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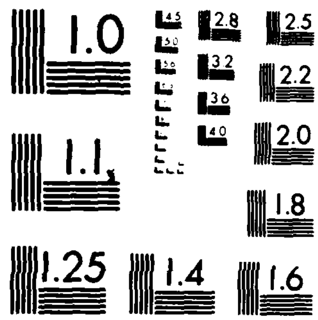

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ONE-DIMENSIONAL STRUCTURED PHASE TRANSFORMATIONS UNDER PRESCRIBED LOADS

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ABSTRACT

We consider a finite one-dimensional elastic body with stored-energy function of the form $W(\gamma) + \varepsilon^2(\gamma')^2$, where γ is the strain, γ' the strain gradient, and ε a small parameter. We assume that $W(\gamma)$ is non-convex, of the type capable of supporting two-phases. We show that when the body is acted on by end loads the only stable strain fields are the single-phase solutions $\gamma \equiv$ constant.

AMS (MOS) Subject Classification: 73G05

Key Words: Elasticity, Phase Transformations

Work Unit Number 2 - Physical Mathematics

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SIGNIFICANCE AND EXPLANATION

epsilon sub 2 (Gamma') sub 2

The authors ~~We~~ consider a one-dimensional elastic bar acted on by end loads. To allow for the possibility of phase transitions they ~~we~~ consider a stored-energy

gamma gamma

function of the form $W(\gamma) + \epsilon^2(\gamma')^2$ with $W(\gamma)$ a nonconvex function of

gamma

strain γ, a function of the type capable of supporting two-phases. The term

gamma

involving the strain gradient γ' is added to model the region of rapidly varying strain in the vicinity of a phase transition.

Epsilon

It is ~~We~~ shown that this problem has smooth solutions which, for ϵ small, correspond to the single-phase and two-phase solutions of the problem with

Epsilon It is

$\epsilon = 0$. ~~We~~ shown further that the only stable solutions of this problem are the single-phase solutions $\gamma \equiv$ constant.

ONE-DIMENSIONAL STRUCTURED PHASE TRANSFORMATIONS UNDER PRESCRIBED LOADS

Jack Carr, Morton E. Gurtin, and Marshall Slemrod

1. Introduction

Consider a bounded one-dimensional body identified with the interval $[0,L]$. We assume that the body is elastic with stored energy $W(\gamma)$ a C^2 function of strain[1] γ, $0 < \gamma < \infty$, and with stress

$$\sigma(\gamma) = \frac{dW(\gamma)}{d\gamma}$$

of the form shown in Figure 1. Thus

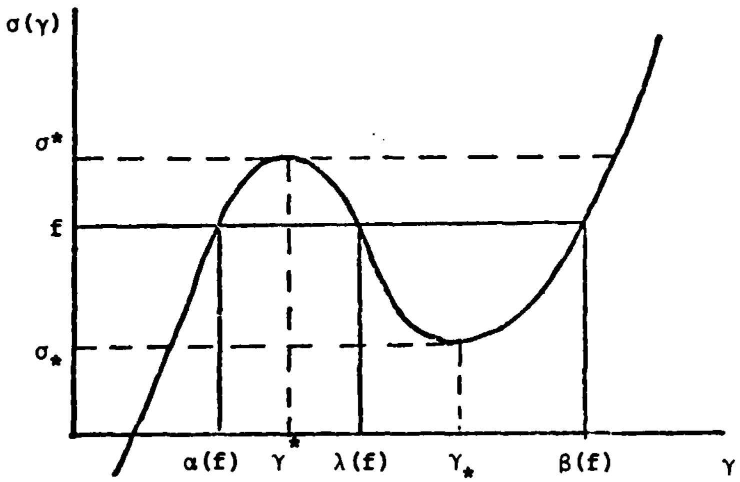

Figure 1. The stress-strain law.

