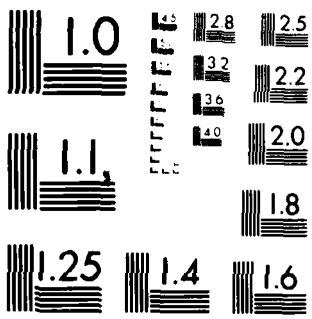
AD-A132 821 STABILIZATION OF SOLUTIONS FOR A CLASS OF PARABOLIC INTEGRO-DIFFERENTIAL EQUATIONS(U) WISCONSIN UNIV-MADISON MATHEMATICS RESEARCH CENTER H ENGLER 1/1

UNCLASSIFIED AUG 83 MRC-TSR-2555 DAAG29-80-C-0041 F/G 12/1 NL

END
DATE
FILMED
10 83
DTIC



MICROCOPY RESOLUTION TEST CHART
NATIONAL BUREAU OF STANDARDS 1963-A

2

AD-A132 821

MRC Technical Summary Report #2555

STABILIZATION OF SOLUTIONS FOR A CLASS OF PARABOLIC INTEGRO-DIFFERENTIAL EQUATIONS

Hans Engler

Mathematics Research Center
University of Wisconsin—Madison
610 Walnut Street
Madison, Wisconsin 53706

August 1983

(Received June 17, 1983)

Approved for public release
Distribution unlimited

Sponsored by

U. S. Army Research Office
P. O. Box 12211
Research Triangle Park
North Carolina 27709

UNIVERSITY OF WISCONSIN-MADISON
MATHEMATICS RESEARCH CENTER

STABILIZATION OF SOLUTIONS FOR A CLASS OF PARABOLIC INTEGRO-DIFFERENTIAL EQUATIONS

Hans Engler

Technical Summary Report #2555
August 1983

ABSTRACT

For parabolic integro-differential equations of the form

$$u_t(\cdot,t) + Au(\cdot,t) + \int_0^t g(t - s,u(\cdot,s))ds = f(\cdot,t) \tag{0.1}$$

on a time-space cylinder $\Omega \times [0,\infty)$ the question of convergence of solutions to a limit solution is studied. Here A is (e.g.) the negative Laplacian, and $g(s,u)$ is typically of the structure

$$g(s,u) = \sum_{i=1}^{N} a_i(s)g_i(u) . \tag{0.2}$$

The kernels $a_i(\cdot)$ have to satisfy certain decay properties, but no assumptions concerning the smallness of the g_i or of their derivatives are made; rather, the main assumption on the g_i is that they be monotone. Uniform convergence of solutions and of their derivatives for general initial and boundary conditions is shown; convergence rates are given which show the dependence on the spectrum of A and on the decay properties of the kernels a_i.

AMS (MOS) Subject Classifications: 35K55, 45K05

Key Words: Parabolic equations, Integro-differential equations, Boundary value problems, Asymptotic behavior, Feed-back control

Work Unit Number 1 (Applied Analysis)

Sponsored by the United States Army under Contract No. DAAG29-80-C-0041.
Supported by Deutsche Forschungsgeineinschaft.

SIGNIFICANCE AND EXPLANATION

Integro-differential equations arise in the description of feed-back control systems, where the control variables are derived from "filtered" observations of the state or where the control mechanism possesses inertia. The author studies ~~We study~~ a model equation for a "distributed" control system (e.g., the state varies over some space-like domain) which contains also some diffusion effects and give conditions under which the state will tend to some limit, as time goes to infinity, regardless of the initial situation. The limit is shown to satisfy an elliptic differential equation. Convergence rates are also given; These ~~they~~ show the "slowing-down" effect of a slow control mechanism on the convergence of the state variable. The problem under study can also be viewed as a natural extension of a type of reaction-diffusion equation that has received wide attention in the literature.

The responsibility for the wording and views expressed in this descriptive summary lies with MRC, and not with the author of this report.

STABILIZATION OF SOLUTIONS FOR A CLASS OF PARABOLIC INTEGRO-DIFFERENTIAL EQUATIONS

Hans Engler

1. Introduction

The purpose of this article is a study of the asymptotic behavior of solutions of semilinear parabolic integro-differential equations

$$\partial_t u(x,t) - \Delta_x u(x,t) + \int_0^t g(t-s,x,u(x,s))ds = f(x,t) \tag{1.1}$$

in a semi-infinite time-space cylinder $\Omega \times [0,\infty)$. Here Ω is a domain in $\mathbb{R}^n$, Δ_x is the Laplacian, and g is a real-valued function, typically of a "semi-separable" structure

$$g(s,x,u) = \sum_{j=1}^{N} a_j(s,x) \cdot k_j(x,u) \ . \tag{1.2}$$

The precise assumptions are stated in Section 2. Equation (1.1) (with g as in (1.2)) can be obtained from the system of parabolic and ordinary differential equations

$$\partial_t u(x,t) - \Delta_x u(x,t) = \vec{b}^T(x) \cdot \vec{v}(x,t) \ , \tag{1.3}$$

$$\frac{d}{dt} \vec{v}(x,t) + A(x)\vec{v}(x,t) = \vec{k}(x,u(x,t)) \tag{1.4}$$

which models a feed-back control system, where the vector $\vec{v}$ of controls depends on the state variable u through a control mechanism with inertia (expressed by the matrix differential equation (1.4)); see [12] for detailed discussions of the corresponding ordinary differential equation and [14]. We want to consider (1.1) as a natural extension of the semilinear parabolic equation

$$\partial_t u(x,t) - \Delta_x u(x,t) + \bar{g}(x,u(x,t)) = f(x,t). \tag{1.5}$$

Sponsored by the United States Army under Contract No. DAAG29-80-C-0041. Supported by Deutsche Forschungsgemeinschaft.

For (1.5) with t-independent Dirichlet-boundary data it is well-known that for $\bar{g}$ satisfying $\frac{\partial}{\partial u}\bar{g}(x,u) \geq 0$ the solution $u(\cdot)$ converges to a steady state solution u_∞ of

(1.6)
$$-\Delta_x u_\infty(x) + \bar{g}(x,u_\infty(x)) = \lim_{t\to\infty} f(x,t) ,$$

if the right-hand side limit exists. Moreover, if, e.g., f is t-independent, then

(1.7)
$$\|u(\cdot,t) - u_\infty(\cdot)\| \leq C \cdot e^{-\lambda_0 t}$$

for some appropriate norm $\|\cdot\|$, where $\lambda_0 > 0$ is the principal eigenvalue of $-\Delta_x$.

A quick proof of this fact (assuming the necessary regularity for u and f to be t-independent) consists in differentiating (1.5) with respect to t and multiplying the result with $e^{2\lambda_0 t} \cdot \partial_t u(\cdot,t)$. After integrating over $\Omega \times [0,t]$ this gives the estimate

(1.8)
$$\frac{1}{2} e^{2\lambda_0 t} \cdot \|\partial_t u(\cdot,t)\|^2_{L^2} - \int_0^t e^{2\lambda_0 s} \cdot \lambda_0 \cdot \|\partial_t u(\cdot,s)\|^2_{L^2} ds$$
$$+ \int_0^t e^{2\lambda_0 s} \|\nabla_x \partial_t u(\cdot,s)\|^2_{L^2} ds \leq \frac{1}{2} \|\partial_t u(\cdot,0)\|^2_{L^2} ,$$

from which the estimate

(1.9)
$$\|\partial_t u(\cdot,t)\|_{L^2} \leq C \cdot e^{-\lambda_0 t}$$

follows by the variational characterization of λ_0,

$$\int_\Omega |\nabla_x w|^2 \geq \lambda_0 \cdot \int_\Omega |w|^2 \quad \text{for all} \quad w \in W_0^{1,2}(\Omega) .$$

The estimate (1.7) then follows (with the L^2-norm). We want to study the same questions for the equation (1.1):

a) What are conditions on g that guarantee the convergence of u to some limit u_∞?

b) What are the convergence rates?

We are also going to use the same approach as sketched above.

A few remarks are in order to indicate what results one can expect and under which conditions.

(i) The equation for the limit u_∞ should be

$$(1.10) \qquad -\Delta_x u_\infty + \int_0^\infty g(s,x,u_\infty(x))ds = \lim_{t\to\infty} f(x,t) \ ,$$

and we would expect to get convergence for any boundary data for which (1.10) has a solution. Also, if we want to deduce estimates like (1.7) (which imply convergence for any initial data), then the limit equation (1.10) should have a (unique) globally stable solution, and of course the integral in (1.10) should converge; i.e. $g(s,u)$ should become "small" as s goes to infinity. If we assume instead that

$$g(\infty,x,u) = \lim_{t\to\infty} g(t,x,u)$$

is not identically zero, then we would only expect a solution u_∞ to exist if additionally (at least)

$$g(\infty,x,\bar{u}(x))|_{\partial\Omega} \equiv 0$$

for the boundary data $\bar{u}$. We shall not deal with the case $g(\infty,\cdot,\cdot) \not\equiv 0$ here.

(ii) In the case of "small" perturbations, e.g.

$$(1.11) \qquad \int_0^\infty e^{-\delta s} \cdot \sup_{x,u} |\partial_u g(s,x,u)|ds < \lambda_0$$

for some $\delta > 0$, stability of the steady state solution u_∞ (from (1.10)) has been shown in various settings (see, e.g., [5]). This would correspond to the condition

$$(1.12) \qquad \sup_{x,u} |\partial_u \bar{g}(x,u)| < \lambda_0$$

for the equation (1.5); we shall here look for conditions that generalize the sign condition $\partial_u \bar{g}(x,u) > 0$.

(iii) We now look at the special case

$$g(s,x,u) = a(s) \cdot (L \cdot u + M) \tag{1.13}$$

with $L \geq 0$, $M \in \mathbb{R}$, $a(\cdot)$ some kernel.

Taking $M = 0$, $L > \lambda_0$, and (formally) $a = \delta_T$ (δ-distribution with unit mass at $t = T$), we see that a solution of the form

$$u(x,t) = \cos(\bar{\omega}t) \cdot \phi_0(x) \tag{1.14}$$

of (1.1), viz. of

$$\partial_t u(x,t) - \Delta_x u(x,t) + L \cdot u(x,t - T) = 0$$

exists on the whole real line, if

$$\begin{cases} \bar{\omega} = \sqrt{L^2 - \lambda_0^2}\ , \\ T = \frac{1}{\bar{\omega}} \text{arc}\ \tan(\frac{\bar{\omega}}{\lambda_0})\ , \end{cases} \tag{1.15}$$

where ϕ_0 is the normalized eigenfunction of $(-\Delta_x)$ for the eigenvalue λ_0. This phenomenon of "oscillations (i.e. instability) introduced by delays" is well-known and will persist if we approximate $a = \delta_T$ by $a_\varepsilon = \varepsilon^{-1} \cdot 1_{[T,T+\varepsilon]}$ for some small ε; cf. [4]. We shall exclude these "Hopf-bifurcation" type phenomena by introducing certain assumptions on the kernel $a(\cdot)$ (convexity).

(iv) Taking $L = 0$, $M \neq 0$ arbitrary, $a(s) = e^{-\alpha s}$ in (1.13), the resulting linear equation (1.1) can be solved "explicitly" for zero Dirichlet boundary data by means of the "variation-of-constants" formula (fundamental solution), cf. [7]. As a result, the solution will converge to the limit u_∞ at the rate $e^{-\beta t}$, where $\beta = \min(\alpha, \lambda_0)$. Hence we expect the convergence rate for general equations (1.1) to be not better than the decay properties of the kernel functions appearing in (1.2), or of some related quantities derived for the function g.

(v) Finally, if $\bar{g}(x,u) = L \cdot u$ in (1.5), then (for $L > -\lambda_0$) the convergence rate of solutions will be improved, namely $\|u(\cdot,t) - u_\infty\|$ will be of the order $e^{-(L+\lambda_0)t}$. That a similar phenomenon can not be expected for equations of the type (1.1), can be seen by looking at the case (1.13) again, with $a(s) = e^{-\alpha s}$, $M = 0$. Let $\phi_n(\cdot)$ be an convergence and existence results require the adaptation of a standard "bootstrapping" technique to prove regularity and boundedness. Sections 2 and 3 only deal with the case that g in (1.1) does not depend on x.

In Section 4 we indicate how to extend the results to arbitrary time-independent Dirichlet boundary data and to x-dependent nonlinearities g. We also indicate how to include more general elliptic operators instead of the Laplacian. In Theorem 9, we state a result that covers the case of (inhomogeneous, time-dependent) boundary data of the third type,

$$\partial_\nu u(x,t) + \alpha(x)u(x,t) = f_1(x,t) \qquad (x \in \partial\Omega) \ , \tag{1.22}$$

where ∂_ν denotes the outer normal derivative. In Theorem 10, we give conditions that imply the same convergence results for solutions of equations of higher order,

$$\partial_t u(x,t) + (-\Delta_x)^m u(x,t) + \int_0^t g(t - s,u(x,s))ds = f(x,t), \tag{1.23}$$

$m > 1$ some integer. We conclude with some remarks about systems and equations of higher order with nonlinear differential operators under the integral and with open questions.

It is a pleasure to acknowledge the hospitality and the stimulating atmosphere at the Mathematics Research Center, University of Wisconsin-Madison, and at the Department of Mathematics, Northwestern University, Evanston, where the author was a visitor while this work was prepared.

