

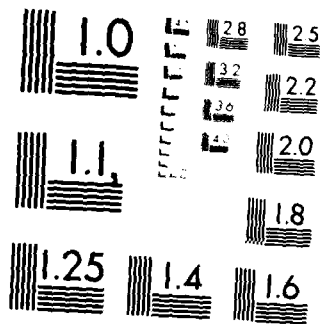
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 -- 23 - TECHNICAL OBJECTIVE: (U) TO IMPROVE THE NAVY'S WEAPONRY BY CREATING
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 -- FINITE SET AND (C) INVESTIGATIONS OF RANDOM PACKING STUDIES IN TWO,
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KINGMAN'S SUBADDITIVE ERGODIC THEOREM

By

J. Michael Steele

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DEPARTMENT OF STATISTICS
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Kingman's Subadditive Ergodic Theorem

By

J. Michael Steele

The objective of this note is to give a proof of Kingman's subadditive ergodic theorem which is perhaps simpler and more direct than those previously given. ([2], [3], [4], [5], [6], [7], [11]).

Theorem. Suppose T is a measure preserving transformation of the probability space $(\Omega, \mathcal{F}, \mu)$ and that $\{g_n, 1 \leq n < \infty\}$ is a sequence of integrable functions which satisfy

$$(1) \quad g_{n+m}(x) \leq g_n(x) + g_m(T^n x) .$$

With probability one we then have the existence of the limit

$$\lim_{n \rightarrow \infty} g_n(x)/n = g(x) \geq -\infty ,$$

where $g(x)$ is an invariant function.

Proof. We first check that $g(x) = \liminf g_n(x)/n$ is an invariant function. Since $g_{n+1}(x)/n \leq g_1(x)/n + g_n(Tx)/n$ we see $g(x) \leq g(Tx)$ which gives $\{g(x) > \alpha\} \subset T^{-1}\{g(x) > \alpha\}$. The fact that T is measure preserving then implies $\{g(x) > \alpha\} = T^{-1}\{g(x) > \alpha\}$ up to null sets. This implies g is measurable with respect to the invariant σ -field and hence is invariant. The function $\phi(x) = \max(t, g(x))$ where $t \in (-\infty, 0)$ is also invariant.

For $\epsilon > 0$, set $A_\ell = \{x: g_\ell(x) \leq \ell(\phi(x) + \epsilon)\}$ and note that $\mu(\bigcup_{\ell=1}^{\infty} A_\ell) = 1$, so we can choose N such that for $B(N) = (\bigcup_{\ell=1}^N A_\ell)^c$ we have $\mu(B(N)) \leq \epsilon$.

Now, by Birkhoff's ergodic theorem, $\frac{1}{n} \sum_{k=1}^n 1_{B(N)}(T^k x)$ converges a.s. to $E(1_{B(N)} | \mathcal{G})$ where $\mathcal{G}$ is the invariant field of T ; so by Chebyshev's inequality

$$\mu(\limsup_{n \rightarrow \infty} \frac{1}{n} \sum_{k=1}^n 1_{B(N)}(T^k x) \geq \lambda) \leq \epsilon/\lambda.$$

Setting

$$C_M = \{x: \sum_{k=1}^n 1_{B(N)}(T^k x) \leq 2n\lambda, \quad \forall n \geq M\}$$

we have for M sufficiently large that $\mu(C_M) \geq 1 - 2\epsilon/\lambda$.

For any $x \in C_M$ and $n \geq M$ we obtain a decomposition for the integer set $[0, n)$ into three classes of intervals by the following algorithm:

Begin with $k = 0$. If k is the least integer in $[0, n)$ not in an interval already constructed then we consider $T^k x$. If $T^k x \in B(N)^c$ then there is an $\ell \leq N$ so that $g_\ell(T^k x) \leq \ell(\phi(T^k x) + \epsilon) \leq \ell(\phi(x) + \epsilon)$ and we take $[k, k+\ell)$ as an element of our decomposition provided $k+\ell \leq n$. If $T^k x \in B(N)$ we take the singleton interval $[k, k+1)$.

This algorithm provides a decomposition of some $[0, n')$ with $n' - N \leq n' \leq n$, and it is extended to a decomposition of $[0, n)$ by adding as many singletons as necessary.

Thus for any $x \in C_M$ we have a decomposition of $[0, n)$ into a set of u intervals $[\tau_i, \tau_i + \ell_i)$, $1 \leq i \leq u$, for which $g_{\ell_i}(T^{\tau_i} x) \leq \ell_i(\phi(x) + \varepsilon)$ together with a set of v singletons $[\sigma_i, \sigma_i + 1)$ for which $1_{B(N)}(T^{\sigma_i} x) = 1$, and a set of w singletons contained in $(n-N, n)$.

