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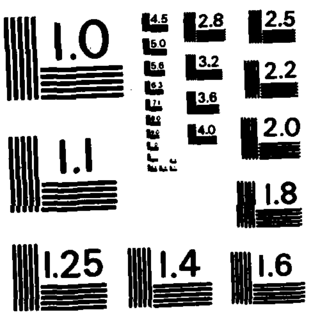

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Nonuniqueness in Wakes and Boundary Layers

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SUMMARY

In streamlined flow past a flat plate aligned with a uniform stream, it is shown that (a) the Goldstein near-wake and (b) the Blasius boundary layer are nonunique solutions locally for the classical boundary layer equations, whereas (c) the Rott-Hakkinen very-near-wake appears to be unique. In each of (a), (b) an alternative solution exists which has reversed flow and which apparently cannot be discounted on immediate grounds. Thus, depending mainly on how the alternatives for (a), (b) develop downstream, the symmetric flow at high Reynolds numbers could have 2, 4 or more, simple steady forms.

Concerning non-streamlined flow, e.g. past a bluff obstacle, new similarity forms are described for the pressure-free viscous symmetric closure of a predominantly slender long wake beyond a large-scale separation. Features arising include nonuniqueness, singularities and algebraic behavior, consistent with non-entraining shear layers with algebraic decay. Nonuniqueness also seems possible in reattachment onto a solid surface and for nonsymmetric or pressure-controlled flows including the wake of a symmetric cascade.

ACKNOWLEDGEMENTS

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1.0 INTRODUCTION

The first part of this work is concerned with a class of basic wake flows which, according to their near-wake properties at least, are found to possess nonunique solutions with or without reversed flow present. The common theme involved is the nature of the steady laminar planar flow past a finite body, fixed in a uniform stream for instance, at large Reynolds number.

At one extreme, if the body is sufficiently streamlined then attached motion can result on the body surface (see Section 5 however) and the thin wake beyond is governed by the viscous boundary layer equations. At the other extreme, for non-streamlined motion, large-scale separation and flow reversal take place in general and the wake is thick with detached viscous shear layers present. Even then there is reason to think that the main recirculatory eddies may well be so long that, on the characteristic wake scale much greater than the body dimensions, the wake closure and subsequent recovery to uniform flow occur in a predominantly slender flow, i.e., the wake is thick but very long. See Sychev (1967), Messiter (1975), Smith (1979). The main issue there centers around the flow features at closure. If these are effectively inviscid then vorticity conservation tends to suggest a strong reversed jet being projected upstream, along the center-line in symmetric flow, possibly ruling out the presence of only a slowish recirculatory motion overall. So the alternative of significant viscous action being brought into play, before or after the main wake closure, to suppress a strong reversed center-line flow upstream, suggests itself. Thus, for example, the main closure could force a detached shear layer to return almost to the center-line by inviscid means but leave any necessary reversed flow to be sealed off by the viscous wake motion holding beyond. The above two extremes provide the motivation for the study of viscous wakes under certain starting conditions. Moreover, in both extremes the induced pressure is usually negligible, since either the body must be thin to be streamlined or the major wake is slender. As a result, although there are some exceptions to this view, one of which is discussed below, we are led to consider for the most part pressure-free viscous wakes.

Streamlined motion is epitomized by the symmetric flow past a finite flat plate at zero incidence. The pressure-free boundary layer equations control the flow on and beyond the plate, apart from a small leading edge zone and a triple-deck at the trailing edge to adjust the Blasius solution (see Section 5) to the near-wake. The near-wake here, described first in Goldstein's remarkable paper of 1930, is reconsidered in Section 2 below. It is something of a surprise that the solution turns out to be nonunique: one solution is Goldstein's, with forward flow; the second solution has reversed center-line flow. Both are formally acceptable as far as we can tell, with the second solution suggesting an account, for the trailing edge adjustment and the downstream wake, quite distinct from the well-known classical account.

Nonuniqueness and other aspects are implied also for nonsymmetric or slightly thicker bodies and these are noted in Section 2. In contrast the Rott-Hakkinen (1965) "very near"-wake within the triple-deck admits only one solution for symmetric flow. However, the extent of nonuniqueness possible in the aligned flat plate problem is enhanced later by Section 5.

The application of pressure-free viscous wake properties to non-streamlined flow and to wake closure in particular is discussed in Section 3, by means of more general near-wake forms. Some of these forms lead to algebraic singularities, linking with the author's (1983) proposal of shear layers exhibiting algebraic decay and no entrainment, and possibly pointing to a self-consistent account of large-scale wake closure. Finally, two extra kinds of near-wakes exhibiting nonuniqueness are examined in Section 4. One concerns the long wake behind a typical body in a symmetric cascade, where again the boundary layer equations hold but, unlike most of the other wakes studied here, the pressure can be significant because of the confinement present. For a wide cascade a solution branching from the uniform flow solution is found. The second extra near-wake considered describes the behavior of slowish eddy motion at the onset of wake closure.

Secondly, Section 5 concerns flow over a fixed surface. Aspects similar to those of Sections 3 and 4 apply generally, but in particular it is conjectured that even the pressure-free boundary layer on an aligned flat plate may have a nonunique solution. For, as well as the Blasius solution, another apparently self-consistent starting solution near the leading edge is possible with a reversed flow coming to rest there. In all these examples of new solutions, however, it remains to be seen whether some external agency must be present, e.g., downstream, to allow a deviation from the usual solutions, or not. If so, the nonuniqueness found could still be of physical significance. But if no external agency is required, then in view of the possible nonuniqueness of the near-wake (Section 2) as well as the Blasius solution (Section 5) there could be 2, or even more, simple steady solutions for that most basic problem of free symmetric flow past a flat plate in a uniform stream.

Throughout these near-wake and boundary layer flows the planar boundary layer equations apply, usually pressure-free and hence classical, although similar features including nonuniqueness can be expected to persist in nonzero pressure fields and in three-dimensional motions. Further details, especially of the contexts from which this work arose, including non-entraining shear layers and the cascade flow problem, are given by Smith (1982a, 1983). Other contexts include the motion of liquid layers and the flow near or above a finite heated plate, for example. The nondimensional scaled variables u, ψ, x, Y, p are used below, denoting the streamwise velocity, the stream function, the streamwise and lateral coordinates, and the pressure, respectively, and Re is the Reynolds number. We have in mind subsonic/incompressible fluid flow, although the implications for supersonic motion are similar. All the cases of nonuniqueness and/or reversed flow and/or singularities found below in near-wakes and on solid surfaces pose quite challenging computational tasks, classical or interactive, for the rest of the wake or boundary layer or for the shorter scale flows implied. It would seem worthwhile addressing these tasks,* nevertheless, e.g. to test the nonuniqueness of flow past a flat plate, the extent of that non-uniqueness, and to continue the study of large-scale wake closure. These problems remain of both fundamental concern and wide application.

*Also, with nonuniqueness present, the stability of the flows and their unsteady development from an initial state assume extra significance.

2.0 THIN NEAR-WAKES

The thin viscous wake of a flat plate ($y = 0$, $-1<x<0$) placed at zero incidence to a uniform stream is governed by the classical boundary layer equations (with $y = Re^{-1/2}Y$)

$$u = \frac{\partial\psi}{\partial Y} \quad , \quad u\frac{\partial u}{\partial x} - \frac{\partial\psi}{\partial x}\frac{\partial u}{\partial Y} = 0 + \frac{\partial^2 u}{\partial Y^2} \qquad (2.1a,b)$$

holding in $x>0$, with the boundary conditions

$$u(x,\infty) = 1, \; \psi(x,0) = \frac{\partial u}{\partial Y}(x,0) = 0, \psi\,(0^+,Y) = \psi_B(Y). \qquad (2.1c\text{-}f)$$

Here $\psi_B(Y)$ in (2.1f) is the Blasius profile (see also Section 5) at the trailing edge $x = 0^-$, satisfying $\psi_B(0) = \psi_B'(0) = 0$ (no slip),

$\psi_B''(0) = \lambda(= 0.33206...)$, $\psi_B'(\infty) = 1$ and $2\psi_B''' + \psi_B\psi_B'' = 0$, while (2.1c-e) are the free stream and symmetry conditions.