2. Statement of Main Results

In this section we study the parabolic integrodifferential equation

$$\partial_t u(x,t) - \Delta_x u(x,t) + \int_0^t g(t-s,u(x,s))ds = f(x,t) \quad (x \in \Omega,\ t \in (0,\infty)) \tag{2.1}$$

with initial and boundary conditions

$$u(x,0) = u_0(x) \qquad (x \in \Omega) \tag{2.2}$$

$$u(x,t) \equiv 0 \qquad (x \in \partial\Omega,\ 0 < t < \infty)\ . \tag{2.3}$$

Here $\Omega \subset \mathbb{R}^n$ is a bounded open domain with smooth boundary $\partial\Omega$ (locally, $\partial\Omega$ should be the graph of a $C^{2+\alpha}$-function, $\alpha > 0$);

$$\Delta_x = \sum_{i=1}^n \frac{\partial^2}{\partial x_i^2} \text{ is the Laplacian },$$

and

$$g : [0,\infty) \times \mathbb{R} \to \mathbb{R}$$

is a given function which is (at least) continuous in both variables. The data u_0 and f are given functions on Ω resp. on $\Omega \times [0,\infty)$.

We are interested in solutions u which are in

$$L^p(0,T;W^{2,p}(\Omega)) \cap W^{1,p}([0,T],L^p(\Omega)) \tag{2.4}$$

for all $1 < p < \infty$ and all $T > 0$. Such solutions will be continuous together with their first x-derivatives on $\bar{\Omega} \times [0,T]$ (see [10]).

For later convenience we define

$$G(s,u) = \int_0^u g(s,r)dr \quad \text{for } u \in \mathbb{R},\ s > 0\ . \tag{2.5}$$

After possibly changing the right-hand side f, we can assume that

$$g(s,0) = 0 \text{ for all } s\ . \tag{2.6}$$

The assumptions on u_0 and f are

$$u_0 \in W^{2,\infty}(\Omega) , \tag{2.7}$$

$$u_0 \in W_0^{1,2}(\Omega) , \tag{2.8}$$

$$f \in L^\infty(\Omega \times (0,T)), \quad \partial_t f \in L^1(0,T;L^2(\Omega)) , \tag{2.9}$$

for all $T > 0$. For g and G, we shall use the assumptions

$$u \mapsto G(s,u) \text{ is convex for all } s , \tag{2.10}$$

$$u \mapsto -G'(s,u) \text{ exists and is convex for all } s , \tag{2.11}$$

$$(s,u) \mapsto g'(s,u) \text{ is continuous} , \tag{2.12}$$

$$g(s,u) \to 0, \text{ as } s \to \infty, \text{ for all } u . \tag{2.13}$$

Here

$$G'(s,u) = \frac{\partial}{\partial s} G(s,u) ;$$

similarly $g''(s,u) = \frac{\partial^2}{\partial s^2} g(s,u)$ etc. Finally, let λ_0 denote the principal eigenvalue for $(-\Delta_x)$ on Ω with zero Dirichlet boundary data.

<u>Theorem 1</u>: Let u be a solution of (2.1)-(2.3) in the sense described above, and let u_0 and f satisfy (2.7)-(2.9) and g resp. G satisfy (2.5), (2.6), (2.10)-(2.13). Assume that there exists a $\kappa > 0$, such that for all u, $g'(\cdot,u)$ is absolutely continuous and for almost all s

$$u \mapsto g''(s,u) + \kappa \cdot g'(s,u) \text{ is non-decreasing} . \tag{2.14}$$

Then

$$\int_\Omega |\partial_t u(\cdot,t)|^2 dx \cdot b(t) \le C_0(u_0,f) \quad \text{for a.e. } t, t \le T , \tag{2.15}$$

$$\int_0^T \int_\Omega |\nabla_x \partial_t u(\cdot,t)|^2 dx \cdot \frac{b(t)}{(\log b(t) + 1)^{1+\delta}} dt \le C_\delta(u_0,f) \tag{2.16}$$

for every T and every $\delta > 0$.

Here

$$b(t) = \exp(t \cdot \min(\kappa, 2 \cdot \lambda_0)) , \tag{2.17}$$

where C_0 (resp. C_δ) depend on Ω and on

(2.18) $$\|u_0\|_{W^{2,2}(\Omega)} + \int_0^T \left(\int_\Omega |\partial_t f(\cdot,t)|^2 \cdot b(t)\right)^{1/2} dt$$

(resp. additionally on $\delta > 0$).

<u>Theorem 2</u>: Let u, u_0, f, g satisfy (2.1)-(2.3), (2.5)-(2.12). Assume that for all $b > 0$, u such that $g(0,u) \neq g(0,u+h)$

(2.19) $$s \mapsto \log(g(s,u+h) - g(s,u)) \text{ exists for all } s \text{ and is convex ;}$$

and define

(2.20) $$-\kappa(s) = \sup\left\{\frac{g'(s,u+h) - g'(s,u)}{g(s,u+h) - g(s,u)} \;\middle|\; h > 0,\ u \in \mathbb{R},\ g(s,u+h) \neq g(s,u)\right\} .$$

Then

(2.21) $$\int_\Omega |\partial_t u(\cdot,t)|^2 \cdot b(t) < C_0(u_0,f) \quad \text{for a.e. } t < T ,$$

(2.22) $$\int_0^T \int_\Omega |\nabla_x \partial_t u(\cdot,t)|^2 dx \cdot \frac{b(t)}{(\log b(t) + 1)^{1+\delta}} dt < C_\delta(u_0,f)$$

for every $T > 0$, $\delta > 0$.

Here,

(2.23) $$b(t) = \exp\left(\int_0^t \min\{\kappa(s), 2 \cdot \lambda_0\} ds\right) ,$$

and C_0, C_δ depend on the same quantity (2.18) (and additionally on $\delta > 0$) as the constants in Theorem 1.

To illustrate the conditions (2.10), (2.11) and (2.14) resp. (2.19), we consider the examples

(2.24) $$g(t,u) = a_0(t) \cdot g_0(u)$$

and

(2.25) $$g(t,u) = \sum_{i=1}^{N} a_i(t) \cdot g_i(u) .$$

Let us assume that $a_0(0) = 1 = a_i(0)$ $(1 \leq i \leq N)$. Then (2.10) and (2.13) hold if

(2.26) $$a_0(s) > 0 ,$$

(2.27) $$a_0(s) \to 0, \text{ as } s \to \infty ,$$
$$g_0 \text{ is monotone non-decreasing} ,$$

resp. if (2.26) and (2.27) hold for all a_i and g_i. The property (2.11) will hold if additionally

(2.28) $$a_0'(s) \leq 0 \text{ resp. } a_i'(s) \leq 0 \quad (0 \leq s \leq T) \text{ for all } i ,$$

and property (2.14) is implied by

(2.29) $$a_0''(s) + \kappa \cdot a_0'(s) \geq 0 \qquad (s \geq 0)$$

resp.

(2.30) $$a_i''(s) + \kappa \cdot a_i'(s) \geq 0 \qquad (s \geq 0,\ 1 \leq i \leq N) .$$

Finally, for g of the structure (2.24), the assumption (2.19) means

(2.31) $$\log a_0(\cdot) \text{ is convex on } [0,\infty) ,$$

and (2.20) amounts to

(2.32) $$\kappa(s) = \frac{-a_0'(s)}{a_0(s)} = -(\log a_0(s))' ,$$

such that in (2.23)

(2.33) $$b(t) = \frac{1}{a_0(t)} , \text{ if } a_0'(s) + 2\lambda_0 a_0(s) \geq 0 \text{ for all } s ,$$

resp. $b(t) = e^{2\lambda_0 t}$ on the interval where $a_0' + 2\lambda_0 a_0 < 0$.

Similarly, if g has the form (2.25), then the assumption (2.19) holds if

(2.34) $$\log a_i(\cdot) \text{ is convex on } [0,\infty) \text{ for all } i ,$$

and in (2.20) we can take

(2.35) $$\kappa(s) = \min_{1 \leq i \leq N} \frac{-a_i'(s)}{a_i(s)} ,$$

such that in (2.23)

$$b(t) = \min_{1 \le i \le N} \frac{1}{a_i(t)}, \tag{2.36}$$

if $a_i'(s) + 2\lambda_0 a_i(s) \ge 0$ for all s and some i, with the obvious modification

$$b(t) = e^{2\lambda_0 t}, \tag{2.37}$$

if on some interval $[0,t_0]$ $a_i'(t) + 2\lambda_0 a_i(t) < 0$ for all i.

Conditions (2.29) resp. (2.30) show further that in Theorem 1 g could be of the structure (2.24) with some of the a_i possessing bounded support, a possibility that is excluded by (2.34), whereas in Theorem 2 some of the a_i could decay like t^{-m_i}, which does not give a T-independent constant κ in (2.14) of Theorem 1. In the typical case where all the a_i are decaying exponentials, both results apply and give the same formula.

From the estimates (2.16) resp. (2.22) we get some information on the asymptotic behavior of solutions.

<u>Corollary 3</u>: Let u be a solution of (2.1)-(2.3) on $\Omega \times [0,\infty)$, and let all the assumptions of Theorem 1 hold. Further assume that the space dimension n is equal to 1 or that

$$|g(0,u)| \le C \cdot (|u|^q + 1) \quad \text{for all} \quad u \in \mathbb{R} \tag{2.38}$$

with some $C > 0$, some $q > 0$, if $n = 2$, and $0 \le q \le \frac{n+2}{n-2}$, if $n \ge 3$, and that

$$\int_0^\infty \Big(\int_\Omega |\partial_t f(x,t)|^2 dx \cdot b(t)\Big)^{1/2} dt < \infty, \tag{2.39}$$

$$\operatorname*{ess\,sup}_{\Omega\times(0,\infty)} |f(x,t)| < \infty \tag{2.40}$$

with $b(\cdot)$ as in (2.17). Then

$$u(\cdot,t) \to u_\infty, \quad \nabla_x u(\cdot,t) \to \nabla_x u_\infty, \quad \text{as } t \to \infty, \tag{2.41}$$

uniformly on Ω, and u_∞ solves

$$-\Delta_x u_\infty(x) + \int_0^\infty g(s,u_\infty(x))ds = f_\infty(x), \qquad (x \in \Omega) \tag{2.42}$$

$$u_\infty(x) = 0 \qquad (x \in \partial\Omega), \tag{2.43}$$

and

$$f_\infty = \lim_{t\to\infty} f(\cdot,t). \tag{2.44}$$

A few remarks about the limit equation (2.42) are in order:

(i) The conditions (2.39) and (2.40) guarantee that the L^p-limit in (2.44) exists for all $p < \infty$, since

$$\|f(\cdot,t) - f(\cdot,s)\|_{L^p} \le \|f(\cdot,t) - f(\cdot,s)\|_{L^p}^{\frac{p-2}{p}} \cdot \|\int_s^t \partial_t f(\cdot,\tau)d\tau\|_{L^2}^{\frac{2}{p}} \tag{2.45}$$

$$\le C \cdot \left(\int_s^t \|\partial_t f(\cdot,\tau)\|_{L^2}^2 \cdot b(\tau)d\tau\right)^{\frac{2}{p}} \cdot \left(\int_s^t \frac{d\tau}{b(\tau)}\right)^{\frac{2}{p}} \to 0,$$

if $s,t \to \infty$. Thus f_∞ exists and is essentially bounded, since it is also the weak-*-limit of $f(\cdot,t)$ in $L^\infty(\Omega)$.

(ii) The solution u_∞ solves the variational problem

$$J(u) = \int_\Omega \{\tfrac{1}{2} |\nabla_x u|^2 + \int_0^\infty G(s,u(x))ds - f_\infty \cdot u\}dx \to \min. \tag{2.46}$$

in $W_0^{1,2}(\Omega)$. Here the integrals

$$\int_0^\infty G(s,v)ds, \quad \int_0^\infty g(s,v)ds \tag{2.47}$$

are finite for all $v \in \mathbb{R}$, since (2.13), (2.14) imply that

$$s \to (G'(s,u) + \kappa G(s,u)) \tag{2.48}$$

is increasing for all u and vanishes for $s \to \infty$. Thus $G'(s,u) + \kappa \cdot G(s,u) \le 0$, and

$$0 \le e^{\kappa s} G(s,u) \le G(0,u) \tag{2.49}$$

for all s and u, which proves the existence of the first integral in (2.47). The existence of the second integral in (2.47) follows with a similar argument.

Standard theory then shows that the solution u_∞ of the Euler equation (2.42) for the problem (2.46) exists, is unique and is in the class $\bigcap_{p<\infty} W^{2,p}(\Omega)$ (cf. [11]): We only have to note that

$$v \mapsto \int_0^\infty g(s,v)ds$$

inherits the monotonicity from the $g(s,u)$.

<u>Corollary 4</u>: Let u be a solution of (2.1)-(2.3) on $\Omega \times [0,\infty)$, and let all the assumptions of Theorem 2 hold. Further assume the growth conditions (2.38) and the boundedness assumptions (2.39) and (2.40) of Corollary 4, with $b(\cdot)$ given by

$$b(t) = \exp(\int_0^t \min\{\kappa(s), 2 \cdot \lambda_0\}ds) \tag{2.50}$$

and κ as in (2.20). Finally let

$$\int_0^\infty \frac{(\log b(s) + 1)^{1+\delta}}{b(s)} ds < \infty \tag{2.51}$$

for some $\delta > 0$. Then

$$\begin{matrix} u(\cdot,t) \to u_\infty \\ \nabla_x u(\cdot,t) \to \nabla_x u_\infty \end{matrix} \qquad \text{as} \quad t \to \infty \tag{2.52}$$

uniformly on Ω. The function u_∞ solves the limit equation (2.42) with boundary condition (2.43). Again, the conditions (2.10) and (2.11) guarantee that

$$(2.53) \qquad u \mapsto \int_0^\infty g(s,u)ds$$

exists and is monotone for all u, due to

$$(2.54) \qquad \operatorname{sign} u \cdot g(s,u) = |g(s,u)| \le |g(0,u)| \cdot \exp\left(\int_0^s -\kappa(\tau)d\tau\right)$$

$$\le |g(0,u)| \cdot \frac{1}{b(s)}$$

and the integrability of $\frac{1}{b(s)}$.