$\sigma' > 0$ on $(0,\gamma^*)$, $\sigma' < 0$ on (γ^*,γ_*), and $\sigma' > 0$ on (γ_*,∞). Let $\sigma_* = \sigma(\gamma_*)$, $\sigma^* = \sigma(\gamma^*)$. Then for $f \in (\sigma_*,\sigma^*)$ there are exactly three values of γ for which

$$\sigma(\gamma) = f .$$

We denote these by

$$\alpha(f) < \lambda(f) < \beta(f) ,$$

omitting the argument f when convenient, and we extend $\alpha(f)$ {respectively $\beta(f)$} in the obvious manner for $f \in (0,\sigma_*]$ {respectively $[\sigma^*,\infty)$}.

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[1] If the underlying deformation u carries material points $x \in [0,L]$ to points $u(x)$, then $\gamma = du/dx$.

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We restrict our attention to loading applied by a soft device[2], so that the load f (= constant) is prescribed. The problem of determining the stable strain field resulting from this load consists in minimizing the corresponding potential energy. This problem can be stated precisely as follows:

(S) Minimize

$$\int_0^L [W(\gamma) - f\gamma]\,dx \tag{1.1}$$

over all $\gamma \in L^1(0,L)$ with $W(\gamma) \in L^1(0,L)$.

This problem has been studied by Ericksen [1975], and we briefly record some of his results. The Euler-Lagrange equation is

$$\sigma(\gamma) = f\ ,$$

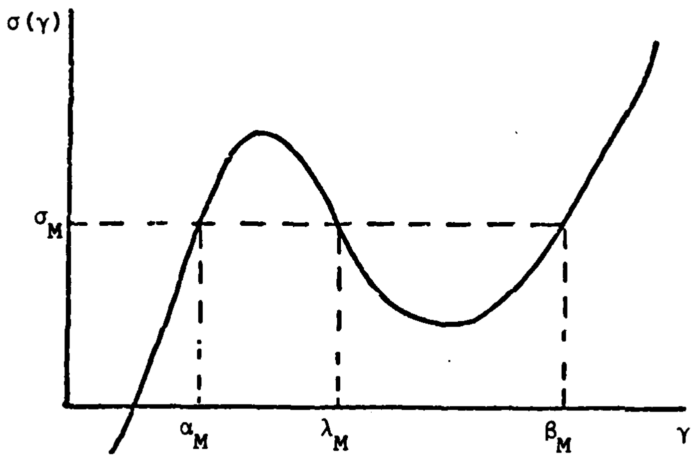

Figure 2. The Maxwell line.

while the form of the minimizer depends on whether or not the load f is smaller than, equal to, or larger than the Maxwell stress σ_M, where σ_M is the unique value of stress for which (Figure 2)

$$W(\beta_M) - W(\alpha_M) = \sigma_M(\beta_M - \alpha_M)$$
$$\alpha_M = \alpha(\sigma_M),\ \beta_M = \beta(\sigma_M)\quad .$$

The strains α_M and β_M are the lower and upper Maxwell strains, respectively.

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2 Cf. Ericksen [1975].

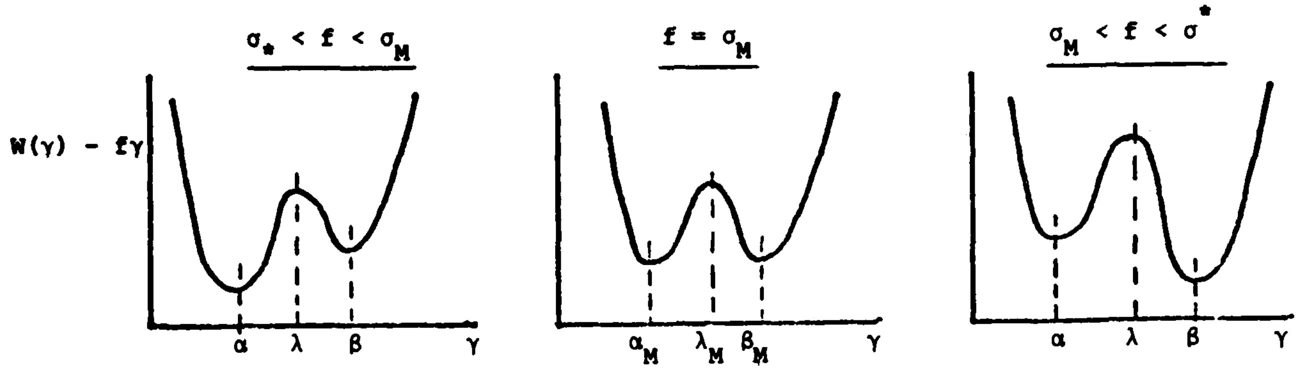

Figure 3. The potential $W(\gamma) - f\gamma$.