The near-wake, for $x \to 0^+$, then has the Goldstein (1930) double structure, to accommodate the nonlinear change from *no slip to symmetry* at $Y = 0$. Across most of the near-wake the expansion

$$\psi = \psi_B(Y) + x^{1/3} A_1 \, \psi_B'(Y) + \ldots \qquad (2.2a)$$

holds, for $0<Y<\infty$, describing a displacement of the oncoming Blasius flow controlled by the unknown constant A_1. Near $Y = 0$, in contrast, ψ takes the form

$$\psi = x^{2/3}\,\bar{G}(\bar{\xi}) + \ldots, \text{ with } \bar{\xi} \equiv Y/x^{1/3} \;, \qquad (2.2b)$$

because of (2.1e), and from (2.1a,b,d-f) $\bar{G}$ satisfies

$$\bar{G}''' + \frac{2}{3}\bar{G}\,\bar{G}'' - \frac{1}{3}\bar{G}'^2 = 0 \tag{2.3a}$$

$$\bar{G}(o) = \bar{G}''(o) = 0 \tag{2.3b,c}$$

$$\bar{G}(\xi) \sim \frac{1}{2}\lambda\,\bar{\xi}^2 + o\,(\bar{\xi}^2) \text{ as } \bar{\xi} \to \infty. \tag{2.3d}$$

The constraint (2.3d) affords the match with (2.2a) as well as (2.1f), since (2.3a) allows $\bar{G}'\sim\lambda\bar{\xi} + c$ with the constant c unknown, so that $\lambda\, A_1 = c$.

The similarity problem (2.3a-d) has a solution due to Goldstein, with

$$\bar{G}'(o) = 1.611\ \lambda^{2/3}, \text{ giving } c = 0.892\ \lambda^{2/3}\ , \tag{2.4}$$

which then fixes $A_1 = \lambda^{-1}c$.

The main issue of this section, however, is that the solution is <u>not</u> unique. A second solution of the nonlinear problem (2.3a-d) exists which may be equally acceptable on present grounds, i.e., in the context of (2.1a-f). The second solution arises as follows. Taking $\bar{G}'(o)$ to be nonzero, since otherwise $\bar{G}(\bar{\xi}) \equiv o$ from (2.3a-c), we set

$$\bar{G} = |\bar{G}'(o)|^{1/2}\, G(\xi),\ \bar{\xi} = |\bar{G}'(o)|^{-1/2}\,\xi, \tag{2.5}$$

which casts the problem for $G(\xi)$ as an initial value one:

$$G''' + \frac{2}{3}G\,G'' - \frac{1}{3}G'^2 = o, G(o) = G''(o) = o, G'(o) = \pm 1, \tag{2.6a-d}$$

where $\pm$ correspond to $\bar{G}'(o) \gtrless 0$, from (2.5). Also, the condition (2.3d) now becomes an unknown asymptotic property,

$$G \sim \frac{1}{2}a\,(\xi + b)^2,\ \xi \to \infty, \tag{2.7}$$

to be acquired by the solution of (2.6a-d), it is hoped, with $\bar{G}'(o)$, c then being found from $|\bar{G}'(o)| = (\lambda/a)^{2/3}$ and $c = a^{1/3}\, b\, \lambda^{2/3}$. See also Weyl (1941, 1942a,b) Jones & Watson (1963).

Analysis as in Section 3 below, supported by a numerical treatment of (2.6a-d) (Fig. 1), shows that both values ±1 for G'(o) yield the behavior (2.7). If G'(o) = 1 we obtain

$$a = 0.489 \ , \ b = 1.132 \ , \tag{2.8a}$$

giving the original Goldstein results (2.4); but if G'(o) = -1 we find

$$a = 6.082 \ , \ b = -3.048 \ , \tag{2.8b}$$

which give the alternative results

$$\bar{G}'(o) = -0.300\ \lambda^{2/3} \ , \ c = -5.564\ \lambda^{2/3} \tag{2.9}$$

for the solution of (2.3a-d). This second solution has the properties that $G'(\xi_1) = 0$, $G(\xi_2) = 0$ where $\xi_1 \doteq 2.129$, $\xi_2 \doteq 3.376$. Hence reversed flow (u<o) is present for $0 \leq Y \leq \xi_1\, |\bar{G}'(o)|^{-1/2}\, x^{1/3}$, near the wake center-line, as $x \to 0^+$, and the separating streamline $\psi = o$ is at $Y \approx \xi_2\, |\bar{G}'(o)|^{-1/2}\, x^{1/3}$. The center-line velocity is reversed, $u(x,o) \approx -0.300\ \lambda^{2/3}\, x^{1/3}$ for small positive x.

The two solutions are shown in Figs. 1(a-d). Figure 1(a) shows these solutions in terms of the normalized velocity functions of Eqs. (2.6a-d) for the region near the wake centerline. The same results are also shown in Fig. 1(b) in terms of the velocity functions of Eqs. (2.2b), (2.4) and (2.9). Also shown in Fig. 1(b) are the rapid approach to the asymptotic solution, Eq. (2.3d). No more (regular) solutions can exist, incidentally, from the above argument. Neither do solutions exist for any negative oncoming shear λ.

It is intriguing to consider the above nonuniqueness in relation to the wake flow further downstream and to the flow nearer the trailing edge. If the first solution (2.4) of Goldstein holds then the wake motion remains entirely forward for all x > 0 and can be determined, necessarily numerically, by forward marching in x from the starting forms (2.2a,b) with (2.4). Far downstream, as $x \to \infty$, the uniform state is achieved in the form

$$\psi \sim Y + \frac{I_o}{\pi^{1/2}} \int_o^{\eta} e^{-\eta^2/4} d\eta \ , \ \eta \equiv Y/x^{1/2} \ , \qquad (2.10a)$$

consistent with the momentum deficit integral

$$I_o = \int_o^{\infty} u(1-u)dY = \text{constant } (> o) \qquad (2.10b)$$

holding throughout $x > o$, from (2.1a-f), and determined by the Blasius form of (2.1f). The motion is also forward in the triple-deck and other regions induced closer to the trailing edge where x is $O(Re^{-3/8})$ and smaller; see Jobe & Burggraf (1974) and references in Messiter (1979), Stewartson (1981), Smith (1982b). An apparently complete description of the flow past the aligned flat plate at large Re is achieved, and comparisons with experiments and numerical Navier-Stokes solutions are favorable at surprisingly low Reynolds numbers.

