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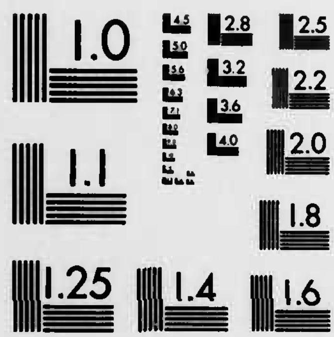

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AN INTRODUCTION TO THE PARABOLIC EQUATION FOR ACOUSTIC PROPAGATION

Alan B. Coppens

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NAVAL POSTGRADUATE SCHOOL
Monterey, California

Rear Admiral John Ekelund
Superintendent

David Schrady
Provost

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This report was prepared by:

ALAN B. COPPENS
Professor of Physics

Reviewed by:

C.N.K. MOOERS, Chairman
Department of Oceanography

J.N. Dyer, Chairman
Department of Physics

Released by:

WILLIAM M. TOLLES
Dean of Research

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The derivation of the parabolic wave equation for acoustic propagation is studied and presented pedagogically for tutorial purposes. The literature is reviewed and modifications to the parabolic equation to increase accuracy are mentioned. Some of the algorithms for computer implementation of the parabolic approximation are discussed qualitatively, and the various approaches to dealing properly with the density change between the water column and the bottom are examined.

ABSTRACT

The derivation of the parabolic wave equation for acoustic propagation is studied and presented pedagogically for tutorial purposes. The literature is reviewed and modifications to the parabolic equation to increase accuracy are mentioned. Some of the algorithms for computer implementation of the parabolic approximation are discussed qualitatively, and the various approaches to dealing properly with the density change between the water column and the bottom are examined.

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Preface

The Environmental Acoustic Research Group at the Naval Postgraduate School is engaged in research to establish beneficial and detrimental environmental effects important to present and future Navy acoustic systems.

Pursuant to the above objectives, environmental and acoustic models are used to interpret and predict the complex results obtained when actual experimental or operational scenarios are utilized. Acoustic parabolic wave-equation models are useful for long-range environmental acoustical studies. However the basic assumptions and certain errors in the use of these models are not always obvious. For this reason the present work was initiated to provide a tutorial introduction mainly for use of students in the Environmental Acoustic Research Group at the Naval Postgraduate School and others interested in obtaining an initial orientation in this field of research.

Presently five professors from the Departments of Oceanography and Physics are involved in this research as well as approximately ten graduate students.

I. The Parabolic Wave Equation

A. Introduction

Historically, the (Leontovich-Fock) parabolic-equation approximation was developed for dealing with electromagnetic propagation; see, for example, Fock (1965). This approximation to the elliptical wave equation made its way into the area of acoustic propagation through the work of Hardin and Tappert (1973), who applied the split-step Fourier transform method for its evaluation.

Since its introduction, the parabolic equation has been subjected to substantial analysis in the acoustics community, and a variety of alternative methods for solving the equation have been developed. It is the purpose of this report to present a fundamental introduction to the parabolic-equation approximation with some discussion of the more viable methods of numerical solution. Some of the advantages and disadvantages of these methods will be noted, but detailed discussion of the problems and requirements in implementing these methods as computer algorithms will not be treated; these lie beyond the simple introduction attempted herein.

For an overview of these and other methods of solving the wave equation and its various approximate forms, a good starting point is DeSanto (1979), and in particular the articles contained therein by DeSanto, and DiNapoli and Deavenport.

B. Derivation

If an acoustic pressure field of constant angular frequency $\omega = 2\pi f$ is assumed, then the source-free linear acoustic wave equation

$$\nabla^2 \underline{p} = c^{-2}(\partial^2 \underline{p}/\partial t^2) \qquad 1.1$$

where the speed of sound c is a function of space reduces to the Helmholtz equation (often termed the "time independent" wave equation). For the spatial factor $\underline{S}$ of the pressure

$$\underline{p} = \underline{S}(\text{space}) \exp(-i\omega t) \qquad 1.2a$$

this becomes

$$\nabla^2 \underline{S} + k^2 \underline{S} = 0 \qquad 1.2^b$$

where

$$k = \omega / c(\text{space}) \qquad 1.2c$$

If we now assume cylindrical symmetry and adopt cylindrical coordinates (r,z) where z represents depth and r is the horizontal distance from the z-axis, (1.2) becomes

$$\partial^2 \underline{S}/\partial r^2 + r^{-1}(\partial \underline{S}/\partial r) + \partial^2 \underline{S}/\partial z^2 + k^2 \underline{S} = 0 \qquad 1.3$$

It is useful to define k in terms of an arbitrary constant value k_o and the index of refraction

$$n = c_o/c(r,z) \quad \text{where} \quad \omega/k_o = c_o \qquad 1.4$$

so that

$$k^2 = n^2 k_o^2 \qquad 1.5$$

Let us now write the solution for $\underline{S}$ in the form

$$\underline{S} = H_o^{(1)}(k_o r)\, \underline{f}(r,z) \tag{1.6}$$

which when substituted into (1.3) results in the equation

$$\partial^2\underline{f}/\partial r^2 + \partial^2\underline{f}/\partial z^2 + \left[\frac{1}{r} + \frac{2}{H_o^{(1)}(k_o r)}\,\frac{dH_o^{(1)}(k_o r)}{dr}\right]\frac{\partial \underline{f}}{\partial r} + (k^2-k_o^2)\underline{f} = 0 \tag{1.7}$$

If we now restrict attention to ranges r such that

$$k_o r >> 1 \quad \text{and} \quad H_o^{(1)}(k_o r) \approx \sqrt{2/(\pi k_o r)}\, \exp[i(k_o r-\pi/4]] \tag{1.8}$$

then [ ] simplifies to $2ik_o$ and (1.7) becomes well-approximated by

$$\frac{\partial^2\underline{f}}{\partial z^2} + \frac{\partial^2\underline{f}}{\partial r^2} + 2ik_o\frac{\partial \underline{f}}{\partial r} + (k^2-k_o^2)\underline{f} = 0 \tag{1.9}$$

The pivotal assumption that $\underline{f}$ varies slowly with respect to range,

$$\left|\partial^2\underline{f}/\partial r^2\right| << \left|2k_o\partial\underline{f}/\partial r\right| \tag{1.10}$$

results in the "parabolic (wave) equation"

$$\partial^2\underline{f}/\partial z^2 + 2ik_o(\partial\underline{f}/\partial r) + (k^2-k_o^2)\underline{f} = 0 \tag{1.11}$$

The assumption (1.10) has some extremely important implications:

(a) The Helmholtz equation, an elliptic equation, has been reduced to a parabolic equation. This means that the entire acoustic field need not be solved for all relevant ranges and depths "simultaneously" subject to boundary conditions on a surface surrounding the volume of interest. Instead, for an outwardly progressing wave, an initial boundary condition can be established for some small r and then solutions for larger r's obtained by increasing r incrementally. This offers the possibility of a substantial saving of computer time and memory. However, the boundary condition (initial condition) assumed for the first range must be carefully selected. This will be discussed later.