As is clear from Figure 1, the potential $W(\gamma) - f\gamma$ has the form shown in Figure 3. For $f < \sigma_*$ the local minimum at β disappears, as does the local minimum at α when $f > \sigma^*$. Therefore for $f \neq \sigma_M$ the solution to Problem S is[1]

$$\gamma(x) \equiv \alpha \quad \text{when} \quad f < \sigma_M \ ,$$

$$\gamma(x) \equiv \beta \quad \text{when} \quad f > \sigma_M \ .$$

For $f = \sigma_M$ the problem is far more interesting. Here any function of the form

$$\begin{aligned} &\gamma(x) = \alpha_M \quad \text{for all} \quad x \in S \ , \\ &\gamma(x) = \beta_M \quad \text{for all} \quad x \in [0,L] \setminus S \ , \qquad (1.2) \\ &S \subset [0,L], \quad S \ \text{measurable} \ , \end{aligned}$$

is a minimizer.

We will use the following terminology: solutions of the form

$$\gamma(x) \equiv \text{constant}$$

will be called <u>single-phase solutions</u>; solutions of the form (1.2) with S

---

1 The energy (1.1) also has a local minimum relative to $L^\infty(0,L)$ at α for $\sigma_M < f < \sigma^*$ and one at β for $\sigma_* < f < \sigma_M$; these constant functions are not local minima relative to $L^1(0,L)$.

and $[0,L]\setminus S$ nontrivial will be called two-phase solutions; in the latter case, when both the number n_1 of nontrivial connected components of S and the number n_2 of nontrivial connected components of $[0,L]\setminus S$ are finite, we will refer to n_1+n_2-1 as the degree of the solution.

Thus for $f \neq \sigma_M$ the solution is single-phase (and unique); for $f = \sigma_M$ there are two single-phase solutions ($\gamma \equiv \alpha_M$ and $\gamma \equiv \beta_M$) and an uncountable infinity of two-phase solutions of each degree ≥ 1.

For the case $f = \sigma_M$ it seems reasonable to ask whether any of the solutions are preferred. One might expect that the single-phase solutions are, in some sense, more stable than the two-phase solutions. We here show that this expectation is indeed true provided we consider the present theory as an approximation to a higher-order theory which allows the strain-gradient to enter the constitutive equation for the stored energy.

A general nonlinear theory of this type was apparently first given by Toupin [1962] and is based on constitutive functions for the stored energy and stress of the forms

$$\hat{W}(\gamma,\gamma') \quad \text{and} \quad \sigma(\gamma,\gamma')$$

with

$$\sigma(\gamma,\gamma') = \frac{\partial}{\partial\gamma}\hat{W}(\gamma,\gamma') \quad .$$

The presence of the strain-gradient $\gamma' = d\gamma/dx$ models situations in which the strain varies rapidly; in this instance the force interactions within the underlying crystal lattice cannot be adequately described by the usual notion of stress and one needs - in addition to the (classical) stress σ - a second-order stress μ which, in the present theory, is related to $\hat{W}$ through

$$\mu(\gamma,\gamma') = \frac{\partial}{\partial\gamma'}\hat{W}(\gamma,\gamma') \quad .$$

We here take the simplest possible generalization of our original theory and write

$$\hat{W}(\gamma,\gamma') = W(\gamma) + \varepsilon^2(\gamma')^2$$

with W <u>the original stored energy</u> and ε a "small" constant (> 0). The stress σ is then independent of γ' and is given by

$$\sigma(\gamma) = \frac{dW(\gamma)}{d\gamma} \quad ,$$

while the second-order stress obeys the simple relation

$$\mu(\gamma') = 2\varepsilon^2\gamma' \quad .$$

Within this framework our problem has the following form:

(S_ε) <u>Minimize</u>

$$\int_0^L [W(\gamma) - f\gamma + \varepsilon^2(\gamma')^2]dx \qquad (1.3)^1$$

<u>over the class of absolutely continuous functions</u> γ <u>on</u> $[0,L]$.