If the second solution (2.9) holds, on the other hand, then reversed flow $u < o$ must be present for at least an O(1) distance x in the wake as depicted in Fig. 1(c). The solving of (2.1a-f), again necessarily numerical, is a more difficult task due to the upstream influence in the reversed motion. The integral (2.10b) is preserved still, suggesting (2.10a) far downstream, but here and below the question of regularity at wake closure, where $u(x,0) \to 0$ with $x \to x_0 > 0$, also has to be addressed (indeed, could there be more than one closure point, implying multiple eddies in the wake?). Local analysis indicates that a close analogue of Goldstein's (1948) singularity for instance cannot occur at a wake closure here, especially with negligible pressure gradient, but a regular description is not likely either. For (2.1b) and its Y-derivatives then suggest that only an identically zero profile is allowed at $x = x_0$. With a pressure gradient acting regular or singular wake closure can take place, e.g., analogous to Goldstein's (1948) forms. A prime example of a wake with non-uniform pressure occurs in a branching flow where the flat plate upstream is sited within a channel, parallel to the walls: Bates (1978), Badr et al (1983). Without a pressure gradient however it may well be that one of the singularities in Section 3 below is encountered as $x \to x_0 -$, or x_0 may be infinite. If there is a non-removable singularity at closure that would tend to suggest that an external agency is required, downstream perhaps, to allow (2.9) to occur in the near-wake, and the free flow past a flat plate adheres to the original form (2.4). If not, however, (2.9) introduces nonuniqueness. Removal of any singularity then presumably would invoke an interactive local account near $x = x_0$. The second solution (2.9) must also be able to serve as a downstream ($X \to \infty$) condition for the triple-deck, the most important of the flow structures nearer the trailing edge, in which $x = Re^{-3/8}X$. There the central point is that the problem of adjusting from the Blasius flow (as $X \to -\infty$) to the near-wake (as $X \to +\infty$) is elliptic in character, because of the local pressure-displacement interaction. Hence there appears to be no obvious reason to discount the second solution (2.9) as a downstream asymptote there (Fig. 1(b)): it is consistent with the controlling equations at least. On both scores, then, for smaller or larger x, the second solution (2.9) does not seem inconsistent yet. It would be interesting to conduct computational studies of the two new nonlinear problems posed by (2.9): the triple-deck and the wake, both having reversed flow of course. Supposing that each problem does have a solution then we have apparently a second description of the flow past an aligned flat plate at large Re. If so, the nonuniqueness would be expected to persist at finite Re and Navier-Stokes calculations then would be of interest. Again, nonuniqueness could continue to hold as, say, slight degrees of nonsymmetry are introduced into the flow field. An alternative, however, is that the triple-deck problem admits the solution (2.9) as a downstream state only when the local trailing edge motion is sufficiently disturbed, say by a small nonzero incidence or body thickness as in Melnik & Chow (1975), Ruban & Sychev (1979), Smith & Merkin (1981), Smith (1982c). As the disturbance increases in size, could the second solution (2.9) (with reversed flow) come into play before or after separation occurs at the trailing edge, for instance? Could (2.9) enable the range of triple-deck solutions with separation to be enlarged? In all such cases it is necessary, for the above nonuniqueness, that the wake problem (2.1a-f) holding in $x > 0$ has a solution with the starting form (2.9) and that should be regarded as a primary concern. Section 5 below is also relevant here, for the flow ahead of the trailing edge.

Another feature concerns the "very-near-" wake within the triple-deck itself, as $X \to 0^+$. There a local similarity form holds again but governed by the Rott-Hakkinen (1965) problem

$$\bar{g}''' + \frac{2}{3}\,\bar{g}\,\bar{g}'' - \frac{1}{3}\,\bar{g}'^2 = \frac{2}{3}\,p_1 \qquad (2.11a)$$

$$\bar{g}(o) = \bar{g}''(o) = 0, \quad \bar{g}' - \lambda_T\,\bar{\xi} \to 0 \text{ as } \bar{\xi} \to \infty \qquad (2.11b\text{-}d)$$

in symmetric flow, with the constant $\lambda_T > 0$. Here the local wake has to be displacement-free, from (2.11d), and the pressure coefficient p_1 is unknown, in contrast with (2.3a-d). The solution of (2.11a-d) is found to be unique (with $\bar{g}'(o)$, p_1 taking Hakkinen & Rott's (1965) values), however, according to a normalization based on $\bar{g}'(o)$ as before and integration for p_1 varying. Figure 1(d) gives the displacement function

$$\Delta \equiv \lim_{\bar{\xi} \to \infty} \left[\bar{\xi} - \frac{\bar{g}'}{\bar{g}''(\infty)} \right]$$

as it varies with p_1 for $\bar{g}'(o) = \pm 1$ as well as the asymptotic solution for $p_1 \to \infty$, corresponding in effect to $g'(o) = 0$. Note that only one solution gives $\Delta = 0$ as required by Eq. (2.11d) and that is Rott and Hakkinen's (1965). Also, no solution producing $\bar{g}''(\infty)$ finite exists to the left of the dotted curves, which both give $|\Delta| \to \infty$ at finite negative values of p_1. Thus nonuniqueness is not compounded by the very-near-wake, in symmetric flow at least.

3.0 THICKER NEAR-WAKES

Thicker bodies or larger angles of incidence, i.e. non-streamlined motions, tend to produce large-scale separation and eddies eventually (e.g., Cheng & Smith 1982) and there the properties of wake closure downstream have to be addressed. We consider symmetric flow, for which the governing equations near the center-line, beyond the major wake closure, are again the classical boundary layer ones. If in addition the main eddies upstream are predominantly slender, as in the extended Kirchhoff approach for example, then the induced pressure is small, being proportional to the eddy slope, and so the pressure-free equations (2.1a,b) apply. Also, from an origin shift, $x > 0$ again, and (2.1c-e) hold. Only the starting profile $u(0^+,Y)$ at $x = 0^+$ remains to be specified. If it has forward velocities only, smooth, nonzero and of order unity, then (2.1a-e) can be integrated forward in x, giving a fairly standard wake phenomenon. If, on the other hand, the wake closure is not quite complete yet, then the starting profile contains some reversed velocities and the wake is non-standard. We consider first whether (a) such reversed flow can be present near the center-line and (b) it can be initially small. Here (a) is meant to correspond to the downstream condition for a smaller scaled, mainly inviscid, process of near-closure in which for instance an incoming detached, forward moving, shear layer is joined to a reversed center-line motion. Condition (b) then allows that reversed motion to remain weak upstream and hence consistent with a slowish (e.g., extended Kirchhoff) eddy there. So the idea is that complete closure is achieved through the action of viscosity downstream when x is finite, whereas a complete closure for small x would be inviscid and force a strong reversed center-line motion inconsistent with the slowish eddy ahead. Further details here are given by Smith (1983). Non-standard wakes can arise also if (c) the starting profile gives forward but small flow anywhere.

The above and other contexts indicate a local viscous similarity solution holding in the near-wake as $x \to 0^+$ for some $Y \geq 0$ of the form $\psi = x^{K/(K+1)}G(\xi)$ with $\xi = Y/x^{1/(K+1)}$. Here K is a constant with $|K| > 1$ so that $u = x^{(K-1)/(K+1)} G'(\xi)$ is small: see later for $|K| < 1$. From (2.1a,b,d,e) the function G satisfies

$$(K+1)G''' + K\,G\,G'' - (K-1)G'^2 = 0 \qquad (3.1a)$$

$$G(o) = G''(o) = 0. \qquad (3.1b,c)$$

Also, from the reasoning in Section 2 (the special case K = 2) and a normalization, two extra conditions are allowable,

$$G'(o) = \pm 1. \tag{3.1d}$$

Hence the possibility of nonuniqueness arises for any starting profile of the kind (a)-(b), or (c).

Computational solutions of (3.1a-d) are presented in Figs. 2(a-e). They fall into line with certain analytical properties below and with the view that as ξ increases from zero only two ultimate trends can be observed. If $K < -1$ a singularity occurs at a finite position $\xi = \xi_o > 0$, with

$$G \sim C_1(\xi_o - \xi)^{-|K|} + O(\xi_o - \xi)^{-1}, \; \xi \to \xi_o -. \tag{3.2a}$$

In contrast, if $K > 1$ the solution continues for all ξ, giving (apart from an origin shift)

$$G \sim C_2 \, \xi^K + O(\xi^{-1}), \; \xi \to \infty. \tag{3.2b}$$

Here C_1, C_2 are unknown constants but both are positive, for an integral of (3.1a-c) gives

$$G'' = \left(\frac{K-1}{K+1}\right) \exp\left\{\frac{-K}{K+1} \int_o^\xi G(\xi_1)d\xi_1\right\} \int_o^\xi G'^2(\xi_2)\exp\left\{\frac{K}{K+1}\int_o^{\xi_2} G(\xi_1)d\xi_1\right\}d\xi_2 \tag{3.3a}$$

which shows that the vorticity is positive,

$$G'' > 0 \;, \text{ for } \xi > 0, \tag{3.3b}$$

since $|K| > 1$. Therefore the velocity $G'(\xi)$ increases with ξ, from the centerline values in (3.1d), after which the ultimate trends (3.2a,b) both produce forward flow, $G' \to +\infty$. It is interesting, as seen in Fig. 2(a), that these trends are inevitable whether $G'(o)$ is positive or negative, for that yields nonuniqueness as the trends (3.2a,b) represent features of the given starting profile.

