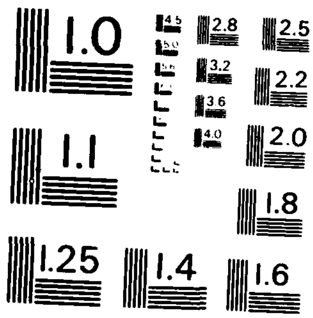
AD-A131 497 PARTS AND SERVICE DEMAND DISTRIBUTION GENERATED BY PRIMARY PRODUCTION(U) ROCHESTER UNIV NY GRADUATE SCHOOL OF MANAGEMENT J KEILSON ET AL. OCT 82 WP-QM8216 1/1

UNCLASSIFIED AFOSR-TR-83-0672 AFOSR-79-0043 F/G 15/5 NL

END
DATE FILMED
8-83
DTIC



MICROCOPY RESOLUTION TEST CHART
NATIONAL BUREAU OF STANDARDS-1963-A

PARTS AND SERVICE DEMAND DISTRIBUTION GENERATED BY PRIMARY PRODUCTION*

by

Julian Keilson
Peter Kubat

Working Paper Series No. QM 8216

October 1982

The Graduate School of Management
The University of Rochester

*This paper was supported in part by and motivated by GTE Laboratories, Waltham, MA 02154, and by the United States Air Force, Office of Scientific Research, under grant No. AFOSR-79-0043.

UNCLASSIFIED

SECURITY CLASSIFICATION OF THIS PAGE (When Data Entered)

REPORT DOCUMENTATION PAGE		READ INSTRUCTIONS BEFORE COMPLETING FORM
1. REPORT NUMBER AFOSR-TR- 83-0672	2. GOVT ACCESSION NO.	3. RECIPIENT'S CATALOG NUMBER
4. TITLE (and Subtitle) PARTS AND SERVICE DEMAND DISTRIBUTION GENERATED BY PRIMARY PRODUCTION		5. TYPE OF REPORT & PERIOD COVERED TECHNICAL
		6. PERFORMING ORG. REPORT NUMBER Working Paper Series #QM8216
7. AUTHOR(s) Julian Keilson and Peter Kubat		8. CONTRACT OR GRANT NUMBER(s) AFOSR-79-0043
9. PERFORMING ORGANIZATION NAME AND ADDRESS The Graduate School of Management University of Rochester Rochester NY 14627		10. PROGRAM ELEMENT, PROJECT, TASK AREA & WORK UNIT NUMBERS PE61102F; 2304/A5
11. CONTROLLING OFFICE NAME AND ADDRESS Mathematical & Information Sciences Directorate Air Force Office of Scientific Research Bolling AFB DC 20332		12. REPORT DATE OCT 82
		13. NUMBER OF PAGES 22
14. MONITORING AGENCY NAME & ADDRESS(if different from Controlling Office)		15. SECURITY CLASS. (of this report) UNCLASSIFIED
		15a. DECLASSIFICATION DOWNGRADING SCHEDULE

16. DISTRIBUTION STATEMENT (of this Report)

Approved for public release; distribution unlimited.

17. DISTRIBUTION STATEMENT (of the abstract entered in Block 20, if different from Report)

18. SUPPLEMENTARY NOTES

19. KEY WORDS (Continue on reverse side if necessary and identify by block number)

20. ABSTRACT (Continue on reverse side if necessary and identify by block number)

Most commercial and military systems subject to failure have an initial period of growth with uncertain time dependent growth rate. Prediction of spare parts demand and service personnel demand is correspondingly uncertain and existant statistical tools are inadequate for the adaptive ad hoc planning need.

In the authors' model, systems subject to failure enter into use at the epochs of a time-inhomogeneous Poisson process of rate $\lambda(t)$. A component or module of each system in use has constant failure rate ν and generates demand (CONTINUED)

DD FORM 1 JAN 73 1473 EDITION OF 1 NOV 65 IS OBSOLETE

UNCLASSIFIED

SECURITY CLASSIFICATION OF THIS PAGE (When Data Entered)

ITEM #20, CONTINUED: for parts and service. The distribution of the cumulative failures n(t) is obtained. Numerical methods and the asymptotic distribution for large t are described.

ABSTRACT

Most commercial and military systems subject to failure have an initial period of growth with uncertain time dependent growth rate. Prediction of spare parts demand and service personnel demand is correspondingly uncertain and existant statistical tools are inadequate for the adaptive *ad hoc* planning needed.

