta^H$ and $v_2 = vR_2$ we have $\tilde{\tilde{\sigma}} = v\tilde{\sigma}^H$ so that Γ is hermitian. Further, since $P^* \ I^{-1}Q^*$ is positive and diagonal, $\Gamma \geq 0$. Then, with $\mathrm{Re}\mu < 0$, $\dot{\mathcal{L}} < 0$ for all $\xi \neq 0$. Thus, $\mathcal{L}$ is a Lyapunov function. That this case represents a rate

output feedback law is easily verified by transforming (83) back to the modal coordinate basis in accordance with relations (7) and (87). □

In other words, for large modal frequency uncertainties and co-located rate sensors ($\tilde{\gamma} = \tilde{\beta}^H$, $\gamma = \beta^H$), the asymptotic control is a rate feedback law proposed by Balas [4] which is stable in the presence of errors in all modal parameters.

Analogous results are to be expected when uncertainties in low frequency modes are small while modelling accuracy degenerates for modes of increasing order. In this instance we anticipate that the control designed according to Eqs. (61) through (63) will necessarily approach the asymptotic form for the high frequency, poorly known modes, while resembling the deterministic plant compensator design for the low order modes.

4. FULL-ORDER COMPENSATION: NUMERICAL PROCEDURES & RESULTS

4.1 Computational Methods

For computational purposes it is advantageous to work with (61) through (63) in the original modal coordinate basis rather than in the complex form given above. Letting:

$$\tilde{P} \triangleq \phi^{-1}H_P\phi^{-1} \quad , \quad \tilde{\hat{P}} \triangleq \phi^{-1}H_{\hat{P}}\phi^{-1}$$

$$\tilde{Q} \triangleq \phi\, Q\, \phi^H \quad , \quad \tilde{\hat{Q}} \triangleq \phi\, \hat{Q}\, \phi^H$$

$$\left.\right\} \quad (85)$$

and

$$\tilde{q} \triangleq \phi^{-1}q$$

$$K = \kappa\phi^{-1}$$

$$F = \phi\, f$$

$$\left.\right\} \quad (86)$$

system (17) assumes the form

$$\dot{x} = (\bar{A} + a(t))x - BKq + w_1$$

$$\tilde{q} = (\bar{A} - \tfrac{1}{2}I)\tilde{q} - BK\tilde{q} + F\, C(x - \tilde{q}) + F\, \tilde{w}_2$$

$$\left.\right\} \quad (87)$$

Using (5) through (8), (14.c) and (85), relations (61) through (63) become (we shall suppress the tildes of $\tilde{P}$, $\tilde{Q}$, etc. in (85)):

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$$K = R_2^{-1} B^T P$$

$$F = Q C^T V_2^{-1}$$

$$\left.\right\} \quad (88)$$

$$0 = \overline{A}_m^T P + P \overline{A}_m + D_P[I, P + \hat{P}] + R_1 - P \Sigma P$$

$$0 = \overline{A}_m Q + Q \overline{A}_m^T + D_Q[I, Q + \hat{Q}] + V_1 - Q \overline{\Sigma} Q$$

$$\left.\right\} \quad (89)$$

$$0 = (\overline{A}_m - Q \overline{\Sigma})^T \hat{P} + \hat{P}(\overline{A}_m - Q \overline{\Sigma}) + P \Sigma P$$

$$0 = (\overline{A}_m - \Sigma P) \hat{Q} + \hat{Q}(\overline{A}_m - \Sigma P) + Q \overline{\Sigma} Q$$

$$\left.\right\} \quad (90)$$

where:

$$\overline{A}_m \triangleq \overline{A} - \tfrac{1}{2}I$$

$$\Sigma \triangleq B R_2^{-1} B^T$$

$$\overline{\Sigma} = C^T V_2^{-1} C$$

$$\left.\right\} \quad (91)$$

and where for any square matrix, M:

$$D_P[I,M] \triangleq \underset{k=1,\ldots,n}{\text{block-diag}} \left[\frac{I_k}{2\bar{\omega}_k}[M_{2k-1,2k-1} + \bar{\omega}_k^2 M_{2k,2k}] \begin{bmatrix} \bar{\omega}_k^2 & 0 \\ 0 & 1 \end{bmatrix} \right]$$

$$D_Q[I,M] \triangleq \underset{k=1,\ldots,n}{\text{block-diag}} \left[\frac{I_k}{2\bar{\omega}_k}[\bar{\omega}_k^2 M_{2k-1,2k-1} + M_{2k,2k}] \begin{bmatrix} 1 & 0 \\ 0 & \bar{\omega}_k^2 \end{bmatrix} \right] \quad \left.\right\} \quad (92)$$

We also note, for future reference, that the asymptotic solution, (78), for large uncertainty levels becomes:

$$P^* = \underset{k=1,\ldots,n}{\text{block-diag}} \left[\frac{1}{4\eta_k\bar{\omega}_k^3}[R_{1_{2k-1,2k-1}} + \bar{\omega}_k^2 R_{1_{2k,2k}}] \begin{bmatrix} \bar{\omega}_k^2 & 0 \\ 0 & 1 \end{bmatrix} \right]$$

$$Q^* = \underset{k=1,\ldots,n}{\text{block-diag}} \left[\frac{1}{4\eta_k\bar{\omega}_k^3}[\bar{\omega}_k^2 V_{1_{2k-1,2k-1}} + V_{1_{2k,2k}}] \begin{bmatrix} 1 & 0 \\ 0 & \bar{\omega}_k^2 \end{bmatrix} \right] \quad \left.\right\} \quad (93)$$

in the modal coordinate basis.

With the above expressions, the discussion of computational techniques may proceed. First, note that the proof of Theorem 7 gives rise to a convergent sequence of approximations to P, Q, $\hat{P}$, $\hat{Q}$ and yields the following algorithm:

Theorem 9

Under the conditions of Theorem 7 and denoting the positive definite solutions of (89) and (90) by P, $\hat{P}$, Q and $\hat{Q}$:

$$\lim_{k \uparrow \infty} (P_k, \hat{P}_k, Q_k, \hat{Q}_k) = (P, \hat{P}, Q, \hat{Q}) \tag{94}$$

where the sequences $\{P_k\}$, $\{\hat{P}_k\}$, $\{Q_k\}$ and $\{\hat{Q}_k\}$ are defined (for $k = 0,\ldots,\infty$) by:

$$0 = \overline{A}_m^T P_k + P_k \overline{A}_m + R_1 + D_P[I, P_k + \hat{P}_k] - P_k \Sigma P_k$$

$$0 = \overline{A}_m Q_k + Q_k \overline{A}_m^T + V_1 + D_Q[I, Q_k + \hat{Q}_k] - Q_k \overline{\Sigma} Q_k \tag{95}$$

$$0 = \hat{P}_{k+1} (\overline{A}_m - Q_k \overline{\Sigma}) + (\overline{A}_m - Q_k \overline{\Sigma})^T \hat{P}_{k+1} + P_k \Sigma P_k$$

$$0 = \hat{Q}_{k+1} (\overline{A}_m - \Sigma P_k)^T + (\overline{A}_m - \Sigma P_k) \hat{Q}_{k+1} + Q_k \overline{\Sigma} Q_k \tag{96}$$

with either:

$$\hat{P}_o = \hat{P}^\ell \quad , \quad \hat{Q}_o = \hat{Q}^u \tag{97}$$

or

$$\hat{P}_o = \hat{P}^u \quad , \quad \hat{Q}_o = \hat{Q}^\ell \tag{98}$$

Furthermore, $\hat{P}^\ell$, $\hat{Q}^\ell$, $\hat{P}^u$ and $\hat{Q}^u$ are defined as the positive definite solutions of

$$0 = \hat{P}^u \, \bar{A}_m + \bar{A}_m^T \, \hat{P}^u + P^u \, \Sigma \, P^u \qquad \text{a.}$$

$$0 = \bar{A}_m^T \, P^u + P^u \, \bar{A}_m + R_1 + D_P[I, \, P^u + \hat{P}^u] - P^u \, \Sigma \, P^u \qquad \text{b.} \left.\begin{array}{c} \\ \\ \\ \\ \\ \end{array}\right\} \text{(99)}$$

$$0 = \hat{Q}^u \, \bar{A}_m^T + \bar{A}_m \, \hat{Q}^u + Q^u \, \bar{\Sigma} \, Q^u \qquad \text{c.}$$

$$0 = \bar{A}_m \, Q^u + Q^u \, \bar{A}_m^T + V_1 + D_Q[I, \, Q^u + \hat{Q}^u] - Q^u \, \bar{\Sigma} \, Q^u \qquad \text{d.}$$

$$0 = \hat{P}^\ell(\bar{A}_m - Q^u \, \bar{\Sigma}) + (\bar{A}_m - Q^u \, \bar{\Sigma})^T \, \hat{P}^\ell + P^\ell \, \Sigma \, P^\ell \qquad \text{a.}$$

$$0 = \bar{A}_m^T \, P^\ell + P^\ell \, \bar{A}_m + R_1 + D_P[I, \, P^\ell] - P^\ell \, \Sigma \, P^\ell \qquad \text{b.} \left.\begin{array}{c} \\ \\ \\ \\ \\ \end{array}\right\} \text{(100)}$$

$$0 = \hat{Q}^\ell(\bar{A}_m - \Sigma \, P^u)^T + (\bar{A}_m - \Sigma \, P^u) \, \hat{Q}^\ell + Q^\ell \, \bar{\Sigma} \, Q^\ell \qquad \text{c.}$$

$$0 = \bar{A}_m \, Q^\ell + Q^\ell \, \bar{A}_m^T + V_1 + D_P[I, \, Q^\ell] - Q^\ell \, \bar{\Sigma} \, Q^\ell \qquad \text{d.}$$

The proof is immediate from that of Theorem 7 and use of the transformations (7), (8), (85) and (86).

The above computational scheme requires at each step, the solution of Lyapunov equations, standard Riccati equations and stochastic Riccati equations of the form considered in [7]. This is obviously the case for (95), (96) and (100). To see that, at most, solution of stochastic Riccati equations is required for computation of the starting values, consider (99). Setting $P_s \triangleq P^u + \hat{P}^u$, addition of (99.a) and (99.b) produces the stochastic Riccati equation:

$$0 = P_s \overline{A}_m + \overline{A}_m^T P_s + R_1 + D_P[I, P_s] \tag{101}$$

This defines $P_s > 0$ uniquely, and (99.b) may be written

$$0 = \overline{A}_m^T P^u + P^u \overline{A}_m + R_1 + D_P[I, P_s] - P^u \Sigma P^u \tag{102}$$

Thus, P^u is the unique positive definite solution of an ordinary Riccati equation. Finally, solution of the Lyapunov equation (99.a) yields $\hat{P}^u$. An analogous scheme may be used for determination of Q^u and $\hat{Q}^u$ from (99.c,d).

Although convergence of the sequences defined by (95) through (100) is assured, the numerical procedure involves stochastic Riccati equations at each step, and these, in turn, demand an iterative method of solution.

In place of the above rather cumbersome method of solution we may introduce a much more direct iterative method based upon the following computational sequence (k = 0, 1, 2,...):

$$
\begin{aligned}
0 &= (\overline{A}_m - \Sigma P_k)^T P_{k+1} + P_{k+1}(\overline{A}_m - \Sigma P_k) + R_1 && \text{a.} \\
&\quad + D_P[I, P_k + \hat{P}_k] + P_k \Sigma P_k && \\
0 &= (\overline{A}_m - \Sigma P_k)\hat{Q}_{k+1} + \hat{Q}_{k+1}(\overline{A}_m - \Sigma P_k)^T + Q_k \Sigma Q_k && \text{b.}
\end{aligned}
\tag{103}
$$

$$0 = (\overline{A}_m - Q_k \overline{\Sigma})Q_{k+1} + Q_{k+1}(\overline{A}_m - Q_k \overline{\Sigma})^T + V_1 \qquad a.$$

$$+ D_Q[I, Q_k + \dot{Q}_k] + Q_k \overline{\Sigma} Q_k \qquad (104)$$

$$0 = (\overline{A}_m - Q_k \Sigma)^T \hat{P}_{k+1} + \hat{P}_{k+1}(\overline{A}_m - Q_k \Sigma) + P_k \Sigma P_k \quad b.$$

with starting values defined by:

$$\hat{P}_o = \hat{Q}_o = 0 \qquad (105)$$

$$\overline{A}^T P_o + P_o \overline{A} + R_1 - P_o \Sigma P_o = 0$$

$$\overline{A} Q_o + Q_o \overline{A}^T + V_1 - Q_o \overline{\Sigma} Q_o = 0 \qquad (106)$$

Note that (103) and (104) entail only the solution of Lyapunov equations. Because of its evident convenience, this approach, in preference to that of Theorem 9, has been implemented computationally. Although proof of convergence remains the object of investigation, the sequence defined by (103) through (106) have been found to be convergent in all numerical studies performed to date.

In the specific implementation of (103) through (106) used to obtain the numerical results discussed in the following sections, the iterative sequence is terminated once all diagonal entries of P and Q converge to within a given tolerance - i.e., given $\varepsilon > 0$, solution of (103) and (104) is carried up to $k = k_T(\varepsilon)$, where

$$\left.\begin{array}{l} |(P_{k_T+1} - P_{k_T})_{ii}| \leq \epsilon \\[2ex] |(Q_{k_T+1} - Q_{k_T})_{ii}| \leq \epsilon \end{array}\right\} \; \Psi \; i \qquad\qquad (107)$$

The Lyapunov equations (103) and (104) are solved by the real Schur vector approach following Kitagawa [21] while the Riccati equations (106) are solved by the real Schur vector technique of Laub [22].

The major computational burden in solving Lyapunov-type equations is in reducing the stability matrix to real Schur forms. Hence, computation is reduced by half when only one reduction is used to solve (103.a) and (103.b) and one for (104.a) and (104.b). Clearly, the solution sequence of (103) and (104) also reduces storage requirements.

4.2 General Description of Design Studies

In the remainder of this report, we discuss application of the stochastic design approach to various simple example problems. This is done mainly with a view toward illustrating the improvement in robustness properties to be expected and the qualitative form of the control, particularly for large levels of modelled frequency uncertainty.

In all examples considered, the system retains the general form of (87):

$$\frac{d}{dt}\begin{pmatrix} x \\ \tilde{q} \end{pmatrix} = \begin{bmatrix} A & -BK \\ FC & \overline{A}_m - BK - FC \end{bmatrix} \begin{pmatrix} x \\ \tilde{q} \end{pmatrix} + \begin{pmatrix} w_1 \\ F\tilde{w}_2 \end{pmatrix} \qquad\qquad (108)$$

except that

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$$A = \begin{bmatrix} A_u & 0 \\ 0 & A_e \end{bmatrix} \tag{109}$$

where A_e represents the elastic modes while A_u now comprises both rigid body and unstable modes. Matrices B, C, K and F are as defined previously, and w_1 and $\tilde{w}_2$ are white noise processes with intensity matrices V_1 and V_2, respectively.

Uncertainty is modelled only in the open-loop frequencies associated with the elastic modes. It is assumed that open-loop frequency deviations are normally distributed random variables. In consequence, if T_m denotes the decorrelation time corresponding to the mth elastic mode, then (32) yields:

$$T_m = \frac{\sqrt{\pi}}{2} (\sigma_m \bar{\omega}_m)^{-1} \tag{110}$$

where σ_m is the standard deviation of the mth elastic mode frequency relative to its nominal value, $\bar{\omega}_m$. For illustrative purposes, we adopt the simple model

$$\sigma_m = \sigma \bar{\omega}_m \tag{111}$$

to reflect a degradation of structural modelling accuracy with increasing nominal modal frequency. Thus, in the following example problems, modal frequency uncertainty levels are uniquely defined by the relative standard deviation, σ, of the first mode.

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Although second-mean stability is proved under the conditions stated in Theorem 7, it is desirable to demonstrate robustness for specific designs by determining stability for a range of parameter variations.

To this end, we conduct sensitivity studies as follows. For each compensator design (corresponding to particular nominal values and a value of σ in (111), we compute the closed-loop poles for a set of perturbed system models obtained by replacing the nominal value of the elastic mode dynamics matrix, $\overline{A}_e$, by:

$$
A_e(\delta) \;=\; \text{block-diag}
\begin{bmatrix}
0 & 1 \\
-\overline{\omega}_k^2 (1+\delta)^2 & -2\eta_k \overline{\omega}_k (1+\delta)
\end{bmatrix}
\tag{112}
$$

where a range of values of the relative frequency deviation, δ, are considered.

