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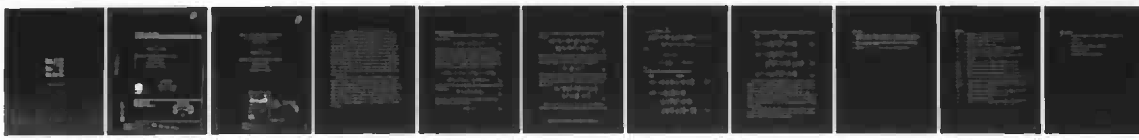

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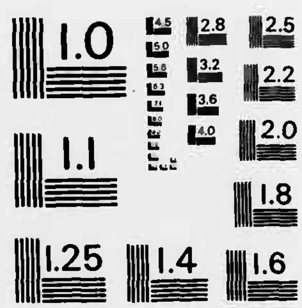

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Numerical Calculation of Variable Property Flows in Curvilinear Orthogonal Coordinates

(AMS Report 1379)

by

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In an earlier communication (dealing with flow in curved pipes) a numerical procedure capable of calculating variable property flows in arbitrary curvilinear orthogonal coordinates was reported[1]. Convective and diffusive base vector variation source terms appearing in the differential momentum equation were tabulated but their derivation was not described. In addition, and because of the cases studied, only brief mention was made with regard to the extra source terms arising in variable property flows. The purpose of this document letter is to provide a concise derivation of the differential momentum equation which is the basis for the numerical procedure described in a previous work ~~reference 1~~ and to tabulate general expressions for all the sources which can arise. From a knowledge of the metric tensor in the curvilinear system of interest the forms of these terms are readily derived from the tables for calculation purposes.

Throughout this letter, standard tensor notation with the summation convention is employed. A detailed exposition of the rules and uses of tensor analysis in fluid mechanics is available in reference 2. Briefly, superscripts refer to contravariant components, subscripts represent covariant components and commas denote covariant differentiation. Any quantity followed by an index within parentheses, such as $v(i)$, denotes the ith (in this case) physical component of that quantity. While the present analysis is limited to steady state forms of the conservation equations, it is readily extended to time dependent situations.

The Momentum Equation

The steady-state vector equation of motion for a variable property flow is given by:

$$\vec{\nabla} \cdot (\rho \vec{V} \vec{V}) = \rho \vec{F} - \vec{\nabla} P + \vec{\nabla} \cdot \vec{T} \tag{1}$$

In equation (1), $\vec{F}$ is a body force field such as that arising due to gravitational, buoyancy, electrostatic effects, etc, $\vec{T}$ is the stress tensor field, and P is the pressure field. The symbol $\vec{\nabla}\cdot$ denotes the divergence operator in vector notation, whereas $\vec{\nabla}$ denotes the gradient operator. An equivalent tensor form in orthogonal coordinates for the ith contravariant component of equation (1), assuming a Newtonian fluid, is given by:

$$\rho\, v^j v^i_{,j} = \rho f^i - g^{ii} P_{,i} + \mu\, g^{jj} v^i_{,jj} + (\lambda + \mu)\, g^{ii} v^k_{,ki} +$$

$$2e^{ij} \mu_{,j} + g^{ii} v^k_{,k} \lambda_{,i} \qquad \text{(no summation on } i\text{)} \tag{2}$$

In equation (2), e^{ij} is the contravariant form of the deformation tensor and is defined by:

$$e^{ij} = \tfrac{1}{2} (g^{ii} v^j_{,i} + g^{jj} v^i_{,j}) \tag{3}$$

The quantity μ is the fluid viscosity and is related to the value λ by Stokes' hypothesis, according to which:

$$\lambda + \frac{2}{3}\mu = 0 \tag{4}$$

Substitution of equation (4) into equation (2) and pre-multiplying by $g_{ii}^{\frac{1}{2}}$ results in:

$$\rho\, g_{ii}^{\frac{1}{2}}\, v^j\, v^i_{,j} = \rho\, g_{ii}^{\frac{1}{2}}\, f^i - \frac{1}{g_{ii}^{\frac{1}{2}}} P_{,i} + \mu\, g_{ii}^{\frac{1}{2}}\, g^{jj}\, v^i_{,jj} + \frac{\mu}{3} \frac{1}{g_{ii}^{\frac{1}{2}}} v^k_{,ki} + 2 g_{ii}^{\frac{1}{2}}\, e^{ij}\, \mu_{,j} - \frac{2}{3} \frac{1}{g_{ii}^{\frac{1}{2}}} v^k_{,k}\, \mu_{,i} \tag{5}$$