(b) The parabolic approximation is equivalent to neglecting any back-scattering, since the solution at some range r_1 is the source for the solution at some larger range r_2 and is independent of any intervening changes in the speed of sound, depth of the water column, etc: As the solution is stepped out, changes at larger r can have no effect on the fields previously obtained for smaller r.

(c) The parabolic assumption can introduce unavoidable errors in the details of the resultant acoustic field. We shall demonstrate this by considering a particularly simple acoustic model which can be solved by both the Helmholtz and parabolic equations. Comparison of the respective solutions will aid in revealing some of the inherent errors resulting from the parabolic-equation approximation.

For use later, notice that for large $k_o r$ (1.6) becomes

$$\underline{S} = \sqrt{2/(\pi k_o r)}\ \underline{f}(r,z)\ \exp[i(k_o r - \pi/4)] \tag{1.12a}$$

and that the acoustic pressure in the parabolic-equation approximation has the form

$$\underline{p} = A \sqrt{2/(\pi k_o r)}\ \underline{f}(r,z) \exp[i(k_o r-\pi/4)] \exp(-i\omega t) \qquad 1.12b$$

with A an arbitrary constant and $\underline{f}(r,z)$ the solution of (1.11). Recall this is valid for $k_o r >> 1$.

C. The Range-Independent Case

Let us assume that the speed of sound is a function only of depth and that any relevant boundaries are also range independent. Direct solution of the Helmholtz equation (1.2) results in the well-known summation of normal modes

$$\underline{p}_m = \exp(-i\omega t) \sum_m \underline{A}_m Z_m(z) H_o^{(1)}(k_m r) \qquad 1.13a$$

where the constants $\underline{A}_m$ are determined by the properties of the Z_m and source depth, and the depth-dependent functions Z_m are solutions of the equation

$$\frac{d^2 Z_m}{dz^2} + \left[k^2(z) - k_m^2\right] Z_m = 0 \qquad 1.13b$$

The eigenfunctions Z_m and the eigenvalues k_m are established by the function k(z), the boundary conditions at the top and bottom of the water column, and the properties of any ocean bottom (if important). In the limit of large r each normal mode has asymptotic behavior

$$\underline{p}_m \rightarrow \exp(-i\omega t) \underline{A}_m Z_m(z) \sqrt{2/(\pi k_m r)} \exp\left[i(k_m r - \frac{\pi}{4})\right] \qquad 1.14$$

Solution of the same problem for the parabolic wave equation proceeds analogously: Let

$$\underline{f}_m = \underline{R}_m(r) Z_m(z) \tag{1.15}$$

Substitute into (1.11) and perform the usual separation of variables. The result is the pair of equations

$$d^2 Z_m/dz^2 + [k^2(z) - k_m^2] Z_m = 0 \tag{1.16a}$$

and

$$2ik_o(d\underline{R}_m/dr) + (k_m^2 - k_o^2)\underline{R}_m = 0 \tag{1.16b}$$

Note that (1.16a) and (1.13b) are identical.

Equation (1.16b) can be solved by direct integration to yield

$$\underline{R}_m = \exp\left(i \frac{k_m^2 - k_o^2}{2k_o} r \right) \tag{1.17}$$

Collection of terms reveals the parabolic-equation solution

$$\underline{p}' = \exp(-i\omega t) \sum_m \underline{A}_m \; Z_m(z) \; H_o^{(1)}(k_o r) \; \exp\left(i \frac{k_m^2 - k_o^2}{2k_o} r \right) \tag{1.18}$$

In the limit of large r, each term has asymptotic form

$$\underline{p}'_m \rightarrow \exp(-i\omega t) \underline{A}_m \; Z_m(z) \left[\frac{2}{\pi(k_o r)}\right]^{1/2} \exp\left[i\left(\frac{k_m^2 + k_o^2}{2k_o} r - \pi/4\right)\right] \tag{1.19}$$

From (1.14), we see that the phase speed c_m of the m-th normal mode is

$$c_m = \omega/k_m \qquad 1.20$$

whereas from (1.19), the equivalent term solving the parabolic approximation is

$$c'_m = \omega 2k_o \ / \ (k_m^2 + k_o^2) \qquad 1.21$$

Were the acoustic field to consist of only a single mode, $m = M$, then the choice $k_o = k_M$ would exactly eliminate the phase speed error, and the solution of the parabolic equation would be identical with that of the Helmholtz equation. Unfortunately, this is not usually the case. However, since k_o is an arbitrary constant, it is clear that if the acoustic field is composed of a set of normal modes whose values of k_m all lie very close together (a "narrow band" of modes), all other modes being negligibly small or absent, then the choice $k_o = \langle k_m \rangle$ where the average $\langle\rangle$ is taken over just this narrow band will tend to minimize phase errors. Even here, however, the errors may not be trivial.

First, we see that except for the special case $k_o = k_M$ the phase speed c_m for the normal mode is different from the analogous phase speed c'_m for the equivalent term in the parabolic solution. This means that the spatial pattern of the phase-coherent combination of the pressure terms will be distorted. There is an additional and equally important effect resulting from the fact that the phase speeds for individual modes do not change proportionately with those for the equivalent terms in the parabolic equation. Differentiation of c_m with respect to k_m yields

$$\Delta c_m/\Delta k_m = -\omega/k_m^2 = -c_m/k_m \qquad 1.22$$

whereas the same operation applied to c_m' results in

$$\Delta c_m'/\Delta k_m = -c_m' 2k_m/(k_m^2 + k_o^2) = -c_m/k_o \qquad 1.23$$

Thus, the analogous phase speeds are not related simply by a constant, which would merely "stretch" or "shrink" the entire interference pattern with respect to range. Instead, the detailed interference pattern of the acoustic field will also be changed. For a clear example of these effects, see Fig. 5 of McDaniel (1975-1).

An estimate of the maximum range for which the parabolic equation retains sufficient accuracy can be obtained following the development of Fitzgerald (1975):

Assume that the acoustic field is made up primarily of a set of strong modes with indices m lying between m(max) and m(min). This could correspond to a field consisting of trapped modes in a mixed layer, a shallow-water channel with a fast bottom, or the deep sound channel.