In our model, systems subject to failure enter into use at the epochs of a time-inhomogeneous Poisson process of rate $\lambda(t)$. A component or module of each system in use has constant failure rate μ and generates demand for parts and service. The distribution of the cumulative failures $N(t)$ is obtained. Numerical methods and the asymptotic distribution for large t are described.

In Section 2, the case of constant Poisson build up where $\lambda(t) = \lambda$ is treated. The cumulative demand distribution at time t is obtained through generating function methods by treating the population of systems in use and the cumulative demand as a bivariate Markov chain. An algorithm for finding the distribution numerically when the cumulative demand at time t is modest is displayed in Section 3. A graphical display shows growing normality for large demand. In Section 4, time dependent entry rate $\lambda(t)$ is treated, and the distribution of cumulative demand N(t) is found. It is shown that this demand is both a compound Poisson distribution and a mixture of Poisson distributions, and that the mean and variance of the demand at time t have simple useful forms. Specifically it is shown that

$$E[N(t)] = \mu \int_0^t \lambda(\tau)(t-\tau)d\tau \tag{1.1}$$

and that

$$\mathrm{Var}[N(t)] = \mu^2 \int_0^t \lambda(\tau)(t-\tau)^2 d\tau + E[N(t)]. \tag{1.2}$$

The asymptotic character of the distribution of demand is exhibited in Section 5, and conditions under which the cumulative demand is normal or Poisson are described. Practical use of the tools developed in this paper is discussed in the final section.

The paper is self-contained. Only a knowledge of probability theory, Poisson processes, the central limit theorem and elementary stochastic process tools as found, e.g., in Ross (1970) is required.

2. Replacement Demand Associated with Constant Poisson Build Up

Consider a random counting process $M(t)$ for "primary" events (e.g., aircraft entering service). We suppose throughout this section that all primary units produced are still functioning at time t. Each primary event generates subsequent secondary Poisson events (e.g., demands for service or part replacement) with rate μ. If T_i is the time of the i-th primary event and if $L(t - T_i)$ is the random number of secondary events up to time t, initiated by the primary event at T_i, then the total number $N(t)$ of events is

$$N(t) = \sum_{i=1}^{M(t)} L(t - T_i) \quad , \tag{2.1}$$

with the usual convention that an empty sum has the value 0. We are interested in the distribution of $N(t)$ when $M(0) = 0$.

We first consider the case for which the primary counting process $M(t)$ is Poisson with rate λ. To find the distribution of $N(t)$ for this case one can use bivariate Markov processes and bivariate generating functions. Let $[M(t), N(t)]$ be the bivariate Markov process on the set of states $\{(m,n);\ 0 \le m,\ 0 \le n\}$ with transition rates

$$\nu_{(m,n)(m+1,n)} = \lambda$$

and (2.2)

$$\nu_{(m,n)(m,n+1)} = \mu m \quad .$$

Let $p_{m,n}(t) = P[M(t) = m,\ N(t) = n \mid M(0) = 0,\ N(0) = 0]$ and let us denote by

$$g(u,v,t) = \sum_{\text{all } m,n} p_{m,n}(t) u^m v^n \tag{2.3}$$

the bivariate probability generating function of $[M(t), N(t)]$.

Since

$$\frac{d\,p_{m,n}(t)}{dt} = -(\lambda + m\mu)p_{m,n}(t) + \lambda p_{m-1,n}(t) + m\mu p_{m,n-1}(t), \quad \text{for} \quad n,m \geq 0 \tag{2.4}$$

one finds

$$\frac{\partial g(u,v,t)}{\partial t} = -\lambda(1-u)g(u,v,t) - \mu u \frac{\partial}{\partial u} g(u,v,t) + \mu uv \frac{\partial}{\partial u} g(u,v,t) \tag{2.5}$$

If $g(u,v,t) = \exp\{Q(u,v,t)\}$ then clearly, for $g(u,v,0) = 1$, one has

$$\frac{\partial Q(u,v,t)}{\partial t} = -\lambda(1-u) - \mu u(1-v)\frac{\partial}{\partial u} Q(u,v,t) \quad . \tag{2.6a}$$

$$Q(u,v,0) = 0 \tag{2.6b}$$

If one seeks a solution linear in u, i.e.,

$$Q(u,v,t) = A(t) + B(v,t)u \tag{2.7}$$

then (2.6) becomes

$$\frac{d}{dt} A(t) + u \frac{\partial B(v,t)}{\partial t} = -\lambda(1-u) - \mu(1-v)u\, B(v,t) \quad . \tag{2.8}$$