The Euler-Lagrange equation for this problem is

$$2\varepsilon^2\gamma'' = \sigma(\gamma) - f \quad , \qquad (1.4)$$

while the natural boundary conditions have the form

$$\gamma'(0) = \gamma'(L) = 0 \quad , \qquad (1.5)$$

or equivalently, $\mu(\gamma') = 0$ at $x = 0,L$.

The boundary conditions (1.5) seem reasonable. Indeed, our wish is to focus on the region of rapidly varying strain that occurs in the vicinity of a phase transition, and it is there that the higher-order stress μ should play an important role. Away from the phase transition we expect the classical

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[1] Energy functionals of this type originate in the classic paper of van der Waals [1893]. Cahn and Hilliard [1958] independently rederived van der Waals' basic theory, and in the last twenty five years gradient theories have become a popular tool in analyzing phase transitions and other physical phenomena. (Cf. Rowlinson's [1979] translation of van der Waal [1893] for a list of selected references.)

theory to apply, and for that reason the boundary conditions (1.5) seem appropriate. [Of course, one might expect it to be difficult if not impossible to achieve this form of boundary condition in practice.]

It is clear that if a single-phase solution γ minimizes the original functional (1.1), γ trivially minimizes (1.3), as $\gamma' \equiv 0$. Thus the complete list of solutions of Problem S_ε consists of the single-phase solutions

$$\begin{aligned} &\gamma(x) \equiv \alpha && \text{for } f < \sigma_M \ , \\ &\left.\begin{aligned}\gamma(x) &\equiv \alpha_M \\ \gamma(x) &\equiv \beta_M\end{aligned}\right\} && \text{for } f = \sigma_M \ , \\ &\gamma(x) \equiv \beta && \text{for } f > \sigma_M \ . \end{aligned}$$

Because of the term $\varepsilon^2(\gamma')^2$ in (1.3), two-phase solutions of the form (1.2) are not members of any function class which renders S_ε a meaningful problem. On the other hand, there are solutions of the Euler-Lagrange equation (1.4) for $f = \sigma_M$ which spend most of their time near α_M and β_M and which correspond to the two-phase solutions of the unstructured theory. This raises the question of whether such structured solutions can be relative minimizers of (1.3). We will show that the answer to this question is no; in fact, we will prove that all nonconstant solutions γ of the problem

$$\begin{aligned} 2\varepsilon^2\gamma'' &= \sigma(\gamma) - f \ , \\ \gamma'(0) &= \gamma'(L) = 0 \end{aligned} \tag{1.6}$$

are unstable.

Remark. Our analysis is also appropriate to a fluid in a one-dimensional chamber acted on by a pressure $p = -f$. In this instance γ is the specific volume, $W(\gamma)$ is the internal energy per unit mass, $x \in [0,L]$ labels material points in a fixed reference configuration with uniform density ρ_0, and the functionals (1.1), (1.3) - multiplied by ρ_0 - represent the total potential energy.

2. Structured solutions.

It is convenient to change length scales. We let

$$t = x/\varepsilon, \quad L_\varepsilon = L/\varepsilon$$

and consider $\gamma(t)$ as a function of t rather than x; the problem (1.6) then takes the form

$$(\hat{S}_\varepsilon)\begin{cases} 2\ddot{\gamma} = \sigma(\gamma) - f \quad , \\ \dot{\gamma}(0) = \dot{\gamma}(L_\varepsilon) = 0 \end{cases} \tag{2.1}$$

with $\dot{\gamma} = d\gamma/dt$, and multiplying $(2.1)_1$ by $\dot{\gamma}$ and integrating leads to the "first integral"

$$\dot{\gamma}^2 = \Phi(\gamma) - \Phi(\gamma_0) \quad , \tag{2.2}$$

where

$$\begin{aligned} \Phi(\gamma) &= W(\gamma) - f\gamma \quad , \\ \gamma_0 &= \gamma(0) \quad . \end{aligned} \tag{2.3}$$