Both (3.2a,b) form self consistent asymptotes for (3.1a), but if $K < -1$ the asymptote $G \sim C_2 \xi^K$ as $\xi \to \infty$ is unobtainable as it requires a change in sign of the vorticity, contradicting (3.3b). A contradiction also arises then from substitution into (3.3a). Similarly the asymptote $G \sim C_1(\xi_0-\xi)^K$ as $\xi \to \xi_0-$ is unobtainable for $K > 1$. Two other asymptotes acceptable for (3.1a) alone are $G \to$ constant $G(\infty)$ as $\xi \to \infty$ and $G \sim 6(\xi-\xi_0)^{-1}$ either as $\xi \to \xi_0-$ or as $\xi \to \infty$. In the first however $G(\infty)$ must be positive, to avoid exponential growth as $\xi \to \infty$, and that contradicts (3.3b). In the second, the vorticity is negative if ξ_0 is finite, while it must change sign if $\xi \to \infty$, so that again neither limit is acceptable for the whole problem (3.1a-d). Substitution into (3.3a) again confirms this. This second asymptote corresponds to an exact solution $G = 6(\xi-\xi_0)^{-1}$ of the governing equation (3.1a), incidentally: see Smith (1983), Appendices A, B, and Section 4(ii) below.

For $K > 1$, then, (3.2b) is achieved. Also the viscous region here ($\xi = O(1)$) is small and contracting as $x \to 0^+$. Outside, for Y of $O(1)$, the starting profile has the form $\psi(0^+,Y) \sim C_2 Y^K$ as $Y \to 0^+$, to match (3.2b), and as shown in Fig. 2(b,c) overall the flow structure and nonuniqueness are similar to those of Section 2. Some limiting analysis is possible: Appendix C.

The properties implied for $K < -1$, where the viscous region is large and expanding as $x \to 0^+$, are quite distinct. There the singularity (3.2a) occurs (see also Appendices). So near $\xi = \xi_0$ a thinner region arises, far from the center-line, wherein $Y - \xi_0 x^{-1/(|K|-1)} \equiv \bar{Y}$ is of $O(1)$ and the starting profile $\psi(0^+,\bar{Y})$ is arbitrary except that $\psi(0^+,\bar{Y}) \sim C_1|\bar{Y}|^{-|K|}$ as $\bar{Y} \to -\infty$, to match (3.2a), and a matching condition is required as $\bar{Y} \to \infty$. The latter depends on how multi-structured the near-wake is: see Figs. 2(d) and (e). The simplest condition is $\partial\psi(0^+, \infty)/\partial\bar{Y} = 1$, which satisfies (2.1c) directly and gives, for the wake displacement $\delta(x)$ defined by $\psi \sim Y - \delta(x)$ as $Y \to \infty$, the growth

$$\delta(x) \sim \xi_0\, x^{-1/(|K|-1)} (\to \infty),\ x \to 0^+. \qquad (3.4)$$

This growth upstream seems sensible physically with regard to both the wake closure anticipated further downstream and to the detached flow properties holding on shorter scales. Also the algebraic decay ($\bar{Y} \to -\infty$) of the $O(1)$ thick region of faster flow above is in line with the algebraic decay attainable in detached shear layers on a shorter length scale: see Smith (1983). Again, other realistic conditions are possible for $\bar{Y} \to \infty$. A decay

$$\psi(0^+, \bar{Y}) \sim C_3 + C_4 \bar{Y}^{-\bar{\nu}}, \quad \bar{Y} \to +\infty, \tag{3.5a}$$

like that for $\bar{Y} \to -\infty$, can be tolerated with $\bar{\nu}(>0)$, C_4, C_3 (a mass flux) constants. That induces a further, thicker, viscous region outside of thickness $\bar{Y} = O(x^{-1/(\bar{\nu}-1)})$ and $\psi - C_3 = O(x^{\bar{\nu}/(\bar{\nu}-1)})$ is small, provided $\bar{\nu} > 1$. In effect (3.1a) holds again with K replaced by $-\bar{\nu}$ and (3.1b-d) are replaced by

$$G \sim C_4(\xi - \xi_o)^{-\bar{\nu}} \text{ as } \xi \to \xi_o +, \tag{3.5b}$$

from (3.5a). As ξ increases further, then slower motion results,

$$G \sim 6\xi^{-1} \text{ or constant, as } \xi \to \infty, \tag{3.5c}$$

consistent with (3.1a); or a singularity like (3.2a) is repeated, and so on. See also Appendices. An analogue of (3.3a) can also be applied. Conversely, if $\bar{\nu} < 1$ then no such outer region arises and a direct join to a further incoming shear layer more removed from the center-line can be made. Thus the solutions with $K < -1$ seems perhaps more relevant to a shorter scale detached flow than those with $K > 1$, if significant reversed flow is to proceed into the viscous wake.

Other points are as follows. First, simple positive power solutions for $G(\xi)$ in (3.1a) exist only for K = 1,2; see also (4.7a,b) below. For K = 1, $G(\xi) = \pm$ constant ξ satisfies (3.1a-c) but gives $u = \pm$ constant, producing a center-line flow not coming to rest, in contrast with a slowish eddy motion in the shorter scale features. For K = 2, $G(\xi) = -d_1\xi^2$ satisfies (3.1a,b) but not (3.1c). Instead a no slip condition is satisfied, a matter taken up in Section 5. Second, a range of solutions of (3.1a), but with no slip $G = G' = 0$ at $\xi = 0$, exists: again see Section 5 below. Third, if a significant pressure gradient, classical or interactive, is acting then more possibilities open up: e.g., there are Kennedy's (1964) classical solutions; Gajjar & Smith (1983) demonstrate branching in liquid layer and hypersonic flows; and ultimate forward flow as in (3.2a,b) can be avoided in special cases, e.g., Goldstein's (1948). The relevance to external wake closure behind a slender eddy is then apparently lost, however. Fourth, alternative accounts of the near-wake are obtainable of course with the center-line motion reversed but strong, i.e. $u(0^+, Y)$ finite and arbitrary for finite Y, with $u(0^+, 0)$ negative say. The eddy flow ahead must then contend with a strong reversed flow, possibly by means of algebraic decay again and non-entrainment (Smith 1983). Fifth, $\delta(\infty) = I_o$ from (2.10b). Sixth, an integral form

$$\frac{\partial u}{\partial Y} = \exp\left\{\int_o^Y V dY\right\} \int_o^Y \frac{u\partial u}{\partial x} \exp\left\{-\int_o^Y V dY\right\} dY \qquad (3.6)$$

of (2.1a-e) with $V = -\partial\psi/\partial x$ confirms the property (3.3b) that $\partial u/\partial Y > 0$ for small $Y > 0$ if the center-line flow is coming to rest, $U(x,o) \to 0$, as $x \to 0^+$. Hence the velocity u has a local minimum at the center-line. Seventh, as in Section 2, numerical studies of the complete wakes stemming from the above starting forms with or without reversed flow would be of interest, as would continued studies of the shorter scale detached motions. Eighth, the energy integral of (2.1a-e) requires

$$\frac{\partial}{\partial x}\{\delta_E\} \equiv \frac{\partial}{\partial x}\left\{\int_o^\infty u(1-u^2)dY\right\} = 2\int_o^\infty \left(\frac{\partial u}{\partial Y}\right)^2 dY. \qquad (3.7)$$

So due to dissipation the energy thickness $\delta_E(x)$ increases monotonically with x after usually being positive at the start of the wake. Finally, the similarity solutions above, and below, apply also to flow at $x = 0^-$, coming to or from rest if $|K| > 1$. There $|x|$ replaces x, so that the nonlinear terms in (3.1a) change sign. Then letting $G \to -G$ reproduces (3.1a-d), hence (3.2a,b) and hence the conclusion that sufficiently far from the center-line the motion must be reversed, whether the center-line velocity is positive or negative. This conclusion can be avoided in special cases, e.g., Goldstein's (1948), only if a non-zero pressure gradient is present.

4.0 ADDITIONAL NEAR-WAKES AND NONUNIQUENESS

Two other kinds of near-wakes of interest, particularly concerning wake closure and the possibility of nonuniqueness, arise in: (i) the wake of a non-streamlined body placed in a wide-spread cascade or wind-tunnel; (ii) the slowish flow between a detached shear layer and a wake center-line. Their properties are summarized below.