To minimize error, set

$$k_o = \langle k_m \rangle$$

or, almost equivalently,

$$c_o = \langle c_m' \rangle$$

since either will be about half way between the values found for m(max) and m(min). Define

$$\Delta c' \approx \left| c'_{m(max)} - c_o \right| \approx \left| c'_{m(min)} - c_o \right| \qquad 1.24a$$

and

$$\Delta k \approx \left| k_{m(max)} - k_o \right| \approx \left| k_{m(min)} - k_o \right| \qquad 1.24b$$

The accuracy requirement can be approximated by the condition

$$\left| \frac{k_m^2 + k_o^2}{2k_o} r - k_m r \right| < \frac{\pi}{2} \qquad m = m\,(max) \text{ or } m(min) \qquad 1.25a$$

Elementary manipulation with (1.24) yields

$$r < \pi k_o/(\Delta k)^2 \qquad 1.25b$$

or, with the help of $c_m^2 c_o \approx c_o^3$,

$$r < \pi c_o^3/[\omega(\Delta c')^2]. \qquad 1.25c$$

Equation (1.25) is somewhat less restrictive than Fitzgerald's result, but serves as a reasonable guideline. Note the explicit frequency dependence in (1.25c): For a given family of excited modes, the maximum permissible range for given accuracy will decrease with increasing frequency.

D. Improving the Accuracy of the Parabolic Equation

(1) The "Pseudoproblem"

In the light of the phase-coherence difficulties discussed above, Brock et. al. (1977) investigated the feasibility of modifying the problem to reduce these effects. They were guided by ray-tracing predictions of the turning points of the rays and the requirement of matching the normal mode and parabolic phase speeds, at least over the narrow band of k_m's required for the validity of the parabolic equation (1.11). Based on these considerations, they determined that an approximate analogous "pseudoproblem" could be formulated for which the index of refraction and the depth at which a particular speed of sound was found were adjusted according to the mapping

$$(n,z) \rightarrow (n^*,z^*) \qquad 1.26a$$

where

$$n^* \approx (2n - 1)^{\frac{1}{2}} \qquad 1.26b$$

$$z^* \approx n^{\frac{1}{2}} z \qquad 1.26c$$

Utilizing this technique, they determined that sensitivity to the choice of c_O was considerably reduced, and the predictions of the transmission loss given by the "pseudoproblem" matched those predicted by the normal mode solution of the original problem much better than the solution of the parabolic equation without the mapping of (n,z).

While these results were obtained for the range-independent case, the authors make the plausible assertion that if in a range-dependent problem the mapping is to be done at each new range step, then the improvement of results over those without the mapping should be about the same as for the range-independent case.

(2) Alternative Equations

It should be pointed out that (1.11) is not the only form that a parabolic approximation to the wave equation can take.

Indeed, several investigators have made rather extensive studies of alternative forms and more accurate approximations. We shall confine our discussion but provide references for the reader interested in pursuing these extensions further.

McDaniel (1975-2) studied several methods of separating solutions to an asymptotic wave equation into outgoing and incoming components. Depending on the method, when any back-reflected (incoming) component is neglected, modified parbolic equations result which, when compared to the asymptotic wave equation, reveal errors of various orders. Of the three cases studied, two led to second-order errors and one led to fourth- order errors. The commonly-encountered parabolic equation (1.11) was one of the second-order cases. In addition, numerical analyses using different algorithms were performed and results were checked for internal consistency by exchanging source and receiver positions and verifying that acoustic reciprocity held to reasonable accuracy.

Palmer (1976) investigated improvements to the Eikonal equation and approximations to normal-mode theory by assuming that the Eikonal equation can be applied in the horizontal plane. This leads to expressions for the normal mode coefficients and the development of an appropriate Green's function. After rather elaborate mathematical development, some modified parabolic equations can be extracted. The thrust of the discussion, however, is toward a further understanding of the plausibility of the physical restrictions necessary to justify the validity of discarding small-order terms in the Helmholtz equation to obtain a parabolic equation.

An investigation by DeSanto (1977) into the mathematical relationship between the solution to the Helmholtz equation, $\underline{S}$, and the solution $\underline{f}$ to the parabolic equation (1.11) yielded a collection of correction terms. DeSanto's approach was to assume that $\underline{S}$ and $\underline{f}$ could be related by an integral,

$$\underline{S}(r,z) = A_o \int_0^{\infty} \underline{f}(y,z)\ R(y,r,z)\ \exp[B(y,r)]dy \qquad 1.27$$

where A_o is constant, R and B are unknown functions, and y is the dummy variable of integration. (The above integral is for cylindrical coordinates, a special case of the more general formulation accomplished by DeSanto.) If this is substituted into the Helmholtz equation and k(r,z) written in the form

$$k(r,z) = k_1(z) + k_2(r,z) \quad \text{where } k_1 \gg k_2 \qquad 1.28$$

then it is possible to obtain B and the functional dependence of $\underline{f}$ on y by requiring self-consistency. What remains is a differential equation for R. DeSanto then shows that the solution $\underline{f}$ to the parabolic equation (1.11) results from the stationary-phase approximation of the integral (1.27). Retaining higher accuracy in evaluating the integral provides correction terms to the parabolic equation and therefore to $\underline{f}$. In a later paper, DeSanto, Perkins, and Baer (1978) begin with a "corrected parabolic approximation" [compare with (1.12a)]

$$S = \sqrt{2/(\pi k_o r)}\ \exp\left[i(k_o r - \pi/4)\right]\left[\underline{f} + (ir/2k)(\partial^2 \underline{f}/\partial r^2)\right] \qquad 1.29$$

derived in the earlier paper, and note that this can be adopted very easily into the algorithms solving for $\underline{f}$. The procedure is to step out the solutions to $\underline{f}$ two range increments from r and then use the numerical approximation for the second derivative with respect to r evaluated at $r + \Delta r$,

$$\left.\frac{\partial^2 \underline{f}}{\partial r^2}\right|_{r+\Delta r, z} \approx \frac{\underline{f}(r+2\Delta r,z) - 2\underline{f}(r+\Delta r,z) + \underline{f}(r,z)}{(\Delta r)^2} \equiv \underline{f}''(r+\Delta r,z) \qquad 1.30$$

and the value of $\underline{f}$ predicted at the first of the two range increments is corrected by

$$f_c(r+\Delta r,z) = f(r+\Delta r,z) + (i\Delta r/2k)\underline{f}''(r+\Delta r,z) \qquad 1.31$$

where the subscript "c" designates the corrected value. Their comparisons between solutions obtained from normal modes, the parabolic approximation, and the "corrected" parabolic approximation suggest that the errors introduced by the parabolic approximation are roughly halved. However, this approach requires about twice the computer time and three times the memory compared to the uncorrected parabolic approximation.

II. Boundary Conditions for the Parabolic Equation

Except for special cases, the parabolic equation, like the Helmholtz equation, does not yield analytical solutions for space-dependent speed of sound profiles or irregular boundaries. Instead, numerical methods must be adopted. There are a number of numerical techniques now available for use with the parabolic wave equation. We shall mention some of those that are currently popular.

The major driving forces developing computer algorithms for numerical solutions are that computers are limited in available memory and computer time is expensive. As a result, emphasis has been placed on fast-running programs which require relatively little memory. Since the parabolic equation is designed to be stepped out in range, it is important to use techniques which allow the largest possible increments in both depth and range. Since each step requires numerical mathematical manipulation of input data and the results of the previous range step, efficient computational schemes are required. In this report we say little about these aspects of the problem; our purpose is to describe the methods rather than discuss the details of their advantages or disadvantages as far as computer implementation is concerned.