Obviously, since (112) implies a change in all modal frequencies by a fixed percentage, robustness for independently random frequency deviations cannot be established. However, perturbations of the above form still provide a convenient and practical means of illustrating relative stability for a subclass of the parameter uncertainties originally postulated. Note also that (112) involves variation directly in the structural mode frequencies as opposed to parameter variations of the form (10). Thus, although uncertainties in structural mode frequencies are closely allied with open-loop frequency uncertainties, unconditional stability under such perturbations may not be expected and (112) provides a good test of the robustness properties of the stochastic design.

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Further important features which we wish to illustrate in the following are the progressive diagonalization of P and Q with increasing uncertainty level and the concomittant reduction of the controller form to rate output feedback in the case of colocated actuators and sensors. To demonstrate diagonalization of P and Q, we introduce the following measures of diagonal dominance:

$$P_i \triangleq \sum_j |P_{ij}|/(P_{ii}P_{jj})^{\frac{1}{2}} - 1$$

$$Q_i \triangleq \sum_j |Q_{ij}|/(Q_{ii}Q_{jj})^{\frac{1}{2}} - 1$$

(113)

Clearly, P_i, $Q_i \geq 0$, with equality if and only if the ith column is zero except for the ith element.

4.3 Numerical Examples

The first two examples discussed here involve low-order systems and are intended to illustrate the robustness properties to be expected under the stochastic design approach. We particularly consider the effect of rigid body and unstable modes and non-colocation of actuators and sensors. The last example demonstrates the asymptotic properties of the control for large uncertainty levels for a fairly high-order system.

A.   Two Mass System

The two mass system shown in Figure 2 provides a simple example to illustrate the relative stability of the stochastic design approach. We assume a force actuator located on mass 1 and suppose that mass 2 is subjected to a white disturbance force, w, of unit intensity. Also, for simplicity, m_1, m_2 and k

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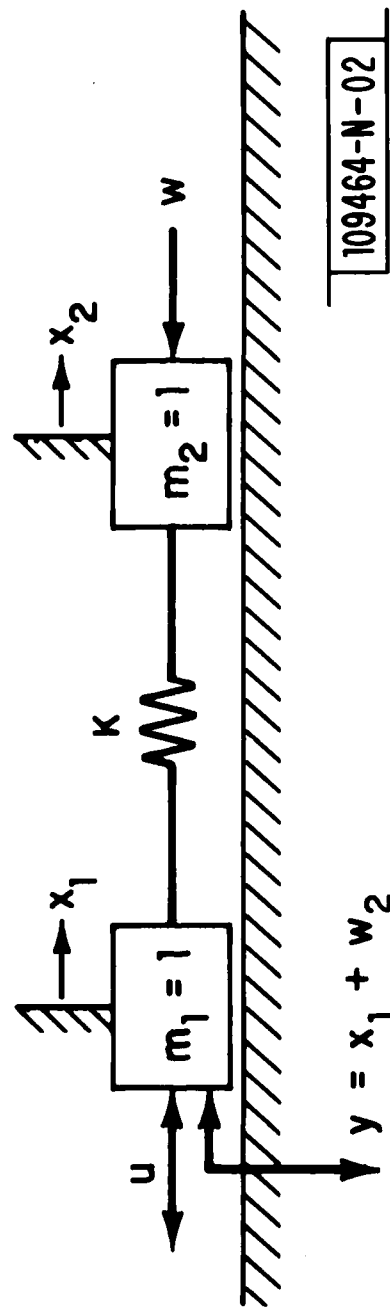
Fig. 2 Two mass system

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$y = x_1 + w_2$

are set equal to unity. Ignoring damping for the moment, the dynamic equations are:

$$\ddot{x}_1 = -x_1 + x_2 + u$$

$$\ddot{x}_2 = x_1 - x_2 + w$$

(114)

It is desired to suppress the mean-square displacement of mass 2. Thus, we choose:

$$\bar{J}_s = E[x_2^2 + u^2]$$

(115)

Finally, two cases of sensor location are considered:

$$\text{non-colocation:} \quad y = x_2 + \tilde{w}_2$$

$$\text{colocation:} \quad y = x_1 + \tilde{w}_2$$

(116)

where $\tilde{w}_2$ is observation noise of unit intensity. Note that in both cases, displacement sensing is assumed.

Recasting this problem in the modal coordinate basis and assuming 0.5 percent modal damping, the various matrices appearing in (108) and characterizing the control formulation (88)-(90) become:

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$$
\overline{A} = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & -2 & -0.014 \end{bmatrix} , \quad B = \begin{bmatrix} 0.00 \\ 0.50 \\ -0.17 \\ 0.17 \end{bmatrix} \qquad \text{(117.a,b)}
$$

Non-colocation: $C = [1, 0, -1, -1]$ a.

Colocation: $C = [1, 0, 1, 1]$ b.   (118)

$$
R_1 = \begin{bmatrix} 1 & 0 & -1 & -1 \\ 0 & 0 & 0 & 0 \\ -1 & 0 & 1 & 1 \\ -1 & 0 & 1 & 1 \end{bmatrix} , \quad R_2 = 1 \qquad \text{(119.a,b)}
$$

$$
V_1 = \begin{bmatrix} 0.00 & 0.00 & 0.00 & 0.00 \\ 0.00 & 0.25 & 0.08 & -0.08 \\ 0.00 & 0.08 & 0.03 & -0.03 \\ 0.00 & -0.08 & -0.03 & 0.03 \end{bmatrix} , \quad V_2 = 1 \qquad \text{(120.a,b)}
$$

These relations define the nominal system model. Uncertainty in the single open-loop elastic mode frequency present in this problem is modelled as in (110) and (111) with σ denoting the standard deviation relative to the nominal frequency $\overline{\omega} = \sqrt{2}$.

Under the above assumptions, we now discuss the non-colocated and colocated sensor cases separately as follows:

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A.1 Sensor at Mass 2 (Non-Colocation)

Using (103)-(106) we first generated three designs corresponding to uncertainty levels $\sigma = 0.0$, 0.5 and 1.0. For each such design, closed loop poles were computed for modal frequency perturbations of the form (112), with δ varying over the range ± 0.90. Note that since there is only one elastic mode, such root-loci fully characterize robustness under modal frequency uncertainties.

The root-loci are shown in Figures 3, 4 and 5, where pole locations for the nominal system are indicated by solid dots. The deterministic plant, LQG design ($\sigma = 0$) shown in Figure 3 is unstable for frequency variations $\delta = -0.22$ to -0.90 and $\delta = 0.10$ to 0.50. It is seen that the nominal system poles at $-0.2 \pm 1.4i$, corresponding to compensator poles, tend to push the closed-loop elastic mode into the right half plane. In contrast, the stochastic design with $\sigma = 0.5$ (Figure 4) places these offending poles at $-1.1 \pm 1.5i$ (for the nominal system), thereby minimizing their effect on the elastic mode. This stabilizes the design for positive frequency variations but the system is still unstable for variations with $\delta = -0.25$ to -0.9. When the modelled uncertainty level is increased further ($\sigma = 1.0$, Figure 5), an increased stability margin is achieved for positive frequency variations but the unstable region for negative variations remains virtually unchanged.

A.2 Sensor at Mass 1 (Colocation)

Here, we again consider three designs corresponding to $\sigma = 0$, $\sigma = 0.5$ and $\sigma = 1.0$, and plot the root-loci for variations of δ from -0.9 to $+0.9$. The results are shown in Figures 6, 7 and 8. In this case, the deterministic design (Figure 6) is unstable for variations $\delta = -0.15$ to -0.60 and for $\delta \geq 0.30$.
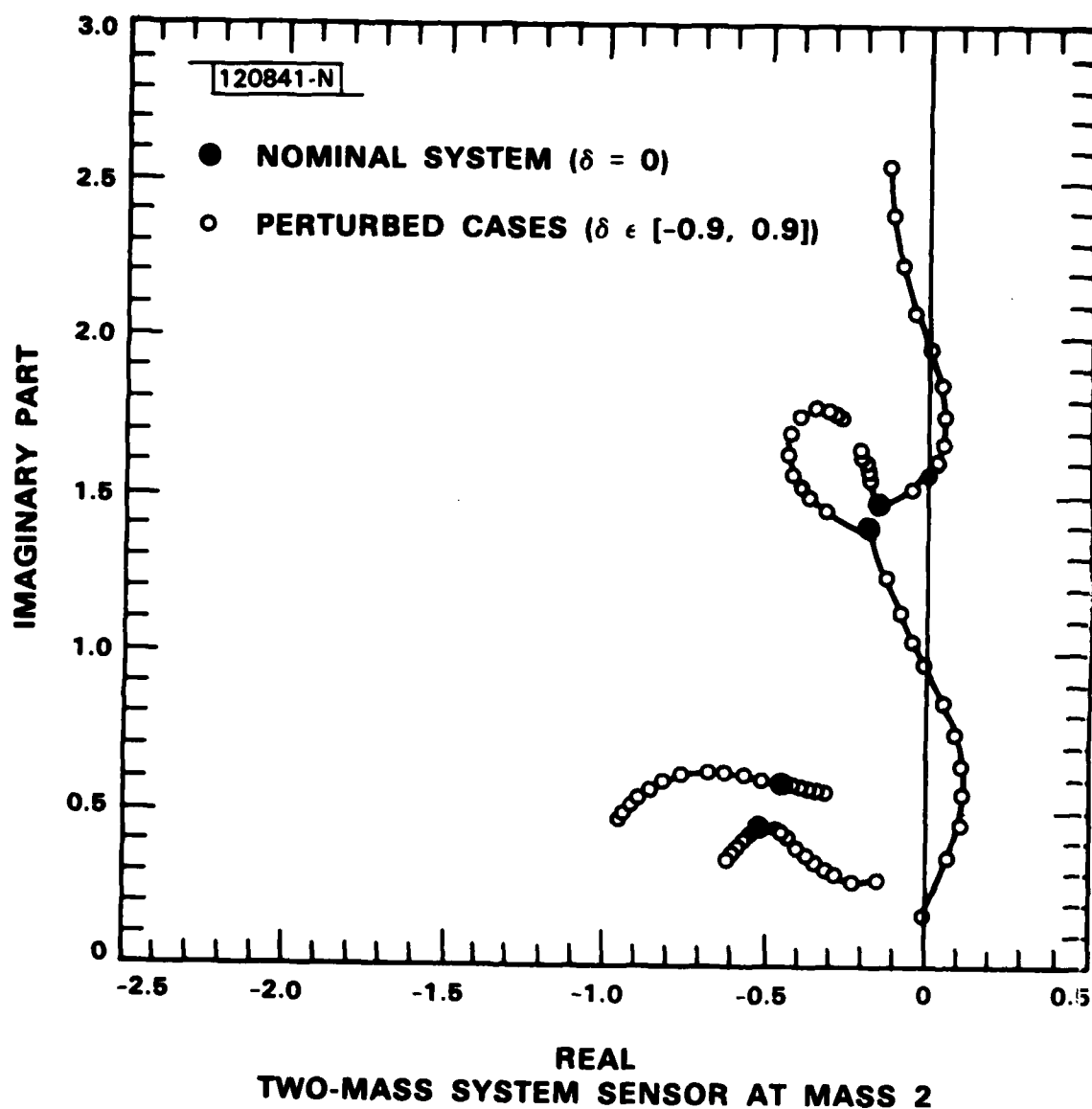
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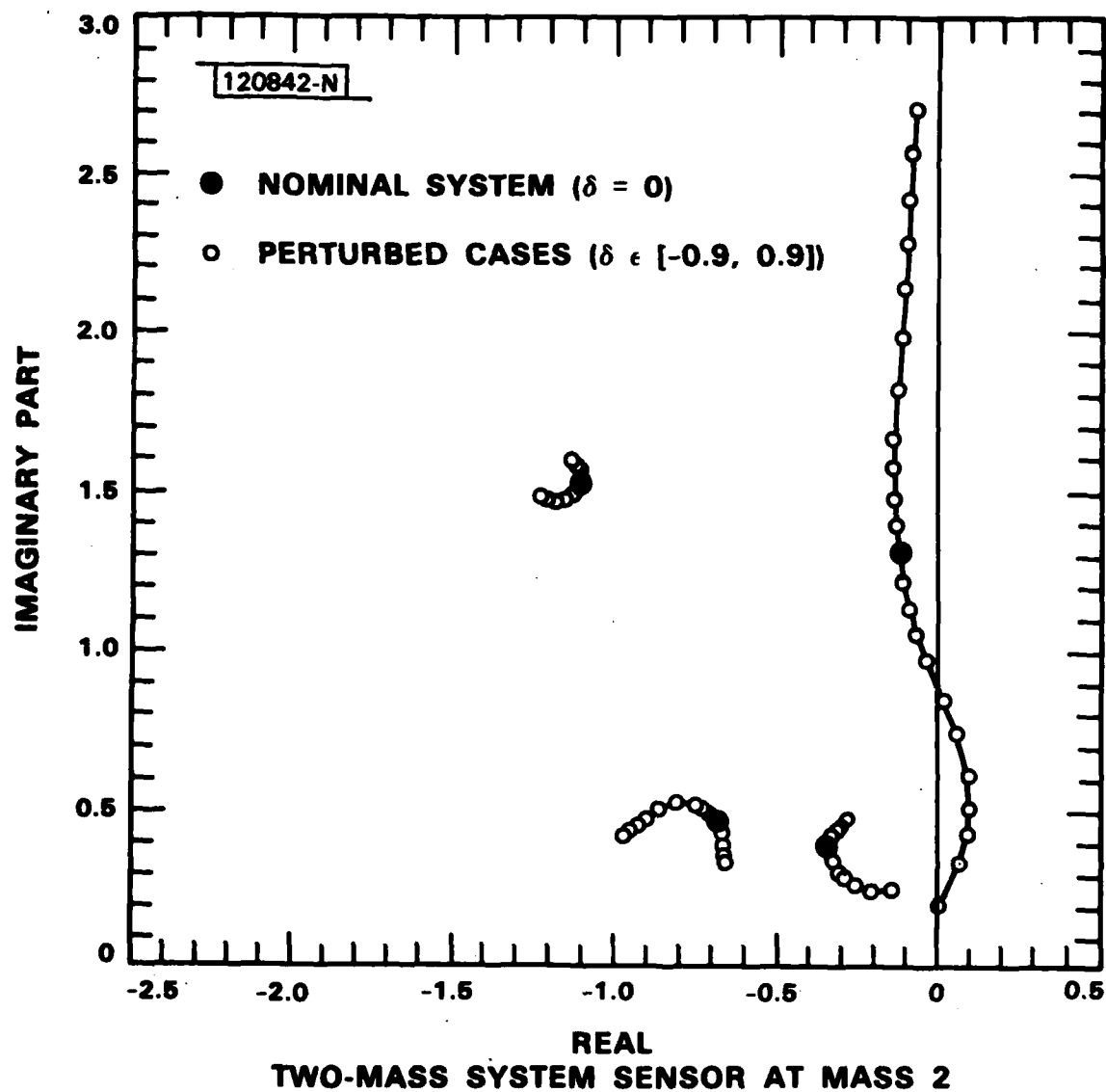Fig. 3 Closed-loop poles for two mass system, non-colocated case, $\sigma = 0$