(i) The wide-spread cascade. Consider the boundary layer equations

$$u = \frac{\partial\psi}{\partial Y}, \quad u\frac{\partial u}{\partial x} - \frac{\partial\psi}{\partial x}\frac{\partial u}{\partial Y} = -\frac{dp}{dx} + \frac{\partial^2 u}{\partial Y^2} \qquad (4.1a)$$

governing the O(Re) long wake of a body forming part of an infinite cascade (Fig. 3), such that the symmetry, mass flux and center-line conditions

$$\psi = \frac{\partial u}{\partial Y} = 0 \quad \text{at } Y = 0; \quad \psi = 1, \frac{\partial u}{\partial Y} = 0 \quad \text{at } Y = 1 \qquad (4.1b,c)$$

hold. Here ψ, x, Y have been normalized with respect to the O(1) upstream mass flux, to Re, and to the O(1) cascade spread, and unlike in most of the other wakes studies in this paper the pressure p is unknown here. For a relatively wide-spread cascade or thin body the starting condition is simply the original uniform stream

$$u = 1, \ \psi = Y \quad \text{at } x = 0^+, \text{ for } 0 < Y < 1 \qquad (4.1d)$$

c.f. a finite-spread cascade would have $u = (1-H)^{-1}$ for $H < Y < 1$, $u = 0$ for $0 < Y < H$, at $x = 0^+$, with H between 0 and 1.

One solution of the wake problem (4.1a-d) is the undisturbed original flow

$$u = 1, \ \psi = Y, \ p = 0, \quad \text{for all } x > 0, Y, \qquad (4.2)$$

of course, meaning that wake closure and recovery have taken place already before the final stage of (4.1a-d) is entered. There is another possibility, however, suggested by an alternative near-wake form of (4.1a-d), as follows.

Suppose the flow near the center-line is small, locally uniform and reversed near $x = 0$, given by $-u = O(x^n)$, $-\psi = O(Yx^n)$, n unknown. To adjust the incoming stream (4.1d) to this, for small x, Y, a thin shear layer is necessary, of the Chapman form and therefore requiring an entrainment $-\psi = O(x^{1/2})$. Hence the reversed motion below the shear layer can supply the entrainment if the shear layer is positioned at $Y = O(x^{1/2-n})$, requiring $n < 1/2$. The majority of the near-wake above the shear layer then must accommodate the $O(x^{1/2-n})$ displacement but preserve mass flux, from (4.1c), and so, effectively from one-dimensional channel flow theory, a pressure drop $O(x^{1/2-n})$ is induced. This agrees with the $O(x^{2n})$ pressure fall necessary to drive the reversed center-line motion if $n = 1/6$. Formally therefore we have the expressions for small x,

$$\psi \sim -\gamma_1 Y\, x^{1/6}\ , \quad u \sim -\gamma_1\, x^{1/6}\ , \quad \text{for } 0 < Y < \gamma_2\, x^{1/3} \qquad (4.3a)$$

below the shear layer, where γ_1, γ_2 are unknown positive constants; the corresponding pressure

$$p \sim -1/2\ \gamma_1^2\, x^{1/3}\ ; \qquad (4.3b)$$

the Chapman shear layer

$$\psi = x^{1/2} f_c(\eta) + \ldots, \quad u = f_c'(\eta) + \ldots, \quad \text{for } Y = \gamma_2 x^{1/3} + x^{1/2}\eta\ , \qquad (4.3c)$$

where $2f_c''' + f_c f_c'' = 0$, $f_c'(\infty) = 1$, $f_c'(-\infty) = 0$, giving $f_c(-\infty) = -\kappa$

with $\kappa = 1.24$; and finally

$$\psi = Y + x^{1/3}\,\frac{\gamma_1^2}{2}\,(Y-1) + \ldots, \quad u = 1 + x^{1/3}\,\frac{\gamma_1^2}{2} + \ldots \qquad (4.3d)$$

above the shear layer, to conserve mass. All of (4.3a-d) satisfy the governing equations (4.1a-d) for small x. Matching (4.3a,c) and then (4.3c,d) gives $\gamma_1\gamma_2 = \kappa$, $\gamma_2 = \gamma_1^2/2$ in turn, yielding

$$\gamma_1 = (2\kappa)^{1/3} \ , \ \gamma_2 = (\kappa^2/2)^{1/3} \ . \qquad (4.3e)$$

The local expansion then appears to be self-consistent and further terms including exponentials can be generated in principle (Smith 1983). The starting form (4.3a-e) represents a second (eigen)solution, branching from the incoming uniform stream and an alternative to the first solution (4.2). Note that the momentum integral equation

$$p(x) + \int_0^1 u^2 dY = 1 \qquad (4.4)$$

obtained from (4.1a-d) is satisfied by (4.3a-e). Indeed, there appears to be no theoretical objection to the second account (4.3a-e) at first sight, nor any obvious reason why regular wake closure and subsequent flow recovery back to the uniform state (4.2) should not take place further downstream, for x > 0, yielding self-consistency as seen in Fig. 3. Here the branching solution for cascade flow is depicted showing the small-x layers $O(x^{1/3})$, $O(x^{1/2})$ flow reversal and the possible eddy closure downstream. Accordingly it would be very interesting to see the results of a numerical study of the branching solution, with reversed flow, stemming from (4.3a-e). The alternative to the structure depicted in Fig. 3 is simply the original uniform stream, u=1, for all x.

(ii) Flow between a detached shear layer and a wake center-line. If such flow is sufficiently slow then it too is controlled by the viscous boundary layer equations but, unlike most other motions considered in this paper, it can also be affected by the small pressure gradient due to a predominantly slender eddy. For the flow to be so slow the entrainment, into the shear layer which shelters the slow flow from faster motion outside, must be obliterated since otherwise the slow moving fluid is effectively inviscid. The mechanism for non-entrainment involves algebraic decay in the shear layer (Smith 1983) and hence an algebraic growth within the slower flow, wherein we have in scaled terms

$$u = \frac{\partial \psi}{\partial Y} \ , \ u\frac{\partial u}{\partial x} - \frac{\partial \psi}{\partial x}\frac{\partial u}{\partial Y} = -p'(x) + \frac{\partial^2 u}{\partial Y^2} \qquad (4.5a,b)$$

$$\psi = \frac{\partial u}{\partial Y} = 0 \quad \text{at } Y = 0 \ , \qquad (4.5c,d)$$

if there is no center-line jet. Here $S(x)$ gives the shear layer position, and for $0 < Y < S(x)$. The algebraic growth requires

$$\psi \sim \frac{6(x-x_1)}{(Y-S(x))}, \quad u \sim -\frac{6(x-x_1)}{(Y-S(x))^2} \quad \text{as } Y \to S(x)-, \qquad (4.5e)$$

consistent with (4.5a,b) and with the shear layer (Smith 1983). The constant x_1 is unknown.

We seek local solutions of (4.5a-e) for eddy closure, as $x \to o \pm$ say, supposing that the slowish flow here can speed up then, i.e., $|u| \to \infty$. Thus $\psi \sim x^{K/(K+1)}G(\xi)$, $Y = \xi x^{1/(K+1)}$ again, $u \sim x^{(K-1)/(K+1)} G'(\xi)$ but now $|K| < 1$, for $x \to 0^+$. So G satisfies (3.1a-d) if the pressure gradient becomes negligible then. Numerical solutions for $|K| < 1$ are shown in Fig. 4, while (3.3a) now requires

$$G'' < 0 \quad \text{for } \xi > 0. \qquad (4.6)$$

As a result the ultimate trends encountered are not covered by (3.2a,b). Instead

$$G \sim 6(\xi - \xi_o)^{-1} \text{ as } \xi \to \xi_o - , \qquad (4.7a)$$

or

$$G \to G(\infty) > 0 \quad \text{as } \xi \to \infty , \qquad (4.7b)$$

or

$$G \sim C_2\, \xi^K + O(\xi^{-1}) \quad \text{as } \xi \to \infty . \qquad (4.7c)$$

Here (4.7a) is attained, for some $\xi_o > 0$, whenever $G'(o) = -1$; (4.7a) also holds if $G'(o) = 1$ when $-1 < K < K_1$, but for $K_1 < K < K_2$, $G'(o) = 1$, (4.7b) is attained, while (4.7c) is found for $G'(o) = 1$, $K_2 < K < 1$. Here $K_1, K_2 \doteqdot 0.5$ and are almost certainly equal, a result suggested by analysis in Appendix B. Further analytical support can be given, e.g., Appendices A-C. Note that the limits $G \sim 6\xi^{-1}$, $\xi \to \infty$, and (3.2a) are not attainable.