One then must have

$$\frac{d}{dt} A(t) = -\lambda$$

$$\frac{\partial}{\partial t} B(v,t) = \lambda - \mu(1-v)B(v,t) \quad .$$

One thus obtains a particular solution of (2.6) with $A(0) = B(0) = 0$:

$$A(t) = -\lambda t \quad ,$$

and

$$B(t) = \lambda t\left[\frac{1 - e^{-\mu t(1-v)}}{\mu t(1-v)}\right] .$$

Hence

$$g(u,v,t) = \exp\{-\lambda t[1 - \frac{u}{\mu t(1-v)} (1 - e^{-\mu t(1-v)})]\} . \qquad (2.9)$$

A simple argument demonstrates that (2.6) has only one solution, which we have found.

The probability generating function for $N(t)$ is then

$$\rho_t(v) = E[v^{N(t)}] = g(1,v,t) \qquad (2.10)$$

which becomes

$$\rho_t(v) = \exp\{-\lambda t[1 - \frac{1 - e^{-\mu t(1-v)}}{\mu t(1-v)}]\} \quad . \qquad (2.11)$$

This is the generating function of a compound Poisson process, whose increments, say V_t, are simply related to a secondary Poisson distribution. Specifically, one can write (2.11) as

$$\rho_t(v) = \exp\{-\lambda t[1 - \alpha_t(v)]\} \qquad (2.12)$$

where

$$\alpha_t(v) = E[v^{V_t}] = \frac{1 - e^{-\mu t(1-v)}}{\mu t(1-v)} \quad . \tag{2.13}$$

$\alpha_t(v)$ is then the generating function of the compounding distribution of V_t. It is easily seen that $\alpha_t(v)$ corresponds to the distribution

$$P[V_t = n] = \frac{1}{\mu t} \sum_{r=n+1}^{\infty} \frac{(\mu t)^r}{r!} e^{-\mu t} \quad . \tag{2.14}$$

The mean and the variance may be obtained directly from (2.11) and are

$$E[N(t)] = \frac{\lambda\mu t^2}{2} , \tag{2.15}$$

$$Var[N(t)] = \lambda t[\frac{(\mu t)^2}{3} + \frac{\mu t}{2}] \quad . \tag{2.16}$$

Note that the variance-mean ratio is

$$\frac{Var[N(t)]}{E[N(t)]} = 1 + \frac{2}{3} \mu t , \tag{2.17}$$

which is always larger than one.

We note that the generating function (2.11) is similar in character to that of a generating function obtained by Berg (1981).

3. Evaluation of the Distribution

We have just shown that the distribution of $N(t)$ is that of a compound Poisson distribution with the underlying Poisson process having rate λ, and with the compounding variable V_t having the distribution (2.14). Thus,

$$p_n(t) = P\{N(t) = n\} = \sum_{k=0}^{\infty} e^{-\lambda t} \frac{(\lambda t)^k}{k!} \alpha_n^k(t) \tag{3.1}$$

where $\alpha_n^k(t)$ is the n-th term of the k-fold convolution of V_t with itself. In other words,

$$\alpha_n^k(t) = P\{V_t^{*k} = n\} , \tag{3.2}$$

where V_t^{*k} denotes the sum of k independent random variables, each having distribution (2.14). The distribution of V_t^{*k} can be obtained easily by convolving iteratively the distribution (2.14).

One can obtain $p_n(t)$ explicitly from (2.11) for small values of n by differentiation. One finds that

$$p_0(t) = \exp(-\lambda t + \frac{\lambda}{\mu}(1 - e^{-\mu t})) , \tag{3.3}$$

$$p_1(t) = \frac{\partial \rho_t(v)}{\partial v}\bigg|_{v=0} = p_0(t) \cdot \frac{\lambda}{\mu}[1 - e^{-\mu t} - \mu t\, e^{-\mu t}] \tag{3.4}$$

and that

$$p_2(t) = \frac{1}{2}\frac{\partial^2}{\partial v^2}\rho_t(v)\bigg|_{v=0} = \frac{\lambda}{\mu} p_0(t)\{\frac{\lambda}{\mu}[1 - e^{-\mu t} - \mu t e^{-\mu t}]^2 + 2(1 - e^{-\mu t}) - 2\mu t e^{-\mu t} - (\mu t)^2 e^{-\mu t}\} \tag{3.5}$$

For larger values of n, say $n \geq 3$, the probabilities $p_n(t)$ can be obtained numerically. The plots of the standardized survival functions, i.e., of

$$P\left\{\frac{N(t) - E[N(t)]}{\sqrt{Var[N(t)]}} > t\right\} ,$$

for various values of parameters $B \equiv \lambda t$ and $A \equiv \mu t$ on normal probability paper are shown in Figure 1. The diagonal straight line in Figure 1 represents the survival function of the standard normal distribution.