For future use we note also that the associated energy (1.3) is now given by

$$E(\gamma) = \int_0^{L_\varepsilon} [\dot{\gamma}^2 + \Phi(\gamma)]dt \quad . \tag{2.4}$$

Remark. Equation $(2.1)_1$ is the equation of motion of an undamped oscillator of "mass" 2 and "potential energy" $-\Phi(\gamma)$; the functional E is the corresponding "Hamiltonian".

We now state two propositions; these allow us to concentrate on nonconstant solutions to Problem $\hat{S}_\varepsilon$ and to $f \in (\sigma_*,\sigma^*)$. The first proposition follows from the fact that $\gamma \equiv$ constant solves $\hat{S}_\varepsilon$ if and only if $\sigma(\gamma) = f$. The second is a direct consequence of the phase portraits for $f \notin (\sigma_*,\sigma^*)$; for such f there are no trajectories with $\dot{\gamma} = 0$ at two distinct places.

Proposition. *The complete list of constant solutions of Problem* $\hat{S}_\varepsilon$ *is as follows:*

$$\left.\begin{array}{ll} \alpha, & 0 < f < \sigma_* \ , \\ \alpha,\lambda, & f = \sigma_* \ , \\ \alpha,\lambda,\beta, & \sigma_* < f < \sigma^* \ , \\ \lambda,\beta, & f = \sigma^* \ , \\ \beta, & f > \sigma^* \ . \end{array}\right\} \qquad (2.5)$$

Proposition. *All nonconstant solutions to Problem* $\hat{S}_\varepsilon$ *correspond to* f *in the range* (σ_*,σ^*).

The graphs of $-\Phi$ for $f \in (\sigma_*,\sigma^*)$ are simply the curves of Figure 3 turned upside-down. In view of the remark, this allows us to easily arrive at the phase diagrams shown in Figure 4.

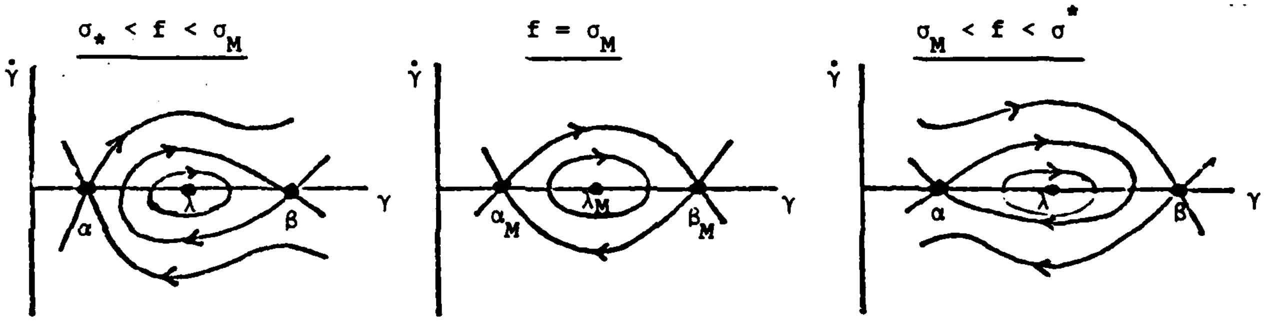

Figure 4. Phase diagrams

Let us agree to use the term *admissible trajectory* for a *nonconstant* solution $\gamma(t)$, $0 < t < T_\gamma$, of the problem

$$2\ddot{\gamma} = \sigma(\gamma) - f \ ,$$

$$\dot{\gamma}(0) = \dot{\gamma}(T_\gamma) = 0 \ .$$

$T_\gamma \in (0,\infty)$ is then the <u>duration</u> of γ, and the number of times t, not including $t = 0$, that $\dot{\gamma}(t) = 0$ is the <u>degree</u> of γ. Problem $\hat{S}_\varepsilon$ (for nonconstant solutions) can then be stated as follows: find an admissible trajectory with duration L_ε.