We also note that in the situation of both corollaries the convergence of higher derivatives of $(u(\cdot,t) - u_\infty(\cdot))$ to zero can be shown, if the right-hand side f and g satisfy suitable smoothness properties. It is enough to show that sufficiently high derivatives of the solution will stay bounded and to interpolate with (2.52).

We finally give an existence result for solutions of equations of the type (2.1)-(2.3) which is sufficient to cover the cases treated above.

Theorem 5: Let u_0, f, and g, G satisfy (2.5)-(2.12). Moreover, assume that $G'(\cdot,u)$ is continuous for every u and that

$$(2.55) \qquad u \mapsto G'(s,u) - G'(s+h,u)$$

is convex for every $0 \le s \le s + h$. Finally assume that

$$(2.56) \qquad |g(0,u)| \le C \cdot (|u|^{q(n)} + 1) \quad \text{for} \quad n \ge 2 ,$$

where C is some constant, $q(n) = \frac{n+2}{n-2}$, if $n \ge 3$, and $q(2)$ is some constant.

Then (2.1)-(2.3) has a solution u in the regularity class (2.4) on $\Omega \times [0,T]$. If g has the structure

$$(2.57) \qquad g(s,u) = \sum_{i=1}^{N} a_i(s) \cdot g_i(u) ,$$

then Theorem 5 applies if the g_i are non-decreasing and satisfy the growth condition (2.38) and if the a_i are positive, non-increasing and convex. It is possible to relax these conditions to

(2.58) $$u \mapsto g_i(u) + M \cdot u \text{ is non-decreasing}$$

for some $M > 0$,

(2.59) $$a_i \in W^{1,\infty}([0,T],\mathbb{R})$$
$$a_i \in BV([0,T],\mathbb{R})$$
$$a_i(0) > 0 \ .$$

It should be noted that the growth condition (2.56) in Theorem 5 is slightly more general than the condition (2.38) in Corollaries 3 and 4, where the "limiting" growth exponent is not permitted.

We finally give a result for the situation with less regular initial data.

Corollary 5.A:

(i) Let all the assumptions of Theorem 5 be satisfied, except that $u_0(\cdot)$ is only required to be in $L^\infty(\Omega)$. Then there exists a weak solution u of (2.1)-(2.3) on $\Omega \times [0,T]$;

(2.60) $$u \in L^\infty(0,T;L^\infty(\Omega)) \ ,$$

(2.61) $$u \in L^p(\varepsilon,T;W^{2,p}(\Omega)) \cap W^{1,p}([\varepsilon,T],L^p(\Omega))$$

for any $\varepsilon > 0$.

(ii) Let all the assumptions of Corollary 3 hold, except that $u_0(\cdot)$ need only be in $L^\infty(\Omega)$. Then all the conclusions of Corollary 3 hold.

(iii) Let all the assumptions of Corollary 4 hold, except that $u_0(\cdot)$ need only be in $L^\infty(\Omega)$. Also, assume that

(2.62) $$\int_0^\infty \frac{ds}{\sqrt{b(s)}} < \infty \ ,$$

with $b(\cdot)$ defined as in (2.23). Then the conclusions of Corollary 4 hold.

3. Proofs

In the proofs of Theorems 1, 2 and 5, use will be made of the following Lemma:

Lemma 6: Let $h : \mathbb{R} \to \mathbb{R}$ be non-decreasing, and let $a, b : [0,T] \to \mathbb{R}$ be such that

(3.1) $$a \text{ is absolutely continuous}, \quad a' \leq 0 \leq a \quad \text{on } [0,T]$$

(3.2) $$b \geq 0 \quad \text{on } [0,T]$$

(3.3) $$b \cdot a \text{ is absolutely continuous}$$

(3.4) $$s \mapsto b(s+\tau) \cdot a'(s), \qquad s \in [0, T-\tau]$$
$$\text{is non-decreasing for a.e. } \tau \in [0,T) .$$

Let $v \in W^{1,1}([0,T],\mathbb{R})$, then for $0 \leq t \leq T$

(3.5) $$\int_0^t v'(s) \cdot b(s) \cdot \frac{d}{ds}\left(\int_0^s a(s-\tau)h(v(\tau))d\tau\right)ds$$
$$\geq a(t)b(t)H(v(t)) - a(0)b(0)H(v(0)) - \int_0^t (a(s) \cdot b(s))'H(v(s))ds$$
$$= \int_0^t a(s) \cdot b(s) \cdot \frac{d}{ds} H(v(s))ds ,$$

where $\frac{d}{dr} H(r) = h(r)$.

Proof: We first assume that $a \in C^2([0,T],\mathbb{R})$, $b \in C^1([0,T],\mathbb{R})$, and that (3.1), (3.2) and (3.4) hold. Then

(3.6) $$\int_0^t b(s) \cdot v'(s) \cdot \frac{d}{ds}\left(\int_0^s a(s-\tau)h(v(\tau))d\tau\right)ds$$
$$= \int_0^t b(s) \cdot a(0) \cdot v'(s) \cdot h(v(s))ds +$$
$$+ \int_0^t b(s) \cdot v'(s) \cdot \int_0^s a'(s-\tau)h(v(\tau))d\tau ds .$$

The first integral on the right-hand side is

$$(3.7) \qquad a(0)\{(b(t)H(v(t)) - b(0)H(v(0))) - \int_0^t b'(s)H(v(s))ds\} \ .$$

The second integral can be transformed to give

$$(3.8) \quad \int_0^t \int_\tau^t b(s)a'(s-\tau)v'(s)ds \ h(v(\tau))d\tau$$

$$= \int_0^t [b(t)a'(t-\tau)v(t)-b(\tau)a'(0)v(\tau) - \int_\tau^t \frac{d}{ds}(b(s)a'(s-\tau)) \cdot v(s)ds] \cdot h(v(\tau))d\tau \ .$$

The monotonicity of h implies that this is

$$\ldots > \frac{1}{\lambda} \cdot \int_0^t \{H(v(\tau)) - H((1 + \lambda b(\tau)a'(0))v(\tau) -$$

$$- \lambda b(t)a'(t - \tau)v(t) + \lambda \cdot \int_\tau^t \frac{d}{ds}(b(s)a'(s - \tau))v(s)ds)\}d\tau \ ,$$

for all positive λ. Choose λ so small that

$$(3.9) \qquad 1 + \lambda b(\tau) \cdot a'(0) > 0 \qquad (0 < \tau < t) \ .$$

Since also

$$(3.10) \qquad -b(t)a'(t - \tau) > 0$$

$$(3.11) \qquad \frac{d}{ds}(b(s)a'(s - \tau)) > 0$$

by (3.4), and

$$(3.12) \qquad 1 + \lambda b(\tau)a'(0) - \lambda b(t)a'(t - \tau) + \lambda \int_\tau^t \frac{d}{ds}(b(s)a'(s - \tau))ds = 1 \ ,$$

the convexity of H now implies that the term in (3.8) can be further estimated by

$$(3.13) \qquad \ldots \geq \int_0^t \{b(t)a'(t-\tau)H(v(t)) - b(\tau)a'(0)H(v(\tau)) -$$

$$- \int_\tau^t \frac{d}{ds}(b(s)a'(s-\tau))H(v(s))ds\}d\tau = b(t)\cdot H(v(t))\cdot(a(t)-a(0))$$

$$+ \int_0^t (b'(s)a(0) - b'(s)a(s) - b(s)a'(s))H(v(s))ds \; .$$

Combining this with (3.7) we then get the first half of (3.5). The second half of (3.5) is simply an integration by parts.

To get the estimate (3.5) for general a's and b's, we approximate a,b by smooth a_n,b_n such that the a_n,b_n still satisfy (3.1)-(3.4) and

$$a_n(0) = a(0), \quad b_n(0) = b(0) \; ,$$

$$(3.14) \qquad \begin{matrix} a_n' \to a \\ b_n' \to b \end{matrix} \quad \text{in } L^1(0,T;\mathbb{R}) \; .$$

Then also

$$(3.15) \qquad (a_n \cdot b_n)' \to (a\cdot b)' \quad \text{in } L^1(0,T;\mathbb{R})$$

and

$$(3.16) \qquad ((s,\tau) \mapsto b_n(s)\cdot a_n'(\tau)) \to b(s)\cdot a(\tau) \quad \text{in } L^1([0,T]^2,\mathbb{R}) \; ;$$

we can thus pass to the limit and get (3.5) in the general situation.

QED.

As a consequence, we get the

<u>Lemma 7</u>: Let $g : [0,T] \times \mathbb{R} \to \mathbb{R}$ be continuous, g' be continuous, and $b : [0,T] \to [0,\infty)$ be absolutely continuous,

(3.17) $\quad g(s,u)$ and $-g'(s,u)$ be non-decreasing in u for all s ,

(3.18) $\quad s \mapsto b(s+\tau)\cdot(g'(s,u+h) - g'(s,u))$ be non-decreasing for every $-\infty < u < u+h < \infty$ and all $\tau \in [0,T)$.

Let $v \in W^{1,1}([0,T],\mathbb{R})$.

Then for $0 \leq t \leq T$

$$(3.19) \qquad \int_0^t v'(s) \cdot b(s) \cdot \frac{d}{ds} \left(\int_0^s g(s-\tau, v(\tau))d\tau\right)ds$$

$$\geq b(t)G(t,v(t)) - b(0)G(0,v(0)) - \int_0^t G_1(s,v(s))ds ,$$

where

$$(3.20) \qquad G(s,v) = \int_0^v g(s,r)dr$$

and

$$(3.21) \qquad G_1(s,v) = \frac{d}{ds}(b(s) \cdot G(s,v)) .$$

Proof: We first assume that g is smooth; i.e. that $g_u' = \frac{\partial}{\partial u}\frac{\partial}{\partial s} g$ is continuous. Then (3.17) is equivalent to

$$(3.22) \qquad g_u(s,u) \geq 0 \geq g_u'(s,u) \quad \text{for all} \quad s,u ,$$

and (3.18) is equivalent to

$$(3.23) \qquad s \mapsto b(s+\tau) \cdot g_u'(s,u)$$

is non-decreasing for every $u \in \mathbb{R}$, $\tau \in [0,T)$.

Now write

$$(3.24) \qquad g(s,u) = g(s,0) + \int_{-\infty}^{\infty} g_u(s,r) \cdot h_r(u)dr$$

with

$$(3.25) \qquad h_r(v) = \begin{cases} -1, & v \leq r \leq 0 \\ 1, & 0 \leq r \leq v \\ 0 & \text{for all other } r,v . \end{cases}$$

Each $h_r(\cdot)$ is non-decreasing, and

$$(3.26) \qquad H_r(v) = \int_0^v h_r(z)dz = \begin{cases} r - v, & v \le r \le 0 \\ v - r, & 0 \le r \le v \\ 0 & \text{for all other } r,v . \end{cases}$$

Inserting the representation (3.24) into the left-hand side of (3.19) and using Lemma 6 for every term then gives

$$(3.27) \qquad \int_0^t v'(s) \cdot b(s) \cdot \frac{d}{ds} (\int_0^s g(s - \tau, v(\tau))d\tau)ds$$

$$\ge \int_{-\infty}^{+\infty} \{b(t)g_u(t,r) \cdot H_r(v(t)) - b(0)g_u(0,r) \cdot H_r(v(0)) -$$

$$- \int_0^t \frac{d}{ds} (b(s)g_u(s,r))H_r(v(s))ds\}dr$$

$$+ b(t)g(t,0)v(t) - b(0)g(0,0)v(0) - \int_0^t \frac{d}{ds} (b(s)g(s,0))v(s)ds .$$

An integration by parts then gives the estimate (3.19).

If g' is only continuous, then the approximation

$$g_\varepsilon(s,u) = \frac{1}{2\varepsilon} \int_{u-\varepsilon}^{u+\varepsilon} g(s,r)dr$$

gives a smooth g_ε for which the estimate (3.19) holds. The passage to the limit as $\varepsilon \downarrow 0$ is straightforward; thus (3.19) holds in the general situation.

QED.

<u>Proof of Theorems 1 and 2</u>:

(i) Let $b : [0,T] \to \mathbb{R}$ be any absolutely continuous function. Define the backward difference operator

$$d_h k(t) = \frac{k(t) - k(t - h)}{h} \quad \text{for } h \le t \le T ,$$

if $k : [0,T] \to \mathbb{R}$ is measurable. We apply d_h to the equation (2.1), multiply with $b(t) \cdot d_h u(t)$ (for $h \le t \le \bar{t}$) and integrate the resulting quantity over $\Omega \times [h,\bar{t}]$.

After an integration by parts, this gives the identity

$$(3.28)\qquad \frac{1}{2}\int_{\Omega} |d_h u(\cdot,\bar{t})|^2 \cdot b(\bar{t}) - \frac{1}{2}\int_{\Omega} |d_h u(\cdot,h)|^2 \cdot b(h)$$

$$- \frac{1}{2}\int_h^{\bar{t}}\int_{\Omega} |d_h u(\cdot,s)|^2 \cdot b'(s)ds + \int_h^{\bar{t}}\int_{\Omega} |\nabla_x d_h u(\cdot,s)|^2 \cdot b(s)ds$$

$$+ \int_h^{\bar{t}}\int_{\Omega} d_h u(\cdot,s) \cdot b(s) \cdot d_h(\int_0^s g(s-\tau,u(\cdot,\tau))d\tau)ds =$$

$$= \int_h^{\bar{t}}\int_{\Omega} d_h u(\cdot,s) \cdot b(s) \cdot d_h f(\cdot,s)ds .$$

Since

$$t \mapsto \int_0^t g(t-s,u(\cdot,s))ds + f(\cdot,t)$$

is in $W^{1,1}([0,T],L^2(\Omega))$ by assumption, we can send h to 0 in (3.28) and obtain

$$(3.29)\qquad \frac{1}{2}\int_{\Omega} |u_t(\cdot,\bar{t})|^2 \cdot b(\bar{t}) - \int_0^{\bar{t}}\int_{\Omega} |\nabla_x u_t(\cdot,s)|^2 \cdot b(s)ds$$

$$+ \int_0^{\bar{t}}\int_{\Omega} u_t(\cdot,s) \cdot b(s) \cdot \frac{d}{ds}(\int_0^s g(s-\tau,u(\cdot,\tau))d\tau)ds =$$

$$= \frac{1}{2} \cdot \{\int_{\Omega} |\Delta_x u_0 + f(\cdot,0)|^2 \cdot b(0) + \int_0^{\bar{t}}\int_{\Omega} |u_t(\cdot,s)|^2 \cdot b'(s)ds\}$$

$$+ \int_0^{\bar{t}}\int_{\Omega} u_t(\cdot,s) \cdot b(s) \cdot f_t(\cdot,s)ds ,$$

where the index t denotes partial differentiation with respect to t. Note that a differentiation of (2.1) with respect to t shows that in fact u_t (and hence $\Delta_x u$) are continuous from $[0,T]$ with values in $L^2(\Omega)$ (cf. [10]).