From the plots, it can be seen that, for about $A \geq 20$ and $B \geq 10$, the central part of the distribution can be well approximated by the standard normal. Note, that the left tail of the distribution of N(t) is heavier than normal so that the normal approximation underestimates the percentage points of the N(t) distribution. On the other hand, the right side of N(t) is lighter and the mass point at the extreme left represents the probability of having no replacements needed. Theoretical justification of the normality will be given in Section 5.

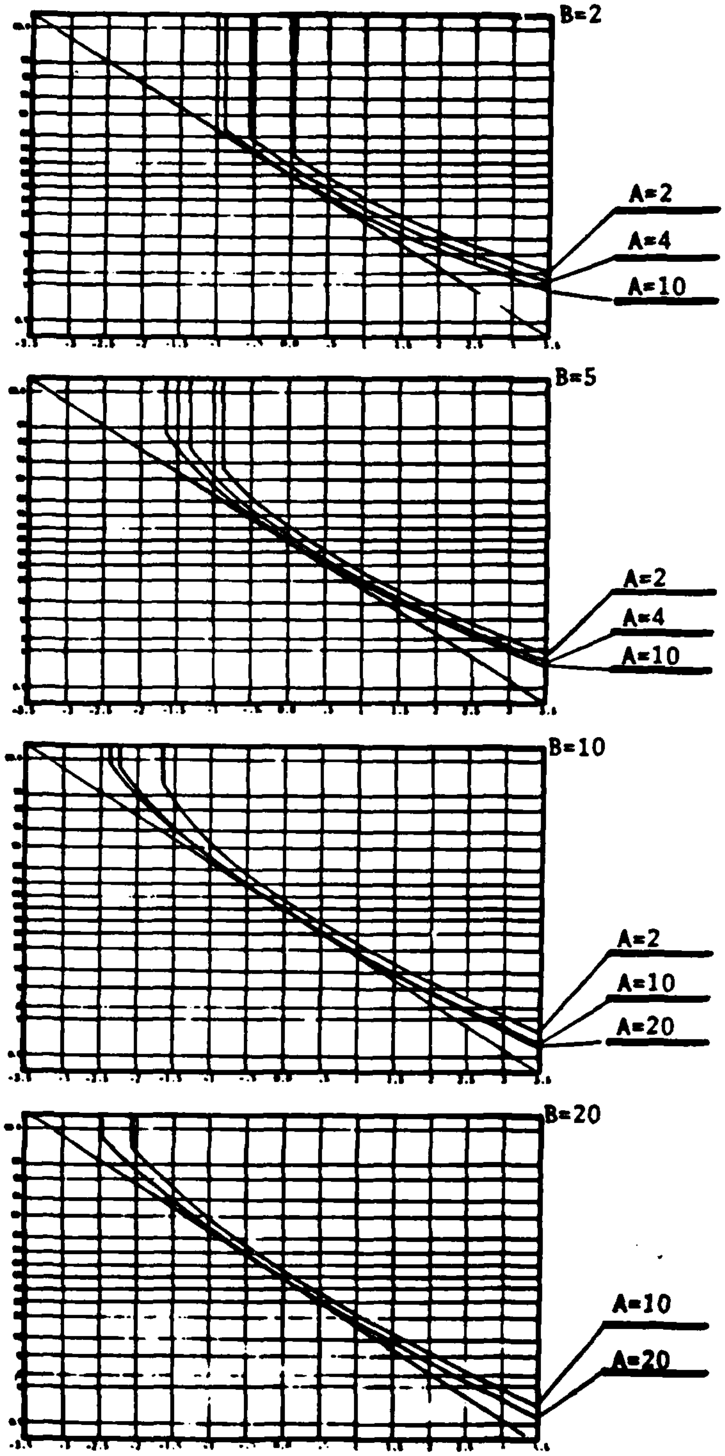


Figure 1: **Standardized Survival Function of N(t) for Constant Poisson Input Stream.**

4. The Distribution of the Cumulative Demand N(t) for More General Primary Input Streams

The process described in Section 2 can be generalized to better model real world situations.

Theorem 4.1. If the sequence of primary events is any point process and if the number of secondary events generated by primary events is Poisson, then N(t) is a mixture of Poisson distributions, i.e.,

$$\rho_t(v) = E[v^{N(t)}] = \int_0^\infty e^{-\theta(1-v)} dG_t(\theta) \tag{4.1}$$

where $G_t(\theta)$ is a c.d.f.