This formulation allows us to easily establish the following existence theorem for Problem $\hat{S}_\varepsilon$.

<u>Theorem</u>. <u>Let</u> $f \in (\sigma_*,\sigma^*)$. <u>Then there is a constant</u> $C = C(f) > 0$ <u>such that</u>:

(i) <u>for</u> $\varepsilon > L/C$, <u>Problem</u> $\hat{S}_\varepsilon$ <u>has only constant solutions</u>;

(ii) <u>for</u> $\varepsilon < L/C$, <u>Problem</u> $\hat{S}_\varepsilon$ <u>has nonconstant solutions of degree</u> n <u>for each</u> $n = 1,2,\ldots,N$, <u>where</u> N <u>is the largest integer with</u> $N < L/(C\varepsilon)$.

<u>Proof</u>. We will establish this result only for the case

$$f = \sigma_M \;;$$

the remaining cases $\sigma_* < f < \sigma_M$ and $\sigma_M < f < \sigma^*$ are completely analogous.

It is clear from the phase portrait that each admissible trajectory γ may be labeled by its initial-value $\gamma(0) = \gamma_0$ and its degree N. Further, each such γ has γ_0 in either (α_M,λ_M) or (λ_M,β_M), and there is an obvious one-to-one correspondence between the two resulting classes of trajectories; this allows us to restrict our attention to trajectories γ with

$$\gamma_0 \in (\alpha_M,\lambda_M) \quad .$$

Given such a trajectory, $\dot{\gamma}(t) > 0$ for $0 < t < t_1$, where $t_1 = T_\gamma/N$ with T_γ the duration and N the degree of γ; further, $\gamma(t_1) = \gamma_1$, where

$$\gamma_1 = \gamma_1(\gamma_0)$$

is the unique solution of

$$\Phi(\gamma_1) = \Phi(\gamma_0)$$

in the interval (λ_M,β_M) (cf. (2.3) and Figure 3). The duration

$$T_\gamma = T(\gamma_0,N)$$

of such a trajectory is then trivially equal to

$$T_\gamma = N \int_0^{T_\gamma/N} \dot{\gamma}(t)^{-1} d\gamma(t) \quad ,$$

and if we change variable of integration from t to γ, we conclude, with the aid of (2.3), that

$$T(\gamma_0,N) = NT(\gamma_0) \quad ,$$

$$T(\gamma_0) = \int_{\gamma_0}^{\gamma_1(\gamma_0)} \frac{d\xi}{\sqrt{\Phi(\xi)-\Phi(\gamma_0)}} \quad . \tag{2.6}$$

$T(\gamma_0)$ has the limits[1]

$$T(\alpha_M^+) = +\infty, \qquad T(\lambda_M^-) = \pi\sqrt{\frac{2}{-W''(\lambda_M)}} > 0 \tag{2.7}$$

(the latter being one-half the period of the oscillator defined by $(2.1)_1$ linearized about the rest point λ_M). Further the integrand in (2.6) is bounded away from zero, as is the interval of integration for $\gamma_0 \in (\alpha_M,\lambda_M-\delta)$. Thus, in view of $(2.7)_2$, C defined by

$$C = \inf\{T(\gamma_0): \alpha_M < \gamma_0 < \lambda_M\}$$

is strictly positive.

Clearly, the equation

$$T(\gamma_0) = L_\varepsilon \tag{2.8}$$

has no solution $\gamma_0 \in (\alpha_M,\lambda_M)$ when $L_\varepsilon < C$, and this implies (i) of the theorem.

On the other hand, for $L_\varepsilon > C$ we conclude from $(2.7)_1$ that (2.8) has at least one solution $\gamma_0 \in (\alpha_M,\lambda_M)$, as does the more general relation

$$nT(\gamma_0) = L_\varepsilon$$

for $n = 1,2,\ldots,N$ with N the largest integer $< L_\varepsilon/C$. This yields (ii).

---

[1] Cf., e.g., Hale [1969], p. 179; Arnold [1978], Chapter 2, Section 4.

