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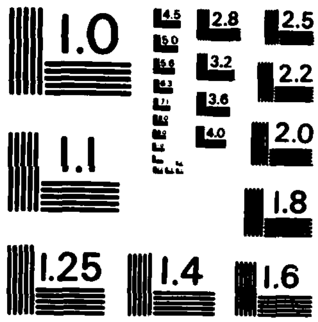

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ON A VISCOMETRIC FLOW OF A SIMPLE FLUID

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ABSTRACT

A class of globally viscometric flows which has relevance to slow flows occurring between two infinite parallel plates rotating with differing angular velocities about a common axis, is studied.

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Key Words: viscometric flow, simple fluid, motions with constant stretch history, Rivlin-Ericksen fluid

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SIGNIFICANCE AND EXPLANATION

Viscometric flows are locally equivalent to steady simple shear flows and in such flows the behavior of a simple fluid can be completely characterized by three scalar functions of a single variable, namely the shear. Most of the familiar flows in the literature, namely Couette flow, Poiseuille flow, etc., belong to the above class. In this paper we investigate a class of viscometric flows which has relevance to the flows occurring between infinite parallel plates rotating about a common axis with different angular velocities.

The responsibility for the wording and views expressed in this descriptive summary lies with MRC, and not with the author of this report.

ON A VISCOMETRIC FLOW OF A SIMPLE FLUID

K. R. Rajagopal*

1. Introduction

In one of his several pioneering papers in the fifites, Rivlin [1] studied the torsional flow between two parallel disks. He considered a velocity field of the form:

$$u = -\psi z y, \quad v = \psi z x \quad \text{and} \quad w = 0, \tag{1}$$

u, v, and w being the velocities in the x, y, and z directions, respectively. The above motion is viscometric (cf. Pipkin [2]) and has relevance to the low Reynolds number flow between rotating disks. The form (1) corresponds to a flow in which each plane parallel to the plates is rotating as though it were rigid, the angular velocity of these plates varying linearly. However, such a linear variation is by no means the only possible one in the case of a simple fluid.

In this paper, I shall consider a generalization of (1) which is applicable for the slow flow of a simple fluid between parallel plates rotating with differing angular velocities about a common axis (see Fig. 1). The assumed form for the velocity field falls into the category of pseudo-plane motions which were studied by Berker [3].

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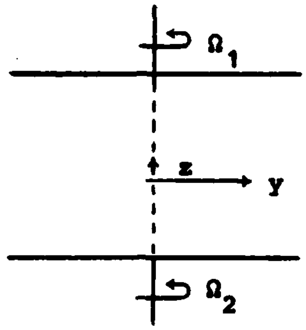

Figure 1.

We shall assume a flow fluid of the form

$$u = -\Omega(z)y, \quad v = \Omega(z)x, \quad w = 0, \tag{2}$$

where $\Omega(z)$ is an arbitrary function z which needs to be determined from the equations of motion for the specific fluid under consideration.

After a brief discussion of the basic definitions and notations that we will need, in the next section, we proceed to show that a motion of the form (2) is viscometric. We conclude with an example of a specific fluid model wherein $\Omega(z)$ need not be linear.

2. Preliminaries

Let $\underset{\sim}{x}$ denote the position of an element $\underset{\sim}{X}$ in the reference state at time t and let $\underset{\sim}{\xi}$ denote the position of $\underset{\sim}{X}$ at time τ. The dependence of $\underset{\sim}{\xi}$ on $\underset{\sim}{x}$, t and τ can be expressed as

$$\underset{\sim}{\xi} = \chi\,(\underset{\sim}{x},\tau). \tag{3}$$

The relative deformation gradient $\underset{\sim}{F}_t(\tau)$ is then defined through

$$\underset{\sim}{F}_t(\tau) = \text{grad}_{\underset{\sim}{x}}\, \chi_t(\underset{\sim}{x},\tau). \tag{4}$$