Proof: Let $\omega = (\tau_1(\omega), \tau_2(\omega), \ldots)$ be a sample sequence of primary epochs. Then the conditional probability generating function is

$$\rho_t(v|_\omega) = e^{-\alpha_t(\omega)(1-v)}$$

and $\alpha_t(\omega)$ is a Poisson parameter. Hence

$$\rho_t(v) = \int \rho_t(v|_\omega)\alpha_t(d\omega) \quad ,$$

where $\alpha_t(d\omega)$ is a measure on the space of sample sequences. The theorem then follows. □

Example 4.2. We will verify the Poisson mixture property for the Poisson input case of Section 2. For notational brevity, let $\lambda t \equiv B$ and $\mu t \equiv A$. When the primary stream is Poisson, (2.11) may be written as

$$\rho_t(v) = \exp\{-B[1 - \frac{1 - e^{-A(1-v)}}{A(1-v)}]\} \quad .$$

Letting 1-v = s, we get

$$\int_0^\infty e^{-\theta s} dG(\theta) = \exp\{-B[1 - \frac{1 - e^{-As}}{As}]\} \quad .$$

But this is a Laplace transform of a compound Poisson process obtained from a uniform distribution over (0,A). Hence the mixing distribution $G(\theta)$ is that compound Poisson distribution.

Remark 4.3. The result of Theorem 4.1 is not altered by phase out of primary sources due to failure, obsolescence, or random utilizations, as we see next.

For most industrial applications, the primary production will also vary with time due to initial buildup and ultimate phaseout. Not only will primary units (e.g., aircraft) disappear because of catastrophic failure or obsolescence but the primary units available may have random utilization arising from economic fluctuations, long downtimes, etc.

For such a more general input streams let $M(t)$ be the total number of primary units in operation at time t, and let $N(t)$ be the cumulative number of replacement parts (service procedures) required up to time t. Let $m_\omega(t)$ be a realization of the input process. For each such realization let $N_\omega(t)$ be the random variable associated with that primary realization. If the secondary events are Poisson with rate μ, then $N_\omega(t)$ has Poisson distribution with parameter

$$I_\omega(t) = \mu \int_0^t m_\omega(y)dy \quad . \qquad (4.2)$$

Equation (4.2) defines a random process $I(t)$ of basic interest. Since

$$I(t) = (\mu t) \cdot \frac{1}{t} \int_0^t M(y)dy \tag{4.2a}$$

we see that $I(t)$ is dimensionless and that $(\mu t)^{-1} I(t)$ is the average population up to time t. Moreover,

$$\rho_t(v) = E[v^{N(t)}] = \int e^{-I_\omega(t)(1-v)} \alpha_t(d\omega) =$$

$$= E[e^{-I(t)(1-v)}] \quad . \tag{4.3}$$

Comparing (4.1) and (4.3) we see that

$$G_t(\theta) = P\{I(t) \le \theta\} \quad . \tag{4.4}$$

The following results are a direct consequence of Theorem 4.1.

Theorem 4.4. (a) $E[N(t)] = E[I(t)] = \mu \int_0^t E[M(y)]dy$ (4.5)

(b) $E[N(t)(N(t)-1)] = E[I^2(t)]$ (4.6)

(c) $\mathrm{Var}[N(t)] = \mathrm{Var}[I(t)] + E[I(t)]$. (4.7)

Note that if $I(t)$ is non-random with $E[I(t)] \equiv \xi(t)$, then we have

$$E[v^{N(t)}] = E[e^{-I(t)(1-v)}] = e^{-\xi(t)(1-v)} \quad , \tag{4.8}$$

and thus $N(t)$ is a nonhomogeneous Poisson process with rate $\xi(t)$, i.e.,

$$p_n(t) = \frac{[\xi(t)]^n}{n!} e^{-\xi(t)} \quad , \quad n = 0,1,2,\ldots \tag{4.9}$$

Example 4.5. Suppose that the primary process is deterministic and the interarrival times between primary events equal T^*. In this case

$$m_\omega(y) = k\ , \quad \text{if} \quad kT^* \le y < (k+1)T^*$$

for $k = 0,1,2,\ldots$, and

$$I(t) = I_\omega(t) = \mu \int_0^t m_\omega(y)dy = \mu K[t - (K+1)\frac{T^*}{2}]\ , \qquad (4.10)$$

where $K = [t/T^*]$.