Before turning to the models, it is necessary to discuss two aspects of the parabolic equation which are common to all methods of solution. In every case, it is necessary to begin the computation with an input data set of the values of $\underline{f}$ at some initial range as a function of depth. This is the <u>initialization</u>

problem. The second aspect is that of treating the boundaries of the water column and bottom. There is little difficulty with the surface, which is represented by a pressure release boundary so that at zero depth $\underline{f}(r,0) = 0$ for all r. However, the bottom, if an important aspect of resultant acoustic field, presents some difficulties.

A. The Initialization Problem

Whatever set of values for $\underline{f}(r_o,z)$ at the initial range r_o are chosen, they must be consistent with the acoustic source generating the pressure field. It is therefore useful to obtain a few results for an omnidirectional point source located at (0,Z) in an ocean whose properties vary only with z. This "boundary condition" can be built into the acoustic wave equation by including a "source term".

If the omnidirectional source has unit pressure amplitude at a distance of 1 m, then the appropriate inhomogeneous wave equation in cylindrical coordinates is

$$\left[\nabla^2 + k^2(z)\right] \underline{S} = -2\delta(r)\ \delta(z-Z)/r \qquad 2.1$$

The presence of the term on the RHS guarantees that

$$\lim_{(r,z)\to(0,Z)} \underline{S} = [r^2+(z-Z)^2]^{-\frac{1}{2}} \exp[ik\sqrt{r^2+(z-Z)^2}\ ] \qquad 2.2$$

Given that conditions exist for the trapping of sound in a channel, it is plausible to perform the separation of variables

$$\underline{S} = \sum_m \underline{R}_m(r) Z_m(z) \qquad 2.3$$

and assume that the Z_m satisfy

$$d^2 Z_m/dz^2 + [\omega^2/c^2(z) - k_m^2] Z_m = 0 \qquad 2.4$$

the appropriate boundary conditions, and are normalized. Then, the Z_m form an orthonormal set of eigenfunctions. Substitution of these results into (2.1) yields

$$\sum_m \left[d^2 \underline{R}_m/dr^2 + r^{-1}(d\underline{R}_m/dr) + k_m^2 \underline{R}_m \right] Z_m = -2\delta(r)\,\delta(z-Z)/r \qquad 2.5$$

If both sides of this equation are multiplied by $Z_n(z)$, integrated over z, and use made of the orthonormality condition

$$\int Z_m\, Z_n\, dz = \delta_{mn} \qquad 2.6$$

then the result is

$$d^2 \underline{R}_m/dr^2 + r^{-1}(d\underline{R}_m/dr) + k_m^2 \underline{R}_m = -2\delta(r) Z_m(Z)/r \qquad 2.7$$

which is solved by

$$\underline{R}_m = i\pi Z_m(Z)\, H_o^{(1)}(k_m r) \qquad 2.8$$

Now, if (2.7) is substituted back into (2.5), we obtain the useful relationship

$$\sum_m Z_m(Z)\, Z_m(z) = \delta(z-Z) \qquad 2.9$$

This expression ignores the collection of continuous eigenfunctions, since these, practically speaking, contribute nothing to the acoustic field at ranges of interest.

(1) The Gaussian Field.

When an omnidirectional source of sound is reasonably distant from either the ocean surface or any bottom, then the source may be approximated by a Gaussian pressure field for the initial set of pressure as a function of depth. The difficulty is to determine the parameters of the Gaussian distribution to "match" the point source to the particular propagation problem under consideration.

In the immediate vicinity of an omnidirectional source which is not too close to any reflective boundary, the amplitude of the pressure must decrease with distance from the source, according to spherical spreading. For such a source located at $(r,z) = (0,Z)$ therefore, from (1.2) the amplitude of the radiated pressure must be given by

$$P' = [r^2+(z-Z)^2]^{-\frac{1}{2}} \qquad 2.10$$

(Recall we are in cylindrical coordinates with radial symmetry and have assumed unit pressure amplitude at a distance of 1 m. This amplitude choice facilitates conversion between source level SL and pressure amplitude for sources of arbitrary strength.)

In the case of an infinite, homogeneous medium the source with amplitude given by (2.10) and angular frequency ω must be described by an appropriate collection of delta functions at (0,Z) However, all the energy radiated from the source does not find its way into the sound field trapped by the sound-speed profile.

Instead, energy can be lost through interaction with the bottom. Thus, rays whose angles of elevation or depression exceed certain limits will be lost to the channel at large ranges, and the source appears as if it is not a point source, but instead possesses vertical directivity.

Brock (1978) presents one way of incorporating this apparent directivity into the parabolic equation initialization:

Assume that in the volume of water surrounding the source at (0,Z) acoustic conditions are relatively uniform. Then we can treat k as constant for $z \sim Z$ and $r \sim 0$. If we then take

$$k = k_o = \omega/c(Z) \qquad 2.11$$

where c(Z) is the speed of sound near the source, the parabolic equation (1.11) simplifies to

$$\partial^2 \underline{f}/\partial z^2 + 2ik_o\left(\partial \underline{f}/\partial r\right) = 0 \qquad 2.12$$

Within the volume for which (2.11) is a reasonable approximation, but still under the condition $k_o r >> 1$, we require that the amplitude of the solution to (2.12) be consistent with the amplitude P' given by (2.10). We can then extrapolate $\underline{f}$ back to r = 0 and obtain an extrapolated equivalent boundary condition along the z-axis. (Physically, this seemingly-artificial approach is equivalent to taking the far-field behavior of a directional source and extrapolating it back to the position of the source, thereby ignoring near-field effects.)

This extrapolation is accomplished by <u>reversing</u> the steps of the argument. <u>Postulate</u> that the equivalent boundary condition for $\underline{f}$ at (0,z) is given by the Gaussian distribution

$$\underline{f}(0,z) = F \exp[-(z-Z)^2/\sigma^2] \qquad 2.13$$

where F and σ are to be determined. With this as an initial condition (and requiring that $\underline{f}$ vanish at large r), according to Brock the solution of (2.12) can be verified to be

$$\underline{f}(r,z) = [F/\sqrt{\underline{g}(r)}]\exp\left\{-(z-Z)^2/[\sigma^2 \underline{g}(r)]\right\} \qquad 2.14a$$

$$\underline{g}(r) = 1 + i\, 2r/(k_o \sigma^2) \qquad 2.14b$$

Now, substitute (2.14) into (1.12) and, since we have two arbitrary constants, set $A = (\pi k_o/2)^{1/2}$ for convenience. Next, take $P = |\underline{p}|$,

$$P = \left|\underline{f}(r,z)\right| / \sqrt{r} \qquad 2.15$$

Brock's result is

$$P^2 = (F^2/r)\left(h/\sqrt{1+h^2}\right) \exp\left\{-2h^2(z-Z)^2/[\sigma^2(1+h^2)]\right\} \qquad 2.16a$$

with

$$h = \frac{1}{2} k_o \sigma^2/r \qquad 2.16b$$

The next step is to expand $(P')^2$ and P^2 in power series. For $(P')^2$, assume that $[(z - Z)/r]^2$ is small. For P^2, assume that the exponent in (2.9a) is small. This leads to two power series in $[(z - Z)/r]^2$, and equating coefficients of the leading two terms yields the results

$$\sigma = \sqrt{2}/k_o \qquad 2.17a$$

$$F = \sqrt{2/k_o}/\sigma = \sqrt{k_o} \qquad 2.17b$$

Thus, the initial values of $\underline{f}$ are given by

$$\underline{f}\,(0,z) = \sqrt{k_o}\,\exp[-k_o^2(z-Z)^2/2] \qquad 2.18$$

for an omnidirectional source with unit pressure amplitude at 1 m.