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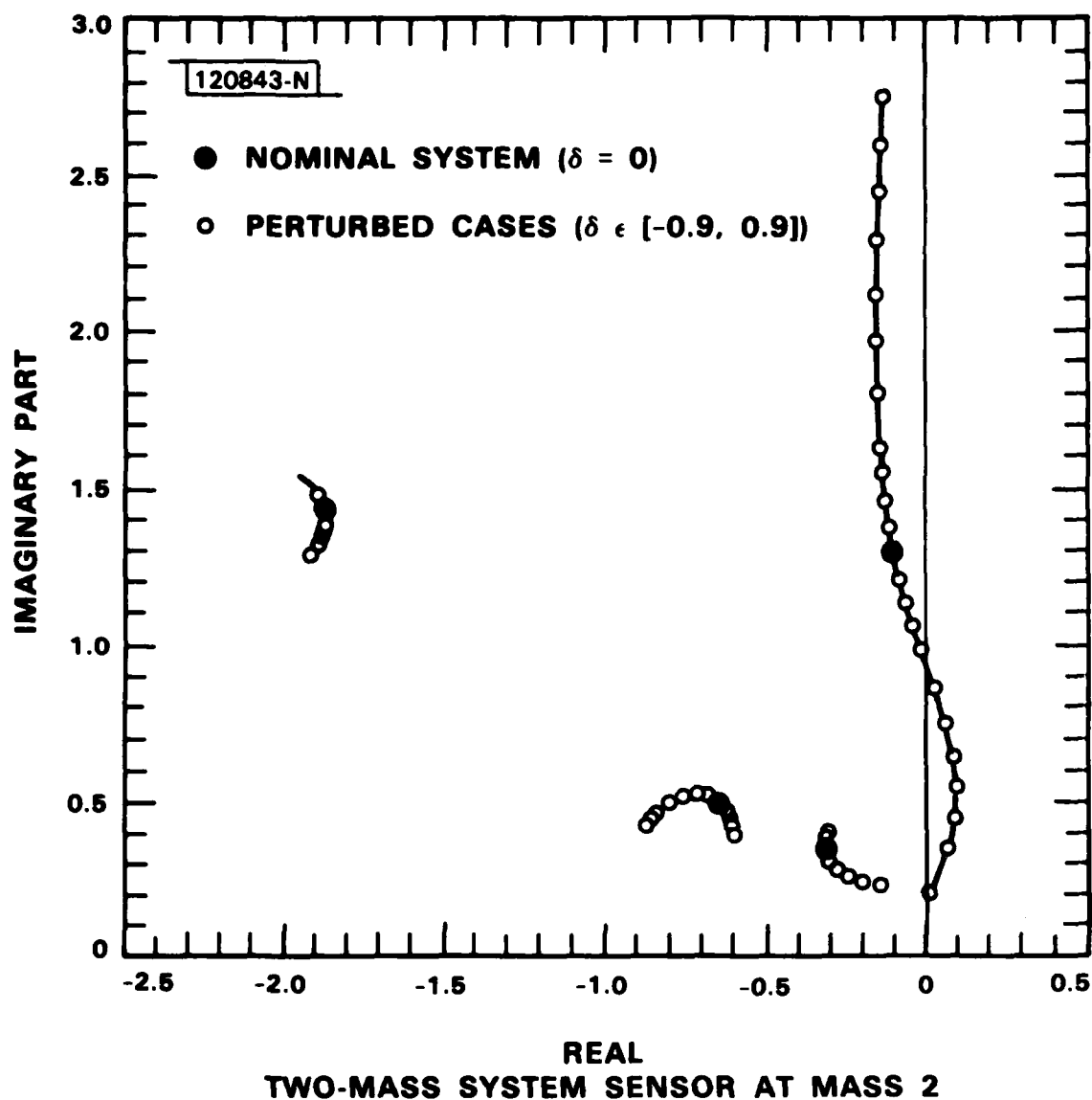Fig. 4 Closed-loop poles for two mass system, non-colocated case, σ = 0.5

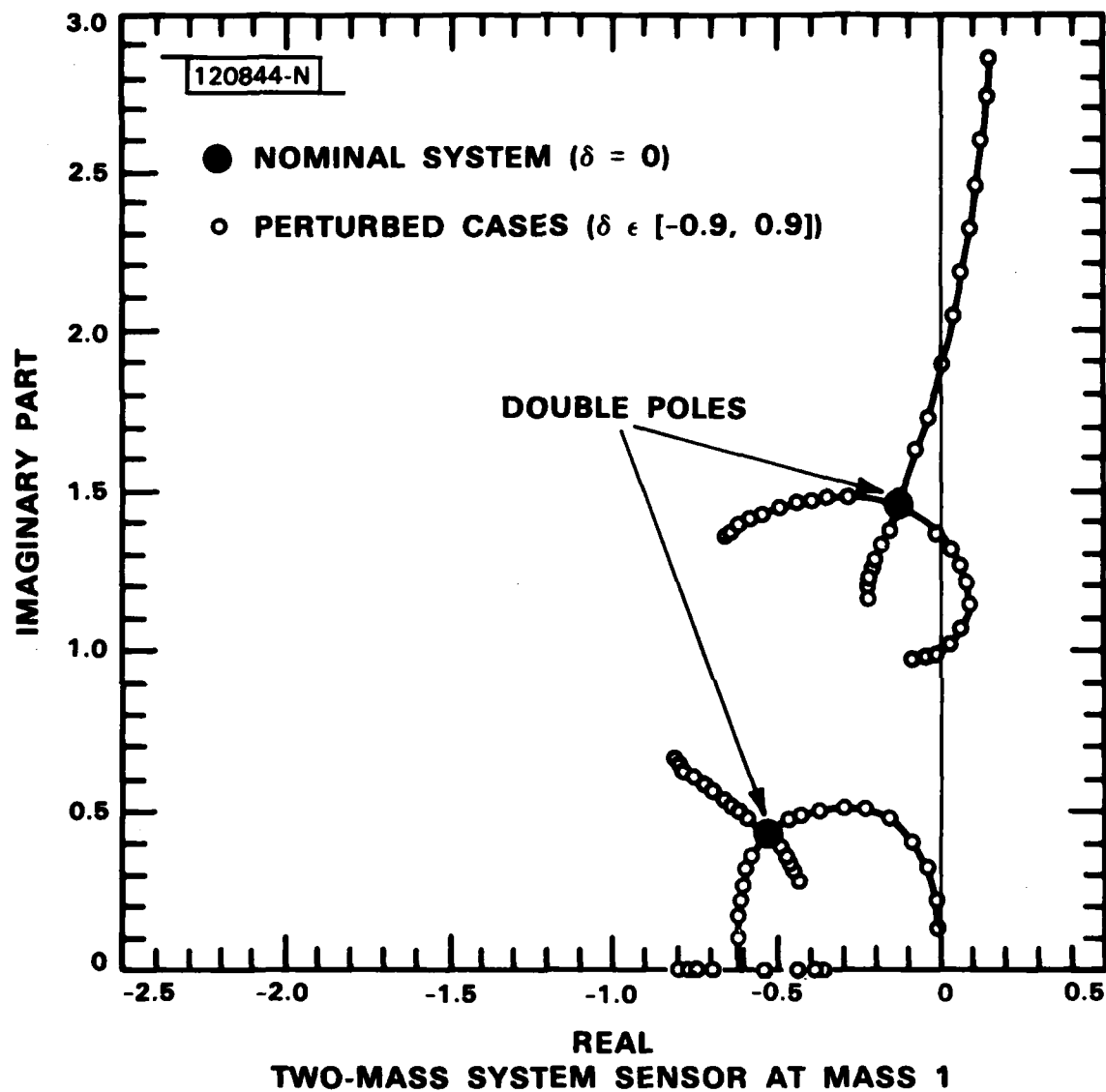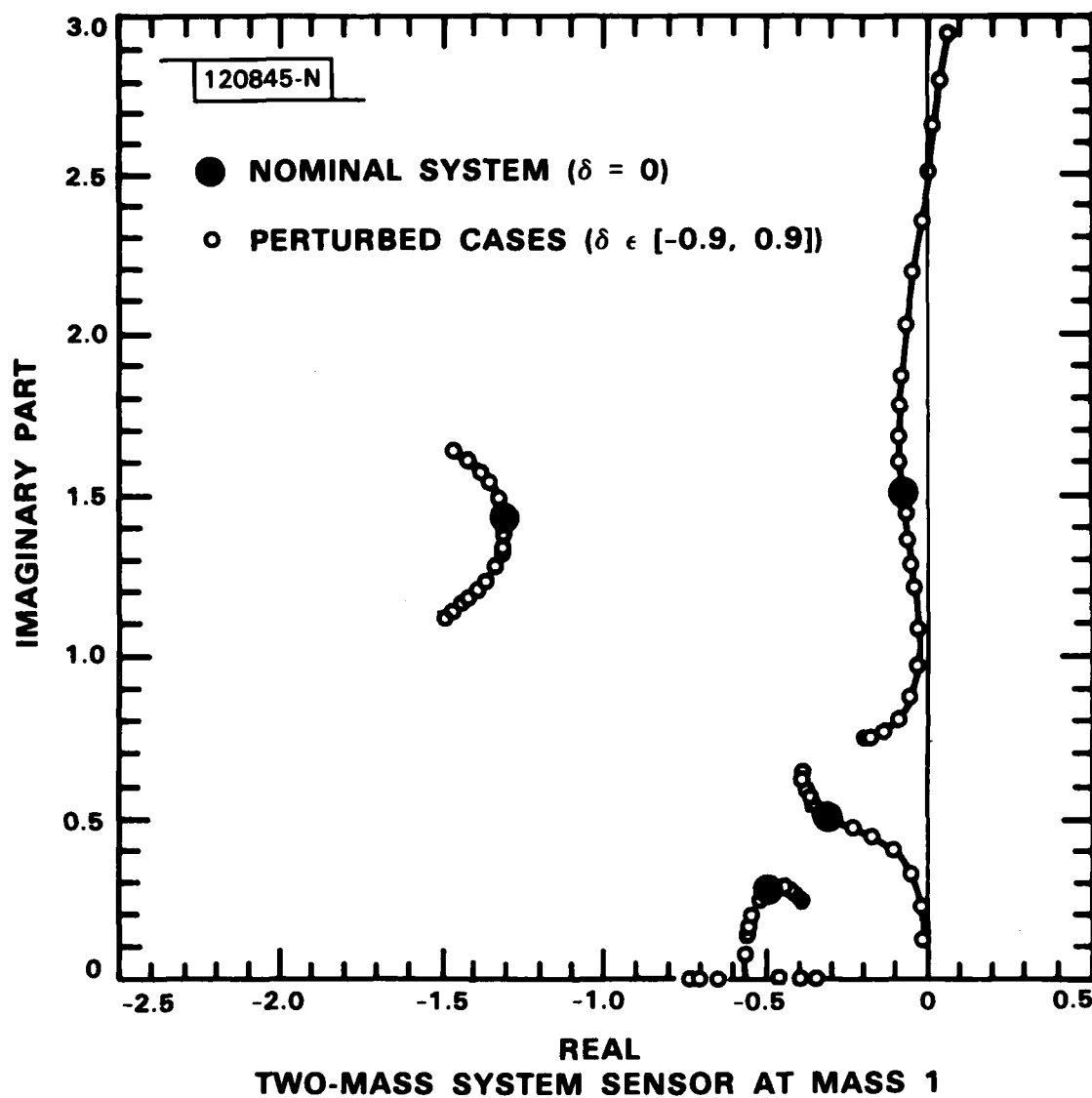Fig. 5 Closed-loop poles for two mass system, non-colocated case, σ = 1.0

Fig. 6 Closed-loop poles for two mass system, colocated case, σ = 0.0

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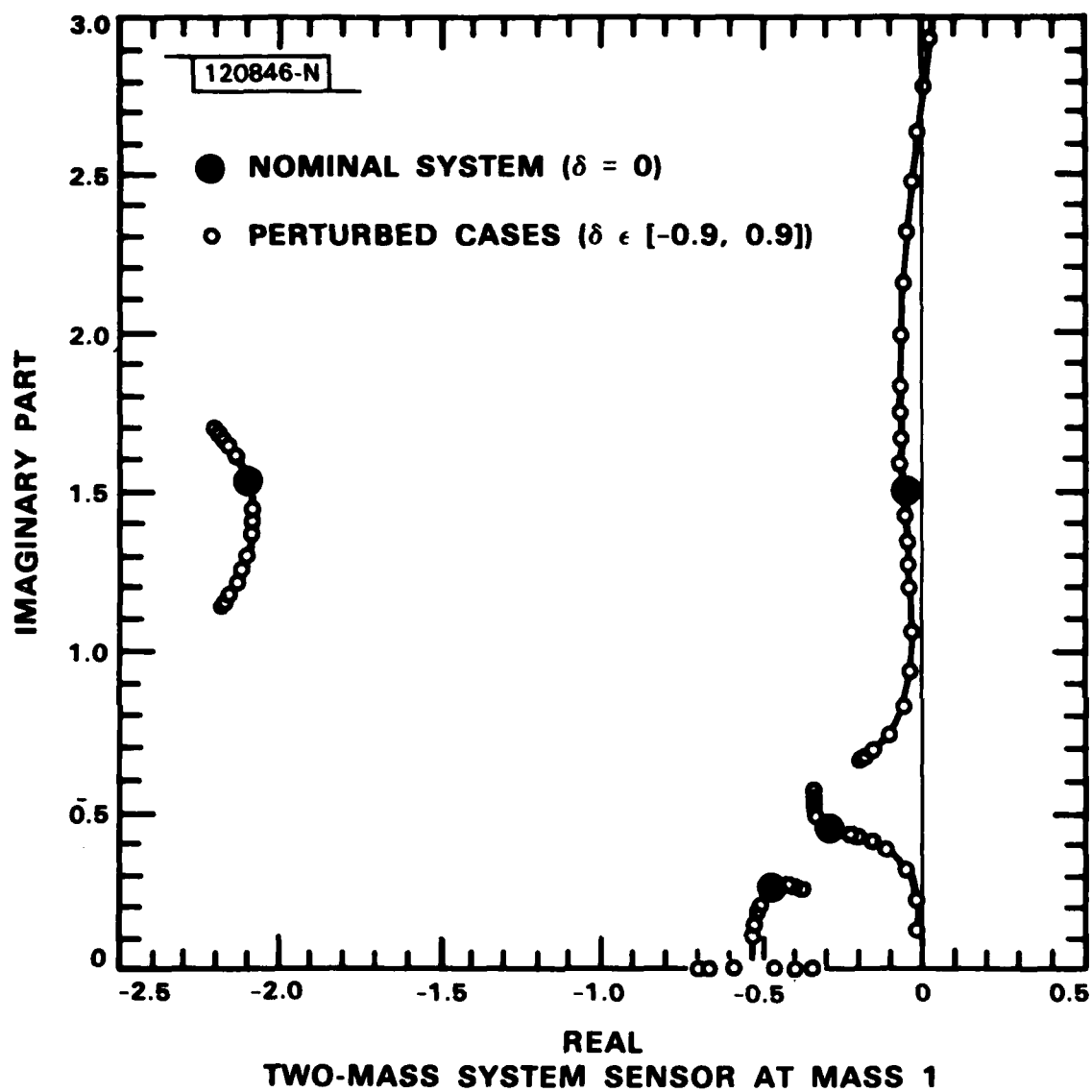Fig. 7 Closed-loop poles for two mass system, colocated case, σ = 0.5

Fig. 8 Closed-loop poles for two mass system, colocated case, σ = 1.0

On the other hand, the stochastic design for $\sigma = 0.5$ (Figure 7) is unstable only for $\delta > 0.70$. This stability region is still further extended to $\delta < 0.90$ by increasing σ to 1.0 (Figure 8). Thus, it is seen that with sensor/actuator colocation, the stochastic design can extend the stability region for both positive and negative frequency variations. Overall stability characteristics may be shown as in Figure 9, where we have combined the previous results with design results corresponding to additional values of σ. This figure shows the stability region (unshaded area) in the δ-σ plane. It is clear that by increasing the level of modelled uncertainty in the stochastic design we may progressively enlarge the stability region.