(ii) We next choose b such that the third integral on the left hand side of (3.29) can be estimated. In the situation of Theorem 1, take

$$(3.30) \qquad b(t) = \exp\{\min\{\kappa, 2\lambda_0\} \cdot t\}$$

as in (2.17) (with $\kappa > 0$ as in (2.14)). Then (2.10), (2.11) and (2.12) show that (3.17) holds. Also, (2.14) together with this choice of b implies that (3.18) holds. Thus Lemma 7 allows to estimate

$$(3.31) \qquad \int_0^{\bar{t}} u_t(x,s) \cdot b(s) \cdot \frac{d}{ds} \left(\int_0^s g(s-\tau, u(x,\tau)d\tau\right)ds \geq$$

$$\geq b(\bar{t})G(\bar{t},u(x,\bar{t})) - b(0)G(0,u(x,0)) - \int_0^{\bar{t}} G_1(s,u(x,s))ds$$

for a.e. $x \in \Omega$, $G_1(s,v) = \frac{d}{ds}(b(s) \cdot G(s,v)) = e^{\bar{\kappa}s} \cdot (\bar{\kappa} \cdot G(s,v) + G'(s,v))$, $\bar{\kappa} = \min\{\kappa, 2\lambda_0\}$. Again, (2.11) and (2.14) imply that

$$e^{\bar{\kappa}s} \cdot (G'(s,v) + \bar{\kappa}G(s,v)) \leq 0 ,$$

and from (3.19) we get that finally

$$(3.32) \qquad \int_0^{\bar{t}} u_t(x,s)b(s) \cdot \frac{d}{ds} \left(\int_0^s g(s-\tau, u(x,\tau))d\tau\right)ds \geq -G(0,u(x,0))$$

for a.e. $x \in \Omega$.

Inserting this into (3.29) gives together with Poincaré's inequality

$$(3.33) \qquad \int_\Omega |u_t(\cdot,\bar{t})|^2 \cdot b(\bar{t}) \leq 2 \int_\Omega G(0,u(\cdot,0)) +$$

$$+ 2 \cdot \int_\Omega (|\Delta_x u(\cdot,0)|^2 + |f(\cdot,0)|^2) + 2 \int_0^{\bar{t}} \left(\int_\Omega |u_t(\cdot,s)|^2 \cdot b(s)\right)^{1/2} \cdot$$

$$\cdot \left(\int_\Omega |f_t(\cdot,s)|^2 \cdot b(s)\right)^{1/2} ds .$$

An application of a version of Gronwall's Lemma ([2]) then results in the estimate (2.15). Inserting this estimate back into (3.29) we arrive at

$$(3.34) \qquad \int_0^{\bar t}\int_\Omega |\nabla_x u_t(\cdot,s)|^2 \cdot b(s)ds \le C(u_0,f) + \frac{1}{2}\int_0^{\bar t}\int_\Omega |u_t(\cdot,s)|^2 \cdot b'(s)ds$$

$$\le C(u_0,f) \cdot \left(1 + \frac{1}{2}\int_0^{\bar t} \frac{b'(s)}{b(s)}\, ds\right) = C(u_0,f) \cdot (1 + \frac{1}{2}\log b(\bar t))$$

for all $\bar t$. Applying Lemma 8 below gives the estimate (2.16), and the proof of Theorem 1 is complete.

(iii) To prove Theorem 2, we take

$$b(t) = \exp\{\int_0^t \min\{\kappa(s), 2\lambda_0\}ds\}$$

as in (2.23), with $\kappa(\cdot)$ as in (2.20). We then have to check whether (3.18) holds with this choice of $b(\cdot)$. Thus let $-\infty < u < u + h < \infty$ and consider (for $\tau > 0$ fixed)

$$(3.35) \qquad s \mapsto b(s + \tau) \cdot (g'(s,u + h) - g'(s,u)) .$$

From (2.19) we infer that this function is identically zero (iff $g(0,u) = g(0,u + h)$) or has bounded variation. Its derivative (as a measure) is

$$b'(s + \tau) \cdot (g'(s,u + h) - g'(s,u)) + b(s + \tau) \cdot (g''(s,u + h) - g''(s,u)) ,$$

and this will be non-negative iff

$$(3.36) \qquad (g''(s,u + h) - g''(s,u)) \ge (g'(s,u) - g'(s,u + h))\left(\frac{b'(s + \tau)}{b(s + \tau)}\right) .$$

Now $-\log b$ is convex (since $\kappa(\cdot)$ is decreasing), and it is enough to verify (3.18) for $\tau = 0$ or

$$(3.37) \qquad (g''(s,u + h) - g''(s,u)) \ge (g'(s,u) - g'(s,u + h))(\kappa(s)) .$$

Recalling the definition of $\kappa(\cdot)$, it is obvious that (3.37) is implied by the log-convexity assumed in (2.19).

We thus get the estimate (3.31) as in the proof of Theorem 1. Also,

$$(3.38) \qquad \frac{d}{ds}(b(s) \cdot G(s,v)) \le 0$$

or equivalently

$$(3.39) \qquad \frac{b'(s)}{b(s)} \cdot G(s,v) \le -G'(s,v) ,$$

which is implied by

$$\frac{b'(s)}{b(s)} < \frac{-g'(s,v)}{g(s,v)}, \quad \text{if} \quad g(s,v) \neq 0, \tag{3.40}$$

and this follows from (2.20) and (2.23).

Continuing as above, the estimate (3.33) implies the first assertion (2.15) of Theorem 2, and (3.34) together with Lemma 8 implies (2.16), which completes the proof of Theorem 2.

QED.

__Lemma 8:__ Let $k : [0,\infty) \to [0,\infty)$, $b : [0,\infty) \to [1,\infty)$ be functions, let b be non-decreasing, and assume that for every $t > 0$

$$\int_0^t k(s) \cdot b(s) ds \leq C_0 \cdot (1 + \log b(t)) \tag{3.41}$$

for some $C_0 > 0$. Then for every $\delta > 0$ there exists some $K(\delta) > 0$ such that

$$\int_0^\infty k(s) \cdot b(s) \cdot (1 + \log b(s))^{-1-\delta} ds \leq K(\delta) \cdot C_0 . \tag{3.42}$$

__Proof:__ If b is bounded, the assertion is clear. Otherwise define

$$t_0 = 0, \tag{3.43}$$

$$t_{i+1} = \sup\{t \mid \frac{1 + \log b(t)}{1 + \log b(t_i)} \leq 2\} \quad \text{for} \quad i \geq 0 . \tag{3.44}$$

Without loss of generality $1 + \log b(t_{i+1}) = 2 \cdot (1 + \log b(t_i))$ for all i.

Now if $t > 0$, such that $t_i \geq t$, and $\delta > 0$,

$$\begin{aligned}
&\int_0^t k(s) \cdot b(s) \cdot (1 + \log b(s))^{-1-\delta} ds \leq \\
&\quad \leq \sum_{j=0}^{i-1} \int_{t_j}^{t_{j+1}} k(s)b(s)(1 + \log b(s))^{-1-\delta} ds \leq \\
&\quad \leq \sum_{j=0}^{i-1} (1 + \log b(t_j))^{-1-\delta} \cdot C_0 \cdot (1 + \log b(t_{j+1})) \\
&\quad \leq C_0 \cdot \sum_{j=0}^{\infty} \frac{1 + \log b(t_{j+1})}{(1 + \log b(t_j))^{1+\delta}} ,
\end{aligned}$$

and the infinite series converges because

$$\frac{1 + \log b(t_{j+1})}{(1 + \log b(t_j))^{1+\delta}} \cdot \frac{(1 + \log b(t_{j-1}))^{1+\delta}}{(1 + \log b(t_j))} = 2^{-\delta} < 1 .$$

QED.

The example $b(s) = e^s = k(s)^{-1}$ shows that (3.42) does not hold for $\delta = 0$.

Thanks are due to Bob Pego for suggesting the argument of this Lemma.

Proof of Corollaries 3 and 4: Let A_p be the Laplacian in $L^p(\Omega)$ with zero Dirichlet boundary condition and its natural domain of definition $D(A_p) = W^{2,p}(\Omega) \cap W_0^{1,1}(\Omega)$ $(1 < p < \infty)$, and let $(S_p(t))_{t \leq 0}$ be the analytic semigroup generated by A_p in $L^p(\Omega)$ (see [7]). It is well-known that

(3.45) $$\|A_p^\alpha S_p(t) w\|_{L^p(\Omega)} \leq C(p,\alpha) \cdot \left(\frac{1}{t^\alpha} + 1\right) \cdot e^{-\lambda_0 t} \cdot \|w\|_{L^p(\Omega)} ,$$

(3.46) $$\|v\|_{C^1(\bar{\Omega})} \leq C \cdot \|A_p^\alpha v\|^{1-\varepsilon}_{L^p(\Omega)} \cdot \|v\|^{\varepsilon}_{W^{1,2}(\Omega)} ,$$

if $w \in L^p(\Omega)$, $v \in D(A_p^\alpha)$, $t > 0$, $\alpha > 0$, such that

(3.47) $$\frac{n}{2p} + \frac{1}{2} < \alpha < 1 ,$$

(3.48) $$0 < \varepsilon < \frac{p \cdot (4\alpha - 2) - 2n}{p \cdot (n + 4\alpha - 2) - 2n} .$$

Here, A_p^α is the fractional power, defined as usually,

(3.49) $$A_p^\alpha = \left(\frac{1}{\Gamma(\alpha)} \int_0^\infty t^{\alpha-1} S_p(t) dt\right)^{-1} .$$

Then the solution u of (2.1)-(2.3) can be written as

$$u(t) = S_p(t)u_0 + \int_0^t S_p(t-s)f(s)ds + \int_0^t S_p(t-s)\int_0^s g(s-\tau,u(\tau))d\tau ds$$

(3.50)

$$= \quad u_1(t) \quad + \quad u_2(t) ,$$

where the argument x has been omitted for convenience. We want to show that for sufficiently large α,p (such that (3.47), (3.48) hold with $\varepsilon > 0$)

$$\|A_p^\alpha u(t)\|_{L^p(\Omega)} \le C \text{ for all } t > 0 . \tag{3.51}$$

Once (3.51) has been shown, it follows from the results of Theorems 1 and 2 and from (3.46) that for any $0 \le s, t < \infty$

$$\begin{aligned} \|u(s) - u(t)\|_{C^1(\bar\Omega)} &\le C \cdot \|u(s) - u(t)\|^{\varepsilon}_{W^{1,2}(\Omega)} \\ &\le C \cdot \|\int_s^t \nabla_x u_t(\tau)d\tau\|^{\varepsilon}_{L^2} \\ &\le C \cdot \left(\int_s^t \frac{(1+\log b(\tau))^{1+\delta}}{b(\tau)} d\tau\right)^{\varepsilon} \to 0 , \end{aligned} \tag{3.52}$$

as $s,t \to \infty$, which will prove (2.41) resp. (2.52).

We proceed to show that (3.51) holds, by deriving the estimate for u_1 and u_2 separately. The estimate for u_1 follows from the regularity assumptions (2.7), (2.8) (which implies $u_0 \in D(A_p^\alpha)$ for all $\alpha < 1$ and all $p < \infty$), from the uniform boundedness of $\|f(\cdot)\|_{L^p(\Omega)}$ on $[0,\infty)$, and from the property (3.45).

Next we note that the $W^{1,2}$-estimate (2.16) for u_t and the integrability condition (2.51) (resp. the definition of b) imply that

$$\sup_{0<t} \|u(\cdot,t)\|_{W^{1,2}(\Omega)} \le C < \infty . \tag{3.53}$$

If Ω is an interval, this implies that u is essentially bounded, and the property

$$|g(s,u)| \le \frac{1}{b(s)} \cdot |g(0,u)| \tag{3.54}$$

show that for all x and t

$$|\int_0^t g(t - s,u(x,s))ds| < C \cdot \int_0^\infty \frac{ds}{b(s)} < \infty .$$

The bound (3.51) then follows for u_2 as it followed for u_1.