By (1) and this decomposition of $[0, n)$ we have on C_M ,

$$(2) \quad g_n(x) \leq \sum_{i=1}^u g_{\ell_i}(T^{\tau_i} x) + \sum_{i=1}^v g_1(T^{\sigma_i} x) + \sum_{i=1}^w g_1(T^{n-i} x) \\ \leq (\phi(x) + \varepsilon) \sum_{i=1}^u \ell_i + \sum_{i=1}^u g_1(T^{\tau_i} x) 1_{B(N)}(T^{\tau_i} x) + \sum_{i=1}^N |g_1(T^{n-i} x)|.$$

Since

$$\sum_{k=1}^{\infty} \mu(|g_1(T^k x)| > \delta k) = \sum_{k=1}^{\infty} \mu(|g_1(x)| > \delta k) < \infty, \text{ for all } \delta > 0,$$

the Borel-Cantelli lemma implies $g_1(T^k x)/k \rightarrow 0$ a.s.. From this one easily sees that almost surely

$$(3) \quad \lim_{n \rightarrow \infty} \frac{1}{n} \sum_{i=1}^N |g_1(T^{n-i} x)| = 0.$$

Also, by Birkhoff's ergodic theorem we have

$$(4) \quad \lim_{n \rightarrow \infty} \frac{1}{n} \sum_{i=1}^n g_1(T^i x) 1_{B(N)} = E(g_1 1_{B(N)} | \mathcal{G}).$$

Finally $n \geq \sum_{i=1}^u \ell_i \geq n - N - 2\epsilon n$ so from (2), (3), (4) we have on C_M that

$$(5) \quad \limsup_{n \rightarrow \infty} g_n(x)/n \leq \phi(x) + 3\epsilon + E(g_1 1_{B(N)} | G) .$$

For $N \rightarrow \infty$, $1_{B(N)} \rightarrow 0$ a.s. so by dominated convergence $E(g_1 1_{B(N)} | G) \rightarrow 0$ a.s.. Therefore, by the arbitrariness of ϵ , t , λ , N , and M we have with probability one that

$$\limsup_{n \rightarrow \infty} g_n(x)/n \leq \liminf_{n \rightarrow \infty} g_n(x)/n ,$$

which completes the proof of convergence.

Remarks. (1). The preceeding proof was motivated by the recent proofs of the Birkhoff ergodic theorem and the Shannon-MacMillan-Breiman theorem given by Paul Shields [8]. That work is in part devoted to the simplification and exposition of some recent work of Ornstein and Weiss [7].

(2). Inspection of the preceeding proof shows that it suffices to assume that just $g_1^+ \in L^1$, instead of $g_n \in L^1$, for all n . That the subadditive ergodic theorem persists under this condition was already observed in Kingman [5, p. 885].

(3). David Aldous has shown that Kingman's subadditive ergodic theorem can be used to give a very brief proof of the ergodic theorem for Banach space due to Maurier [8]. If $\{X_1\}$ is a stationary process with values in a Banach space F , we first note there is no loss in assuming $E(X_1 | G) = 0$ where G is the invariant σ -algebra. Also, we can find a linear operator θ on F with finite dimensional range such that $E\|X_1 - \theta X_1\| \leq \epsilon$. Now

Birkhoff's ergodic theorem (applied to linear functionals) shows that $\frac{1}{n} \sum_{i=1}^n \theta(X_i)$ converges a.s. and in L^1 to $E\theta(X_1)$. The L^1 convergence guarantees $\overline{\lim} E \left\| \frac{S_n}{n} - E\theta(X_1) \right\| \leq \epsilon$ from which it follows that $\lim E \|S_n/n\| = 0$. But since $\|S_n\|$ is a subadditive process $\|S_n/n\|$ converges a.s., and now necessarily converges a.s. to zero.

(4) Andrés del Junco has pointed out that there is a useful device of Akcoglu and Sucheston [1] which can be used to circumvent the estimations of the last two terms in equation (2). The idea is that $g'_m = g_m(x) - \sum_{i=0}^{m-1} g_1(T^i x)$ defines a (negative) subadditive process. The last two terms in equation (2) applied to g'_m would then simply not appear.

The proof given above was retained in order to maximize conceptual simplicity (at the cost of a little extra computation).

Acknowledgement. I would like to thank Paul Shields for making available to me his manuscript [10]. I also owe a debt David Aldous, Andrés del Junco and Joe Marhoul for their comments on an earlier draft and their permission to incorporate the remarks given above.

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