Direct substitution of (2.17a) into the exponent of (2.16a) reveals that the exponent is small if $[(z - Z)/r]^2$ is small. Thus, the physical implication of the approximations leading to (2.18) from (2.16) is that the omnidirectional source is approximated by an equivalent directional source whose acoustic axis lies in the horizontal plane and whose beamwidth satisfies the requirement of relatively small angles of elevation and depression. It must be noticed that the Gaussian-field approach may run into difficulties if the source is too close to the water surface or the water-bottom interface. In either of these situations, the Gaussian field may intercept the boundary before it has become negligibly small. As a rough criterion, the source should be removed by several σ from any boundary.

(2) Normal Modes

While the Gaussian field initialization has the advantage of beginning with a field which varies rather smoothly with depth and, if fairly narrow in width, appears to be a reasonable approximation to a point source (at least to the eye), it is not universally accepted as being without inherent error. See, for example, Wood and Papadakis (1980). An approach based on normal modes can be conceptualized as follows:

For ranges near the source, outward propagation can be described by assuming that the speed of sound profile and bottom properties are uniform throughout the medium (this includes the depth). The Helmholtz equation (2.4) can be solved numerically to obtain the depth-dependent eigenfunctions $Z_m(z)$. Since the Z_m form an orthonormal set, they can be used to represent a point source. Then, for a point source at (0,Z) of unit pressure amplitude at r = 1 m, we have from (2.9) and (2.8)

$$S(r,z) = i\pi \sum_m{}' H_o^{(1)}(k_m r)\, Z_m(Z)\, Z_m(z) \qquad 2.19$$

(1.6) allows us to form an initial expression for $\underline{f}(0,z)$,

$$\underline{f}(0,z) = i\pi \sum_m Z_m(Z)\, Z_m(z) \qquad 2.20$$

Because of the small angular-aperture assumption implicit in the parabolic equation, and also because in most situations of practical interest only modes corresponding to rays with small angles of elevation and depression with respect to the horizontal are trapped, the summation over m can be restricted just to that subset of Z_m satisfying these conditions. Notice that since only this subset is to be retained, the Helmholtz equation must be solved only to obtain this subset which results in a substantial saving in computer running time.

Given the initial expression (2.20), the discrete depth values z_m are specified, the associated $\underline{f}(0,z_m)$ found, and the solutions for increasing ranges stepped out. If practical aspects of the problem preclude starting at $r = 0$, the normal modes will have to be allowed to propagate out to the minimum range r_o at which the parabolic equation can be implemented. In such a case, errors introduced as a result of approximations necessitated in utilizing the normal mode problem must be studied carefully. For example, the bottom may have to be unrealistically smoothed out to r_o.

For an example utilizing a normal-mode starter (and displaying some of the difficulties encountered), see Guthrie and Gordon (1977).

(3) Ray Tracing

Another approach to obtaining an initial set of values for $\underline{f}$ at some finite range r_o from the source involves phase-coherent ray tracing. From a practical point of view, this has appeal, since interactions of any ray with the bottom can be dealt with by introducing the appropriate plane-wave reflection coefficient as long as the source is several wavelengths away from the bottom. The required family of rays is traced out to the desired initial range, with all phase information retained (including that arising from surface and bottom reflections), and then combined to give the resultant pressure distribution at

(r_o, z). From this, $\underline{f}(r_o, z)$ can be obtained and the $\underline{f}$'s for larger ranges found by stepping out with whatever algorithm is used to solve the parabolic equation. For an example, see Guthrie and Gordon (1977).

B. Treatment of the Bottom

Adequate representation of the bottom has been a difficulty with the parabolic wave equation. The seriousness of the problem and how it is dealt with depends on the algorithm used in stepping the solution out in range.

We can say that many of the approaches fall into three categories:

(a) Reflection of the equivalent rays from the bottom allows the reflective loss to be given as a function of the apparent angle of incidence of the ray on the bottom. This can be either specified as input, or calculated according to some formula. The possibility of the bottom depth being a function of range is included. (If attention is restricted to sufficiently large range, of course, all rays less grazing than critical can be neglected since these correspond to bottom bounce paths with appreciable transmission into the bottom at each interception.) However, the determination of what particular rays are striking the bottom at each specified range is a nontrivial process, and the models and techniques used in this determination often contain rather sweeping approximations and may impose questionable requirements on the depth dependence of the speed of sound and the absorption coefficient in the bottom. An example of this approach can be found in the report by Stieglitz et al. (1979).

(b) The speed of sound and the absorption coefficient can be specified as functions of depth by a complex index of refraction for the bottom, and this information built into the algorithm. Depending on the computational scheme, it may be necessary to prohibit any discontinuity in $c(z)$ at the water-bottom interface by allowing $c(z)$ to change rapidly but smoothly over some small interval in depth about the interface, and to have the bottom absorption coefficient rise with depth from zero at the interface to finite value at larger depths. In this method it may be necessary to introduce an artificial pressure-release boundary at some appreciable depth in the bottom. This step is not particularly troublesome, if done correctly: For an absorbing or non-absorbing bottom, the acoustic field will eventually begin to decay quasi-exponentially so that the energy found at all but shallow depths in the bottom becomes quite small; if this is allowed to be reflected back up through the bottom and into the water, the error introduced into the resultant acoustic field is negligible if the reflecting surface is deep enough. See, for example, Williams (1975) and the references he cites.

(c) If the algorithm permits, the water-bottom interface can be built into the computer code directly in terms of the boundary conditions of continuity of pressure and continuity of the normal component of the particle velocity. Let the water be labeled fluid 1 and the bottom fluid 2, and assume a flat bottom at depth $z = Z_B$. Then (for a flat horizontal bottom) the boundary conditions on $\underline{f}$ become

$$\underline{f}_1(Z_B) = \underline{f}_2(Z_B) \qquad 2.21$$

and

$$\rho_1^{-1}(\partial \underline{f}_1/\partial z)_{Z_B} = \rho_2^{-1}(\partial \underline{f}_2/\partial z)_{Z_B} \qquad 2.22a$$

Notice that if the bottom is not flat, so that Z_B is itself a function of range, (2.21) remains the same but the continuity of normal particle velocity takes the more general form

$$\rho_1^{-1} \hat{n} \cdot (\nabla \underline{f}_1)_{Z_B} = \rho_2^{-1} \hat{n} \cdot (\nabla \underline{f}_2)_{Z_B} \qquad 2.22b$$

where $\hat{n}$ is the unit normal to the bottom at coordinate (r, Z_B).