If the space dimension is bigger than one, (3.53) still holds. We apply A_p^α to both sides of the equation that defines u_2 ($\alpha < 1$ to be chosen later, but big enough such that (3.47) and (3.48) hold) and write

$$w(t) = \operatorname*{ess\,sup}_{0<s<t} \|A_p^\alpha u(s)\|_{L^p(\Omega)} \tag{3.55}$$

for short. Then (3.45) and (3.54) give the integral inequality

$$\begin{aligned} w(t) &< \int_0^t C(p,\alpha) \cdot ((t - s)^{-\alpha} + 1) \cdot e^{-\lambda_0(t-s)} \cdot \int_0^s \frac{1}{b(s - \tau)} \|g(0,u(\tau))\|_{L^p} d\tau ds \\ &= \int_0^t B(t - s)\|g(0,u(s))\|_{L^p} ds \end{aligned} \tag{3.56}$$

with some $B \in L^1(0,\infty)$. If $n > 2$, then by (2.38) for some q

$$\|g(0,u(s))\|_{L^p(\Omega)} < C \cdot (1 + \|u(s)\|^q_{L^{p\cdot q}(\Omega)}) , \tag{3.57}$$

and for $n = 2$

$$\|g(0,u(s))\|_{L^p(\Omega)} < C \cdot (1 + \|u(s)\|^q_{W^{1,2}(\Omega)}) , \tag{3.58}$$

since the imbedding $W^{1,2}(\Omega) \to L^{p\cdot q}(\Omega)$ is bounded. Thus $w(\cdot)$ is uniformly bounded also in the case $n = 2$ which proves (3.51) again. Finally, if $n > 3$, we pick α such that

$$1 > \alpha > (q - 1) \cdot \frac{n - 2}{4} , \quad q \text{ as in (2.38)}, \tag{3.59}$$

and $r < \infty$ such that

$$\frac{1}{p} - (q - 1) \cdot \frac{n - 2}{2n} = \frac{1}{r} - \frac{1}{n} > \frac{1}{p} - \frac{2\alpha}{n} . \tag{3.60}$$

Then from a standard calculus inequality (see [7])

$$\|v\|^q_{L^{p\cdot q}(\Omega)} < C \cdot \|v\|_{W^{1,r}(\Omega)} \cdot \|v\|^{q-1}_{W^{1,2}(\Omega)} , \tag{3.61}$$

$$(3.62)\qquad \|v\|_{W^{1,r}(\Omega)} \le C\cdot \|A_p^\alpha v\|^\beta_{L^p(\Omega)}\cdot \|v\|^{1-\beta}_{W^{1,2}(\Omega)}$$

$$\le \varepsilon\|A_p^\alpha v\|_{L^p(\Omega)} + C(\varepsilon)\cdot \|v\|_{W^{1,2}(\Omega)}$$

for all suitable v, for some $\beta < 1$, and thus for all $\varepsilon > 0$ (with some $C(\varepsilon) > 0$). Inserting (3.61) and (3.62) into (3.57) and the result into (3.56), we get from the $W^{1,2}$-bound for u and the L^∞-bound for u_1 that

$$(3.63)\qquad w(t) \le \int_0^t B(t-s)\cdot (C(\delta) + \delta\cdot w(s))ds\ ,$$

and δ can be made arbitrarily small. If $\delta < \frac{1}{2}\cdot (\int_0^\infty B(s)ds)^{-1}$, this shows the essential boundedness of $w(\cdot)$, which was needed to complete the proof of the Corollaries.

QED.

Proof of Theorem 5: We apply a standard fixed point argument. For $p < \infty$, denote by $X_p(T)$ the space

$$(3.64)\qquad L^p(0,T;W^{2,p}(\Omega)\cap W_0^{1,1}(\Omega))\cap W^{1,p}([0,T],L^p(\Omega))$$
$$= L^p(0,T;D(A_p))\cap W^{1,p}([0,T],L^p(\Omega))\ ,$$

with A_p defined as above. For $0 \le \sigma \le 1$ and $v \in X_p(T)$, let $w = K(\sigma,v)$ be the unique solution of

$$(3.65)\quad \partial_t w(x,t) - \Delta_x w(x,t) + \sigma\cdot\int_0^t g(t-s,v(x,s))ds = f(x,t)\qquad (x\in\Omega,\ 0<t<T)\ ,$$

$$(3.66)\qquad w(x,t) \equiv 0\qquad (x\in\partial\Omega,\ 0\le t\le T)\ ,$$

$$(3.67)\qquad w(x,0) \equiv u_0(x)\qquad (x\in\Omega)\ .$$

Since $X_p(T)\hookrightarrow C^0(\bar\Omega\times[0,T],\mathbb{R})$, if $p > \frac{n}{2}+1$, with compact imbedding, w is well-defined, and $K : [0,1]\times X_p(T)\to X_p(T)$ is completely continuous (see [10]). Obviously, $u\in X_p(T)$ is a solution of (2.1)-(2.3) iff $K(1,u) = u$; and since K has range in $\bigcap_{p<\infty} X_p(T)$, u will automatically be in the regularity class (2.4). Let now $p > \frac{n}{2}+1$

be fixed; we show that for some large constant M and for any $0 \leq \sigma \leq 1$ the conclusion holds

$$(3.68) \qquad K(\sigma,u) = u \implies \|u\|_{X_p(T)} \leq M .$$

Since there is a unique solution of $K(0,u) = u$, a Leray-Schauder degree argument then shows that there must exist a solution of $K(1,u) = u$ which is bounded by M in $X_p(T)$ (see [17]). We proceed to show (3.68) and assume without loss of generality that $\sigma = 1$. Thus let u be a solution of (2.1)-(2.3). Take difference quotients in t of (2.1), multiply with the corresponding difference quotient of u, integrate over $\Omega \times [0,\bar{t}]$ and let the differences tend to zero. After an integration by parts we get the identity (3.29) (see the proof of Theorems 1 and 2) with $b \equiv 1$.

Next observe that this choice of b allows to estimate the term

$$\int_0^{\bar{t}}\int_\Omega u_t(\cdot,s) \cdot \frac{d}{ds}\left(\int_0^s g(s-\tau,u(\cdot,\tau))d\tau\right)ds$$

from below by

$$(3.69) \qquad \int_\Omega G(\bar{t},u(x,\bar{t}))dx - \int_\Omega G(0,u(x,0))dx - \int_0^{\bar{t}}\int_\Omega G'(s,u(\cdot,s))ds \geq -\int_\Omega G(0,u_0(x))dx$$

by Lemma 7. Inserting this into (3.29), we get an estimate

$$(3.70) \qquad \int_0^T\int_\Omega |\nabla_x u_t(\cdot,s)|^2 ds \leq C(u_0,f) .$$

If the space dimension is $n = 1$, this implies a uniform bound for $|u(x,t)|$ and thus for $|g(t-s,u(x,s))|$ for all x,s,t; thus the regularity theory in [10] gives the bound in (3.68).

If the space dimension is $n = 2$, (3.70) implies that

$$(3.71) \qquad \sup_{0\leq t\leq T} \|u(\cdot,t)\|_{L^r(\Omega)} \leq C(r,u_0,f)$$

for all $r < \infty$. Picking r big enough such that $r > p \cdot q$, q the growth exponent in (2.56), this implies a bound

$$\sup_{t,s} \|g(t-s,u(\cdot,s))\|_{L^p} \leq C(u_0,f,q) \,, \tag{3.72}$$

since (2.11) implies that

$$|g(s,v)| = g(s,v) \cdot \operatorname{sign} v = [g(0,v) + \int_0^s g'(\tau,v)d\tau] \cdot \operatorname{sign} v \tag{3.73}$$

$$\leq g(0,v) \cdot \operatorname{sign} v = |g(0,v)|$$

for all s and v. The bound (3.72) again allows to conclude (3.68).

Finally, if $n > 3$, we get for any $0 \leq t \leq T$ (see [10])

$$\|u\|^p_{X_p(t)} \leq C(u_0,f) + C \cdot \int_0^t \|\int_0^s g(s-\tau,u(\cdot,\tau))d\tau\|^p_{L^p} ds \tag{3.74}$$

$$\leq C \cdot (1 + \int_0^t\int_0^s \|u(\cdot,\tau)\|^r_{L^r} d\tau ds)$$

with $r = p \cdot \frac{n+2}{n-2}$ by (2.56). Using the calculus inequality

$$\|v\|^r_{L^r} \leq C \cdot \|v\|^p_{W^{2,p}(\Omega)} \cdot \|v\|^{r-p}_{W^{1,2}(\Omega)} \tag{3.75}$$

for this choice of r and the bound (3.70) for u, (3.74) implies an integral inequality of the form

$$\|u\|^p_{X_p(t)} \leq C \cdot (1 + \int_0^t \|u\|^p_{X_p(s)} ds) \tag{3.76}$$

for all $0 \leq t \leq T$, C depending only on the data and on T. Gronwall's Lemma then allows to complete the conclusion (3.68). This finishes the proof of Theorem 5.

QED.

Proof of Corollary 5.A: To prove part (i), we first solve (2.1)-(2.3) on some small strip $\Omega \times [0,\varepsilon]$ by means of a standard fixed point argument in $L^\infty(0,\varepsilon;L^\infty(\Omega))$. From standard regularity results, the solution w will be, e.g., in

$$L^p(\delta,\varepsilon;W^{2,p}(\Omega)) \cap W^{1,p}([\delta,\varepsilon],L^p(\Omega))$$

for all $\delta > 0$, $p < \infty$ (see [10]). We then re-write (2.1) on $\Omega \times [\varepsilon,T]$ as

$$\partial_t u(x,t) - \Delta_x u(x,t) + \int_\varepsilon^t g(t-s,u(x,s))ds = f(x,t) - \int_0^\varepsilon g(t-s,w(x,s))ds \; . \tag{3.77}$$

The boundedness of w on $\Omega \times [0,\varepsilon]$ then implies that the right-hand side of (3.77) is in $L^\infty(\varepsilon,T;L^\infty(\Omega)) \cap W^{1,1}(\varepsilon,T;L^2(\Omega))$. Also, without loss of generality

$$\frac{1}{h}\int_{\varepsilon-h}^{\varepsilon} w(\cdot,s)ds \to w(\cdot,\varepsilon), \quad \text{as} \quad h \downarrow 0 \tag{3.78}$$

in all $W^{2,p}(\Omega)$ $(p < \infty)$. Indeed, for fixed p (3.78) will hold for almost every t replacing ε, $0 < t < \varepsilon$. Thus (3.78) will still hold (for almost every t replacing ε) in all $W^{2,N}(\Omega)$, N *a natural number.* Decreasing ε if necessary, we find that (3.78) holds. We now apply Theorem 5 to the equation (3.77) with boundary condition $u(\cdot,t) \in W_0^{1,2}(\Omega)$ and with initial condition $u(\cdot,\varepsilon) = w(\cdot,\varepsilon)$. From the property (3.78) it follows that the function defined by

$$\begin{aligned} &w(x,t), \quad \text{if} \quad t < \varepsilon \\ &u(x,t), \quad \text{if} \quad t > \varepsilon \end{aligned}$$

is a distributional solution of (2.1)-(2.3) on all $\Omega \times [0,T]$; thus it is a solution for the full problem (by the uniqueness of weak solutions of linear parabolic equations) in the regularity class (2.61).

To prove (ii), we again write the equation (2.1) as

$$\partial_t u(x,t) - \Delta_x u(x,t) + \int_\varepsilon^t g(t-s,u(x,s))ds = f(x,t) - \int_0^\varepsilon g(t-s,u(x,s))ds \; , \tag{3.79}$$

and note that as above some small ε can be found which is a Lebesgue point of $u(\cdot,t)$ in all $W^{2,p}(\Omega)$ $(p < \infty)$. (Note that the solution will be in the regularity class (2.61) for all $\varepsilon > 0$, $T < \infty$, by the same argument as used in the proof of part (i) above). To apply Corollary 3 on $\Omega \times [\varepsilon,\infty)$, it is sufficient to show that

$$(3.80) \qquad (x,t) + \int_0^{\varepsilon} g(t - s,u(x,s))ds \in L^{\infty}(\varepsilon,\infty;L^{\infty}(\Omega))$$

and that

$$(3.81) \qquad \int_{\varepsilon}^{\infty} \left(\int_{\Omega} \left|\int_0^{\varepsilon} g'(t - s,u(x,s))ds\right|^2 dx \; b(t)\right)^{1/2} dt < \infty \, .$$

Again, it is immediate that u will be essentially bounded on $\Omega \times [0,\varepsilon]$, if we choose ε small enough; the property (3.80) then follows from the fact that $g(s,u)$ is increasing in u and $g(s,u) \cdot \operatorname{sign} u$ is decreasing in s. Property (3.81) follows since $g'(s,u)$ is decreasing in u and $g'(s,u) \cdot \operatorname{sign} u$ is increasing in s (and negative); thus, if K is a bound for $|u|$ on $\Omega \times [0,\varepsilon]$, and writing

$$w(s) = |g'(s,K)| + |g'(s,-K)| \, ,$$

$$W(s) = |g(s,K)| + |g(s,-K)| \, ,$$

we have

$$(3.82) \qquad \int_{\varepsilon}^{\infty} \left(\int_{\Omega} \left|\int_0^{\varepsilon} g'(t - s,u(x,s))ds\right|^2 dx \; b(t)\right)^{1/2} dt$$

$$\leq \int_{\varepsilon}^{\infty}\int_0^{\varepsilon} w(t - s)ds \cdot b^{1/2}(t)dt \cdot |\Omega|^{1/2}$$

$$\leq C \cdot \int_{\varepsilon}^{\infty} (W(t - \varepsilon) - W(t)) \cdot b^{1/2}(t)dt < \infty \, ,$$

since

$$|g(s,K)| < \frac{1}{b(s)} \cdot |g(0,K)| \tag{3.83}$$

and b is an exponential function.

To prove (iii), we use exactly the same arguments. Since in this case $b(\cdot)$ is not necessarily an exponential function, we need the integrability condition (2.62) to deduce (3.81) from (3.83).

QED.

4. Extensions and Generalizations

The initial-boundary value problem (2.1)-(2.3) represents a typical, however special situation, in which the "energy-type" estimate leading to the inequalities (2.15), (2.16) can be applied. Several extensions are immediate which we want to mention here, without stating them as separate results.