<u>Theorem 4.6</u>. If the primary production stream is a nonhomogeneous Poisson process $M(t)$ with rate $\lambda(t)$ continuous in t then the number of secondary events $N(t)$ has a compound Poisson distribution.

<u>Proof</u>: We have seen in (4.2) that $N(t)$ is a Poisson mixture $K_{I(t)}$ with random mixing parameter

$$I(t) = \mu \int_0^t M(\tau)d\tau = \mu S(t)\ , \qquad (4.11)$$

where $S(t) = \int_0^t M(\tau)d\tau$ is a stochastic integral. One then has to first order in $d\tau$,

$$E[e^{-sdM(\tau)}] = e^{-\lambda(\tau)d\tau[1-e^{-s}]}$$

and

$$\rho_t(v) = E[v^{N(t)}] = \int_0^\infty e^{-\tau(1-v)\mu}\, dF_{S(t)}(\tau) = \phi_{S(t)}(\mu(1-v)) \qquad (4.12)$$

where

$$\phi_{S(t)}(s) = E[e^{-sS(t)}]\ .$$

Hence, $S(t) = \int_0^t dy \int_0^y dM(\tau) = \int_0^t (t-\tau)dM(\tau)$ for every sample path. For any Riemann sequence of partitions P_j of the interval $[0,\tau)$ whose maximum interval $\delta_j = \max_k \delta_{jk}$ goes to zero

$$S(t) = \lim_{\delta_j \to 0} \sum_k (t-\tau_k)[M(\tau_{k+1}) - M(\tau_k)]$$

and

$$\phi_{S(t)}(s) = \lim_{\delta_j \to 0} \prod_k \exp\{-\lambda(\tau_k)\,\delta_{jk}[1-e^{-(t-\tau_k)s}]\}$$

$$= \lim_{\delta_j \to 0} \exp\{-\sum_k \lambda(\tau_k)\,\delta_{jk}[1-e^{-(t-\tau_k)s}]\}$$

$$= \exp\{-\int_0^t \lambda(\tau)[1-e^{-(t-\tau)s}]d\tau\}$$

$$= \exp\{-\Lambda(t)[1 - \int_0^t e^{-(t-\tau)s}\frac{\lambda(\tau)}{\Lambda(t)}d\tau]\} \tag{4.13}$$

where $\Lambda(t) = \int_0^t \lambda(\tau)d\tau$. Thus (4.12) becomes

$$\phi_t(v) = \exp\{-\Lambda(t)[1 - \int_0^t -e^{-(t-\tau)\mu(1-v)}\frac{\lambda(\tau)}{\Lambda(t)}d\tau]\}$$

$$= \exp\{-\Lambda(t)[1-\alpha_t(v)]\}. \tag{4.14}$$

This is the generating function of a compound Poisson distribution, with

$$\alpha_t(v) = \int_0^t e^{-(t-\tau)\mu(1-v)}\frac{\lambda(\tau)}{\Lambda(t)}d\tau \;. \tag{4.15}$$

Remark 4.7. For the homogeneous Poisson case with $\lambda(t) = \lambda$, (4.15) reduces to $\alpha_t(v)$ of (2.13).

5. Asymptotic Normality and Asymptotic Poisson Character of the Cumulative Demand

Consider first the time homogeneous process of Section 2. When the parameters $A = \mu t$ and $B = \lambda t$ are small, one expects that the distribution of $N(t)$ will be Poisson. One also expects that when A and B are large the distribution of $N(t)$ will be close to the normal distribution.

Let $L_{A,B}$ be the number of secondary events $N(t)$ with parameters A and B let K_γ be a Poisson r.v. with parameter γ; let N_{μ,σ^2} be a normal r.v. with mean μ and variance σ^2.

The Poisson character of $L_{A,B}$ for A and B small is described in the following theorem.

Theorem 5.1. For fixed $\alpha > 0$ $L_{A,\alpha/A} \xrightarrow{d} K_{\alpha/2}$ as $A \to 0+$.

Proof: The generating function for $L_{A,\alpha/A}$ is

$$\rho_t(v) = \exp\{-(\alpha/A)[1 - \frac{1 - e^{-A(1-v)}}{A(1-v)}]\} .$$

As $A \to 0^+$ we find from L'Hospital rule that $\lim_{A \to 0^+} \rho_t(v) = \exp\{-\frac{\alpha}{2}(1-v)\}$ which is the generating function of $K_{\alpha/2}$.