<u>Summary</u>: Note that despite the presence of a rigid body mode (the effects of which were not explicitly treated in the preceding theoretical developments), no difficulties were experienced with the convergence of (103)-(106). Although increase of the modelled uncertainty level in the non-colocated sensor case does progressively increase the overall region of stability, the stability boundary for negative frequency deviations is not appreciably affected for large uncertainty levels. This reflects an inherent limitation in non-colocated systems in the presence of rigid body modes. On the other hand, in the colocated sensor case, as Figure 9 demonstrates, the region of stable frequency deviations increases approximately in proportion to the modelled uncertainty level. Thus, it appears that by modelling the frequency uncertainty with sufficient conservatism, an arbitrarily large stability margin may be secured.
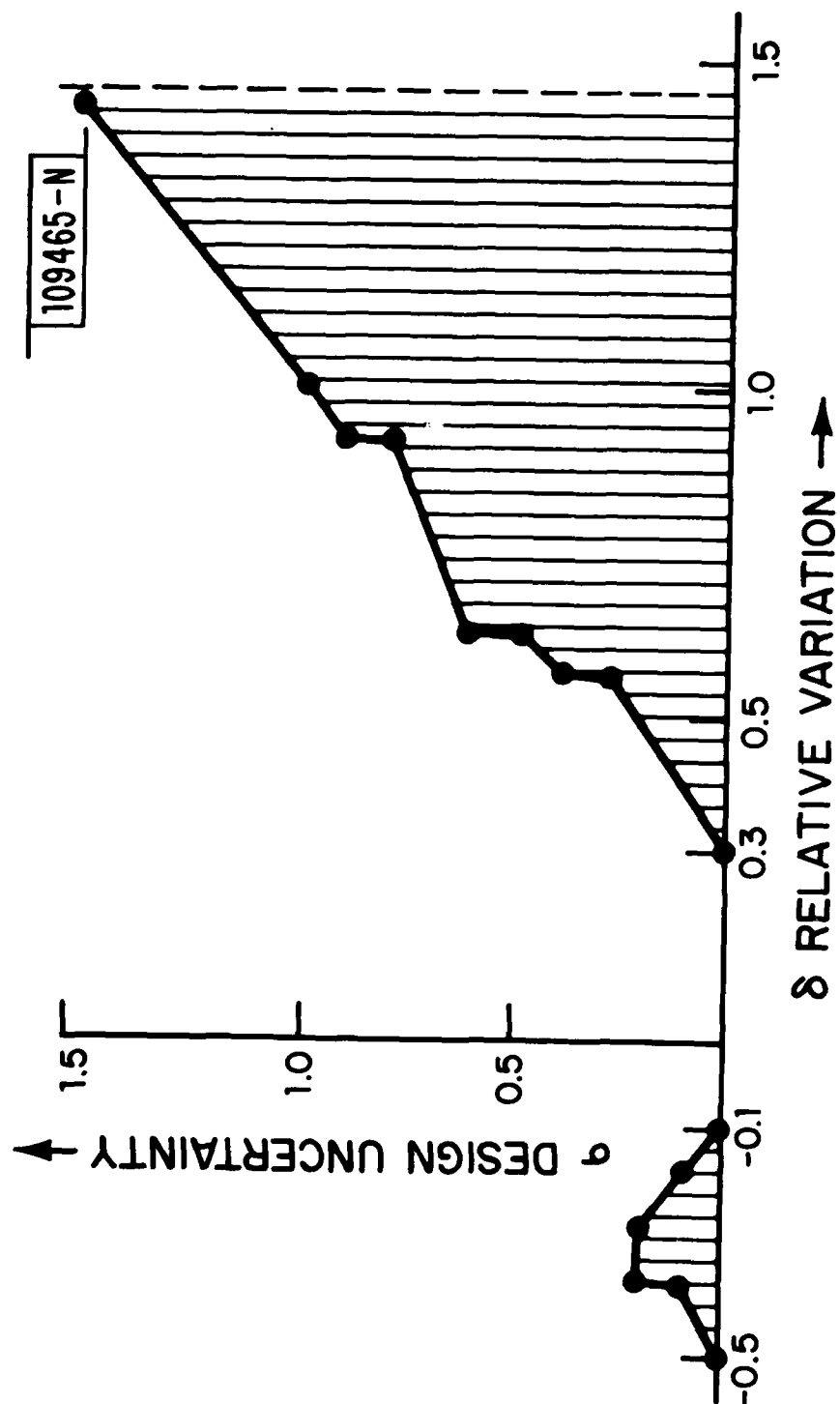
Fig. 9 Stability boundaries for two mass system colocated case (unstable region shaded)

B.   Inverted Pendulum

To illustrate the effect of unstable modes, closed-loop con-
trol of an inverted pendulum, shown in Figure 10, is investigated.
This system was previously considered by Martin [23] and consists
of a flexible beam column pinned to a motorized cart. The beam
acts as an inverted pendulum which can be stabilized by acceler-
ating the cart horizontally with input force u. Measurements of
the cart position and the slope of the beam at its base are
assumed available.

The system model, including all relevant nominal system
parameters are those given by Martin [23] and need not be
repeated here. It suffices to note, however, that the model
encompasses one rigid body mode, one unstable mode and two
elastic modes. The unstable mode frequency is ±3.22 rad/sec
while the first two bending mode frequencies are 3.055 and
22.05 rad/sec. The inherent damping factor for the bending
modes is assumed to be 0.005. Finally, the state weighting
matrix is diagonal:

$$R_1 = \text{diag } [0.001, 0, 0.4, 0.1, 0, 0.1, 0] \tag{121}$$

Uncertainties only in the bending mode frequencies are con-
sidered here and these are modelled in accordance with (110) and
(111).

With these assumptions, three designs were computed corre-
sponding to $\sigma = 0$, 0.2 and 0.5, where σ is the first elastic
mode standard deviation. Root-loci for elastic mode frequency
deviations of the form (112) with δ in the range ±0.30 are
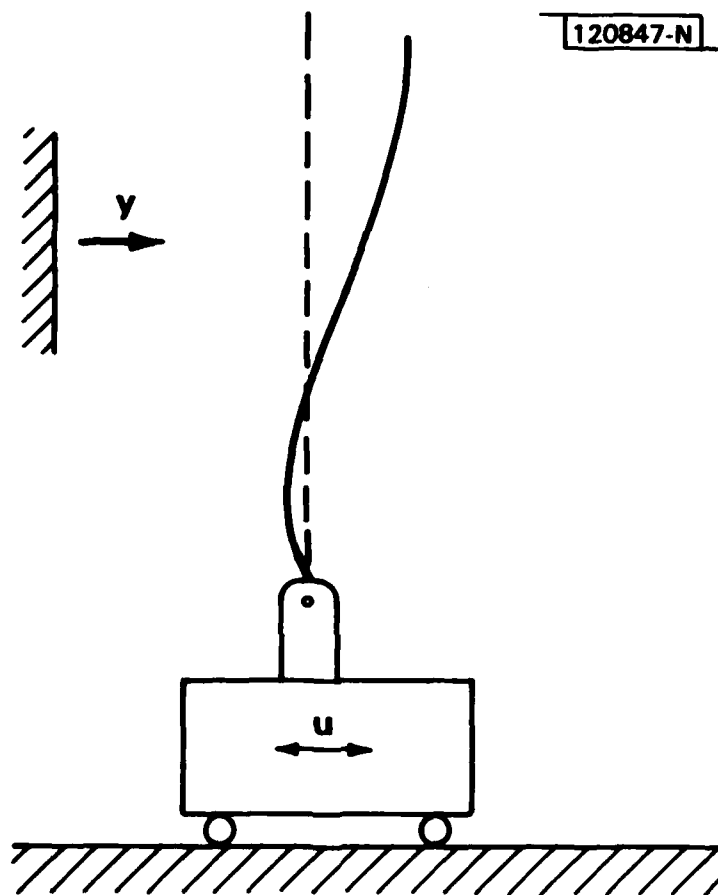shown in Figures 11 through 13. Notice that, in the deterministic
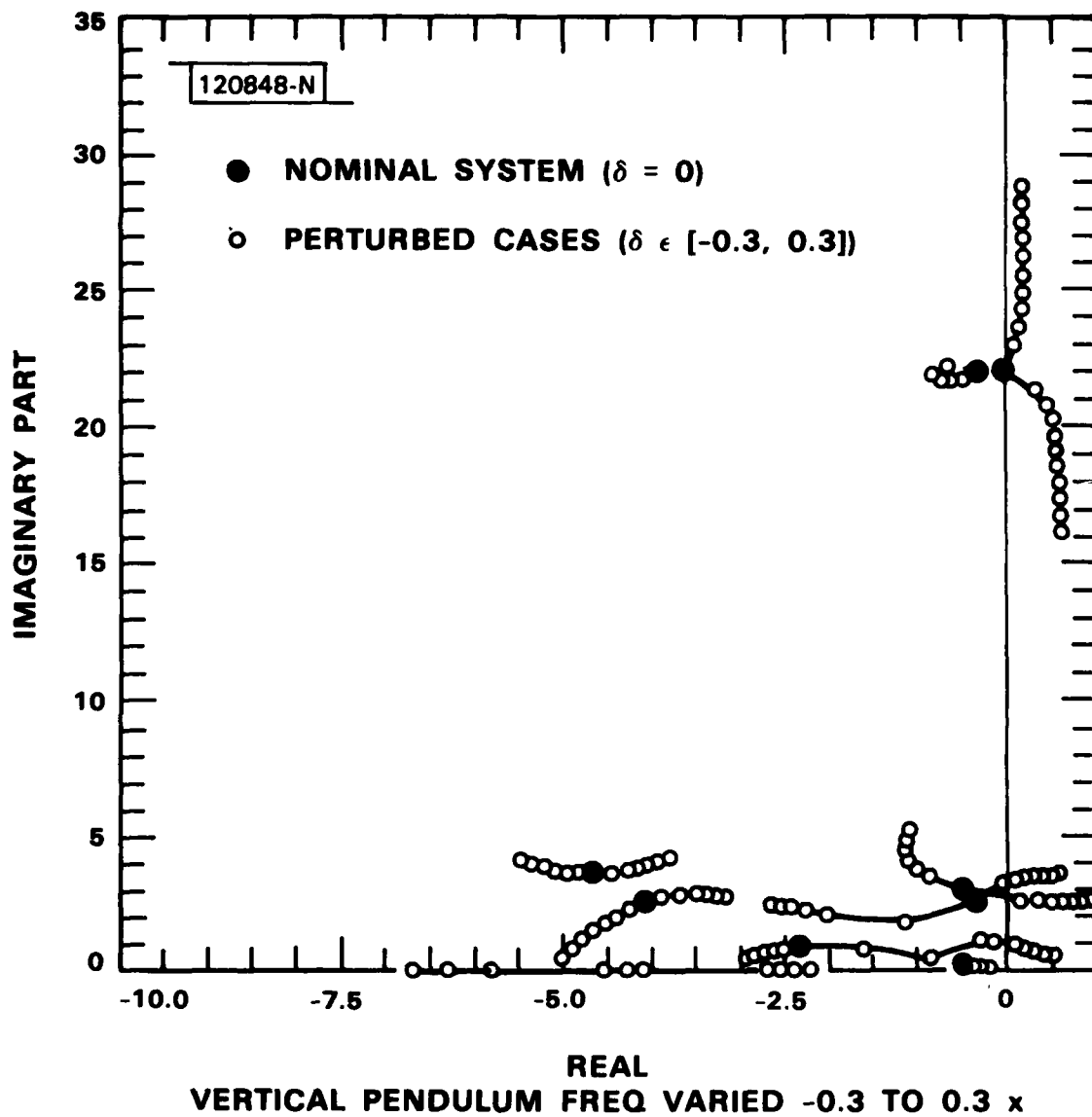
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Fig. 10 Inverted pendulum system

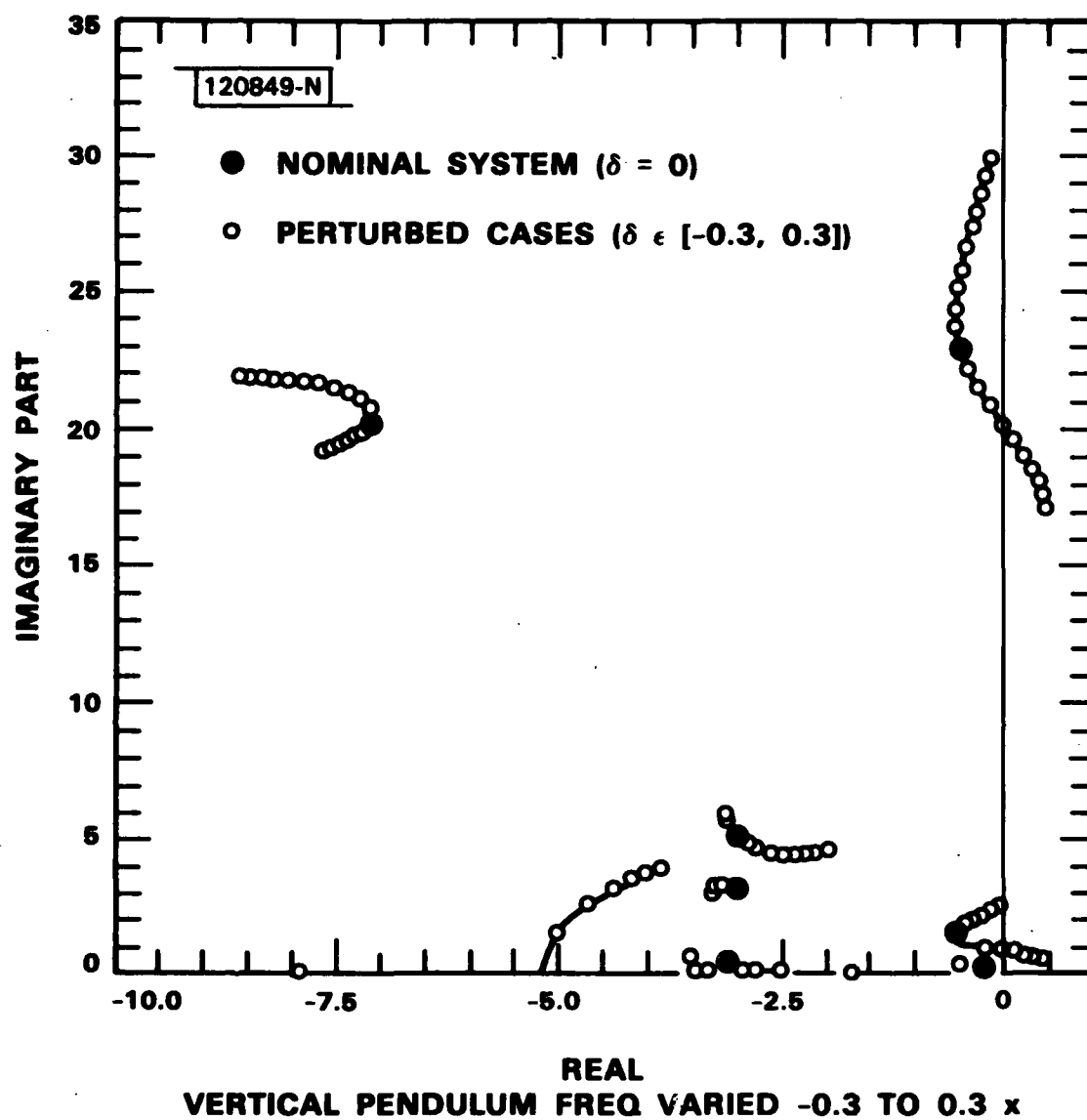Fig. 11   Closed-loop poles for inverted pendulum,
$\sigma = 0.0$
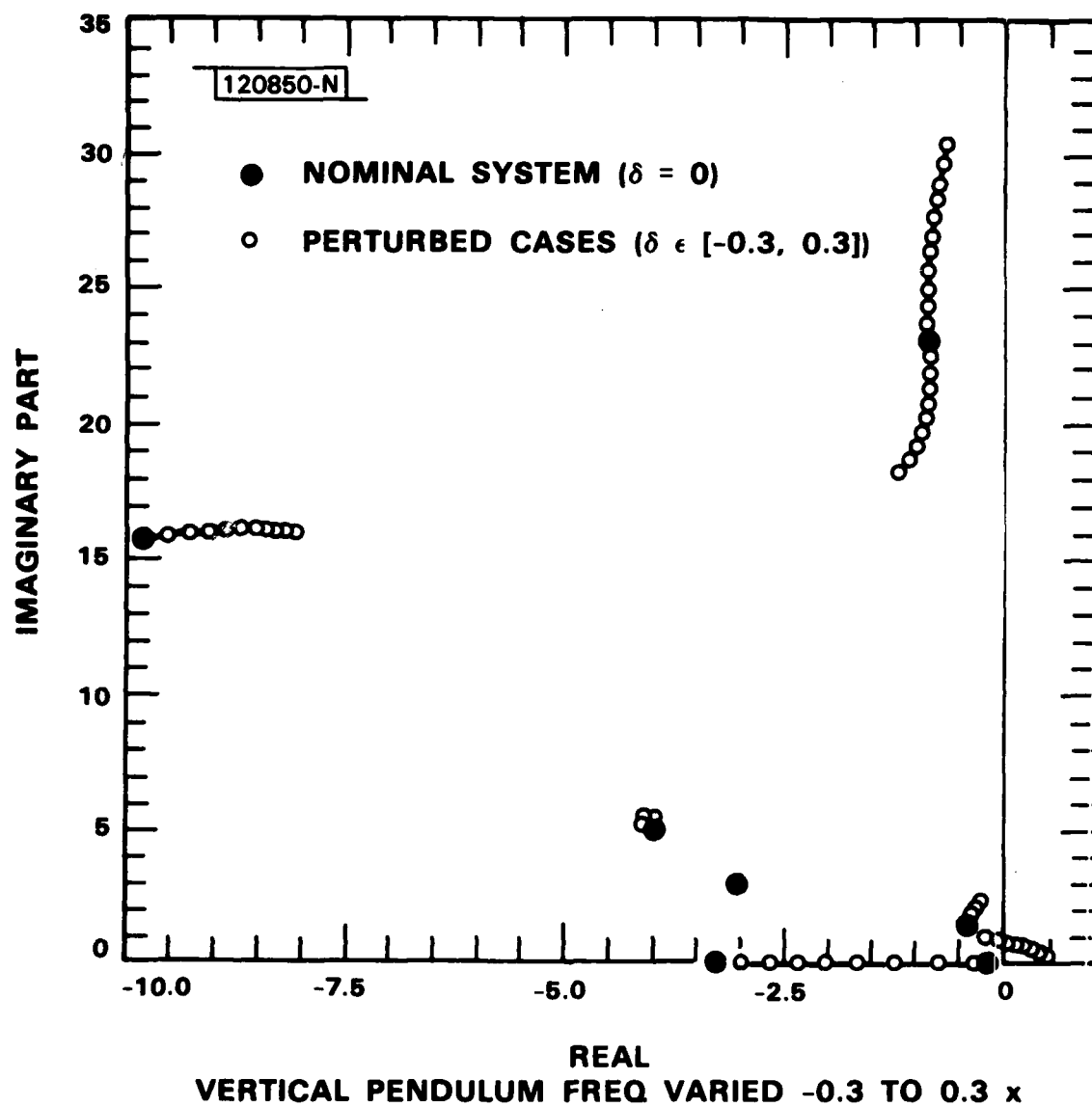
Fig. 12 Closed-loop poles for inverted pendulum, σ = 0.2