First, it is clear that the case of inhomogeneous, but constant boundary data, in which (2.3) is replaced by

$$u(x,t) \equiv u_0(x), \qquad (x \in \partial\Omega,\ t > 0) \tag{4.1}$$

with $u_0 \in W^{2,\infty}(\Omega)$, presents no additional difficulties. Solutions of (2.1), (2.2), (4.1) will still be in the class (2.4), and all the results of Section 2 are valid without any modification, since it was only assumed in the proofs that $\partial_t u|_{\partial\Omega}(\cdot,t) \equiv 0$ in $H^{-1/2}(\partial\Omega)$. By considering $v(x,t) := u(x,t) - u_0(x)$, this problem can be reduced to a problem of type (2.1)-(2.3) with an x-dependent function g in the integral term; this leads to the modification of (2.1):

$$\partial_t u(x,t) - \Delta_x u(x,t) + \int_0^t g(t-s,x,u(x,s))ds = f(x,t) \qquad (x \in \Omega,\ t \in (0,\infty)) . \tag{4.2}$$

If $g(\cdot,\cdot,\cdot)$ is continuous in all three arguments, and if all the properties assumed for g hold uniformly with respect to x, similar results to Theorems 1, 2, 5 and Corollaries 3, 4, 5A are immediate, since only the pointwise version of Lemma 7 was used in the proofs. To show Theorems 1 and 2 it is, in fact, only necessary to assume that

$$|g(s,u,x)| \leq \mu(x) \cdot M(T,R) \quad \text{for all } s \leq T,\ |u| \leq R ,$$

with $\mu \in L^1(\Omega)$, and that g be measurable in x for fixed s,u and continuous in (s,u) for a.e. x. To show the convergence of solutions (Corollaries 3, 4) or the existence (Theorem 5), it will be necessary to assume that $\mu \in L^r(\Omega)$ for some $r > 1$ and to relate r to the growth conditions formulated in (2.38) resp. in (2.56). Since these generalizations are not immediately motivated and more or less straightforward, we shall not give the details.

The question of t-dependent Dirichlet boundary data seems to be more delicate; we do not have a satisfactory way of generalizing our results to this case.

It is also immediate from the proofs that all results remain valid if $-\Delta_x$ is replaced in (2.1) by a second order operator in divergence form,

$$A_0 : u \to \sum_{i,j=1}^{n} \partial_{x_i}(a_{ij}(x)\partial_{x_j} u) , \tag{4.3}$$

with $C^{1+\alpha}(\bar{\Omega})$-coefficients a_{ij}, which is uniformly elliptic. It is even possible to include a general operator on a possibly unbounded domain Ω,

$$A : u \to \sum_{i,j} \{\partial_{x_i}(a_{ij}(x)\partial_{x_j} u + b_i(x)u) + C_j(x)\partial_{x_j} \cdot u\} + d(x) \cdot u , \tag{4.4}$$

as long as a Garding-type inequality

$$\langle Au,u\rangle_{L^2(\Omega)} \geq \lambda_1 \cdot \|\nabla_x u\|^2_{L^2(\Omega)} \tag{4.5}$$

holds with $\lambda_1 > 0$. The constant λ_1 then replaces the eigenvalue λ_0 in (2.17) and (2.23) and in the corresponding conclusions.

We next indicate how to obtain generalizations of these results to some related problems.

Boundary conditions of the third type.

The problem (2.1), (2.2) with the boundary condition

$$\partial_\nu u(x,t) + \alpha(x) \cdot u(x,t) = f_1(x,t), \quad x \in \partial\Omega, \ t > 0 \tag{4.6}$$

can also be treated. Here ∂_ν is the (outer) normal derivative, $\alpha(\cdot) \geq 0$ is some C^1-function, and $f_1 : \partial\Omega \times [0,\infty) \to \mathbb{R}$ is a given function. We assume that

$$\alpha \not\equiv 0 \tag{4.7}$$

which implies that the principal eigenvalue λ_1 of the negative Laplacian with a homogeneous boundary condition (4.6) is positive:

$$\lambda_1 = \inf\{\int_\Omega |\nabla_x w|^2 + \int_{\partial\Omega} \alpha \cdot w^2 | w \in W^{1,2}(\Omega), \int_\Omega w^2 = 1\} > 0 . \tag{4.8}$$

We state the corresponding result on the existence and asymptotic behavior of solutions as

<u>Theorem 9</u>:

a) Let u_0, f, g, and G satisfy (2.5)-(2.7), (2.9)-(2.12), (2.55) and (2.56). Also, let

(4.9) $$f_1 \in L^\infty(0,\infty;W^{1,\infty}(\partial\Omega)) \cap W^{1,\infty}([0,\infty),L^\infty(\partial\Omega))$$

and assume that

(4.10) $$\partial_\nu u_0 + \alpha \cdot u_0 = f_1(\cdot,0) \quad \text{on} \quad \partial\Omega \ .$$

Then the problem (2.1), (2.2), (4.6) has a solution u on $\Omega \times [0,\infty)$ which is for all $T > 0$ and all $p < \infty$ in the regularity class

(4.11) $$\tilde{X}_p(T) = L^p(0,T;W^{2,p}(\Omega)) \cap W^{1,p}([0,T],L^p(\Omega)) \ .$$

b) If also (2.13) and (2.14) hold for some $\kappa > 0$ and

(4.12) $$\int_0^\infty \left(b(t) \cdot \int_\Omega |\partial_t f(\cdot,t)|^2\right)^{1/2} dt < \infty \ ,$$

(4.13) $$\int_0^\infty \left(\int_{\partial\Omega} |\partial_t f_1(\cdot,s)|^{2r}\right)^{\frac{1}{r}} \cdot b(s)ds < \infty \ ,$$

with $r = 1$, if $n = 1$, $r > 1$ arbitrary, if $n = 2$, $r = \frac{n-1}{n}$, if $n > 2$, and

(4.14) $$b(s) = \exp(t \cdot \min\{\kappa, 2 \cdot (\lambda_1 - \varepsilon)\}) \ ,$$

where λ_1 is defined in (4.8) and $\lambda_1 > \varepsilon > 0$, then

(4.15) $$\int_\Omega |\partial_t u(x,t)|^2 dx \cdot b(t) \leq C_0 \quad \text{for all} \quad t > 0 \ ,$$

(4.16) $$\int_0^\infty \int_\Omega |\nabla_x \partial_t u(x,t)|^2 dx \cdot \frac{b(t)}{(\log b(t) + 1)^{1+\delta}} dt \leq C_\delta \quad \text{for all} \quad \delta > 0 \ .$$

Here C_0, C_δ depend on u_0 and on the quantities in (4.12) and (4.13).

Also, if g satisfies the sharper growth condition (2.38), then

(4.17) $$u(\cdot,t) \to u_\infty \quad \text{in} \quad C^1(\bar{\Omega}), \quad \text{as} \quad t \to \infty \ ,$$

where $u_\infty(\cdot)$ solves the elliptic equation

$$
(4.18)\quad \begin{cases} -\Delta u_\infty(x) + \int_0^\infty g(s,u_\infty(x))ds = \lim_{t\to\infty} f(x,t) , \\ \\ \partial_\nu u_\infty(x) + \alpha(x) \cdot u_\infty(x) = \lim_{t\to\infty} f_1(x,t) . \end{cases}
$$

c) If in addition to the conditions in a) (2.19) holds and with $\kappa(\cdot)$ as in (2.20), $\lambda_1 > \varepsilon > 0$,

$$
(4.19)\quad b(t) = \exp(\int_0^t \min\{\kappa(s), 2 \cdot (\lambda_1 - \varepsilon)\}ds) ,
$$

the data satisfy (4.12) and (4.13), then again (4.15) and (4.16) hold. If b satisfies for some $\delta > 0$

$$
(4.20)\quad \int_0^\infty \frac{(\log b(s) + 1)^{1+\delta}}{b(s)} ds < \infty ,
$$

and if *the growth condition* (2.38) holds for g, then also the convergence conclusion (4.17) is true, and the limit u_∞ satisfies the equation (4.18).

Before we indicate the proof, it should be remarked that the limit

$$
\lim_{t\to\infty} f_1(\cdot,t) \in W^{1,\infty}(\partial\Omega)
$$

exists due to the assumptions (4.9), (4.13), and (4.18). From the general theory for elliptic boundary value problems (cf. [10]) it follows that $u_\infty \in W^{2,p}(\Omega)$ for all $p < \infty$.

Proof of Theorem 9: The main new problems in the proof, compared with the arguments given in Section 3, arise from the fact that the boundary data in (4.6) now depend on t. Hence we shall only give the details for the changes that are necessary for this phenomenon, and remain sketchy in the unchanged parts of the proof. Also, since no new differences appear between the proofs for b) and c), only the former statement will be proven.

We start with showing the estimates (4.15) and (4.16) for a solution u that belongs already to the class (4.11) for all $p < \infty$ and all positive T. From the definition of λ_1 and from standard trace and imbedding theorems, cf. [10], it follows that for all $w \in W^{1,2}(\Omega)$ and all $\varepsilon > 0$

$$(4.21) \qquad \int_\Omega |\nabla_x w|^2 + \int_{\partial\Omega} \alpha \cdot w^2 \geq (\lambda_1 - \varepsilon) \cdot \int_\Omega w^2 + C(\varepsilon) \cdot \left(\int_{\partial\Omega} |w|^{2r'}\right)^{\frac{1}{r'}} ,$$

with $r' = \frac{2r}{2r-1}$, r as in (4.13), resp.

$$(4.22) \qquad \int_\Omega |w_x|^2 + \alpha(a) \cdot w^2(a) + \alpha(b) \cdot w^2(b) \geq$$

$$\geq (\lambda_1 - \varepsilon) \cdot \int_\Omega w^2 + C(\varepsilon) \cdot (w^2(a) + w^2(b)) ,$$

if $n = 1$ and $\Omega = [a,b] \subset \mathbb{R}$. Here $C(\varepsilon) > 0$, if $\varepsilon > 0$. Define $b(\cdot)$ as in (4.18) and proceed as in the proof of Theorem 1, i.e. differentiate the equation (2.1) (formally) with respect to t, multiply with $b(t) \cdot \partial_t u(\cdot,t)$, and integrate over $\Omega \times [0,\bar{t}]$. We then arrive at an identity of the form (3.29), with additional terms

$$(4.23) \qquad \int_0^{\bar{t}} \int_{\partial\Omega} \alpha(x) \cdot |\partial_t u(x,s)|^2 do(x) \cdot b(s) ds$$

appearing on the left-hand side and

$$(4.24) \qquad \int_0^{\bar{t}} \int_{\partial\Omega} \partial_t u(x,s) \cdot \partial_t f_1(x,s) do(x) \cdot b(s) ds$$

appearing on the right-hand side. The quantity (4.24) can be estimated by

$$(4.25)\qquad \int_0^{\bar t} \Big(\int_{\partial\Omega} |\partial_t u(\cdot,s)|^{2r'}\Big)^{\frac{1}{2r'}} \cdot \Big(\int_{\partial\Omega} |\partial_t f_1(\cdot,s)|^{2r}\Big)^{\frac{1}{2r}} \cdot b(s)ds$$

$$\le \frac{1}{2} \cdot C(\varepsilon) \cdot \int_0^{\bar t} \Big(\int_{\partial\Omega} |\partial_t u(\cdot,s)|^{2r'}\Big)^{\frac{1}{r'}} \cdot b(s)ds +$$

$$+ K \cdot \int_0^{\infty} \Big(\int_{\partial\Omega} |\partial_t f_1(\cdot,s)|^{2r}\Big)^{\frac{1}{r}} \cdot b(s)ds$$

with $C(\varepsilon) > 0$ from the trace inequality (4.21). Using now (4.13) and the term (4.25) on the left-hand side and estimating all other quantities as in the previous proof, we get again the inequality (3.33) (with the modified $b(\cdot)$), with some additional constants appearing on the right-hand side. The rest of the proof then remains unchanged, and the estimates (4.15) and (4.16) follow. In the case of one space dimension the procedure only needs some formal modification. Note that $\partial_t u$ is (at least) in $L^2(0,T;W^{1,2}(\Omega))$, so that the trace of $\partial_t u$ actually exists.

Next, we give the necessary extension of the arguments for the existence proof. The previous arguments show that under the assumptions stated in part a), any solution of (2.1), (2.2), (4.6) satisfies an a priori estimate

$$(4.26)\qquad \int_0^T \int_\Omega |\nabla_x \partial_t u(\cdot,t)|^2 dxdt \le C$$

for all $T > 0$, C only depending on the data. This estimate is to be exploited in a bootstrapping argument, with the proper modification for the inhomogeneous boundary conditions. We write the solution as $u = w_1 + w_2$, where

$$(4.27)\qquad \partial_t w_1(\cdot,t) - \Delta_x w_1(\cdot,t) + \int_0^t g(t-s,(w_1+w_2)(\cdot,s))ds = f(\cdot,t) - \partial_t w_2(\cdot,t) \;,$$

$$(4.28)\qquad w_1(\cdot,0) = u_0 - w_2(\cdot,0) \;,$$

$$(4.29)\qquad \partial_\nu w_1 + \alpha \cdot w_1 \equiv 0 \quad \text{on} \quad \partial\Omega \times [0,T] \;,$$

(4.30) $$-\Delta w_2(\cdot,t) = 0 \ ,$$

(4.31) $$\partial_\nu w_2(\cdot,t) + \alpha(\cdot) \cdot w_2(\cdot,t) = f_1(\cdot,t) \quad \text{on} \quad \partial\Omega \times [0,T] \ .$$

Using the same type of degree argument as in the proof of Theorem 5 in Section 3, it is sufficient to show that w_1 and w_2 are a priori bounded in any $\tilde{X}_p(T)$ (see (4.11)). Starting with w_2, we denote by B the operator that maps boundary data v to solutions $w = Bv$ of

(4.32) $$\begin{aligned} -\Delta_x w &= 0 \quad \text{in} \quad \Omega \\ \partial_\nu w + \alpha \cdot w &= v \quad \text{on} \quad \partial\Omega \ . \end{aligned}$$

Since $\lambda_1 > 0$, the well-known results on non-homogeneous boundary value problems apply, see [13],

(4.33) $$B : W^{1-\frac{1}{p},p}(\partial\Omega) \to W^{2,p}(\Omega)$$

is bounded for all $p < \infty$, where the fractional order Sobolev spaces are defined as in [13]. Since $W^{1,\infty}(\partial\Omega) \hookrightarrow W^{1-\frac{1}{p},p}(\partial\Omega)$ for all p, the condition (4.9) implies that $w_2 \in L^p(0,T;W^{2,p}(\Omega))$ for all p. Also, from [13],

(4.34) $$B : L^p(\partial\Omega) \to W^{1+\frac{1}{p}-\delta,p}(\Omega)$$

is bounded for every $\delta > 0$, which shows that also $w_2 \in W^{1,\infty}([0,T],L^\infty(\Omega)) \subset W^{1,p}([0,T],L^p(\Omega))$ for all $p < \infty$.