It is well known that K_γ is close to normal when γ is large. Theorem 5.1 then say that the distribution will look Poisson when $E[N(t)] = \lambda\mu t^2/2$ is modest and λt is small. The asymptotic normality for the time homogeneous case will be treated in the following discussion for the nonhomogeneous case.

From (4.2) and (4.11) we know that $N(t) = K_{\mu S(t)}$, where $S(t) = \int_0^t M(\tau)d\tau$, with $M(\tau)$ being the total number of primary events up to time τ generated by a nonhomogeneous Poisson process. Differentiating

(4.13) with respect to s at s = 0 we find

$$\bar{S}_t = E[S(t)] = \int_0^t \lambda(\tau)(t-\tau)d\tau = \lambda(t)*t \ . \tag{5.1}$$

Differentiating twice one finds that

$$E[S^2(t)] = \lambda(t)*t^2 + \{\lambda(t)*t\}^2. \tag{5.2}$$

Hence,

$$\sigma_t^2 = \mathrm{Var}[S_t] = \int_0^t \lambda(\tau)(t-\tau)^2 d\tau = \lambda(t)*t^2 \tag{5.3}$$

For (5.1), (5.2) and (5.3), $\lambda(t)$ must be integrable. For the practical cases of interest $\lambda(t)$ will be continuous and differentiable. We assume as much smoothness as needed for what follows.

<u>Lemma 5.2</u>. Let $t/\sigma_t \to 0$. Then

$$(S(t) - \bar{S}_t)/\sigma_t \xrightarrow{d} N_{0,1} \text{ as } t \to \infty.$$

<u>Proof</u>: First note that

$$E[e^{-s(S(t)-\bar{S}_t)}] = \exp\{\frac{s^2}{2} \int_0^t \Psi(s,\tau,t)(t-\tau)^2\lambda(\tau)d\tau\}$$

where

$$\Psi(s,\tau,t) = \frac{e^{-(t-\tau)s}-1+(t-\tau)s}{(t-\tau)^2 s^2/2} \quad .$$

Then

$$E[e^{-s(S(t)-\bar{S}_t)/\sigma_t}] = e^{s^2/2}\exp\{-\frac{s^2}{2}\int_0^t \frac{[1-\Psi(\frac{s}{\sigma_t},\tau,t)]\lambda(\tau)(t-\tau)^2}{\sigma_t^2}d\tau\}$$

(5.4)

It is easy to show that

$$1 - \Psi(\frac{s}{\sigma_t},\tau,t) \le \frac{1}{3}\frac{st}{\sigma_t}, \text{ for } 0 < s < \infty .$$

For s fixed, one then has

$$0 \le \frac{1}{\sigma_t^2}\int_0^t [1-\Psi(\frac{s}{\sigma_t},\tau,t)]\lambda(\tau)(t-\tau)^2 d\tau \le \frac{1}{3}\frac{st}{\sigma_t}. \quad (5.5)$$

As $t \to \infty$, the second exponent in (5.4) goest to zero, since $t/\sigma_t \to 0$ and the normality follows.

From $N(t) = K_{\mu S(t)}$ one finds easily that

$$\bar{N}_t = E[N(t)] = E(\mu S(t)] = \mu\bar{S}_t \quad (5.6)$$

and

$$\xi_t^2 = \mathrm{Var}[N(t)] = \mu^2\sigma_t^2 + \mu\bar{S}_t \quad (5.7)$$

We can now prove the main theorm of this section.

Theorem 5.3. If $\sigma_t^2/\bar{S}_t \to \infty$ as $t \to \infty$ then

$$(N(t) - \bar{N}_t)/\xi_t \xrightarrow{D} N_{0,1} .$$

Proof: It follows from (4.12) that

$$E[e^{-uN(t)}] = E[e^{-\mu(1-e^{-u})S(t)}]$$

and hence

$$E[e^{-u(N(t)-\bar{N}_t)/\xi_t}] = E[\exp\{-\mu(1-e^{-u/\xi_t})S(t) + u\mu\bar{S}_t/\xi_t\}]$$

$$= E[\exp\{-\mu(1-e^{-u/\xi_t})(S(t)-\bar{S}_t)\}]$$

$$\cdot E[\exp\{-\mu\bar{S}_t(1-e^{-u/\xi_t} - \frac{u}{\xi_t})\}]. \quad (5.8)$$

From Lemma 5.2, we see that as $t \to \infty$ the first expectation in (5.8) converges to $u^2/2$ and the second expectation to one. This proves the normality.