Fig. 13   Closed-loop poles for inverted pendulum,
σ = 0.5

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design ($\sigma = 0$, Figure 11), both bending modes become unstable for $|\delta| > 0.03$. Increase of the modelled uncertainty to $\sigma = 0.2$ (Figure 12) pushes the compensator poles deeper into the left half plane and dramatically increases the stability region to $\delta \in [-0.09, 0.29]$. A still larger value of σ ($\sigma = 0.5$, Figure 13) completely stabilizes the second bending mode over the range of frequency deviations considered. On the other hand, while the first bending mode is also stabilized for all positive frequency variations, it becomes unstable for $\delta < -0.1$. In other words, increasingly large leve_s of modelled uncertainty result in relatively minor increases in the stability margin for negative frequency variations.

Summary: Although the theoretical developments of earlier chapters assumed a stable open-loop system, the computational scheme of (103)-(106) converged without difficulty in this case to produce stable, robust designs. However, the above results do indicate that the presence of an unstable mode limits the ability of the stochastic design approach to improve the stability margin for negative frequency deviations - particularly for the elastic mode in closest proximity to the unstable mode. In part, this may be ascribed to the fact that in modelling uncertainty only in the elastic modal frequencies we ignore the influence of such uncertainties on the unstable mode so that robustness improvement is mainly restricted to those modes (i.e., the second bending mode, in this case) which are accurately represented in the stochastic design model. Nevertheless, the stochastic design approach yields an enormous enlargement of the stability region over that obtained by the conventional LQG design.

C.   Simply Supported Beam

Here we consider a simply-supported Bernoulli-Euler beam with normalized span-wise coordinate as shown in Figure 14. A force actuator and colocated rate sensor are assumed at $\xi = \xi_a = 2/43$. The nominal system dynamics matrix in the modal coordinate basis is block-diagonal with diagonal blocks of the form (3). With appropriate non-dimensionalization of the equations of motion, we may write:

$$\overline{\omega}_k = k^2 \quad ; \quad k = 1,\ldots,n \tag{122}$$

Furthermore, we set:

$$\eta_k = 0.005 \quad , \quad \forall\, k \tag{123}$$

and B and C assume the form:

$$B = C^T \quad R^{2n\times1}$$

$$B_{k1} = \tfrac{1}{2}(1 + (-1)^k) \, \sin \tfrac{\pi}{4} k \, (1 + \xi_a) \tag{124}$$

$$k = 1,\ldots,2n$$

For this example, an "energy" state weighting is chosen:

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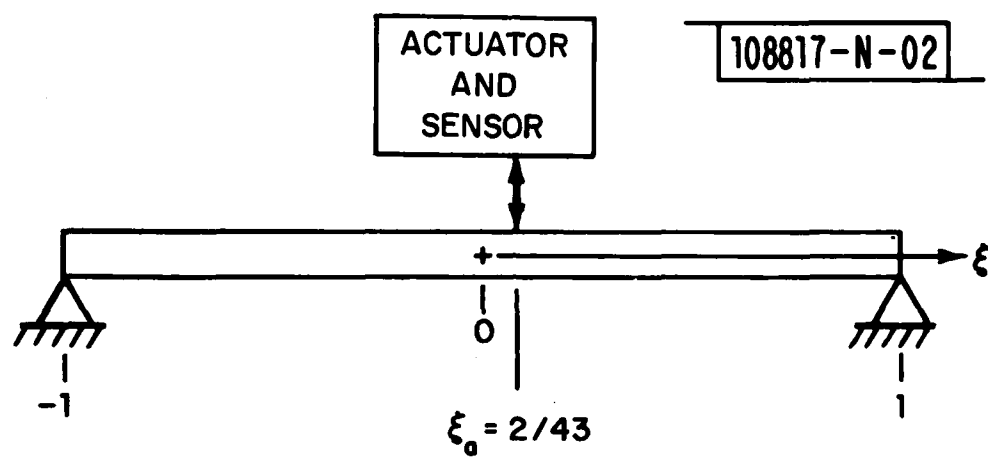
Fig. 14 Simply-supported beam with colocated force actuator and rate sensor

$$R_1 = \underset{k=1,\ldots,n}{\text{block-diag}} \begin{bmatrix} \overline{\omega}_k^2 & 0 \\ 0 & 1 \end{bmatrix} \tag{125}$$

and we suppose the beam to be excited by a spatially and tempo-rally white disturbance force with unit intensity, i.e.,:

$$V_1 = \underset{k=1,\ldots,n}{\text{block-diag}} \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} \tag{126}$$

Since only one actuator and sensor are used, R_2 and V_2 are positive scalars which we denote by ρ_r and ρ_o, respectively.

As before, open-loop frequency uncertainties are modelled according to (110) and (111) so that uncertainty levels are com-pletely specified by the standard deviation, σ, of the first mode. Clearly, (111) also entails a fairly rapid increase of uncer-tainty level with increasing modal order.

Under the above conditions, stochastic designs were com-puted for a range of values of σ, ρ_r and ρ_o. For each such design, root-loci were determined for system perturbations of the form (112). It suffices to note that both stochastic and deterministic designs were stable for all frequency variations considered. Thus, improvement in robustness is not an important issue in this example problem.

What this example shows most clearly is the distinctive form of the control provided by the stochastic approach. Figures 15 through 20 contrast the deterministic design ($\sigma = 0$) with a stochastic design ($\sigma = 0.2$), both with $\rho_r = 0.1$ and $\rho_o = 0.1$ and 15 modes retained in the design model.

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Figures 15 and 16 show the diagonal dominance indicators defined by (113). Recall that P_k, $Q_k = 0$ implies that only the diagonal element of the kth row of P and Q, respectively, are non-zero. Consequently, the results for the stochastic design show that except for the sub-blocks corresponding to the first ~8 modes, P and Q are nearly diagonal. In contrast, the indicators P_k and Q_i for the deterministic design remain significantly above zero for all modes considered.

Diagonalization of P and Q in the stochastic design results in suppression of displacement loops as can be seen from the regulator and observer gains displayed in Figures 17-20. While regulator position gains increase with modal order in the deterministic design (Figure 17), the stochastic position gains are negligible beyond the 8th mode. At the same time, velocity gains (Figure 18) are nearly the same in both designs. The same general tendencies can be seen for the observer gains (Figures 19 and 20). Thus, displacement loops are almost completely suppressed for the higher-order, relatively more uncertain modes and the solution for the corresponding portions of P and Q reduces to the asymptotic forms given by Theorem 8.

It should also be noted from Figures 17-20 that gains for the first three modes (corresponding to more than 20% closed-loop damping for these modes) are nearly the same for both deterministic and stochastic designs. This indicates a region of "quasi-deterministic" control for low order modes.

Summary: This case conforms to the conditions assumed in the theorems of Chapter 3 and it is no surprise that robust stability is readily achieved. Moreover, it is evident that the stochastically designed control for modes beyond the 8th closely approximates the asymptotic form given by Theorem 8 and Lemma 8. In fact, since the asymptotic control is rapidly approached with
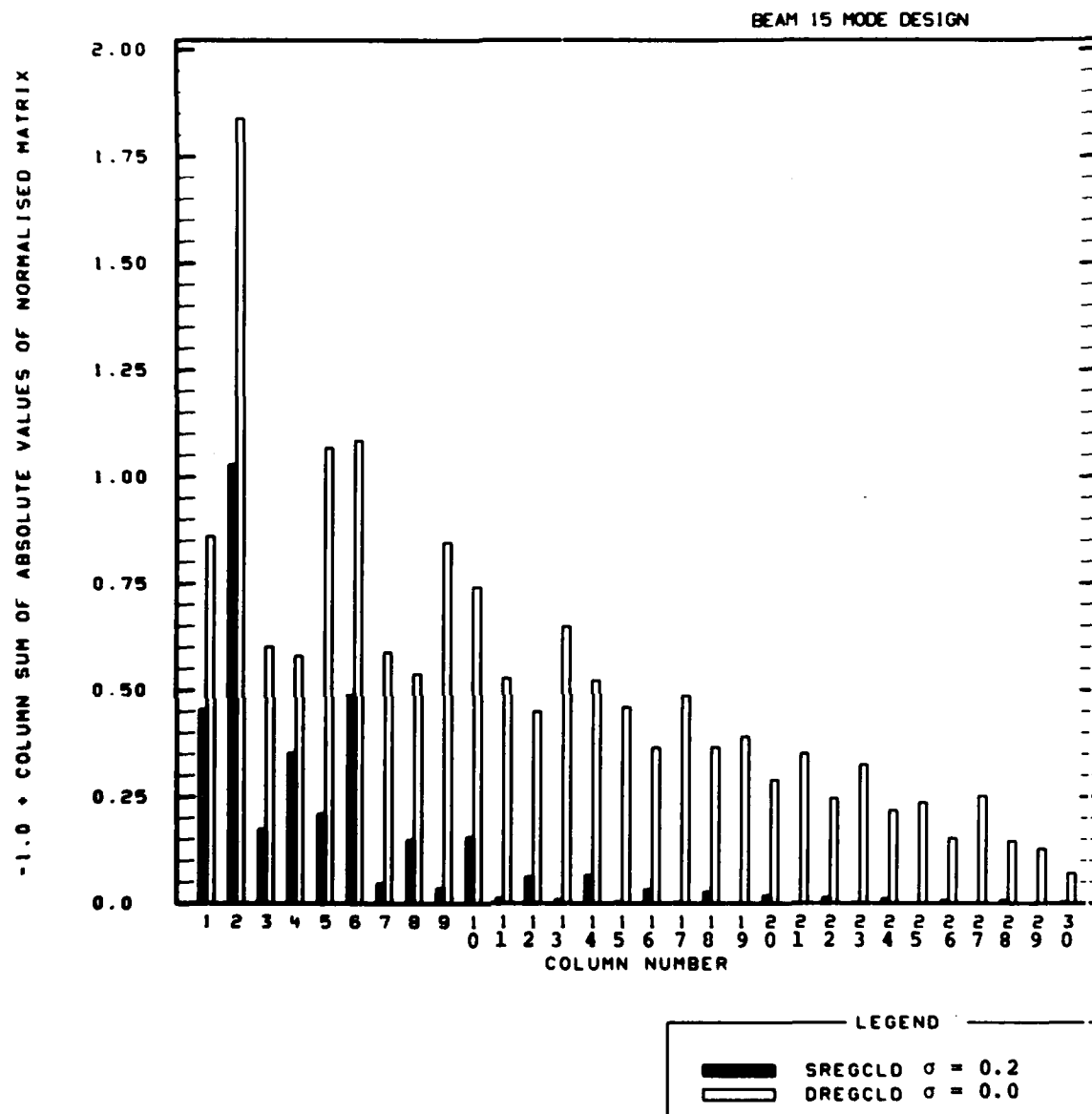
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BEAM 15 MODE DESIGN
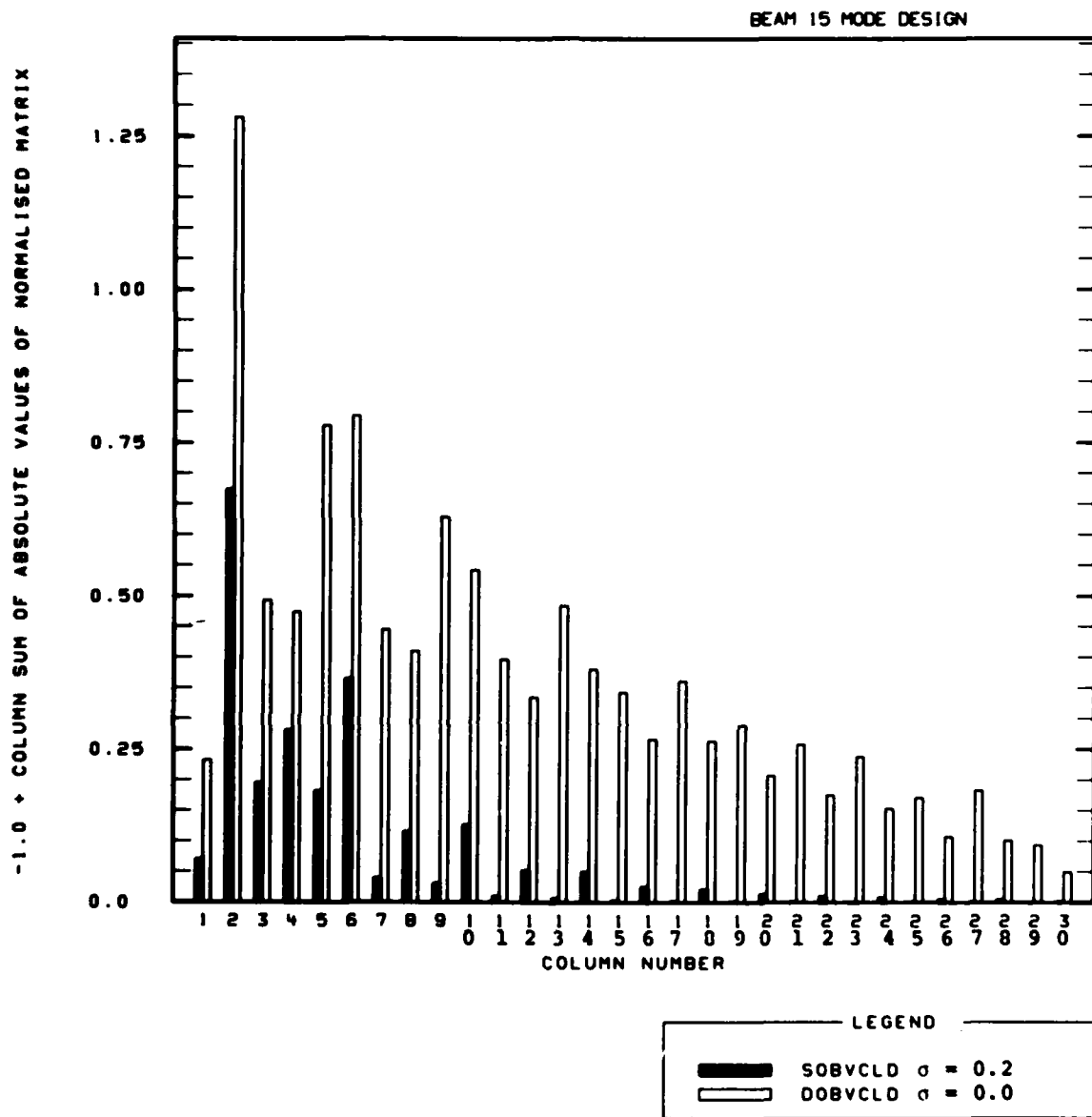
Fig. 15 Diagonal dominance indicator for regulator
cost matrix