Remark. Let $f = \sigma_M$ and consider the solution of degree 1 starting in (α_M, λ_M). For ε small this solution will be close to the heteroclinic orbit from α_M to β_M and hence will spend most of its time near α_M and β_M. In this sense the solution represents the structured analog of the unstructured $(\varepsilon=0)$ two-phase solutions of degree 1 which have α_M followed by β_M. Similarly, the structured solution of degree 2 starting in (α_M, λ_M) is the analog of the unstructured two-phase solutions of degree 2 which have α_M followed by β_M followed by α_M. Similar assertions apply to the other nonconstant solutions.

3. Stability.

Throughout this section the energy

$$E(\gamma) = \int_0^{L_\varepsilon} [\dot{\gamma}^2 + W(\gamma) - f\gamma]dt$$

is considered as a functional with domain

$$\Gamma = \{\gamma \in H^1(0,L_\varepsilon): \ \gamma > 0 \ \text{ on } \ [0,L_\varepsilon]\} \ .$$

In addition, we assume that W is a C^∞ function,[1] so that, by $(2.1)_1$, solutions of $\hat{S}_\varepsilon$ are C^∞ on $[0,L_\varepsilon]$.

Let γ be such a solution. Then

(i) γ is globally stable if

$$E(\gamma) \le E(\omega) \quad \text{for all} \quad \omega \in \Gamma \ ;$$

(ii) γ is locally stable if there is a neighborhood Ω of γ in Γ such that

$$E(\gamma) \le E(\omega) \quad \text{for all} \quad \omega \in \Omega \ ;$$

(iii) γ is unstable if given any norm on the space $C^\infty[0,L_\varepsilon]$ and any neighborhood Ω of γ in this normed space there is an $\omega \in \Omega$ with lower energy:

$$E(\omega) < E(\gamma) \ .$$

Theorem.[2] All nonconstant solutions of Problem $\hat{S}_\varepsilon$ are unstable.

The proof is based on the following lemma, which is well known.

Lemma. Let γ be a solution of $\hat{S}_\varepsilon$. Suppose there exists an $\eta \in C^\infty[0,L_\varepsilon]$ such that

$$I(\gamma,\eta) = \int_0^{L_\varepsilon} [2\dot{\eta}^2 + W''(\gamma)\eta^2]dt < 0 \ .$$

Then γ is unstable.

---

[1] C^2 would suffice provided $C^\infty[0,L_\varepsilon]$ in (iii) is replaced by $C^2[0,L_\varepsilon]$.

[2] Within a slightly different context a similar theorem has been established by Chafee [1975]. (Cf. Casten and Holland [1978] and Matano [1979].) We arrived at our results before discovering the above references.

Proof. Let

$$\varphi(\tau) = E(\gamma + \tau\eta)$$

for all sufficiently small τ. Then a simple calculation, based on (2.1), shows that

$$\varphi'(0) = 0, \quad \varphi''(0) = I(\gamma,\eta) \quad .$$

Since $I(\gamma,\eta) < 0$, this tells us that φ has a strict relative maximum at $\tau = 0$; hence

$$E(\gamma_\tau) < E(\gamma), \quad \gamma_\tau = \gamma + \tau\eta$$

for all sufficiently small $\tau \neq 0$. Moreover, as

$$\|\gamma_\tau - \gamma\| = |\tau| \, \|\eta\|$$

($\|\cdot\|$ arbitrary), every $\|\cdot\|$ - neighborhood contains functions γ_τ, $\tau \neq 0$, and the proof is complete.