Remark 5.4. The condition $\sigma_t^2/\bar{S}_t \to \infty$ is not very restrictive. It holds for a wide range of rates $\lambda(t)$. It holds, for example, when $\lambda(t)$ is a positive constant on a finite interval, or when $\lambda(t) = t^{\alpha}$, $-\infty < \alpha < \infty$, or when $\lambda(t) = \exp\{-\gamma t\}$, $\gamma > 0$. It does not hold, for instance, for $\lambda(t) = \exp\{\beta t\}$, $\beta > 0$.

Remark 5.5. When the primary stream is homogeneous Poisson we have $\lambda(t) \equiv \lambda$ and

$$\bar{S}_t = \lambda \int_0^t (t-\tau)d\tau = \lambda t^2/2$$

and

$$\sigma_t^2 = \lambda \int_0^t (t-\tau)^2 d\tau = \lambda t^3/3 \ .$$

Consequently

$$\bar{N}_t = \mu \bar{S}_t = \mu\lambda t^2/2$$

and

$$\bar{\xi}_t^2 = \mu^2 \lambda t^3/3 + \mu\lambda t^2/2 \ ,$$

which we already obtained in Section 2. Clearly,

$$\frac{\sigma_t^2}{\bar{S}_t} = \frac{\lambda t^3/3}{\lambda t^2/2} = \frac{2}{3} t \to \infty$$

as $t \to \infty$ and the asymptotic normality is justified.

6. Practical Application of the Results

When a new system comes into use, its future growth is often uncertain at least to some degree. For military systems growth may depend on field performance of the systems, on the perception of real costs and on the uncertainities in future government funding. For commercial systems, market acceptance, time dependent price as influenced by volume and competition contribute to the uncertainty of future growth. The primary input rate $\lambda(t)$ is correspondingly unknown. Moreover the failure rate μ of a perticular system component may only emerge from experience. To avoid premature capital outlay costs and storage costs, scheduled incremental acquisition and/or production will often be appropriate. In a typical application, planning will be adaptive and *ad hoc*.

The reader may question the legitimacy of the assumed Poisson rate $\lambda(t)$ for introduction of a system into service. This assumption is made to reflect randomness in the arrival process and to assure tractability. For a commercial system, such as a commercial aircraft or large computer, where market place acceptance has a strong role, the arrival process has a character similar to that of rumor propagation (cf. Dietz (1967), or Bailey (1975)) and the Poisson assumption seems indicated. Even in military contexts where a fixed schedule of introduction is hoped for, randomness in shipping times, training times, etc, encourages the Poisson assumption. In these few cases where introduction at fixed intervals occurs (cf. Example 4.5) the formulae needed are obtained easily.

The adaptive *ad hoc* planning will have only one realization (sample path) ω_0 available. At time t one will have a history for that sample path of cumulative initiations $M(\omega_0,t')$, $0 \le t' \le t$, to date. One will also have for each part type A a history of cumulative replacements $N_A(\omega_0,t')$, $0 \le t' \le t$. A residual number $R_A(t)$ of spare parts on hand will be known. Suppose the failure rate μ is known. Let $t_H = t + \Delta_H$ be some future horizon time of concern. A projected initiation rate $\lambda(t')$ in the period $t \le t' \le t_H$ is assumed available. One can then establish either from (3.1) or from the result of Theorem 5.3 a probability of spare part depletion by time t_H in the absence of new spare parts production in (t,t_H). The production quantities and the production schedule may then be established from lot size techniques such as developed for time varying deterministic demand (cf. Peterson and Silver (1979, pp. 300-341)).

If the failure rate μ is not initially known, it may be estimated at time t from cumulative failure history, as for instance, in Keilson and Sumita (1982). In the latter paper distributed arrival times corresponding to $\lambda(t)$ are incorporated.

Acknowledgement

The authors wish to thank U. Sumita for helpful technical comments.

REFERENCES

Bailey, N. T. J. (1975). The Mathematical Theory of Infectious Diseases and Its Application. Griffin, London.

Berg, M. (1981). Stochastic models for spread of motivating information. Naval Res. Logist. Quart., 28, 133-145.

Dietz, K. (1967). Epidemics and rumours: A survey. J. R. Statist. Soc. A, 130, 505-528.

Keilson, J. and Sumita, U. (1982). Extrapolation of the mean lifetime of a large population from its preliminary survival history. Working Paper Series No. QM-8210, Graduate School of Management, University of Rochester.

Peterson, R. and Silver, E. A. (1979). Decision Systems For Inventory Management and Production Planning. J. Wiley & Sons, New York.

Ross, S. M. (1970). Applied Probability Models with Optimization Applications. Holden-Day, San Francisco.

FILM

9