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Fig. 16 Diagonal Dominance indicator for observer cost matrix
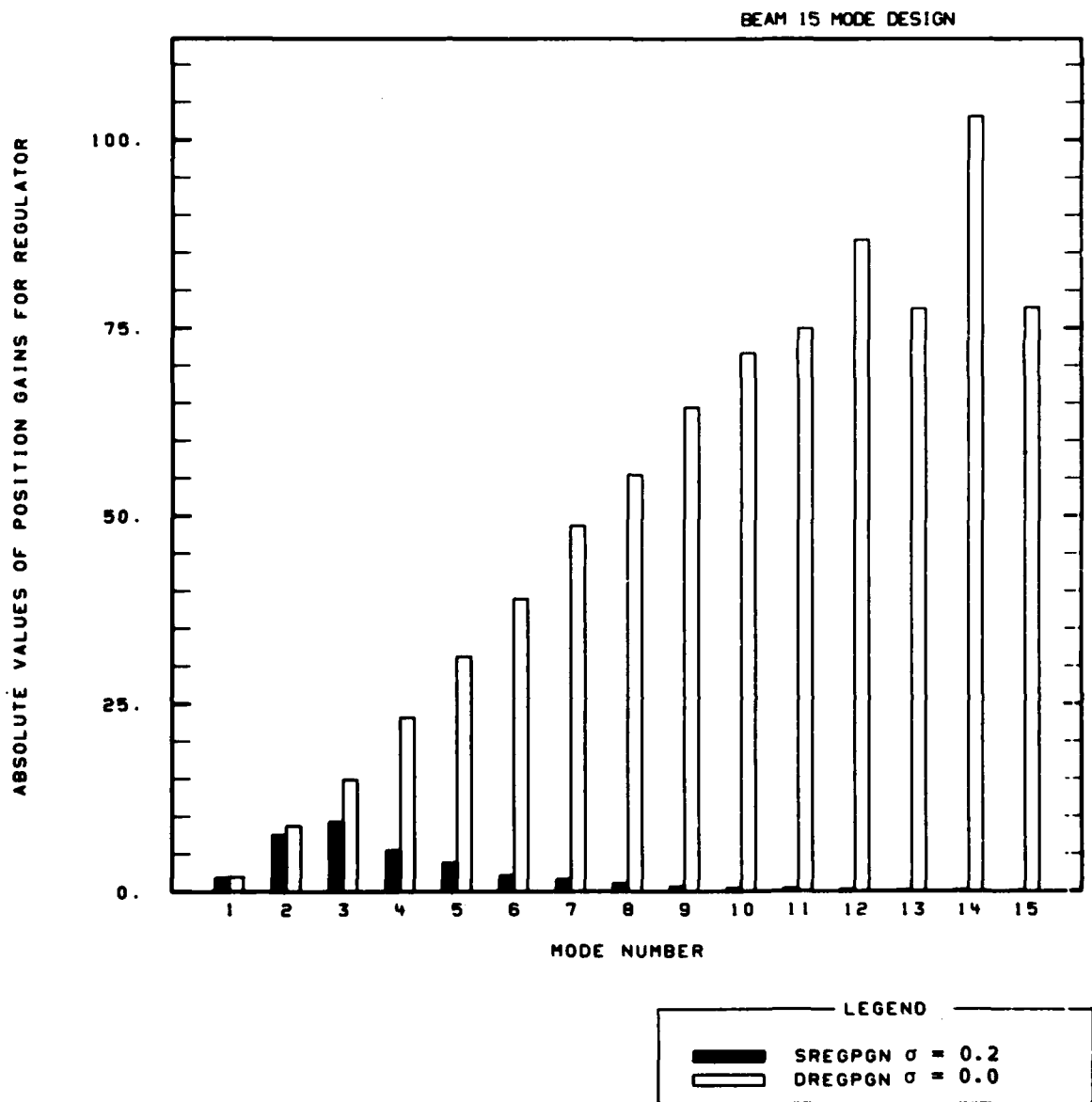
Fig. 17 Position gains for stochastic and deterministic
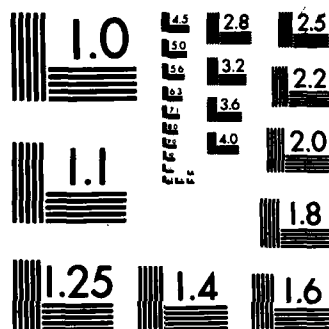regulator

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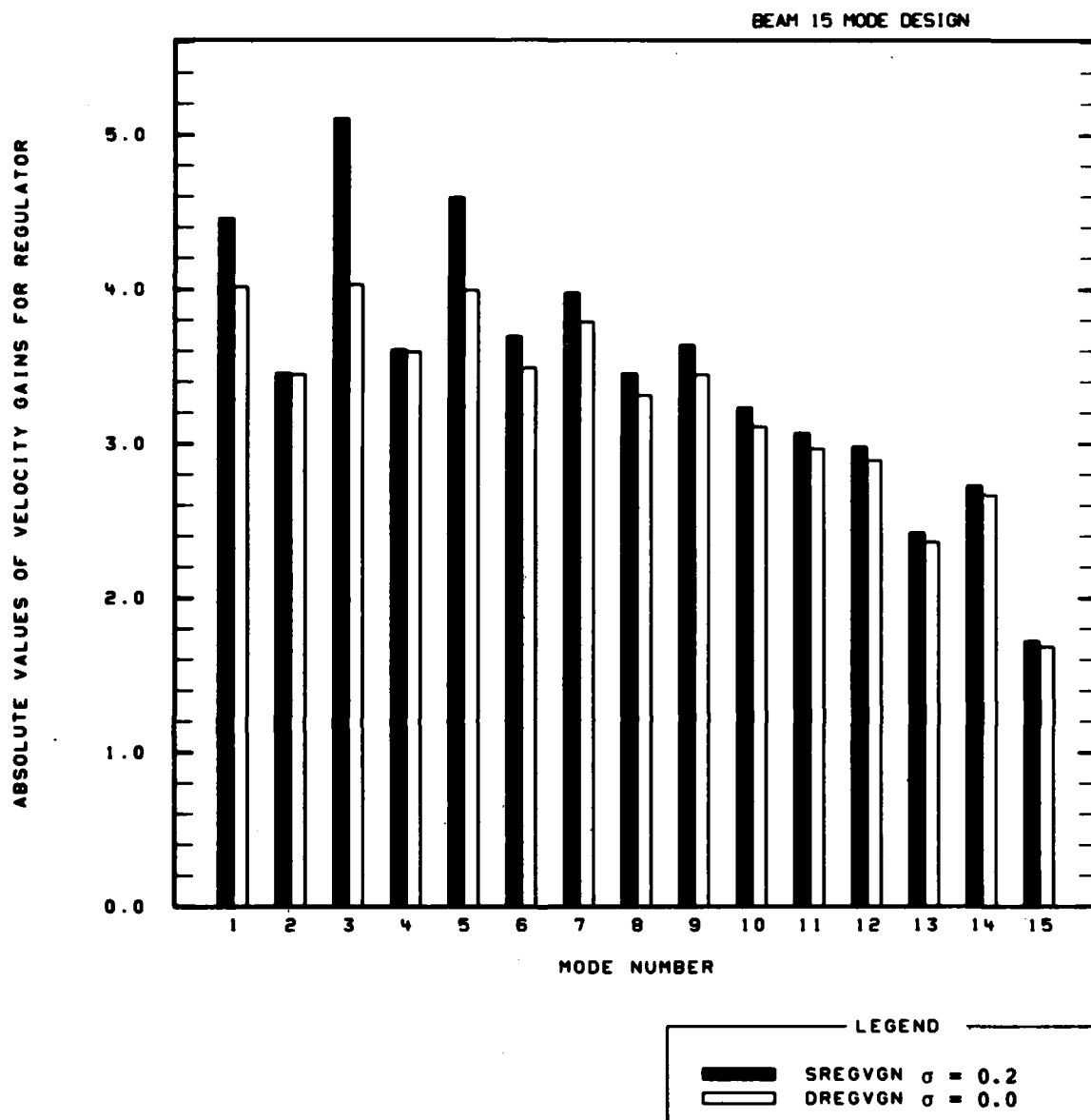
Fig. 18 Velocity gains for stochastic and deterministic regulator
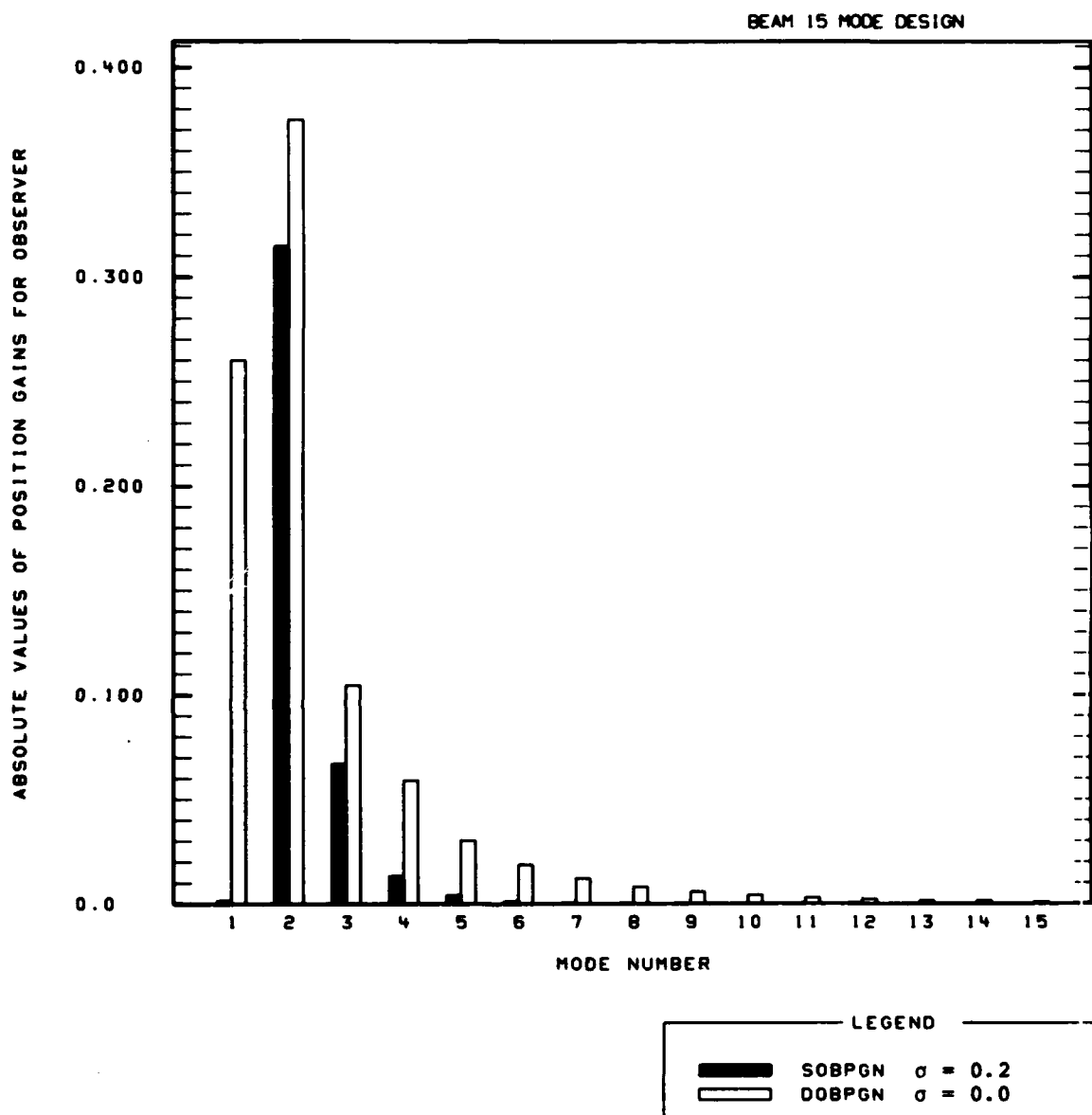
Fig. 19 Position gains for stochastic and deterministic observer

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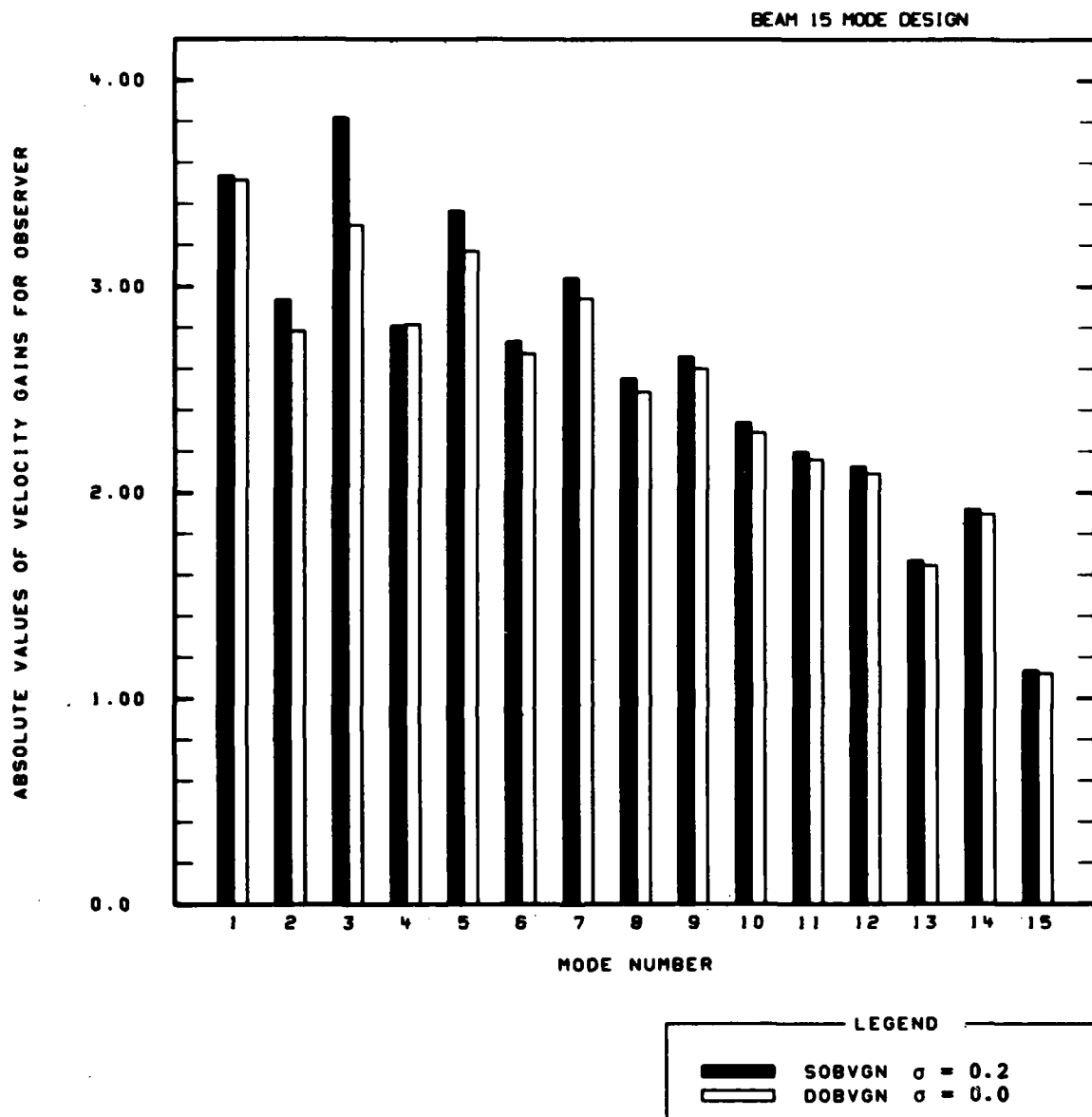
Fig. 20 Velocity gains for stochastic and deterministic observer

increasing modal order, (93) may be used to determine the gains for as many modes beyond the 15th as desired with no significant computational burden. Because of the properties noted above for the control of low-order modes, we may say in summary that the stochastic design approach automatically produces a high authority, essentially deterministic design for low-order, well-known modes and a low authority, rate output feedback control for high-order, very uncertain modes. These two regimes are seen to exist as limiting qualitative features of a unified, globally stable design.