The relative right Cauchy-Green tensor is defined through

$$\underset{\sim}{C}_t(\tau) = \underset{\sim}{F}_t^T(\tau)\underset{\sim}{F}_t(\tau), \tag{5}$$

the velocity gradient tensor $\underset{\sim}{L}(t)$ through

$$\underset{\sim}{L}(t) = \frac{d}{d\tau}\underset{\sim}{F}_t(\tau)\Big|_{\tau=t} . \tag{6}$$

and the Rivlin-Ericksen tensors (cf. Rivlin and Ericksen [4]) through

$$\underset{\sim}{A}_1 = \underset{\sim}{L} + \underset{\sim}{L}^T , \qquad (7)_1$$

$$\underset{\sim}{A}_n = \frac{d\underset{\sim}{A}_{n-1}}{dt} + \underset{\sim}{A}_{n-1}\underset{\sim}{L} + \underset{\sim}{L}^T\underset{\sim}{A}_{n-1} , \quad n=2,3,\ldots \qquad (7)_2$$

A motion is said to be viscometric* (cf. Coleman [5]) if at that given material point, the right relative Cauchy-Green tensor can be expressed as

$$\underset{\sim}{C}_t(t-s) = \underset{\sim}{1} - s\underset{\sim}{A}_1 + \frac{s^2}{2}\underset{\sim}{A}_2 , \tag{8}$$

for all t and if relative to some orthonormal basis $\hat{\underset{\sim}{e}}_i$, the Rivlin-Ericksen tensors have the following matrix representation

$$\underset{\sim}{A}_1 = \begin{pmatrix} 0 & 0 & \kappa \\ 0 & 0 & 0 \\ \kappa & 0 & 0 \end{pmatrix}, \qquad \underset{\sim}{A}_2 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 2\kappa^2 \end{pmatrix} , \tag{9), (10}$$

where κ is usually referred to as the "shear rate"

---

*We choose to use the above definition for a viscometric flow since we shall find the need to employ the kinematical tensors $\underset{\sim}{A}_1$ and $\underset{\sim}{A}_2$ used in the above definition, later on. A flow is viscometric (cf. Coleman, Markovitz and Noll [6]) if

$$\underset{\sim}{F}_t(t-\tau) = \underset{\sim}{R}(t-\tau)(\underset{\sim}{1}-(t-\tau)\underset{\sim}{M}),$$

where $\underset{\sim}{R}(t-\tau)$ is orthogonal with $\underset{\sim}{R}(0) = \underset{\sim}{1}$ and $\underset{\sim}{M}$ is a tensor which has the following matrix representation with respect to a suitable axis

$$\begin{pmatrix} 0 & 0 & \kappa \\ 0 & 0 & 0 \\ \kappa & 0 & 0 \end{pmatrix} .$$

3. The Flow Field

Consider the motion represented by (2), i.e.,

$$u = -\Omega(z)y,$$
$$v = \Omega(z)x, \quad \text{and}$$
$$w = 0,$$

where u, v, and w denote the x, y, and z components of the velocity, respectively. The motion represented by (10) is isochoric. Let us denote by $\boldsymbol{\xi}$ the 3-tuple (ξ, η, ζ). Then (10) implies that

$$\dot{\xi} = -\Omega(\zeta)[\eta], \tag{11$_1$}$$
$$\dot{\eta} = \Omega(\zeta)[\xi], \tag{11$_2$}$$
$$\dot{\zeta} = 0. \tag{11$_3$}$$

with

$$\xi(t) = x, \ \eta(t) = y, \ \text{and} \ \zeta(t) = z. \tag{12}$$

A straightforward computation yields

$$\xi(\tau) = x \cos[(\Omega(z))(t-\tau)] + y \sin[(\Omega(z))(t-\tau)], \tag{13$_1$}$$
$$\eta(\tau) = -x \sin[(\Omega(z))(t-\tau)] + y \cos[(\Omega(z))(t-\tau)], \tag{13$_2$}$$
$$\zeta(\tau) = z \tag{13$_3$}$$