The algebraic growth anticipated, (4.5e), (4.7a), can be achieved for all $|K| < 1$, then, giving $S(x) \sim \xi_o x^{1/(K+1)}$ tending to zero as at a closure point at $x = 0^+$. Closure at $x = 0^-$ has $|x|$ replacing x above, G changes sign in (3.1a) and again (4.7a) can be achieved.

Alternative local solutions are possible with a significant pressure gradient (Appendix C), with ψ, u regular, or with x_1 zero (or $x \to x_1\pm$). Further details concerning (i), (ii) above are given by Smith (1983).

5.0 NONUNIQUENESS OF THE BLASIUS SOLUTION, AND SURFACE REATTACHMENT

This second part of the report comes as a by-product of Section 3, stemming from the exact solution $G = -d_1\xi^2$ of (3.1a) with reversed flow, for $K = 2$, which gives

$$u = -2d_1Y, \quad \psi = -d_1Y^2 . \qquad (5.1a)$$

The constant d_1 is assumed positive here. The local solution (5.1a) for small x satisfies the following conditions. (a) No slip holds at $Y = 0$. (b) The Chapman entrainment $\psi = -\kappa x^{1/2}$ is supplied at the position

$$Y = (\kappa/d_1)^{1/2} x^{1/4} + \ldots, \qquad (5.1b)$$

around which the thinner $O(x^{1/2})$ Chapman form (see Section 4(i)) applies. (c) The reversed velocities u induced between $Y = 0$ (the solid surface) and the shear layer at (5.1b) are small, of order $x^{1/4}$ as $x \to 0$, in keeping with a slowish eddy flow upstream. (d) The pressure response required remains negligible.

The small-x form (5.1a,b) as depicted in Fig. 5 satisfies the boundary layer equations (2.1a-c) with (2.1d-f) replaced by

$$\psi = u = 0 \quad \text{at } Y = 0, \qquad (5.2a)$$

$$u = 1 \quad \text{at } x = 0 \quad \text{for } Y > 0 . \qquad (5.2b)$$

Further terms in the small-x expansion stemming from (5.1a,b) (see also Appendix D) can be generated at will, integral properties can be derived, and once again it would be interesting to follow the subsequent downstream development, of (2.1a-c) with (5.2a,b), which necessarily is a numerical and challenging task since reversed flow is present.

The starting form (5.1a,b) and its induced displacement

$$\delta(x) \sim (\kappa/d_1)^{1/2}\, x^{1/4} \;, \; x \to 0^+ \qquad (5.3)$$

describe, first, what is in effect (from 2.1a-c), (5.2a,b)) the flow over an aligned flat plate* in a uniform stream, and so provide an alternative to the Blasius solution $\psi \sim x^{1/2}\, \psi_B'(\eta)$, $\eta = Y/x^{1/2}$ for which

$$\delta(x) \propto x^{1/2} \;, \; x \to 0^+ \;. \qquad (5.4)$$

Hence the suggestion is that even this most basic flow problem may exhibit non-uniqueness. If so, then for some d_1 (5.1a) - (5.3) must constitute also a multi-structured downstream asymptote for a smaller $O(Re^{-2/5})$ interactive zone close to the leading edge, again an alternative to the Blasius asymptote for the more orthodox Navier-Stokes $O(Re^{-1})$ zone. Whether (5.1a) - (5.3) can apply for the whole flat plate, or whether an external agency is required further downstream say, remains unknown as yet, and the question of a removable or nonremovable singularity downstream also arises, despite a Goldstein (1948)-like one being ruled out by the comments following (5.5a-c) below and as in Section 2. However, the above suggests a novel form of leading-edge separation and, coupled with the suggestions of Section 2, there is now the chance of 2,4 or even more, simple solutions being found for the high Reynolds number steady flow past a finite flat plate at zero incidence.

The second main application of (5.1a) - (5.3) implied is to the start of viscous pressure-free reattachment onto a solid surface, downstream of the gross separation and eddy(ies) produced by an obstacle on the surface or by a ramp for instance. The starting profile can be more general than (5.2b), incidentally, provided it is positive at $Y = 0^+$. As in the near-wakes before, a pressure-displacement interaction does not have to be present, neither is a strong center-line flow produced (from (5.1a,b)). The induced pressure affects the flow significantly only within a shorter length scale and then only near the surface at first where the reversed eddy motion is slowest, from (5.1a,b). There (Smith 1983) the possibility of secondary separation occurs, depending on the interaction present.

*When compared with Section 2 there is an origin shift $x \to x - 1$.

The simple starting form (5.1a) is in fact a special case of the solutions of (3.1a) but with no slip holding,

$$G(o) = G'(o) = 0 \qquad (5.5a,b)$$

$$G''(o) = \pm 1. \qquad (5.5c)$$

Comments similar to those near (3.1a-d) apply here, with $\psi = x^{K/(K+1)}G(\xi)$ and $Y = x^{1/(K+1)}\xi$. Solutions are given in Fig. 6 for Eq. (3.1a) with the constraints of (5.5a-c), corresponding to eddy closure on a solid surface, as K varies with $G''(o) = \pm 1$. For our main interest, i.e. fluid coming to rest as $x \to 0^+$, $|K| > 1$ again and then (3.2a,b) are repeated when $K < -1, K > 1$ in turn, with but one range of exceptions: K = 2 for which (5.1a) holds ($G = \pm 1/2\ \xi^2$), allowing forward or reversed flow, and the cases $1 < K < 2$ with $G''(o) = -1$ which give (4.7a) and ultimate reversed motion. All other cases by contrast give forward flow ultimately, from (3.2a,b), including K = 3; c.f., a nonzero pressure gradient would allow Goldstein's (1930), (1948) separation/reattachment singularity to occur for K = 3 with reversed flow if $x \to 0^+$ or forward flow if $x \to 0^-$ (see Fig. 6). So a generalization of the Chapman match (5.1b) seems unlikely, although further properties akin to those of Section 3 are found. See also Appendices A-C.

As in Section 4(ii), the solutions with $|K| \leq 1$ have a different application concerning eddy closure and different features. Solutions are shown in Fig. 6, comments similar to those of Section 4(ii) apply and some properties can be obtained from Appendices A-C.

As in Section 3, the corresponding similarity properties of flow over a solid surface as $x \to 0-$ can be deduced by change of sign of G, yielding ultimate reversed flow if $|K| > 1$ except for the range $1 < K < 2$ described above.

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APPENDIX A

SPECIAL CASES OF NEAR-WAKES

Helpful results for the near-wake similarity form (3.1a-d) are obtained in certain special cases.

First we consider $K \to 1 \pm$ with $G'(o) = +1$. Then $G = q\xi + \epsilon G_1 + O(\epsilon^2)$ if $K = 1 + \epsilon$. Here $q = 1$, $|\epsilon|$ is small and G_1 satisfies a linear equation with $G_1(o) = G_1'(o) = G_1''(o) = 0$ from (3.1a-d). The solution has

$$G_1'' = \frac{1}{2} q^2 e^{-q\xi^2/4} \int_o^{\xi} e^{q\xi_1{}^2/4} \, d\xi_1 \, . \qquad \text{(A1)}$$

so that $G_1'' \sim \xi^{-1}$ as $\xi \to \infty$. Hence $G_1 \sim \xi \ln \xi$, giving $G \sim \xi^{1+\epsilon}$, $\xi \to \infty$, in line with (3.2b) for $K = 1^+$ and (4.7c) for $K = 1^-$.