Expressions for $v^i_{,j}$ and $v^i_{,jj}$ are needed to put equation (5) into a more useful form. The quantity $v^i_{,j}$ is defined as the jth covariant derivative of the ith contravariant component of the vector velocity, and is given by [2]

$$v^i_{,j} = \frac{\partial v^i}{\partial x^j} + \left\{ {i \atop m\ j} \right\} v^m \tag{6}$$

The expression $\left\{ {i \atop m\ j} \right\}$ represents the second kind of Christoffel symbol and is not a tensor but a function of the metric tensor, g^{ij}; it arises due to the variation of coordinate base vectors[2]. Similarly, the form $v^i_{,jj}$ can be shown to be[3]:

$$v^i_{,jj} = \frac{g_{jj}}{g^{\frac{1}{2}}} \frac{\partial}{\partial x^j} \left[g^{\frac{1}{2}}\, g^{jj} \frac{\partial v^i}{\partial x^j} + g^{\frac{1}{2}}\, g^{jj} \left\{ {i \atop k\ j} \right\} v^k \right] + g_{jj}\, g^{kk} \left\{ {i \atop j\ k} \right\} \left[\frac{\partial v^j}{\partial x^k} + \left\{ {j \atop m\ k} \right\} v^m \right] \tag{7}$$

Substitution of equations (6) and (7) into equation (5), followed by considerable algebraic manipulation*, and expressing the final result in terms

*Details of this derivation are available in reference 3 of physical components, yields:

$$\frac{1}{g^{\frac{1}{2}}}\frac{\partial}{\partial x^j}\left(g^{\frac{1}{2}}\,\rho\,v(i)\,\frac{v(j)}{h_j}\right) - \frac{1}{g^{\frac{1}{2}}}\frac{\partial}{\partial x^j}\left(\mu\, g^{\frac{1}{2}}\,\frac{1}{h_j^2}\,\frac{\partial v(i)}{\partial x^j}\right) = S_F(i) + S_P(i) +$$

$$S_D(i) - S_C(i) + S_\rho(i) + S_\mu(i) + S_{\rho\mu}(i) \qquad (8)$$

where:

$$S_F(i) = \rho\, f(i) \qquad (9)$$

$$S_P = -\frac{1}{h_i}\frac{\partial P}{\partial x^i} \qquad \text{(no sum on } i\text{)} \qquad (10)$$

and:

(a) <u>Terms arising in non-rectangular geometries only</u>

<u>Term</u> <u>Formula</u>

$$S_C(i) = \rho\left[h_i\,\frac{v(j)}{h_j}\,\frac{v(m)}{h_m}\left\{{i \atop m\;j}\right\} - \frac{v(j)}{h_j}\,\frac{v(i)}{h_i}\,\frac{\partial h_i}{\partial x^j}\right] \qquad (11)$$

$$S_D(i) = \mu\, h_i\left[\frac{1}{g^{\frac{1}{2}}}\frac{\partial}{\partial x^j}\left(\frac{g^{\frac{1}{2}}}{h_j^2}\left\{{i \atop k\;j}\right\}\frac{v(k)}{h_k}\right)\right.$$

$$+\frac{1}{h_k^2}\left\{{i \atop k\;j}\right\}\left(\frac{\partial h_j^{-1}\,v(j)}{\partial x^k} + \left\{{j \atop m\;k}\right\}\frac{v(m)}{h_m}\right)$$

$$\left.+\frac{1}{h_j^2}\,\frac{\partial v(i)}{\partial x^j}\,\frac{\partial h_i^{-1}}{\partial x^j} + \frac{1}{g^{\frac{1}{2}}}\frac{\partial}{\partial x^j}\left(\frac{g^{\frac{1}{2}}}{h_j^2}\,v(i)\,\frac{\partial h_i^{-1}}{\partial x^j}\right)\right] \qquad (12)$$