5. CONCLUSION

In the foregoing developments, the minimum information approach to the modelling of uncertain structural systems has been applied to the mean-square optimal design of full-order dynamic compensation. It was seen that judicious choice of the compensator dynamics matrix permits the optimality conditions to be reduced to relatively simple forms. Under fairly mild restrictions, these optimality conditions were shown to possess unique solutions for which closed-loop stochastic stability is guaranteed. Furthermore, the control reduces to a simple, inherently robust asymptotic form for sufficiently high levels of modelled uncertainty.

A simple, straightforward computational scheme was devised for numerical solution of the optimality conditions. The first two numerical examples considered in the last chapter not only show this algorithm capable of handling rigid body and open-loop unstable modes but also suggest that the conditions assumed in the existence and uniqueness result of Chapter 3 can be considerably relaxed. Moreover, these example problems show that the stochastic design approach can enormously increase stability margins over what can be achieved by a conventional, LQG design.

Furthermore, the last numerical example discussed in Section 4.3 suggests that when reciprocal decorrelation times increase monotonically with modal order, closed-loop modes may be divided into two qualitative regimes: the "coherent" and "incoherent" systems (to use the terms introduced in Ref. [7]). Incoherent modes are associated with great *a priori* uncertainty and are mutually uncorrelated and uncorrelated with the coherent system composed of low-order, relatively well-known modes. Since the mean-square optimal control for the incoherent system is known in advance (by virtue of Theorem 8 and expressions (93)),

these qualitative features permit the solution of (89) and (90) for very high-order systems by combination of (93) with the solution of the reduced-order forms of (89) and (90) for the coherent system.

Such a scheme would <u>reduce the computational burden to that associated with the relatively few well-known modes</u>. Of course, this would obviate the difficulties of design computation but not of implementation. Although the asymptotic form of control for the incoherent system approximates a rate output feedback law, thereby aiding practical implementation, a completely satisfactory treatment must await the extension of the stochastic design approach to fixed-order dynamic compensation.

APPENDIX 1

Proof of Theorem 4

First, we write (46.c) and (49) in accordance with the state transformation (50). Using (51), and the notation:

$$\zeta \triangleq \overline{\mu}_m - \beta\kappa - f\gamma - \alpha \qquad (A.1.1)$$

equations (46.c) and (49.a) yield

$$0 = (\overline{\mu}_m - \beta\kappa)^H \tilde{P}_\xi + \zeta^H \tilde{P}^H_{\xi q} + \tilde{P}_\xi (\overline{\mu}_m - \beta\kappa) + \tilde{P}_{\xi q}\zeta \qquad \text{a.}$$

$$+ r_1 + \kappa^H R_2 \kappa + I\{\tilde{P}_\xi + \tilde{P}_{\xi q} + \tilde{P}^H_{\xi q} + \tilde{P}_q\}$$

$$0 = (\overline{\mu}_m - \beta\kappa)^H \tilde{P}_{\xi q} + \zeta^H \tilde{P}_q + \tilde{P}_\xi \beta\kappa + \tilde{P}_{\xi q}(\beta\kappa + \alpha) \qquad \text{b.} \qquad (A.1.2)$$

$$- \kappa^H R_2 \kappa$$

$$0 = \kappa^H \beta^H \tilde{P}_{\xi q} + (\beta\kappa + \alpha)^H \tilde{P}_q + \tilde{P}^H_{\xi q}\beta\kappa + \tilde{P}_q(\beta\kappa + \alpha) \qquad \text{c.}$$

$$+ \kappa^H R_2 \kappa$$

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$$0 = (\overline{\mu}_m - \beta\kappa)\; \tilde{Q}_\xi + \beta\kappa\tilde{Q}_{\xi q}^H + \tilde{Q}_\xi(\overline{\mu}_m - \beta\kappa)^H \qquad \text{a.}$$

$$+ \tilde{Q}_{\xi q}\kappa^H\beta^H + I\{\tilde{Q}_\xi\} + v_1$$

$$0 = (\overline{\mu}_m - \beta\kappa)\; \tilde{Q}_{\xi q} + \beta\kappa\; \tilde{Q}_q + \tilde{Q}_\xi\zeta^H + \tilde{Q}_{\xi q}(\alpha + \beta\kappa)^H \qquad \text{b.} \left.\begin{array}{c}\\ \\ \\ \\ \\ \\ \end{array}\right\} \quad \text{(A.1.3)}$$

$$+ I\{\tilde{Q}_\xi\} + v_1$$

$$0 = \zeta\tilde{Q}_{\xi q} + (\alpha + \beta\kappa)\; \tilde{Q}_q + \tilde{Q}_{\xi q}^H\zeta^H + \tilde{Q}_q(\alpha + \beta\kappa)^H \qquad \text{c.}$$

$$+ I\{\tilde{Q}_\xi\} + v_1 + f\; v_2 f^H$$

and (49.b,c) become:

$$\kappa(\tilde{Q}_\xi - \tilde{Q}_{\xi q} - \tilde{Q}_{\xi q}^H + \tilde{Q}_q)$$

$$= R_2^{-1}\beta^H\; [\tilde{P}_\xi(\tilde{Q}_\xi - \tilde{Q}_{\xi q}) + \tilde{P}_{\xi q}(\tilde{Q}_{\xi q}^H - \tilde{Q}_q)] \qquad \text{(A.1.4)}$$

$$\tilde{P}_q f = [\tilde{P}_{\xi q}^H \tilde{Q}_\xi + \tilde{P}_q \tilde{Q}_{\xi q}^H]\; \gamma^H\; v_2^{-1} \qquad \text{(A.1.5)}$$

while the identity, (49.d), assumes the form:

$$\tilde{P}_{\xi q}^H(\tilde{Q}_{\xi q} - \tilde{Q}_\xi) + \tilde{P}_q(\tilde{Q}_q - \tilde{Q}_{\xi q}^H) = 0 \qquad \text{(A.1.6)}$$

Now taking $\zeta = 0$ (i.e., choosing α as in (53)), we show that (54) through (57) identically satisfy (A.1.2) through (A.1.6).

First note that with $\tilde{P}_{\xi q} = 0$, (A.1.6) becomes

$$\tilde{P}_q(\tilde{Q}_q - \tilde{Q}_{\xi q}^H) = 0$$

which is identically satisfied by (54.b). With (54), (A.1.4) and (A.1.5) assume the forms

$$[\kappa - R_2^{-1}\beta^H \tilde{P}_\xi] (\tilde{Q}_\xi - \tilde{Q}_q) = 0$$

$$\tilde{P}_q[f - \tilde{Q}_q\gamma^H v_2^{-1}] = 0$$

and these are satisfied by (55). By virtue of (55.b), it is readily checked that with $\zeta = 0$, (A.1.3.b) and (A.1.3.c) yield $Q_q = Q_{\xi q}^H$ in accordance with (54.b). Thus, $Q_{\xi q}$ and equation (A.1.3) need not be considered further. Moreover, with (54.a) and (55.a), (A.1.2.b) becomes:

$$0 = \zeta^H P_q$$

which is satisfied if ζ vanishes.

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In consequence, only (A.1.2.a), (A.1.2.c), (A.1.3.a) and (A.1.3.c) remain for consideration. With $\zeta = 0$ and (54) these equations become (56) and (57).

Thus, Eqs. (54) through (57) identically satisfy the stationary conditions (46.c) and (49).

APPENDIX 2

Proof of Theorem 5

Since only elastic modes are considered and $\eta_k > 0$ for all k, it follows that $(\bar{\mu}, \sigma^{\frac{1}{2}})$ is stabilizable and $(r_1^{\frac{1}{2}}, \bar{\mu})$ is detectable. As $\hat{P}$ is presumed to exist as a positive semi-definite matrix, $((r_1 + I\{P\})^{\frac{1}{2}}, \bar{\mu})$ is also detectable by Theorem 3.6 of Ref. [20]. In other words (62.a) assumes the form of the stochastic Riccati equation for the regulator problem:

$$0 = \bar{\mu}_m^H P + P\bar{\mu}_m + I\{P\} + s - P\,\sigma\,P$$

$$s \stackrel{\Delta}{=} r_1 + I\{P\} \geq 0$$

$$(\bar{\mu},\ \sigma^{\frac{1}{2}}) \quad \text{stabilizable}$$

$$(s^{\frac{1}{2}}, \bar{\mu}) \quad \text{detectable}$$

(A.2.1)

Then by virtue of Theorem 12 of Ref. [7], $\bar{\mu} - \sigma P$ is asymptotically stable. But $-I/2$ is non-positive and diagonal so that $\bar{\mu}_m - \sigma P$ is also asymptotically stable. Analogous reasoning for (62.b) establishes the asserted stability properties of $(\bar{\mu} - Q\,\bar{\sigma})$ and $(\bar{\mu}_m - Q\,\bar{\sigma})$ and completes the proof of part A.

Consider system (58) in the absence of disturbance and observation noise:

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$$\dot{\xi} = \mu\xi - \beta\kappa q$$

$$\dot{q} = (\overline{\mu}_m - \beta\kappa)q + f\gamma(\xi - q)$$

$$\left.\right\} \quad \text{(A.2.2)}$$

with κ and f given by (61). To prove part B we show exponential asymptotic stability of the second moment response of this system. Partitioning the second moment matrix of $\begin{pmatrix} \xi \\ q \end{pmatrix}$ in accordance with (48), so that

$$Q \triangleq E\left[\begin{pmatrix} \xi \\ q \end{pmatrix}(\xi^H, q^H)\right] = \begin{bmatrix} Q_\xi & Q_{\xi q} \\ Q_{\xi q}^H & Q_q \end{bmatrix} \qquad \text{(A.2.3)}$$

the Lyapunov equation for Q yields:

$$\dot{Q}_\xi = \overline{\mu}_m Q_\xi - \beta\kappa \, Q_{\xi q}^H + Q_\xi \overline{\mu}_m^H - Q_{\xi q}\kappa^H\beta^H + I\{Q_\xi\} \qquad \text{a.}$$

$$\dot{Q}_{\xi q} = \overline{\mu}_m Q_{\xi q} - \beta\kappa \, Q_q + Q_\xi \gamma^H f^H + Q_{\xi q}(\overline{\mu}_m - f\gamma - \beta\kappa)^H \qquad \text{b.} \quad \left.\right\} \text{(A.2.4)}$$

$$\dot{Q}_q = f\gamma \, Q_{\xi q} + (\overline{\mu}_m - f\gamma - \beta\kappa)Q_q + Q_{\xi q}^H\gamma^H f^H + Q_q(\overline{\mu}_m - f\gamma - \beta\kappa)^H \qquad \text{c.}$$

Alternately, letting:

$$\tilde{Q} \triangleq \begin{bmatrix} \tilde{Q}_\xi & \tilde{Q}_{\xi q} \\ \tilde{Q}_{\xi q}^H & \tilde{Q}_q \end{bmatrix} = T \, Q \, T^T \qquad \text{(A.2.5)}$$

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with T defined by (50), we have:

$$\dot{\tilde{Q}} = (\bar{\mu}_m - \beta\kappa)\tilde{Q}_\xi + \beta\kappa\tilde{Q}_{\xi q}^H + \tilde{Q}_\xi(\bar{\mu}_m - \beta\kappa)^H + \tilde{Q}_{\xi q}\kappa^H\beta^H + I\{\tilde{Q}_\xi\} \quad \text{a.}$$

$$\dot{\tilde{Q}}_{\xi q} = (\bar{\mu}_m - \beta\kappa)\tilde{Q}_{\xi q} + \beta\kappa\tilde{Q}_q + \tilde{Q}_{\xi q}(\bar{\mu}_m - f\gamma)^H + I\{\tilde{Q}_\xi\} \quad \text{b.} \quad \left.\right\} \quad (A.2.6)$$

$$\dot{\tilde{Q}}_q = (\bar{\mu}_m - f\gamma)\tilde{Q}_q + \tilde{Q}_q(\bar{\mu}_m - f\gamma)^H + I\{\tilde{Q}_\xi\} \quad \text{c.}$$

Assuming P and $\hat{P}$ to be positive semi-definite solutions to (62) and (63), consider the non-negative quantity:

$$\pounds \triangleq \text{tr}\left[\begin{bmatrix} P & 0 \\ 0 & \hat{P} \end{bmatrix} \tilde{Q}\right] \qquad (A.2.7)$$

Use of (62.a), (63.a) and (A.2.7) yields:

$$\dot{\pounds} = -\text{tr}[\tilde{Q}_\xi r_1 + [I_{2n}, -I_{2n}]\begin{bmatrix} \tilde{Q}_\xi & \tilde{Q}_{\xi q} \\ \tilde{Q}_{\xi q}^H & \tilde{Q}_q \end{bmatrix}\begin{bmatrix} I_{2n} \\ -I_{2n} \end{bmatrix} P \sigma P]$$

$$= -\text{tr}[Q_\xi r_1 + Q_q P \sigma P] < 0 \qquad (A.2.8)$$

where the last line follows from (A.2.5). Since $Q_q P \sigma P$ is simply $E[q^H\kappa^H R_2\kappa q]$, we conclude from (A.2.8) that for all $Q_o \triangleq Q(t=0) \geq 0$, $\beta\kappa q$ converges exponentially to zero in the mean-square as t increases without bound. Furthermore:

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$$\left|(\beta\kappa Q^H_{\xi q})_{kj}\right| \le \left((E(\beta\kappa q \ q^H\kappa^H\beta^H))_{kk}(E(\xi \ \xi^H))_{jj}\right)^{\frac{1}{2}} \qquad (A.2.9)$$

In view of (A.2.9) and because $\Delta_{Q\xi}[\bar{\mu}_m Q_\xi + \dot{Q}_\xi \ \bar{\mu}_m^H + I\{Q_\xi\}]$ is asymptotically stable (see proof of Theorem 8, Ref. [7]), (A.2.4.a) shows that $Q_\xi = \tilde{Q}_\xi$ must converge exponentially to zero.

Now consider (A.2.6.b,c). The term $I\{\tilde{Q}_\xi\}$ in (A.2.6.c) converges exponentially to zero while $(\bar{\mu}_m - f\gamma)$ is asymptotically stable. Thus, the solution, $\tilde{Q}_q$ of (A.2.6.c) converges exponentially. Since both $(\bar{\mu}_m - \beta\kappa)$ and $(\bar{\mu}_m - f\gamma)$ are asymptotically stable, similar reasoning applied to (A.2.6.b) shows that $\tilde{Q}_{\xi q}$ also converges exponentially. Consequently, all elements of Q converge exponentially to zero as t increases without bound and system (A.2.2) is second mean stable. Almost sure exponential stability is then directly implied.

Finally, since (61) through (63) imply the stationary condition and the control given by (61) is admissible, the conclusion of part C follows by Lemma 2.