Proof of Theorem. Let γ be a nonconstant solution of $\hat{S}_\varepsilon$, and let

$$\eta(t) = \dot{\gamma}(t) + \delta\beta(t)$$

with

$$\beta(t) = \frac{L_\varepsilon - t}{L_\varepsilon} \tag{3.1}$$

and δ an arbitrary constant. Then, writing $I_\delta = I(\gamma,\eta)$,

$$I_\delta = A + 2\delta B + \delta^2 C \quad , \tag{3.2}$$

where

$$A = \int_0^{L_\varepsilon} [2\ddot{\gamma}^2 + W''(\gamma)\dot{\gamma}^2]dt \quad ,$$

$$B = \int_0^{L_\varepsilon} [2\ddot{\gamma}\dot{\beta} + W''(\gamma)\dot{\gamma}\beta]dt \quad ,$$

$$C = I(\gamma,\beta) \quad .$$

By $(2.1)_2$,

$$2\dddot{\gamma} = W''(\gamma)\dot{\gamma} \quad ,$$

and hence, using $(2.1)_2$,

$$A = 2\int_0^{L_\varepsilon} (\ddot{\gamma}\,\dot{\gamma})^{\bullet} dt = 0 \quad ,$$

$$B = 2\int_0^{L_\varepsilon} (\ddot{\gamma}\,\beta)^{\bullet} dt = -2\ddot{\gamma}(0) \quad . \tag{3.3}$$

Clearly, $\ddot{\gamma}(0) \neq 0$, for otherwise, by (2.1), γ would be constant; thus $B \neq 0$. We therefore conclude from (3.2) and (3.3) that for some $\delta \neq 0$,

$$I_\delta < 0 \quad ,$$

and the proof is complete.

<u>Remark</u>. It is interesting to note that the last theorem is independent of the particular form of the stored energy W.

To complete our study we have only to investigate the stability of the *constant solutions* (2.5); but this is not difficult.

<u>Proposition</u>. <u>The constant solutions</u> (2.5) <u>have the following properties</u>: α <u>is globally stable for</u> $0 < f < \sigma_M$ <u>and locally stable for</u> $\sigma_M < f < \sigma^*$; β <u>is globally stable for</u> $\sigma_M < f < \infty$ <u>and locally stable for</u> $\sigma_* < f < \sigma_M$; λ <u>is unstable for</u> $\sigma_* < f < \sigma^*$.

<u>Proof</u>. Let γ be a constant solution. Then for any $\omega \in \Gamma$,

$$E(\omega) - E(\gamma) = \int_0^{L_\varepsilon} [\Phi(\omega) - \Phi(\gamma) + \dot{\omega}^2] dt$$

and the global stability of α follows from the fact that Φ has a global minimum at α for $0 < f < \sigma_M$. Also, for $\sigma_M < f < \sigma^*$, α is a local minimum for Φ; hence there is a $\delta > 0$ such that $\Phi(\alpha) < \Phi(\tau)$ and $\tau > 0$ for all τ with $|\tau-\alpha| < \delta$. Further

$$\Omega = \{\omega \in \Gamma : \ |\omega(t)-\alpha| < \delta \ \text{ for all } \ t \in [0,L_\varepsilon]\}$$

is a neighborhood of $\gamma(t) \equiv \alpha$ in $H^1(0,L_\varepsilon)$ and $E(\alpha) < E(\omega)$ for all $\omega \in \Omega$; hence α is locally stable for $\sigma_M < f < \sigma^*$.

The assertions concerning β are established in a similar manner.

Finally, for $\sigma_* < f < \sigma^*$, λ is not a local minimum for Φ and we can find points τ which are arbitrarily close to λ and have $\Phi(\tau) < \Phi(\lambda)$. Thus we can find constant functions ω which - in any norm on $C^\infty[0,L_\varepsilon]$ - are arbitrarily close to $\gamma(t) \equiv \lambda$ and have $E(\omega) < E(\gamma)$. Therefore λ is unstable.

Acknowledgement. We are grateful to Professor J. Hale for pointing out the papers of Chafee, Casten and Holland, and Matano. The United States Government is authorized to reproduce and distribute reprints for government purposes not withstanding any copyright herein.

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We consider a finite one-dimensional elastic body with stored-energy function of the form $W(\gamma) + \varepsilon^2(\gamma')^2$, where γ is the strain, γ' the strain gradient, and ε a small parameter. We assume that $W(\gamma)$ is non-convex, of the type capable of supporting two-phases. We show that when the body is acted on by end loads the only stable strain fields are the single-phase solutions $\gamma \equiv$ constant.

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