(b) Terms arising in all geometries for variable density and/or viscosity

Term	Formula

$$S_\rho(i) = \frac{\mu}{3}\,\frac{1}{h_i}\left[\frac{\partial}{\partial x^i}\left[\frac{\partial h_k^{-1}\,v(k)}{\partial x^k} + \left\{{k \atop m\;k}\right\}\frac{v(m)}{h_m}\right]\right] \tag{13}$$

$$S_\mu(i) = h_i\,\frac{\partial\mu}{\partial x^j}\left[\frac{1}{h_i^2}\left[\frac{\partial h_j^{-1}\,v(j)}{\partial x^i} + \left\{{j \atop m\;i}\right\}\frac{v(m)}{h_m}\right] + \frac{1}{h_j^2}\left[\frac{\partial h_i^{-1}\,v(i)}{\partial x^j} + \left\{{i \atop m\;j}\right\}\frac{v(m)}{h_m}\right] - \frac{1}{h_i\,h_j^2}\,\frac{\partial v(i)}{\partial x^j}\right] \tag{14}$$

$$S_{\rho\mu}(i) = -\frac{2}{3}\,\frac{1}{h_i}\,\frac{\partial\mu}{\partial x^i}\left[\frac{\partial h_j^{-1}\,v(j)}{\partial x^j} + \left\{{j \atop m\;j}\right\}\frac{v(m)}{h_m}\right] \tag{15}$$

Equation (8) is the physical component form of the momentum equation in general curvilinear orthogonal coordinates and corresponds to equation (4) in reference 1. It has been formulated here such that effects due to variation in physical properties appear in explicitly defined terms, $S_\rho(i)$, $S_\mu(i)$ $S_{\rho\mu}(i)$. These terms represent additional contributions to the momentum balance and are volume integrated in the same manner as $S_D(i)$ and $S_C(i)$ in reference 1 to obtain curvilinear coordinate forms of the difference equations in variable property flows.

Specific forms for the source terms in a curvilinear system of interest are readily derived from the tabulated expressions simply from a knowledge of the metric tensor g^{ij} for that curvilinear system.

References

1. Humphrey, J.A.C., "Numerical Calculation of Developing Laminar Flow in Pipes of Arbitrary Curvature Radius," Can. J. Chem. Engng., accepted for publication (1978). Also, AMS Report No. 1362, Princeton University.

2. Aris, R., Vectors, Tensors and the Basic Equations of Fluid Mechanics. Prentice-Hall, Inc. (1962).

3. Humphrey, J.A.C., Ph.D. Thesis, University of London (1977).

Notation

e^{ij}	:	deformation tensor in contravariant form (i,j = 1, 2, 3)
f^{i}	:	component of body force in tensor notation (i = 1, 2, 3)
$f(i)$	:	physical component of body force (i = 1, 2, 3)
$\vec{F}$	:	body force field in vector form
$g^{\frac{1}{2}}$	:	($\equiv (h_1\ h_2\ h_3)^{\frac{1}{2}}$) square root of the metric tensor determinant
g^{ij}	:	metric tensor in contravariant form (i,j = 1, 2, 3)
h_i	:	scale factor in curvilinear orthogonal coordinates (i = 1, 2, 3)
P	:	pressure
$S_C(i)$	:	convection source term in momentum equation (i = 1, 2, 3)
$S_D(i)$	:	diffusion source term in momentum equation (i = 1, 2, 3)
$S_F(i)$	:	body force source term in momentum equation (i = 1, 2, 3)
$S_P(i)$	:	pressure source term in momentum equation (i = 1, 2, 3)
$S_\mu(i)$	:	source term due to variable viscosity in momentum equation (i = 1, 2, 3)
$S_\rho(i)$	:	source term due to variable density in momentum equation (i = 1, 2, 3)
$S_{\rho\mu}(i)$	:	source term due to variable density and viscosity in momentum (i = 1, 2, 3)
$\vec{T}$	:	stress tensor in vector notation
v^{i}	:	velocity component (contravariant) in tensor notation (i=1,2,3)
$v(i)$	:	physical component of velocity (i = 1, 2, 3)
$\vec{V}$	:	velocity in vector notation
x^{i}	:	coordinate component (contravariant) in tensor notation (i=1,2,3)

Other Symbols

λ	:	constant related to μ by Stokes' Hypothesis in equation (4)
μ	:	viscosity
ρ	:	mass density
$\vec{\nabla}(\ )$	:	gradient operator (Hamilton's operator)
$\vec{\nabla}\cdot(\ )$	:	divergence operator
$\{{}_{j}{}^{i}{}_{k}\}$	:	Christoffel symbol of second kind

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