Second, suppose $K \to 1^{\pm}$ but with $G'(o) = -1$, a less simple case. For ξ of order unity G expands as above but with $q = -1$, so that (A1) gives exponential growth,

$$G_1'' \sim \frac{1}{2} \pi^{1/2} e^{\xi^2/4} \text{ as } \xi \to \infty \, . \qquad \text{(A2)}$$

A nonlinear phase enters for large ξ, therefore, with $\xi = a + a^{-1}\tilde{\xi}$ where the equation $|\epsilon| \exp (a^2/4) = a^3$ determines positive $a \gg 1$, $\tilde{\xi}$ is $O(1)$ and $G = a\,\tilde{G}$ with $\tilde{G}$ of $O(1)$. Hence $\tilde{G}(\tilde{\xi})$ satisfies the Chapman/Blasius equation, from (3.1a),

$$2\tilde{G}''' + \tilde{G}\,\tilde{G}'' = 0, \qquad \text{(A3)}$$

with $$\tilde{G} \sim -1 \pm 2\pi^{1/2} e^{\tilde{\xi}/2} \text{ as } \tilde{\xi} \to -\infty, \qquad \text{(A4)}$$

for $\epsilon \gtrless 0$, to match (A2). Here (A3), (A4) pose a shooting problem for $\tilde{G}$. If $\epsilon > 0$ the perturbation in (A4) is positive and (A3), (A4) yield in effect the Chapman solution for which $\tilde{G}'(-\infty) = 0$ and

$$\tilde{G}' \to \text{constant } \tilde{G}'(\infty) > 0 \text{ as } \tilde{\xi} \to \infty \, . \tag{A5}$$

Hence $\tilde{G} \sim a^2\tilde{G}'(\infty)\ (\xi - a)$ for $\xi > a$ to leading order, in keeping with (3.2b) again. If $\varepsilon < 0$ the perturbation in (A4) is negative however and then, since $\tilde{G}'' < 0$, (A5) is unobtainable. Instead $\tilde{G}$ acquires the counterpart of (4.7a) at a finite value of $\tilde{\xi}$.

Third, if K = 0 then (3.1a-d) reduces to solving

$$H'' + H^2 = 0 \tag{A6}$$

$$H(o) = \pm 1, \; H'(o) = 0 \tag{A7,A8}$$

for $H \equiv G'$. Here (A6) gives $3H'^2 + 2H^3 = \pm 2$ on integration, so that since $H' < 0$

$$-\left(\frac{2}{3}\right)^{1/2} \xi = \int_{\pm 1}^{H} \frac{d\,h}{(\pm 1 - h^3)^{1/2}} \, . \tag{A9}$$

Therefore (4.7a) is obtained ($H \to -\infty$) in both cases, with

$$\left(\frac{2}{3}\right)^{1/2} \xi_o = \int_{-\infty}^{\pm 1} \frac{d\,h}{(\pm 1 - h^3)^{1/2}} \tag{A10}$$

fixing the values of ξ_o in turn. This agrees with Section 4(ii) and Fig. 4.

Fourth, suppose $K = -1 + \varepsilon$ with $|\varepsilon|$ small. Then the viscous region thins, $\hat{\xi} = |\varepsilon|^{-1/2}\xi$ being the characteristic scale, and $G \sim |\varepsilon|^{1/2}\hat{G}$. So we obtain from (3.1a-d)

$$\pm\hat{G}''' - \hat{G}\,\hat{G}'' + 2\,G'^2 = 0 \tag{A11}$$

$$\hat{G}(o) = \hat{G}''(o) = o, \; \hat{G}'(o) = \pm 1, \tag{A12}$$

leading to the singularity $\hat{G} \propto |\hat{\xi}_0 - \hat{\xi}|^{-1} \ln |\hat{\xi}_0 - \hat{\xi}|$ as $\hat{\xi} \to \hat{\xi}_0-$ for some positive ξ_0 or as $\xi \to \infty$, in every case. The thinning and the singular response here reflect the behavior of the solutions of Sections 3 and 4(ii) for K near -1. In addition the case (A11), (A12) corresponds to an $\exp(m_1 x)$ ($|x| \to \infty$) or $\exp(-m_2/|x|^{m_3})$ ($|x| \to 0$) form of x-dependence for ψ, Y with m_1, m_2, m_3 being constants.

Fifth, for large $|K|$ the ultimate form is initially exponential, in line with the asymptotic injection problem (c.f. Jones & Watson 1963), before algebraic behavior as in (3.2a,b) takes over.

Perturbations about the two solutions of Section 2 for $K \to 2 \pm$ also confirm the trends of (3.2a,b) holding them.

If, on the other hand, a pressure gradient is present then (3.1a-d) becomes more like the Falkner-Skan problems. Some solutions with bounded $G'(\infty)$ exist as in Jones & Watson (1963), Kennedy (1964), while Gajjar & Smith (1983) note solutions with algebraic growth. Special cases can also be analyzed as in (A1) - (A12).

APPENDIX B

OTHER ASPECTS OF THE SIMILARITY PROBLEMS

There is some value in regarding the similarity problems (3.1a-d) and others in a phase plane, with G versus G'. For the possible forms of ultimate behavior of G from the differential equation (3.1a) itself can be listed as (3.2a) ($K < -1$ only), (3.2b) ($K > -1$ only), (4.7a), (4.7b) and

$$G \sim 6\ \xi^{-1} \text{ as } \xi \to \infty\ , \tag{B1}$$

and it is useful to decide which forms can be achieved from the boundary conditions (3.1b-d).

If $|K| > 1$ as in Section 3 then (3.3b), coupled with $G' \equiv dG/d\xi$, implies that as ξ increases the solution of (3.1a-d) must move to the right in the G'- G plane, and upward in the right half ($G' > 0$) and downward in the left ($G' < 0$) as depicted in Fig. B1. Here the typical phase plane G'- G for equations (3.1a-d) is shown when $|K| > 1$. The asymptotes are designated by solid arrows 1a, 1b, 2, 3, and 4 while the open arrows denote the general direction of the solutions in view of equation (3.3b). Hence for $K > 1$ and $G'(o) = \pm 1$ the forms (3.2a), (4.7a,b), (B1) are unattainable and instead (3.2b) is inevitable. This includes the case $K = 2$ of Section 2. For $K < -1$ however only (3.2a) can be reached, given the movement above.

If $|K| < 1$, on the other hand, as in Section 4(ii), (4.6) applies instead and the movement for increasing ξ must be to the left, although upward/downward as before. So all of (4.7a,b), (3.2b) are attainable in principle for $G'(o) = 1$, but only (4.7a) is accessible for $G'(o) = -1$, as in Fig. B2. Here the phase plane G'- G is depicted for equations (3.1a-d) when $|K| < 1$ and $G'(o) = \pm 1$. Again the open arrows give the general directions for increasing ξ and the solid arrows denote the limits. Note that infinitely many solutions approach the limits 1a and 3. For $G'(o) = 1$ limiting analysis (Appendix A) shows that (4.7a) holds for $K = 0$ but (4.7c), i.e., (3.2b), for $K = 1-$. Between them there is a separatrix, believed to be for the value of K in (B4) below, at which (4.7b) holds since above this value (4.7a) is inaccessible. Perturbations about (B4) below behave similarly to (A3) - (A5) above and support the response near the separatrix.

A similar view in a phase plane can be adapted to verify the properties achieved in Section 5 and Fig. 6, for the no slip conditions.