Thus, the relative deformation gradient has the following matrix representation:

$$\underset{\sim}{F}_t(\tau) = \begin{pmatrix} \cos[(\Omega(z))(t-\tau)] & \sin[(\Omega(z))(t-\tau)] & \begin{matrix} -x(t-\tau)\Omega'(z)\sin[(\Omega(z))(t-\tau)] \\ +y(t-\tau)\Omega'(z)\cos[(\Omega(z))(t-\tau)] \end{matrix} \\ -\sin[(\Omega(z))(t-\tau)] & \cos[(\Omega(z))(t-\tau)] & \begin{matrix} -x(t-\tau)\Omega'(z)\cos[(\Omega(z))(t-\tau)] \\ -y(t-\tau)\Omega'(z)\sin[(\Omega(z))(t-\tau)] \end{matrix} \\ 0 & 0 & 1 \end{pmatrix} \tag{14}$$

Hence, the right relative Cauchy-Green strain history takes the simple form

$$\underset{\sim}{C}_t(t-s) = \begin{pmatrix} 1 & 0 & sy(\Omega'(z)) \\ 0 & 1 & -sx(\Omega'(z)) \\ y(\Omega'(z))s & -x(\Omega'(z))s & 1+[(\Omega'(z))s]^2(x^2+y^2) \end{pmatrix} \tag{15}$$

We now proceed to compute the Rivlin-Ericksen tensors $\underset{\sim}{A}_n$. First, it follows from (8), the velocity gradient $\underset{\sim}{L}$ is given by

$$\underset{\sim}{L} = \begin{pmatrix} 0 & -\Omega(z) & -y\Omega'(z) \\ \Omega(z) & 0 & x\Omega'(z) \\ 0 & 0 & 0 \end{pmatrix}. \tag{16}$$

Thus, the first two Rivlin-Ericksen tensors are given by

$$\underset{\sim}{A}_1 = \begin{pmatrix} 0 & 0 & -y\Omega'(z) \\ 0 & 0 & x\Omega'(z) \\ -y\Omega'(z) & x\Omega'(z) & 0 \end{pmatrix}, \tag{17}$$

and

$$\underset{\sim}{A}_2 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 2[(y\Omega'(z))^2+(x\Omega'(z))^2 \end{pmatrix} . \tag{18}$$

We also provide the matrix representations of $\underset{\sim}{A}_1^2$ and $\underset{\sim}{A}_1\underset{\sim}{A}_2$ which will be useful later on.

$$\underset{\sim}{A}_1^2 = \begin{pmatrix} [\Omega'(z)y]^2 & -xy(\Omega'(z))^2 & 0 \\ -xy(\Omega'(z))^2 & [x\Omega'(z)]^2 & 0 \\ 0 & 0 & \{[y\Omega'(z)]^2+[x\Omega'(z)]^2\} \end{pmatrix}, \tag{19}$$

$$\underset{\sim}{A}_1\underset{\sim}{A}_2 = \begin{pmatrix} 0 & 0 & -2[\Omega'(z)]^3y[x^2+y^2] \\ 0 & 0 & 2[\Omega'(z)]^3x[x^2+y^2] \\ 0 & 0 & 0 \end{pmatrix} . \tag{20}$$

It is easy to verify that the Rivlin-Ericksen tensors $\underset{\sim}{A}_1$ and $\underset{\sim}{A}_2$ can be expressed in the form (9) and (10) where the new basis $\hat{\underset{\sim}{e}}_i$ (i=1,2,3) is related to the old cartesian basis $\underset{\sim}{e}_i$ (i=1,2,3) through

$$\hat{\underset{\sim}{e}}_1 = \frac{-y\Omega'(z)}{\kappa}\underset{\sim}{e}_1 + \frac{x\Omega'(z)}{\kappa}\underset{\sim}{e}_2 ,$$

$$\hat{\underset{\sim}{e}}_2 = \frac{-y\Omega'(z)}{\kappa}\underset{\sim}{e}_2 - \frac{x\Omega'(z)}{\kappa}\underset{\sim}{e}_1 ,$$