III. Methods of Solution

There are, of course many different ways in which solutions to the parabolic equation (1.11) can be attempted. As with the Helmholtz equation, closed-form or analytical solutions are available only for certain special cases. As a consequence, solutions are accomplished numerically with algorithms resulting in efficient computer use. As might be expected, the more complicated the case, the more difficult the solution. Certain methods of solution are useful and of sufficient accuracy only when the bottom is not important to the problem, as in the long-range propagation in the deep sound channel when the bottom of the channel lies well above the ocean floor, or when the bottom is sufficiently well-matched that it can be treated by fairly simple approximations. More recent methods have been developed which allow more realistic inclusion of a bottom. Our approach here will be to outline a few of the methods in roughly chronological order of appearance, but leaving detailed discussion and the consideration of fine points and subtleties for the interested reader to discover from the references.

Before discussing the methods, we shall develope a single formalism to unify the discussion. Rearrange (1.11) to isolate the derivative with respect to range,

$$\partial \underline{f}/\partial r = \left[\tfrac{1}{2}i(k^2-k_o^2)/k_o + (\tfrac{1}{2}i/k_o)(\partial^2/\partial z^2)\right]\underline{f} \qquad 3.1$$

Following the fundamental lemma of calculus

$$\frac{\partial \underline{f}}{\partial r} = \lim_{\Delta r \to 0} \frac{\underline{f}(r+\Delta r,z) - \underline{f}(r,z)}{\Delta r} \qquad 3.2$$

we can turn this into an equation involving the increment Δr in range, which provides the basis for the numerical incrementation of range,

$$\underline{f}(r+\Delta r,z) = \left[1+(\Delta r)\tfrac{1}{2}ik_o(n^2-1) + (\Delta r)(\tfrac{1}{2}i/k_o)(\partial^2/\partial z^2)\right]\underline{f}(r,z) \qquad 3.3$$

If we define the operators

$$\underline{a} = \tfrac{1}{2}i(k^2-k_o^2)/k_o = \tfrac{1}{2}ik_o(n^2-1) \qquad 3.4a$$

$$\underline{b} = i/(2k_o) \qquad 3.4b$$

$$D^2 = \partial^2/\partial z^2 \qquad 3.4c$$

then (3.3) assumes the form

$$\underline{f}(r+\Delta r,z) = \left[1+(\Delta r)\underline{a}+(\Delta r)\underline{b}D^2\right]\underline{f}(r,z) \qquad 3.5$$

The operator $\underline{a}$ describes the refractive properties of the medium (and may include those of the bottom). It is thus a function of range, depth, and the constant k_o. The product of operators $\underline{b}D^2$ describes the depth dependence of $\underline{f}$.

If we make use of the expansion of an exponential,

$$\exp(x) = 1 + x + O(x^2) \qquad 3.6$$

then it is clear that we can write (3.5) symbolically in the form

$$\underline{f}(r+\Delta r,z) = e^{\Delta r(\underline{a}+\underline{b}D^2)}\underline{f}(r,z) \qquad 3.7$$

if $|\Delta r[\underline{a} + \underline{b}D^2]\underline{f}| << |\underline{f}|$. The reason for this representation is that it will allow manipulation of the operators to be brief and simplified. For example, note that the manipulation

$$e^{A+B} = e^A e^B$$

is identical with

$$1 + A + B \doteq (1 + A)(1 + B)$$

through terms of first order in A and B. The difference between the two sides of the above equation is 2AB, which is of second order. As can be seen, for small Δr the operators $\underline{a}$ and $\underline{b}D^2$ acting on $\underline{f}$ produce terms which are substantially smaller then $\underline{f}$ itself, so this symbolic formalism is useful.

A. the Split-step Fourier Transform

This approach makes use of the Fourier transform and its inverse to eliminate the second partial with respect to z. There are two forms in which this can be done; we first consider the form which appears to be used most widely.

Let us define the symmetric complex Fourier transform pair

$$\underline{F}_s(\ ) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} (\ )\, e^{isz}\, dz \qquad 3.8a$$

$$\underline{F}_s^{-1}(\ ) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} (\ )\, e^{-isz} ds \qquad 3.8b$$

where ( ) represents any reasonably well-behaved function of s in the transform (3.8a) and any reasonably well-behaved function of z in the inverse transform (3.8b). (For the required mathematical properties defining "reasonably well-behaved", consult any standard text on transform theory.)

Let us apply the transform pair to the RHS of (3.7) after utilizing the approximation

$$e^{\Delta r(\underline{a} + \underline{b}D^2)} \approx e^{(\Delta r)\underline{a}} e^{(\Delta r)\underline{b}D^2} \qquad 3.9$$

Now,

$$\underline{F}_s\left(e^{\Delta r\underline{b}D^2}\underline{f}\right) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{\Delta r\ \underline{b}D^2}\underline{f}\ e^{isz}dz \qquad 3.10$$

and

$$e^{\Delta r\underline{b}D^2}\underline{f}e^{isz} \approx (1 + \Delta r\underline{b}\frac{\partial^2}{\partial z^2})\ \underline{f}\ e^{isz}$$

$$\approx (1 - \Delta r\underline{b}s^2)\ \underline{f}\ e^{isz}$$

$$\approx e^{-\Delta r\underline{b}s^2}\ \underline{f}\ e^{isz} \qquad 3.11$$

Because the integration in (3.10) is over z, $\exp(-\Delta r\underline{b}s^2)$ can be factored out,

$$\underline{F}_s\left[\exp(\Delta r\underline{b}D^2)\underline{f}\right] \approx \exp(-\Delta r\underline{b}s^2)\underline{F}_s(\underline{f}) \qquad 3.12$$

and the inverse transform (3.8b) applied,

$$\exp(\Delta r\underline{b}D^2)\underline{f} \approx \underline{F}_s^{-1}\left[\exp(-\Delta r\underline{b}s^2)\underline{F}_s(\underline{f})\right] \qquad 3.13$$

It then follows at once from (3.9) that

$$e^{[\Delta r(\underline{a}+\underline{b}D^2)]} \approx \exp(\Delta r\underline{a})\underline{F}_s^{-1}\left[e^{(-\Delta r\underline{b}s^2)}\underline{F}_s(\underline{f})\right] \tag{3.14}$$

and insertion of this into (3.7) yields the split-step Fourier transform result

$$\underline{f}(r+\Delta r,z) = \exp(\Delta r\underline{a})\underline{F}_s^{-1}[\exp(-\Delta r\underline{b}s^2)\underline{F}_s(\underline{f})] \tag{3.15}$$

where $\underline{a}$ and $\underline{b}$ are given by (3.4a) and (3.4b). Notice that they yield purely multiplicative factors. Examination of (3.15) reveals that the algorithm can be viewed as dealing with <u>diffractive</u> effects by the exponent multiplying $\underline{F}_s$ and <u>refractive</u> effects separately by the exponent multiplying the inverse transform.