To show that $w_1 \in \tilde{X}_p(T)$ for all $p < \infty$ it is enough to invoke the general regularity results in [8] for solutions of linear evolution equations of "parabolic" type. Also, observe that the right-hand side $f - \partial_t w_2$ in (4.27) is in $L^p(0,T;L^p(\Omega))$ for all $p < \infty$ and that the initial data for w_1 is in all $W^{2,p}(\Omega)$ and satisfies the compatibility condition

$$\partial_\nu w_1(\cdot,0) + \alpha \cdot w_1(\cdot,0) = 0$$

by construction. The $\tilde{X}_p(T)$-bound for W_1 then is obtained by means of the same "bootstrapping" argument as in the proof of Theorem 5, making use of the growth conditions on g. It remains to show the convergence

$$\left.\begin{array}{l} u(\cdot,t) \to u_\infty \\ \nabla_x u(\cdot,t) \to \nabla_x u_\infty \end{array}\right\} \text{uniformly on } \bar{\Omega} . \tag{4.35}$$

Define $\tilde{A}_p$ as the Laplacian in $L^p(\Omega)$ with

$$D(\tilde{A}_p) = \{w \in W^{2,p}(\Omega) | \partial_\nu w + \alpha w = 0 \text{ on } \partial\Omega\} ,$$

and let $(\tilde{S}_p(t))_{t\geq 0}$ be the analytic semigroup generated by $\tilde{A}_p$ in $L^p(\Omega)$ (see [7]). Then analogous estimates of (3.45), (3.46) hold, the decay rate λ_0 being replaced by λ_1 from (4.8). We write $u = w_1 + w_2$, w_2 the solution of (4.30), (4.31), w_1 defined as the solution of (4.27)-(4.29), i.e. as

$$w_1(t) = \tilde{S}_p(t)(u_0 - w_2(0)) + \int_0^t \tilde{S}_p(t-s)(f(s) - \partial_s w_2(s))ds$$
$$+ \int_0^t \tilde{S}_p(t-s)[\int_0^s g(s-\tau, w_1(\tau) + w_2(\tau))d\tau]ds , \tag{4.36}$$

where the x-dependence has been suppressed for convenience. The convergence result (4.17) then follows as in the proof of Corollaries 3 and 4 if it can be shown that

$$\sup_{t\geq 0} \|w_2(\cdot,t)\|_{W^{2,p}(\Omega)} < \infty , \tag{4.37}$$

$$\sup_{t\geq 0} \|\tilde{A}_p^\alpha w_1(\cdot,t)\|_{L^p(\Omega)} < \infty \tag{4.38}$$

for α sufficiently close to 1 and p sufficiently large $(p > n)$. The estimate (4.37) again follows from (4.33) and the conditions on f_1; (4.38) is derived in exactly the same way as the corresponding estimates for the solution of the Dirichlet problem in (3.53)-(3.63). This concludes the proof of Theorem 9.

QED.

We remark that the conditions leading to the existence of solutions can be weakened to resemble those in Theorem 5. The necessary modifications in the statement and in the proof are obvious.

Mixed boundary conditions, nonlinear boundary conditions.

The boundary conditions (2.3) resp. (4.6) for the integro-differential equation (2.1) can be further combined and generalized. We briefly indicate two possibilities.

First, assume that $\partial\Omega$ can be split into two parts, $\partial\Omega = \Gamma_1 \cup \Gamma_2$, which we assume to be closed and disjoint (to avoid regularity problems at the interface points). Replacing the boundary conditions (2.3) resp. (4.6) by

$$\begin{cases} u(\cdot,t) \equiv 0 \quad \text{on } \Gamma_1 \text{ for } 0 < t \\ \partial_\nu u(\cdot,t) + \alpha(\cdot)\cdot u(\cdot,t) = f_1(\cdot,t) \quad \text{on } \Gamma_2 \text{ for } 0 < t \end{cases} \tag{4.39}$$

then gives a mixed boundary value problem that can obviously be treated in the same manner as the problems discussed further above. The eigenvalue

$$\lambda_2 = \inf\{\int_\Omega |\nabla_x w|^2 + \int_{\Gamma_2} \alpha \cdot w^2 \,|\, w \in W^{1,2}(\Omega),\ w|_{\Gamma_1} = 0,\ \int_\Omega w^2 = 1\}$$

then gives the decay rate for the associated homogeneous heat equation and replaces λ_1 in Theorem 9, giving the same sort of convergence result.

It is also possible to replace (4.6) by a nonlinear boundary condition

$$\partial_\nu u(\cdot,t) + \beta(u(\cdot,t)) = 0 \quad \text{on } \partial\Omega, \quad t > 0 \tag{4.40}$$

with $\beta : \mathbb{R} \to \mathbb{R}$ being a (locally) Lipschitz-continuous function that is monotonically increasing. Under the assumption that

$$\beta'(r) > \alpha_0 > 0 \quad \text{for a.e. } r \tag{4.41}$$

convergence results can be established, showing that the integro-differential equation has solutions that converge to an equilibrium state u_∞. The role of the eigenvalues λ_0, λ_1 is played by

$$\lambda_3 = \inf\{\int_\Omega |\nabla_x w|^2 + \alpha_0 \cdot \int_{\partial\Omega} w^2 \,|\, w \in W^{1,2}(\Omega),\ \int_\Omega w^2 = 1\} .$$

However, in this situation it is not possible to employ directly results from linear semigroup theory to show, e.g., the existence of a solution in the class described in (2.4) or the convergence $u(\cdot,t) \to u_\infty(\cdot)$ in $C^1(\bar{\Omega})$. Rather, direct methods for the "unperturbed" problem $(g(s,u) \equiv 0)$ will have to be carried over; cf. [6].

Equations of higher order

For the evolution equation

$$(4.42)\qquad \partial_t u(x,t) + (-\Delta_x)^m u(x,t) + \int_0^t g(t-s,u(x,s))ds = f(x,t) \qquad (x \in \Omega,\ 0 < t)$$

with initial and boundary conditions

$$(4.43)\qquad u(\cdot,0) = u_0(\cdot) \quad \text{in} \quad \Omega\ ,$$

$$(4.44)\qquad u(\cdot,t)|_{\partial\Omega} \equiv \partial_\nu u(\cdot,t)|_{\partial\Omega} \equiv \ldots \equiv \partial_\nu^{m-1}u(\cdot,t)|_{\partial\Omega} \equiv 0$$

similar convergence (and existence) results can be derived. In (4.42), $m > 1$ is some integer, and in (4.44) ∂_ν^k denotes the k-th normal derivative (with respect to the outer normal of $\partial\Omega$). We assume here that $\partial\Omega$ is in the class $C^{2m+\alpha}$, $\alpha > 0$.

Semilinear equations of the type (4.42), with the integral term replaced by $g_0(u(x,t))$, have been discussed by various authors; cf. [9] for a general existence and regularity result. We define (for fixed m)

$$\lambda_4 := \inf\{\int_\Omega |\nabla_x^m w|^2 | w \in W_0^{m,2}(\Omega),\ \int_\Omega w^2 = 1\}\ .$$

Then $\lambda_4 > 0$ gives the decay rate for solutions of the "unperturbed" equation (4.42) ($f \equiv 0$, $g \equiv 0$), as can be seen from the theory of analytic semigroups ([7]). For (4.42), we have the following result that is parallel to the previous theorems:

Theorem 10: Let g,G satisfy (2.5), (2.6), (2.10)-(2.12), and (2.55). Assume that for $n > 2m$

$$(4.45)\qquad |g(0,w)| \le C \cdot (|w|^{p(n,m)} + 1) \quad \text{for all} \quad w \in \mathbb{R}\ ,$$

C being some constant, $p^{(2m,m)}$ being some positive number, and

$$(4.46)\qquad p^{(n,m)} < \frac{n+2m}{n-2m}\ , \quad \text{if} \quad n > 2m\ .$$

Also, let

$$(4.47)\qquad u_0 \in W^{2m,\infty}(\Omega) \cap W_0^{m,2}(\Omega)$$

and let f satisfy (2.34).

Then the problem (4.42)-(4.44) has a solution u on $\Omega \times [0,\infty)$ which is in the class

(4.48) $$Y_p(T) = L^p(0,T;W^{2m,p}(\Omega)) \cap W^{1,p}([0,T],L^p(\Omega))$$

for all $p < \infty$ and all positive T.

b) If g also satisfies (2.13), (2.14) for some $\kappa > 0$ and if

(4.49) $$\int_0^\infty \left(b(t) \cdot \int_\Omega |\partial_t f(\cdot,t)|^2\right)^{1/2} dt < \infty$$

with

(4.50) $$b(t) = \exp(t \cdot \min(\kappa,2\lambda_4)) ,$$

then (4.15) and

(4.51) $$\int_0^\infty \int_\Omega |\nabla_x^m \partial_t u(x,t)|^2 dx \cdot \frac{b(t)}{(\log b(t) + 1)^{1+\delta}} < C_\delta$$

hold for all $\delta > 0$. Here C_δ depends on u_0 and on the quantity in (4.49). Also, if a strict inequality holds for the growth exponent in (4.46) and if f is essentially uniformly bounded on $\Omega \times (0,\infty)$, then

(4.52) $$u(\cdot,t) \to u_\infty \text{ in } C^{2m-1}(\bar{\Omega}), \text{ as } t \to \infty ,$$

where $u_\infty(\cdot)$ solves the elliptic problem

(4.53) $$\begin{cases} (-\Delta_x)^m u_\infty(x) + \int_0^\infty g(s,u_\infty(x))ds = \lim_{t\to\infty} f(x,t) \\ u_\infty|_{\partial\Omega} \equiv \dots \equiv \partial_\nu^{m-1} u|_{\partial\Omega} \equiv 0 . \end{cases}$$

c) If in addition to the conditions in a) (2.19) and (2.20) hold for g and if the data f satisfies (4.49), $b(\cdot)$ defined by

(4.54) $$b(t) = \exp(\int_0^t \min\{\kappa(s),2\lambda_4\}ds) ,$$

$\kappa(\cdot)$ as in (2.20), then again (4.15) and (4.51) hold. If b satisfies (4.18) for some $\delta > 0$, if strict inequality holds for the growth exponent in (4.46), and if f is essentially uniformly bounded on $\Omega \times (0,\infty)$, then also the convergence (4.52) holds, and u_∞ solves the elliptic problem (4.53).

The proof of this theorem does not differ from the arguments given in Section 3; hence we do not give its details and only point out the necessary changes. First, under the assumptions of part b) and c), the estimates (4.15) and (4.51) follow in precisely the same way as in the proofs of Theorems 1 and 2 - the term $(-\Delta_x)^m u$ in the equation (4.42) leading to an estimate for the $W^{m,2}$-norm of $\partial_t u$ rather than for the $W^{1,2}$-norm as in, e.g., estimate (2.16). Similarly, under the conditions stated in part a), an estimate

$$\int_0^T \int_\Omega |\nabla_x^m \partial_t u(\cdot,t)|^2 dt \le C(\text{data}, T) \tag{4.55}$$

follows. Next, to prove existence (in part a)) and the *convergence results in part* b) and c), it is again convenient to re-write the solution u by means of a variation-of-constants formula,

$$u(t) = \bar{S}_p(t)u_0 + \int_0^t \bar{S}_p(t-s)f(s)ds + \int_0^t \bar{S}_p(t-s)\int_0^s g(s-\tau,u(\tau))d\tau ds ,$$

where x-arguments have been omitted. Here $\bar{S}_p(\cdot)$ is the analytic semigroup generated by $(-\Delta_x)^m = \bar{A}_p$ in $L^p(\Omega)$ under zero *Dirichlet boundary* conditions (4.44). Then (3.45) holds, instead of (3.46) we have for all $n < p < \infty$

$$\|v\|_{C^{2m-1}(\bar{\Omega})} \le C \cdot \|\bar{A}_p^{\alpha} v\|^{1-\varepsilon}_{L^p(\Omega)} \|v\|^{\varepsilon}_{W^{m,2}(\Omega)} \tag{4.57}$$

if $v \in D(\bar{A}_p^{\alpha})$, $1 > \alpha > 0$, such that $1 - \alpha$ and $\varepsilon > 0$ are sufficiently small (cf. [7]), depending on the size of p.

As in the proof of Corollaries 3 and 4, it is enough to prove that under the conditions of Theorem 10

$$(4.58) \qquad \|\tilde{A}_p^{\alpha} u(t)\|_{L^p(\Omega)} \le C$$

for some α and p such that (4.57) holds. This is again done by a "bootstrapping" argument for $n > 2m$ and by direct arguments for $n \le 2m$. First, if $n < 2m$, then the a priori estimate (4.55) together with suitable embedding theorems ([10]) shows that $u(\cdot,t)$ is uniformly (in t) bounded in any $L^p(\Omega)$, $p \le \infty$, if $n < 2m$, $p < \infty$, if $n = 2m$. The estimate (3.54) together with the growth condition (4.45) (if $n = 2m$) shows that the same type of bound also holds for the nonlinear term

$$\int_0^t g(t - s, u(\cdot,s))ds \ .$$

Using (3.54) and this bound in the variation of constants formula (4.56) then implies the bound (4.58), in the same way as in the case of an elliptic operator of second order.