APPENDIX 3

As a preliminary step to the proof of Theorem 6 we show:

Lemma A.3.1

Under the conditions of Theorem 6 suppose that P_i, $Q_i > 0$. Then:

$$(\bar{\mu}_m, \sigma^{\frac{1}{2}}) \quad , \quad (\bar{\mu}_m, v_1^{\frac{1}{2}}) \qquad \text{controllable}$$

$$(r_1^{\frac{1}{2}}, \bar{\mu}_m) \quad , \quad (\bar{\sigma}^{\frac{1}{2}}, \bar{\mu}_m) \qquad \text{reconstructible}$$

and

$$((P_i \sigma P_i)^{\frac{1}{2}}, \bar{\mu}_m - Q_i \bar{\sigma}) \qquad \text{reconstructible}$$

$$(\bar{\mu}_m - \sigma P_i, (Q_i \bar{\sigma} Q_i)^{\frac{1}{2}}) \qquad \text{controllable}$$

Proof

$(\bar{\mu}, \sigma^{\frac{1}{2}})$ is controllable if and only if (see Ref. [20], p. 45)

$$\text{rank } [\bar{\mu} - \lambda 1, \ \sigma^{\frac{1}{2}}] \ = \ 2n$$

for all λ defined on the spectrum of $\bar{\mu}$. Since $\bar{\mu}$ and I are both diagonal and the modal frequencies, $\bar{\omega}_k$, $k = 1, \ldots, n$ are assumed distinct, this implies:

$$\text{rank } [\bar{\mu}_m - \lambda 1, \ \sigma^{\frac{1}{2}}] \ = \ 2n$$

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for all λ on the spectrum of $\bar{\mu}_m$. Therefore $(\bar{\mu}_m, \sigma^{\frac{1}{2}})$ is controllable. The stated conclusions for $(\bar{\mu}_m, v_1^{\frac{1}{2}})$, $(r_1^{\frac{1}{2}}, \bar{\mu}_m)$ and $(\bar{\sigma}^{\frac{1}{2}}, \bar{\mu}_m)$ can be shown in a similar manner.

Now consider the matrix $[\bar{\mu}_m^H - \lambda 1, P_i\beta]$ with λ on the spectrum of $\bar{\mu}_m^H$. Writing this in the modal coordinate basis (in which the nominal system map, $\bar{\mu}$, has the matrix block-diag $\begin{pmatrix} 0 & 1 \\ -\bar{\omega}_k^2 & -2\eta_k\bar{\omega}_k \end{pmatrix}$) and noting that $(\bar{\mu}_m^H, \beta)$ is controllable, it is seen that the rank of $[\bar{\mu}^H - \lambda 1, P_i\beta]$ is less than 2n only if β is orthogonal to more than one row of P_i. But since $P_i > 0$ this is impossible. Thus.

$$\text{rank } [\bar{\mu}_m^H - \lambda 1, \ P_i \ \beta] \ = \ 2n$$

for all λ on the spectrum of $\bar{\mu}_m^H$. This implies:

$$\sum_{k=0}^{2n-1} (\bar{\mu}_m^H)^k p = C^{2n}$$

where p denotes the image of $P_i\beta$. In consequence, denoting the image of γ^H by G:

$$\sum_{k=0}^{2n-1} (\bar{\mu}_m^H - \bar{\sigma}Q_i)^k p \ = \ \sum_{k=0}^{2n-1} (\bar{\mu}_m^H)^k p + \sum_{k=0}^{2n-2} (\bar{\mu}_m^H)^k G$$

$$= \ C^{2n}$$

i.e., $(\bar{\mu}_m^H - \bar{\sigma}Q_i, P_i\beta)$ is controllable. Since $R_2 > 0$ this means $(\bar{\mu}_m^H - \bar{\sigma}Q_i, (P_i\sigma P_i)^{\frac{1}{2}})$ is also controllable. Therefore $((P_i\sigma P_i)^{\frac{1}{2}}, \bar{\mu}_m - Q_i\bar{\sigma})$ is reconstructible. In exactly analogous fashion, $(\bar{\mu}_m - \sigma P_i, (Q_i\bar{\sigma}Q_i)^{\frac{1}{2}})$ may be shown to be controllable.

The following helps to show that all members of the sequences defined by (67) and (68) can be made positive definite:

Lemma A.3.2

Under the conditions of Theorem 6, $\hat{P}_o > 0$ and $\hat{Q}_o > 0$ together with (67) and (68) imply:

$$P_i, Q_i > 0 \quad ; \quad i \geq 0$$

$$(\bar{\mu}_m - \sigma P_i) \ \& \ (\bar{\mu}_m - Q_i\bar{\sigma}) \quad \text{stable} \ , \ i \geq 0$$

$$\hat{P}_i, \hat{Q}_i > 0 \quad ; \quad i \geq 1$$

Proof

Suppose that $\hat{P}_i > 0$ and $\hat{Q}_i > 0$. Then by Theorem 3.6 of Ref. [20] $((r_1 + I\{\hat{P}_i\})^{\frac{1}{2}}, \bar{\mu})$ and $(\bar{\mu}, (v_1 + I\{\hat{Q}_i\})^{\frac{1}{2}})$ are reconstructible and controllable, respectively. By virtue of Lemma 6 and (67), P_i and Q_i are positive definite and $(\bar{\mu}_m - \sigma P_i)$ and $(\bar{\mu}_m - Q_i\bar{\sigma})$ are asymptotically stable.

Now $\hat{P}_i, \hat{Q}_i > 0$ implies $((P_i\sigma P_i)^{\frac{1}{2}}, \bar{\mu}_m - Q_i\bar{\sigma})$ reconstructible and $(\bar{\mu}_m^H - \sigma P_i, (Q_i\bar{\sigma}Q_i)^{\frac{1}{2}})$ controllable by the previous Lemma. Thus, by virtue of Lemma 4.A and (68), $\hat{P}_{i+1}$ and $\hat{Q}_{i+1}$ are positive definite. The stated conclusion then follows by induction on i. $\quad\square$

As a final preliminary step, we specifically establish quantities which will serve as upper and lower bounds for the sequences defined by (67) and (68).

Lemma A.3.3

Positive definite P^u, $\hat{P}^u$, $\hat{Q}^u$, Q^u are uniquely defined by:

$$0 = \hat{P}^u \overline{\mu}_m + \overline{\mu}_m^H \hat{P}^u + P^u \sigma P^u \qquad \text{a.}$$

$$0 = \overline{\mu}_m^H P^u + P^u \overline{\mu}_m + r_1 + I\{P^u\} + I\{\hat{P}^u\} - P^u \sigma P^u \qquad \text{b.}$$

$$\left.\right\} \text{(A.3.1)}$$

$$0 = \hat{Q}^u \overline{\mu}_m^H + \overline{\mu}_m \hat{Q}^u + Q^u \overline{\sigma} Q^u \qquad \text{a.}$$

$$0 = \overline{\mu}_m Q^u + Q^u \overline{\mu}_m^H + v_1 + I\{Q^u\} + I\{\hat{Q}^u\} - Q^u \overline{\sigma} Q^u \qquad \text{b.}$$

$$\left.\right\} \text{(A.3.2)}$$

Likewise, the equations:

$$0 = \hat{P}^\ell (\overline{\mu}_m - Q^u \overline{\sigma}) + (\overline{\mu}_m - Q^u \overline{\sigma})^H \hat{P}^\ell + P^\ell \sigma P^\ell \qquad \text{a.}$$

$$0 = \overline{\mu}_m^H P^\ell + P^\ell \overline{\mu}_m + r_1 + I\{P^\ell\} - P^\ell \sigma P^\ell \qquad \text{b.}$$

$$\left.\right\} \text{(A.3.3)}$$

$$0 = \hat{Q}^\ell (\overline{\mu}_m - \sigma P^u)^H + (\overline{\mu}_m - \sigma P^u) \hat{Q}^\ell + Q^\ell \overline{\sigma} Q^\ell \qquad \text{a.}$$

$$0 = \overline{\mu}_m Q^\ell + Q^\ell \overline{\mu}_m^H + v_1 + I\{Q^\ell\} - Q^\ell \overline{\sigma} Q^\ell \qquad \text{b.}$$

$$\left.\right\} \text{(A.3.4)}$$

uniquely define positive definite P, $\hat{P}^\ell$, Q and $\hat{Q}^\ell$ with:

$$P^\ell \leq P^u \ , \ P^\ell \leq P^u$$

$$\left. \begin{array}{l} \\ \\ \end{array} \right\} \quad (A.3.5)$$

$$Q^\ell \leq Q^u \ , \ Q^\ell \leq Q^u$$

Moreover, $(\overline{\mu}_m - Q^u \overline{\sigma})$, $(\overline{\mu}_m - Q^\ell \overline{\sigma})$, $(\overline{\mu}_m - \sigma P^u)$ and $(\overline{\mu}_m - \sigma P^\ell)$ are stable.

Proof

First consider (A.3.1). From (A.3.1.a) and the assumption $\lambda_R(\overline{\mu}) < 0$:

$$I\{\hat{P}^u\} < \{P^u \sigma P^u\}$$

Then, by Lemma 6 and (A.3.1.b) it follows that $P^u \leq P*$ where:

$$0 = \overline{\mu}_m^H P* + P* \overline{\mu}_m + r_1 + I\{P*\} + \{P* \sigma P*\} - P* \sigma P* \qquad (A.3.6)$$

Hence the diagonal portion of P* satisfies:

$$0 = \overline{\mu}^H\{P*\} + \{P*\}\overline{\mu} + \{r_1\}$$

As $\lambda_R(\bar{\mu}) < 0$, $\{r_1\} \geq 0$, this uniquely defines $0 \leq \{P*\} < \infty$. Since $P*$ is hermitian, it follows that $P*$ and hence P^u is bounded. Likewise, from (A.3.1.a), P^u is bounded and positive definite.

With the aid of Lemmas 4 and 6, the sequences $\hat{P}_i^u$ and $\hat{P}_i^u$ ($i \geq 0$) defined by

$$
\left.
\begin{aligned}
0 &= \hat{P}_{i+1}^u \bar{\mu}_m + \bar{\mu}_m^H \hat{P}_{i+1}^u + P_i^u \sigma P_i^u \\[2mm]
0 &= \bar{\mu}_m^H P_i^u + P_i^u \bar{\mu}_m + r_1 + I\{P_i^u + \hat{P}_i^u\} - P_i^u \sigma P_i^u \\[2mm]
\hat{P}_o^u &= 0
\end{aligned}
\right\}
\qquad (A.3.7)
$$

may be shown to be positive definite, monotone nondecreasing and, by the previous argument, bounded from above. Use of Lemma 3 then implies that (A.3.1) possesses positive definite solutions for P^u and $\hat{P}^u$. To show uniqueness, let $(P_1^u, \hat{P}_1^u)$ and $(P_2^u, \hat{P}_2^u)$ be two sets of solutions. Manipulation of (A.3.1) produces:

$$
0 = \bar{\mu}_m^H Z + Z \bar{\mu}_m + I\{Z\}
$$

where Z denotes $(P_1^u + \hat{P}_1^u - P_2^u - \hat{P}_2^u)$. It is easily seen that $\Delta_Z[\bar{\mu}_m^H Z + Z\bar{\mu}_m + I\{Z\}]$ is asymptotically stable, whence $Z = 0$ uniquely by Lemma 5. Thus $P^u + \hat{P}^u$ in unique, and (A.3.1.b) assumes the form:

$$0 = \bar{\mu}_m^H P^u + P^u \bar{\mu}_m + r_1 - P^u \sigma P^u$$

$$\tilde{r}_1 \triangleq r_1 + I\{P^u + \hat{P}^u\} \geq 0$$

where $\tilde{r}_1$ is uniquely defined and $(\tilde{r}_1^{\frac{1}{2}}, \bar{\mu}_m)$ is reconstructible. In consequence, the above Riccati equation uniquely defines P^u as a positive definite matrix.

Thus (A.3.1) possesses unique positive definite solutions for P^u and $\hat{P}^u$. Moreover, application of Lemma 6 shows that $(\bar{\mu}_m - \sigma P^u)$ is asymptotically stable. The stated results for (A.3.2) follow analogously.

Finally consider (A.3.3). Application of Lemma 6 to (A.3.3.b) and Lemmas 4 and A.3.1 to (A.3.3.a) suffice to show the existence and uniqueness of positive definite solutions and the stability of $(\bar{\mu}_m - \sigma P^\ell)$. The results for (A.3.4) follow similarly.

Note that since $\hat{P}^u > 0$, comparison of (A.3.1.b) and (A.3.3.b) yields $P^\ell \leq P^u$ by Lemma 6. In consequence, use of Lemma 4 on (A.3.1.a) and (A.3.3.a) suffices to show $\hat{P}^\ell \leq \hat{P}^u$. Analogous proof of the remaining properties, (A.3.5), is straightforward. □

Now we are in a position to prove Theorem 6.

Proof of Theorem 6

Suppose that

$$\hat{P}^u \geq \hat{P}_i \geq \hat{P}^\ell \quad , \quad 0 < \hat{Q}_i \leq \hat{Q}^u \tag{A.3.8}$$

From the previous Lemma $\hat{P}^\ell > 0$ so that $\hat{P}_i > 0$ by Lemma A.3.2. Hence, Lemma 6 and comparison of (A.3.3.b) and (67.a) yields:

$$P_i > P^\ell$$

Similarly:

$$Q_i > Q^\ell$$

Also, since $\hat{Q}_i \leq \hat{Q}^u$, it follows that

$$Q_i \leq Q^u$$

from Lemma 6 and comparison of (A.3.4.b) and (67.b). Similarly, $P_i \leq P^u$.

Manipulation of (68.a) and (A.3.3.a) gives:

$$0 = (\hat{P}_{i+1} - \hat{P}^\ell)(\bar{\mu}_m - Q^u\bar{\sigma}) + (\bar{\mu}_m - Q^u\bar{\sigma})^H (\hat{P}_{i+1} - \hat{P}^\ell)$$
$$+ (\hat{P}_i - \hat{P}^\ell) \, \sigma \, (\hat{P}_i - \hat{P}^\ell) + S \tag{A.3.9}$$

where

$$S \triangleq \hat{P}_{i+1}(Q^u - Q_i)\,\bar{\sigma} + \bar{\sigma}\,(Q^u - Q_i)\hat{P}_{i+1}$$

$$+ P^{\ell}\sigma P_i + P_i \sigma P^{\ell} \qquad\qquad (A.3.10)$$

Now $Q^u - Q_i \geq 0$ and $\bar{\sigma} \geq 0$ and, since $\hat{P}_{i+1} > 0$ by Lemma A.3.3, the positive semi-definiteness of $[\hat{P}_{i+1}(Q^u - Q_i)\bar{\sigma} + \bar{\sigma}(Q^u - Q_i)\hat{P}_{i+1}]$ follows by use of Lemma 4. Similarly, $(P^{\ell}\sigma P_i + P_i \sigma P^{\ell}) \geq 0$ so that $S \geq 0$. Also, because $P_i - P^{\ell} > 0$, $(((P_i - P^{\ell})\sigma(P_i - P^{\ell}))^{\frac{1}{2}}$, $\bar{\mu}_m - Q^u\bar{\sigma})$ is reconstructible by Lemma A.3.1. Then, since $(\bar{\mu}_m - Q^u\bar{\sigma})$ is asymptotically stable, Lemma 4.A gives

$$\hat{P}_{i+1} > \hat{P}^{\ell}$$

Repetition of the same sort of argument yields $\hat{Q}_{i+1} > \hat{Q}^{\ell}$, $\hat{P}_{i+1} \leq \hat{P}^u$ and $\hat{Q}_{i+1} \leq \hat{Q}^u$. In summary, we have shown that (A.3.8) implies:

$$P^{\ell} < P_i \leq P^u \quad , \quad Q^{\ell} < Q_i \leq Q^u$$

$$\hat{P}^{\ell} < \hat{P}_{i+1} \leq \hat{P}^u \quad , \quad \hat{Q}^{\ell} < \hat{Q}_{i+1} \leq \hat{Q}^u$$

Induction on i completes the proof. $\square$

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The minimum information approach to active control of structural systems seeks inherently robust designs by use of mean-square optimization conjoined with a stochastic system model which presumes as little as possible regarding *a priori* information on modal parameter statistics. This report extends earlier results for the regulator problem to the case of full-order dynamic compensation with nonsingular observation noise. Optimality conditions along with sufficient conditions for existence and uniqueness of solutions and for closed-loop stochastic stability are presented. Results concerning asymptotic properties for large uncertainty levels are also given. Numerical results for various simple examples indicate improved robustness properties over standard LQG designs and suggest the possibility that, under the minimum information stochastic approach, the burden of design computation may be reduced to that associated with the relatively well known or "coherent" modes.

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