Detailed investigation of the errors introduced by use of the exponential approximations is involved and results in rather difficult expressions to interpret. However, Jensen and Krol (1975) have investigated the results for situations typically encountered in the ocean and obtained relatively simple criteria: For cases of practical interest, the errors introduced in predicting the behavior of $\underline{f}$ over the range increment Δr are bounded by the <u>larger</u> of

$$\left|2(\omega/c_o)\,(\partial n/\partial z)\Delta r\,(\partial \underline{f}/\partial z)\right| \tag{3.16a}$$

or

$$\left| \left[(\omega/c_o)(\partial n/\partial z)\Delta r \right]^2 \underline{f} \right| \qquad 3.16b$$

These reveal that if $\underline{f}$ is not sufficiently slow in its depth variation, then the error increases linearly with frequency and linearly with the range increment. For sufficiently slowly varying $\underline{f}$, however, the error depends quadratically on both these quantities.

When (3.15) is implemented on a computer, the Fourier transform is replaced by its numerical counterpart, the FFT. This results in a fast-running algorithm which deals with a finite number of terms. The FFT requires a finite depth over which it is applied, so an artificial bottom (pressure release) must be inserted at some distance below the water-bottom interface. Given this interval in z, the depth increment can be selected for application of the FFT. It must be noted, however, that running time increases fairly dramatically as the number of depth increments N is increased. Guthrie and Gordon (1977) estimate that the time will increase approximately as log (N^N). This can place rather drastic restrictions on the variation of the index of refraction n(r,z) with depth, since the number of points in depth must be great enough to reproduce with fairly high accuracy the details of the speed of sound gradients.

An additional problem with the split-step Fourier transform approach is that the change in density between water and the bottom cannot be considered. While the reflection coefficient at at the water-bottom interface is considerably more sensitive to changes in the speed of sound than to changes in the density, if the change in the latter is too great, realistic predictions cannot be expected from the algorithm. Attempts to surmount this difficulty by using equivalent rays and specifying the reflective loss (and phase angle) at the interface have had mixed success. Indeed, pessimism has been expressed as to the utility of this algorithm in situations for which the bottom is an important element of the propagation problem; see, for example, Jensen and Krol (1975) and Brock (1978). Further, inclusion of a bottom increases running time considerably. Jensen and Krol (1975) have compared computer times for split-step parabolic, normal mode, and ray programs for deep water (for which the bottom is not important) and shallow water for which the bottom is important. There is considerable variation in running times, but in particular for their case the shallow water problem led to extremely long running times for the split-step approach.

The second form of the split-step Fourier transform approach has been described by McDaniel (1975). This is based on an alternative formulation of the RHS of (3.7) presented by Tappert and Hardin (1974),

$$\exp[\Delta r(\underline{a}+\underline{b}D^2)] \approx \exp(\Delta r\underline{a}/2)\exp(\Delta r\underline{b}D^2)\exp(\Delta r\underline{a}/2)$$

Application and manipulation of the Fourier transform, its inverse, and the exponential terms proceeds much as before, and the details are uninteresting. The result is

$$f(r+\Delta r,z) \approx e^{\Delta r \underline{a}/2} \underline{F}_s^{-1} \left[e^{-\Delta r \underline{b} s} \underline{F}_s \left(e^{\Delta r \underline{a}/2} \underline{f} \right) \right] \tag{3.17}$$

Comparison of this result with the analogous (3.15) reveals that the index of refraction has been treated a little more carefully. When (3.17) is analyzed for the dependence of errors on the range increment, it appears that the error depends on $(\Delta r)^3$ which is an improvement over the errors resulting from the approximations leading to (3.15). What restrictions must be placed on Δz, however, by the introduction of $n(r,z)$ into the Fourier transform over z do not seem to have been isolated.

B. Alternatives to the Split-step Method

Within the last few years, there have been a number of alternative approaches to developing efficient algorithms for solving the parabolic equation (1.11). We will mention three closely related articles which provide a rather interesting line of evolution. All of these discard the split-step technique with its associated use of the FFT direct and inverse numerical method. Instead, the second derivative with respect to depth is calculated numerically from the depth increments by the use of a central finite-difference approximation. Since the approaches based on this technique require more detailed specification of the coordinates of any spatial point on the range and depth mesh, we will adopt a convenient formalism which, although somewhat unconvential, is easily followed and succinct. Since the depth below the ocean surface can be written as $z_m = m\,\Delta z$ where Δz is the chosen depth increment, and the range can be written as $r_n = n\,\Delta r$, let us define the shorthand

$$\underline{f}(r_n, z_m) = \underline{f}(n\Delta r, m\Delta z) \equiv f[n,m] \qquad 3.18$$

(Note that n need not begin with n = 0. If the parabolic equation is initialized at some finite range, then the appropriate finite value of the initial n can be used.) The underbar on the RHS has been suppressed for notational convenience. It is to be understood that f[n,m] is complex.

Now, it is straightforward to see that the second depth derivative can be estimated numerically by the formula

$$\left.\frac{\partial^2 f}{\partial z^2}\right|_{n,m} \approx \frac{f[n,m+1] - 2f[n,m] + f[n,m-1]}{(\Delta z)^2} \qquad 3.19$$

By use of (3.19), Lee and Papadakis (1980) cast (1.11) into the form

$$df[n,m]/dr = \underline{a}[n,m]f[n,m] + [\underline{b}/(\Delta z)^2]\{f[n,m+1] - 2f[n,m] + f[n,m-1]\} \qquad 3.20$$

where the operators a and b have been defined previously, (3.4). (It is worth recalling, for this discussion and what follows, that b is simply a constant and that a depends functionally on depth and may also depend on range through the square of the index of refraction.) This equation reveals that the first range derivative of f at some point [n,m] depends on the values of f at the adjacent mesh points characterized by [n,m+1], [n,m], [n,m-1]. Thus, (3.20) provides a set of first-order ordinary differential equations, each of which is coupled to its immediate neighbors in depth. While this may appear to require a considerable amount of storage when these matrices are programmed into the computer, the fact that the coupling involves only adjacent depth values collapses the matrix representing the RHS of (3.20) to a "tridiagonal" form for which all elements are zero except those whose depth indices are m, m $\pm$ 1. The depth incrementation is halted at the bottom, and Lee and Papadakis let the element f[n,M+1] represent the value of f at the bottom.