If $n > 2m$, we define $w(\cdot)$ as in (3.55) (with $\tilde{A}_p$ replacing A_p) and derive the estimate (3.56) (with λ_4 instead of λ_0). Then pick α such that

$$(4.59) \qquad 1 > \alpha > (p(n,m) - 1) \cdot \frac{n - 2m}{4m} ,$$

$p(n,m)$ as in (4.46), and close enough to 1 such that (4.57) holds. Next choose $r < \infty$ such that

$$(4.60) \qquad \frac{1}{p} - (p(n,m) - 1) \cdot \frac{n - 2m}{2n} = \frac{1}{r} - \frac{2m - 1}{n} > \frac{1}{p} - \frac{2\alpha m}{n} .$$

This implies ([7])

$$(4.61) \qquad \|v\|^{\tilde{p}}_{L^{p\cdot\tilde{p}}} \le C \cdot \|v\|_{W^{2m-1,r}} \cdot \|v\|^{\tilde{p}-1}_{W^{m,2}}$$

and

$$(4.62) \qquad \|v\|_{W^{2m-1,r}} \le \varepsilon \cdot \|\tilde{A}_p^{\alpha} v\|_{L^p} + C(\varepsilon)\|v\|_{W^{m,2}}$$

for all suitable v, for all $\varepsilon > 0$ (with some $C(\varepsilon) > 0$), and with the abbreviation $\tilde{p} = p(n,m)$. From (4.61) and (4.62) and from the variation of constants formula, we get an integral inequality for $w(\cdot)$ which can be chosen such that it has only bounded solutions. This completes the proof of (4.58) and thus of parts b) and c) of Theorem 10.

For the proof of part a), a fixed point argument as in the proof of Theorem 5 is used, together with optimal regularity estimates in the spaces $Y_p(T)$ (cf. (4.48)); see [8] for the necessary estimates for linear problems. Details are left to the reader.

QED.

Again, as in the proofs of Corollaries 3 and 4 as compared to the proof of Theorem 5, using only the "sub-optimal" regularity estimates that are contained in (3.45) excludes the "limiting" growth exponent $\frac{n+2m}{n-2m}$, but displays the asymptotic behavior of the solution semigroup more clearly.

For the semilinear version of (4.42)-(4.44) (the integral term replaced by $g_0(u(x,t))$, it is known that global solutions u exist, if g_0 has, e.g., the same growth properties that are assumed for $g(0,\cdot)$ in Theorem 10.a). However, in the case of the limiting growth exponent $\frac{n+2m}{n-2m}$, regularity is much harder to prove, cf. [9].

Also, for the limit equation (4.53), the existence and regularity of solutions has been shown in [16], again for the case that corresponds to the limiting growth exponent $\frac{n+2m}{n-2m}$, by means of direct methods for elliptic equations. It is not clear how to extend the convergence result of Theorem 10.b, c to the case of a limiting growth exponent, except for "small" nonlinearities ($g(s,\cdot)$ multiplied by $\varepsilon \cdot g(s,\cdot)$). To do this, one uses the "optimal" growth estimates in the spaces $Y_p(T)$ and notices that the assumption on the spectrum of $(-\Delta)^m$ $(\lambda_4 > 0)$ allows to get T-independent constants in the corresponding regularity estimates.

We conclude with a few remarks concerning systems of (second order) equations with integral terms of the form in (2.1) and higher order equations of the form (4.42) with nonlinear integro-differential operators

$$(4.62) \qquad \int_0^t \sum_{|\alpha|<m} \partial_x^\alpha g_\alpha(t-s,(\partial_x^\beta u(\cdot,s))_{|\beta|<m})\,ds$$

instead of the polynomial type integral operator in (4.42).

In the case of a system, it is still possible to handle finite sum integral operators of the form

$$(4.63) \qquad \sum_{i=1}^{M} \int_0^t a_i(t-s)\vec{g}_i(\vec{u}(\cdot,s))ds$$

with scalar kernels $a_i(\cdot)$ and vector functions $\vec{g}_i(\xi) = \nabla_\xi G_i(\xi)$, the G_i being convex, by utilizing a suitable vector-valued version of Lemma 6 (which holds almost without change in the wording). Similarly, for integral operators of the type (4.62) having the form

$$(4.64) \qquad \int_0^t \sum_{i=1}^{M} a_i(t-s) \sum_{|\alpha|<m} \partial_x^\alpha g_\alpha^i(t-s,(\partial_x^\beta u(\cdot,s))_{|\beta|<m})ds$$

similar results can be derived, if the g_α^i are of the form

$$(4.65) \qquad g_\alpha^i(\xi) = (-1)^{|\alpha|} \cdot \frac{\partial}{\partial \xi_\alpha} G^i(\xi),\ G^i \text{ convex },$$

and satisfy suitable growth estimates.

Open Questions.

a) It is not clear whether solutions of (1.1) or of the related problems discussed above will be unique under the assumptions that have been used, in contrast to the situation for the semilinear parabolic equation (1.5), where monotonicity of $\bar{g}(x,\cdot)$ implies uniqueness. A sufficient condition for the uniqueness of solutions of (1.1) is that $g(s,x,\cdot)$ be locally Lipschitz-continuous in u uniformly for (x,s) in some $\Omega \times [0,\epsilon]$. On the other hand, it is also well-known that solutions of the limit equation (2.42) are unique (with the boundary conditions discussed above), if

$$u \to \int_0^\infty g(s,u)ds$$

is merely monotone non-decreasing.

b) For the semilinear equation (1.5), the existence of regular solutions, convergence rates as (1.9), and the convergence $u(\cdot,t) \to u_\infty(\cdot)$ in $C^1(\bar{\Omega})$ follows without growth restrictions on the nonlinear term $\bar{g}$, since it is possible to bound solutions uniformly on $\Omega \times (0,\infty)$ by means of the maximum principle. It is not clear whether these properties still hold for solutions of (1.1), if one drops the growth restrictions (2.38). The existence of oscillatory solutions with positive initial data (cf. Section 1) shows that a comparison principle does not hold.

c) Using the regularizing effects of the non-linear semigroup that is generated by

$$u \to -\Delta_x u + \bar{g}(u)$$

in $L^p(\Omega)$ $(1 \le p \le \infty)$, it is still possible to show the convergence $u(\cdot,t) \to u_\infty(\cdot)$ for less regular initial data (see [18]): One shows that for some $\varepsilon > 0$ the solution u of (1.5) will be in $L^\infty(\frac{\varepsilon}{2},\varepsilon;L^\infty(\Omega))$ and hence (via linear semigroup theory) $u(\cdot,\varepsilon) \in C^2(\bar{\Omega})$. Taking $u(\cdot,\varepsilon)$ as an initial value, one sees that the regularity of $u(\cdot,0)$ plays a minor role for the convergence behavior. For the integro-differential equation (1.1), the regularity of u on $\Omega \times (0,\varepsilon)$ does play a role, since the effect of irregular initial data is not "forgotten" due to the presence of the integral term. Thus it will in general not be possible to extend the class of initial data for which convergence in $C^1(\bar{\Omega})$ can be shown beyond $L^\infty(\Omega)$ (as described in Corollary 5A). It is conceivable that convergence in some $L^p(\Omega)$ can be shown if $g(s,\cdot)$ satisfies some polynomial growth conditions and u_0 is in some $L^r(\Omega)$, $r < \infty$.

d) For solutions of the semilinear equation (1.5) as well as of the limit equation (2.42), the solution can vanish identically on some parts of Ω, if, e.g.,

$$\bar{g}(u) \sim u^{\alpha}, \quad \text{as} \quad u \to 0 ,$$

with $0 \leq \alpha < 1$, and the boundary data and the size of Ω are in some sense compatible ([19]). It is not clear whether this free-boundary phenomenon still holds for solutions of the integro-differential equation.

REFERENCES

1. V. Barbu: Integro-differential Equations in Hilbert Spaces. An. Sti. Univ. "Al. I. Cuza", Iasi, 19 (1973), 365-383.
2. H. Brezis: Operateurs Maximaux Monotones. Amsterdam 1973.
3. M. G. Crandall, S.-O. Londen, J. A. Nohel: An Abstract Nonlinear Volterra Integrodifferential Equation. JMAA 64 (1978), 701-735.
4. J. Cushing: Integrodifferential Equations and Delay Models in Population Dynamics. Berlin-Heidelberg-New York 1977.
5. H. Engler: Functional Differential Equations in Banach Spaces: Growth and Decay of Solutions. Crelle's J.f.d. Reine Angew. Mathematik 322 (1981), 53-73.
6. L. C. Evans: Regularity Properties for the Heat Equation Subject to Nonlinear Boundary Constraints. Nonlinear Analysis TMA 1 (1977), 593-602.
7. A. Friedman: Partial Differential Equations. New York 1969.
8. P. Grisvard, G. da Prato: Sommes d'Operateurs Linéaires et Equations Différentielles Opérationelles. J. Math. Pures Appl. 54 (1975), 305-387.
9. H. Kielhöfer: Global Solutions of Semilinear Evolution Equations Satisfying an Energy Inequality. JDE 36 (1980), 188-222.
10. O. A. Ladyzenskaja, V. A. Solonnikov, N. N. Ural'ceva: Linear and Quasilinear Equations of Parabolic Type. Providence, Rh. I. 1968.
11. O. A. Ladyzenskaja, N. N. Ural'ceva: Equations aux Derivées Partielles de Type Elliptique. Paris 1968.
12. S. Lefschetz: Stability of Nonlinear Control Systems. New York 1965.
13. J. L. Lions, E. Magenes: Problemi ai Limiti Non Omogenei. V. Ann. della Scuola Norm. Sup. Pisa 16 (1962), 1-44.
14. J. L. Lions: Optimal Control of Systems Governed by Partial Differential Equations. Berlin, Heidelberg, New York 1971.
15. S.-O. Londen, J. A. Nohel: A Nonlinear Volterra Integrodifferential Equation Occurring in Heat Flow. MRC Technical Summary Report #2440, September 1982.

16. S. Luckhaus: Existence and Regularity of Weak Solutions to the Dirichlet Problem for Semilinear Elliptic Systems of Higher Order. Crelle's J.f.d. Reine Angew. Mathematatik 306 (1979), 192-207.

17. J. T. Schwartz: Nonlinear Functional Analysis. New York 1969.

18. L. Veron: Coercivité et Proprietés Régularisantes des Semi-groupes Non Linéaires dans les Espaces de Banach. Tours, 1977.

19. H.-W. Alt, D. Phillips: To appear.

HE:scr

SECURITY CLASSIFICATION OF THIS PAGE (When Data Entered)

REPORT DOCUMENTATION PAGE

READ INSTRUCTIONS BEFORE COMPLETING FORM

1. REPORT NUMBER 2555	2. GOVT ACCESSION NO. AD-A132 821	3. RECIPIENT'S CATALOG NUMBER
4. TITLE (and Subtitle) STABILIZATION OF SOLUTIONS FOR A CLASS OF PARABOLIC INTEGRO-DIFFERENTIAL EQUATIONS		5. TYPE OF REPORT & PERIOD COVERED Summary Report - no specific reporting period 6. PERFORMING ORG. REPORT NUMBER
7. AUTHOR(s) Hans Engler		8. CONTRACT OR GRANT NUMBER(s) DAAG29-80-C-0041
9. PERFORMING ORGANIZATION NAME AND ADDRESS Mathematics Research Center, University of Wisconsin 610 Walnut Street Madison, Wisconsin 53706		10. PROGRAM ELEMENT, PROJECT, TASK AREA & WORK UNIT NUMBERS Work Unit Number 1 - Applied Analysis
11. CONTROLLING OFFICE NAME AND ADDRESS U. S. Army Research Office P.O. Box 12211 Research Triangle Park, North Carolina 27709		12. REPORT DATE August 1983 13. NUMBER OF PAGES 53
14. MONITORING AGENCY NAME & ADDRESS(if different from Controlling Office)		15. SECURITY CLASS. (of this report) UNCLASSIFIED 15a. DECLASSIFICATION/DOWNGRADING SCHEDULE

16. DISTRIBUTION STATEMENT (of this Report)

Approved for public release; distribution unlimited.

17. DISTRIBUTION STATEMENT (of the abstract entered in Block 20, if different from Report)

18. SUPPLEMENTARY NOTES

19. KEY WORDS (Continue on reverse side if necessary and identify by block number)

Parabolic equations
Integro-differential equations
Boundary value problems
Asymptotic behavior
Feed-back control

20. ABSTRACT (Continue on reverse side if necessary and identify by block number)

For parabolic integro-differential equations of the form

$$u_t(\cdot,t) + Au(\cdot,t) + \int_0^t g(t-s,u(\cdot,s))ds = f(\cdot,t) \tag{0.1}$$

on a time-space cylinder $\Omega \times [0,\infty)$ the question of convergence of solutions to a limit solution is studied. Here A is (e.g.) the negative Laplacian,

DD FORM 1473 1 JAN 73 EDITION OF 1 NOV 65 IS OBSOLETE

UNCLASSIFIED

SECURITY CLASSIFICATION OF THIS PAGE (When Data Entered)

20. ABSTRACT - cont'd.

and $g(s,u)$ is typically of the structure

$$g(s,u) = \sum_{i=1}^{N} a_i(s)g_i(u) \quad (0.2)$$

The kernels $a_i(\cdot)$ have to satisfy certain decay properties, but no assumptions concerning the smallness of the g_i or of their derivatives are made; rather, the main assumption on the g_i is that they be monotone. Uniform convergence of solutions and of their derivatives for general initial and boundary conditions is shown; convergence rates are given which show the dependence on the spectrum of A and on the decay properties of the kernels a_i.