$$\hat{\underset{\sim}{e}}_3 = \underset{\sim}{e}_3 ,$$

with

$$\kappa = [(y\Omega'(z))^2 + (x\Omega'(z))^2]^{1/2} .$$

It then follows from equations (15), (17), (18) and the definition of a viscometric flow that the motion (2) under consideration is indeed viscometric. Furthermore, a simple computation yields

$$\underset{\sim}{A}_n = 0, \quad \forall \quad n \geq 3.$$

4. Discussion

It is easy to verify by virtue of (17)-(20) that a velocity field of the form (10) given by*

$$u(x,y,z) = -[\frac{(\Omega_2-\Omega_1)}{h} z + (\frac{\Omega_1 + \Omega_2}{2})]y ,$$

$$v(x,y,z) = [(\frac{\Omega_2 - \Omega_1}{h} z + (\frac{\Omega_1 + \Omega_2}{2})]x,$$

$$w(z,y,z) = 0 ,$$

satisfies the equation of motion for the non inertial flow of the classical linearly viscous fluid and the Rivlin-Ericksen fluids of second and third grade**. In the case of the linearly viscous fluid the above solution is the unique solution to the "Stokes flow" problem. In the case of the incompressible Rivlin-Ericksen fluids of the second and third grade, the above flow would be the unique solution under certain conditions if the fluids are required to be thermodynamically compatible*** (cf. Fosdick and Rajagopal [9]).

---

* This is Rivlin's [4] result extended to the case when both the top and bottom plates are rotating.

** The stress constitutive equations for the linearly viscous fluid and the incompressible Rivlin-Ericksen fluids of second and third grade are given by (cf. Truesdell and Noll [7]):

$$\mathbf{T} = -p\mathbf{1} + \mu\mathbf{A}_1 ,$$

$$\mathbf{T} = -p\mathbf{1} + \mu\mathbf{A}_1 + \alpha_1\mathbf{A}_2 + \alpha_2\mathbf{A}_1^2 ,$$

$$\mathbf{T} = -p\mathbf{1} + \mu\mathbf{A}_1 + \alpha_1\mathbf{A}_2 + \alpha_2\mathbf{A}_1^2 + \beta_1\mathbf{A}_3 + \beta_2[\mathbf{A}_1\mathbf{A}_2 + \mathbf{A}_2\mathbf{A}_1] + \beta_3(\mathrm{tr}\mathbf{A}_1^2)\mathbf{A}_2 .$$

*** The fluid is said to be thermodynamically compatible if it meets the Clausius-Duhem inequality in all its motions and if the specific Helmholtz free energy is a minimum when the fluid is at rest under isothermal conditions. The uniqueness result is not a consequence of Tanner's therorem [10] as the flow in question is not plane.

However, the flow (2) is by no means the only one possible in a general simple fluid. We give below an example of a simple fluid which is properly frame invariant in which an infinity of solutions is possible for the above problem. Of course, the fluid model may not be a realistic one. It should however be noted that one could easily construct fluid models wherein the stress is expressible as polynomials of the gradients of velocity and the $(n-1)^{th}$ accelerations, the class of models studied by Rivlin [1], where non-unique solutions for $\Omega(z)$ are possible.

Let us consider a fluid model whose Cauchy stress $\underset{\sim}{T}$ is given by

$$\underset{\sim}{T} = -p\underset{\sim}{1} + \frac{1}{\left(\mathrm{tr}\underset{\sim}{A}_1^2\right)} \underset{\sim}{A}_2 \, .$$

Such a fluid model is definitely permissible under the class of simple fluids (cf. Wineman and Pipkin [11]). A trivial computation, for the problem in question, verifies that

$$\frac{1}{\left(\mathrm{tr}\underset{\sim}{A}_1^2\right)} \underset{\sim}{A}_2 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix} .$$

It then follows that any smooth $\Omega(z)$ which is such that it is Ω_1 at the top and Ω_2 at the bottom would be permissible!

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