Note that M may itself be a function of n, so that a sloping or irregular ocean-bottom interface is allowed. The range-depth mesh does not project into the bottom. In this sense this approach is incomplete, but the authors avoid this difficulty to some extent by invoking a boundary condition

$$\nabla \underline{p} \cdot \hat{n} + \underline{\alpha}(s)\underline{p} = 0 \qquad 3.21$$

where $\hat{n}$ is the local normal to the interface and s specifies the line of the interface in (r,z) space. It can be quickly seen for monofrequency sound that this is equivalent to the condition

$$-i\omega\vec{\underline{u}}\cdot\hat{n} + \underline{\alpha}(s)\underline{p} = 0 \qquad 3.22a$$

which can be rearranged to yield the form

$$\underline{p}/(\vec{\underline{u}}\cdot\hat{n}) = i\omega/\underline{\alpha}(s) \qquad 3.22b$$

Thus, $i\omega/\underline{\alpha}$ represents the ratio of the pressure $\underline{p}$ to the component of the particle velocity $\vec{\underline{u}}$ locally normal to the bottom. For the case of a flat horizontal bottom, (3.21) takes the form

$$\partial \underline{p}/\partial z + \underline{\alpha}(r)\underline{p} = 0 \qquad \text{at the bottom} \qquad 3.23$$

Equation (3.23) can be solved by $\underline{p} = \exp(-\underline{\alpha}\Delta z)$, and the numerical equivalent for specifying the interface value f[n,M+1] is

$$f[n,M+1] = f[n,M]\exp\left\{-\underline{\alpha}[n]\Delta z\right\} \qquad 3.24a$$

At the ocean surface, of course, the pressure must vanish and the surface boundary condition is

$$f[n,0] = 0 \qquad 3.24b$$

The authors solve two cases: a perfectly-rigid, flat, horizontal bottom and a perfectly-rigid, flat, sloping bottom. For both, $\mathcal{Z} = 0$. Results are compared respectively with a normal mode solution (horizontal bottom) and a method of images solution (sloping bottom), and are found to be in very good agreement.

In a later paper, Lee, Botseas, and Papadakis (1981) reinvestigated the solution of (3.20), this time using an implicit finite difference method rather than the linear and nonlinear multistep methods used in the Lee and Papadakis (1980) article. The practical advantage of the implicit finite-difference technique is that (at the present time) while it is, according to the authors, about equally as fast as solving (3.20) by the multistep methods, it requires significently less memory. Several propagation examples are worked out, using range independent, upslope, and downslope cases. The bottoms were fast and absorptive with densities up to 50% greater than found in the water column. Results were compared with a normal mode program (SNAP) and a split-step parabolic equation (PAREQ), both from SACLANT. Propagation loss curves for all three programs are shown for the various cases, but detailed comparisons and discussions were not attempted. Again, the boundary conditions were handled according to use of (3.24a) for the horizontal bottom, that equation's generalization for the range dependent bottoms, and the surface was represented by (3.24b). The authors also point out the time-saving advantage of not having to run the parabolic solution <u>into</u> the bottom.

McDaniel and Lee (1982) attacked the boundary condition at the bottom more directly. If the bottom is found at depth $m_b \Delta z$, assumed to be flat and horizontal, and to have properties independent of range, then application of continuity of pressure and the normal component of particle velocity across the interface yields

$$f_1[n,m_b] = f_2[n,m_b] \qquad 3.25a$$

$$\rho_1^{-1}\partial f_1[n,m_b]/\partial z = \rho_2^{-1}\partial f_2[n,m_b]/\partial z \qquad 3.25b$$

where the subscripts 1 and 2 refer to the water column and ocean bottom respectively. After performing some Taylor series expansions for the second derivative of f with depth in both media and making use of the boundary conditions, the authors are able to write a parabolic equation which must be satisfied at the interface,

$$\frac{\partial f[n,m_b]}{\partial r}\left(1+\frac{\rho_1}{\rho_2}\right) = \left(a_1[n,m_b] + \frac{\rho_1}{\rho_2}a_2[n,m_b]\right)u[n,m_b] + \frac{2b}{(\Delta z)^2}\left\{\frac{\rho_1}{\rho_2}u[n,m_b+1] - \left(1+\frac{\rho_1}{\rho_2}\right)u[n,m_b] + u[n,m_b-1]\right. \qquad 3.26$$

Away from the interface, of course, (3.20) must hold in both media. Thus, (3.26) is solved for $m = m_b$ but (3.20) is solved for all other m. With the bottom included in the problem, through application of the boundary conditions (3.25), it is necessary to allow the depth z to penetrate at least one increment Δz below the interface. With the single exception of (3.26) for $m = m_b$, the problem is run according to any of the above-mentioned methods; if the split-step FFT method is used, however, it must be assumed that there is no density change from water into bottom. An example is worked out: a two gradient concave speed of sound profile overlies an isospeed fast bottom with density slightly more than twice that of the water. The problem is solved using a normal mode program, the split-step FFT method, and implicit finite-difference methods with and without the interface condition (3.26). For the parabolice equation solutions, a Gaussian-field initialization was used. As seen in Fig. 3 of the article, the agreement between normal mode and the implicit finite difference with (3.26) methods is strikingly good.

IV. Comments

While the parabolic equation approximation introduces an unavoidable phase-speed error, the effects of this error can be mitigated either by solving the pseudoproblem according to the mapping $(n,z)\to(n^*,z^*)$ developed by Brock et al. (1977), or by the modified equation (1.29) obtained by DeSanto, Perkins, and Baer (1978).

Concerning the split-step FFT algorithm, the depth increment must be chosen small enough that abrupt changes in the speed of sound profile do not introduce significant errors. This is not too serious in the water column, but the number of increments can become prohibitive in terms of computational time if there is a sharp change in the speed of sound between the water column and the bottom. Further, this approach cannot deal with the change of density between water and bottom. The various approaches attempting to accommodate the density change do not appear to be very satisfactory, particularly when the bottom is an important part of the propagation problem.

The use of central-difference methods for treating the second partial of $\underline{f}$ with depth has opened up an avenue of approach which allows a much more realistic, and physically satisfying, treatment of a fluid bottom. Further extension along these lines is to be anticipated.

The parabolic equation method is based on monofrequency, continuous wave signals. This means that the multipath problems encountered with transient signals are not considered, and to date it has not been possible to isolate time-separated contributions at the receiver _via_ the parabolic equation. While in principle Fourier synthesis over the required band of frequencies is possible, implementation at this time would appear to be extremely prohibitive.

The assumption of cylindrical symmetry is important and restrictive whenever an irregular bottom is a significant feature of the case of interest. For example, if propagation over a sea-mount is considered, the assumption of cylindrical symmetry converts the mount into a "ring" whose center of symmetry is the z axis. This results in the neglect of any azimuthal reflection or scattering of the incident sound field from the sides of the mount, and can seriously affect comparisons between the cylindrically symmetric parabolic equation predictions and experimental results.

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(JASA is an abbreviation for J. Acoust. Soc. Am.)

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