Another aspect concerns the limiting forms (B1), (4.7a), the exact solution $G = 6\xi^{-1}$ of (3.1a), and their accessibility. With the origin shift in ξ absorbed if necessary, we set $G - 6\xi^{-1}$ small and then (3.1a) shows that generally

$$G \sim 6\xi^{-1} + a|\xi|^q + o(|\xi|^q) , \qquad \text{(B2)}$$

for $|\xi| \to 0$ or $|\xi| \to \infty$, where $(K+1)q^3 + 3(K-1)q^2 + 2(4K-5)q + 12K = 0$. Therefore either $q = -2$, corresponding again to an origin shift, or $(K+1)q^2 + (K-5)q + 6K = 0$, giving

$$q = \left[5 - K \pm (25-34K - 23K^2)^{1/2}\right] / (2K + 2) . \qquad \text{(B3)}$$

There are usually two roots for q. They are both real if $25 - 34K - 23K^2 > 0$, i.e., for $K^- < K < K^+$ where

$$K^{\pm} = \frac{-17 \pm 12\sqrt{6}}{23} = \begin{cases} 0.53886 \ldots & \text{(B4)} \\ -2.01725 \ldots & \text{(B5)} \end{cases}$$

but the roots are complex conjugates if $K < K^-$ or $K > K^+$ in which case

$$\text{real } (q) = (5 - K)/(2K + 2) \qquad \text{(B6)}$$

from (B3). The roots or their real parts are given in Fig. B3. Because of (B2) the ξ - large asymptote (B1) is possible only if $q < -1$ or real $(q) < -1$, in (B3), (B6) respectively, and this restricts K to the range $-7 < K < -1$. Outside that range, for $K < -7$ or $K > -1$, the ξ -finite asymptote (4.7a) is a candidate instead. This holds apart from the special values $K = K^{\pm}$ where (B3) has equal roots, suggesting inaccessibility in line with Fig. 4. Also, for $K \to -1$ the logarithmic behavior noted in Appendix A takes over.

We note that provided $-7 < K < -1$ it is possible for a solution of (3.1a) to proceed smoothly from the form (3.2a), at $\xi = \xi_0-$ say, to the form (B2) for $\xi \to -\infty$, as ξ decreases, according to (B3), (B6) and the phase plane. This property is used by Smith (1983) in the nonuniqueness of shear layers.

APPENDIX C

SPECIAL CASES OF NO-SLIP FLOWS

These arise in the similarity forms (3.1a) with (5.5a-c) considered in Section 5. C.f. Appendix A.

First, suppose $K \to 2 \pm$ with $G''(o) = +1$. Letting $K = 2 + \varepsilon$ with $|\varepsilon|$ small, we have $G = 1/2 q\xi^2 + \varepsilon G_1 + O(\varepsilon^2)$ where $q = 1$. Hence (3.1a), (5.5a-c) yield the solution.

$$G_1'''(\xi) = \frac{1}{3} \exp\left(-\frac{q\xi^3}{9}\right)\left\{q^2 \int_o^{\xi} \xi_1 \exp\left(\frac{q\xi_1^{\,3}}{9}\right) d\xi_1 + p_1\right\}, \quad \text{(C1)}$$

with $p_1 = 0$ here. So $G_1''' \sim \xi^{-1}$ as $\xi \to \infty$, giving $G_1 \sim 1/2\ \xi^2 \ln \xi$ and $G \sim 1/2\ \xi^{2+\varepsilon}$, in line with (3.2b) for $K = 2\pm$.

Second, if $K = 2 + \varepsilon$ with $|\varepsilon| < < 1$ again but $G''(o) = -1$, then $q = -1$ above and (C1) gives

$$G_1''' \sim I \exp(\xi^3/9) \text{ as } \xi \to \infty \quad \text{(C2)}$$

where $I = 3^{-2/3}(-1/3)! > 0$. The growth (C2) leads to a new nonlinear stage at large $\xi = b + b^{-2}\tilde{\xi}$, with $\tilde{\xi}$ of O(1), $|\varepsilon| \exp(b^3/9) = b^8$ fixes $b >> 1$, and $G = b^2 \tilde{G}$. There the shear layer problem

$$\tilde{G}''' + \frac{2}{3} \tilde{G}\tilde{G}'' - \frac{1}{3} \tilde{G}'^2 = 0, \quad \text{(C3)}$$

$$\tilde{G} \sim -1/2 \pm 27 I \exp(\tilde{\xi}/3) \text{ as } \tilde{\xi} \to -\infty, \quad \text{(C4)}$$

holds, from (3.1a), (C2), for $\varepsilon \gtrless 0$. If $\varepsilon > 0$ the positive perturbation in (C4) is consistent with the shear layer solution in which $\tilde{G}'(-\infty) = 0$, $\tilde{G}''(+\infty) > 0$ (see Stewartson 1981, Smith 1982), yielding $G \sim (1/2)b^6\tilde{G}''(\infty)(\xi-b)^2$ for $\xi > b$, in agreement with (3.2b) for $K \to 2^+$. But if $\varepsilon < 0$ the negative disturbance in (C4) causes $\tilde{G}$ to become singular at a finite value of $\tilde{\xi}$ in agreement with (4.7a). Both these cases agree with the properties of Fig. 6 near $K = 2$ when $G''(o) = -1$.

Third, we consider K = 0, for which (3.1a), (5.5a-c) require (A6) but with

$$H(o) = 0 \ , \ H'(o) = \pm 1 \ , \qquad \text{(C5,C6)}$$

where $H \equiv G'$. Hence $3H'^2 + 2H^3 = 3$, and if $H'(o) = -1$ we have $H \leq 0$ and

$$\xi = -\int_o^H \frac{dh}{(1-2/3h^3)^{1/2}} \ , \qquad \text{(C7)}$$

giving the singularity (4.7a) with $H \to -\infty$ and

$$\xi_o = \int_{-\infty}^o \frac{dh}{(1-2/3h^3)^{1/2}} \ . \qquad \text{(C8)}$$

If $H'(o) = {}^+1$, however, $H \geq 0$ for $0 \leq \xi \leq 2\xi_1$ and there is symmetry about $\xi = \xi_1$. In $0 \leq \xi \leq \xi_1$ the first sign in (C7) in reversed and $H' \geq 0$. Here

$$\xi_1 = \int_o^{(3/2)^{1/3}} \frac{dh}{(1-2/3\ h^3)^{1/2}} \ . \qquad \text{(C9)}$$

At $\xi = \xi_1$, H attains its maximum $(3/2)^{1/3}$, while in $\xi \geq \xi_1$

$$\xi - \xi_1 = -\int_{(3/2)^{1/3}}^H \frac{dh}{(1-2/3h^3)^{1/2}} \ . \qquad \text{(C10)}$$

So H passes through zero at $\xi = 2\xi_1$ before acquiring the singular behavior (4.7a) again, with

$$\xi_o = \xi_1 + \int_{-\infty}^{(3/2)^{1/3}} \frac{dh}{(1-2/3h^3)^{1/2}} \ . \qquad \text{(C11)}$$

The results (C8), (C11) are in line with those of Fig. 6 for K = 0.

Comments concerning $K \to -1$ and $|K| \to \infty$, as well as pressure gradients, are similar to those of Appendix A. Note that as in Gajjar & Smith (1983) the pressure term p_1 in (C1) for instance can be adjusted to avoid the growth (C2) and hence avoid (4.7a) if necessary. A like effect can be achieved in (A1), (A2) and elsewhere. Finally, perturbations about the two solutions for $K = 1$, one the Blasius and the other yielding (4.7a), confirm the occurrence of (4.7a,c) for $K \to 1\pm$.

APPENDIX D

FURTHER TERMS IN THE ALTERNATIVE FLAT PLATE SOLUTION

Of the further terms generated in the small-x expansion (5.1a,b) the most "dangerous" and interesting are exponentially small ones within the reversed motion. Algebraically small ones overall pose little difficulty. The exponentials stem from the behavior of the Chapman solution at its lower extremes,

$$f_c(\eta) \sim -\kappa + \hat{A}e^{\kappa\eta/2} \qquad \text{(D1)}$$

as $\eta \equiv [Y-(\kappa/d_1)^{1/2}x^{1/4}]x^{-1/2} \to -\infty$, where $\hat{A}$ is a positive constant. As a result the slower reversed motion between the Chapman layer and the surface must have the underlying development

$$\psi = -d_1\chi^2x^{1/2} + \ldots + x^M A(\chi) \exp[g(\chi)/x^{1/4}] \,. \qquad \text{(D2)}$$

Here M is a constant, A,g are unknown functions of χ, but it is required that the Real (g) < 0 for exponential decay as $x \to 0^+$, and $\chi \equiv x^{-1/4}Y$ with $0 < \chi < (\kappa/d_1)^{1/2}$.

Substituting (D2) into the governing equations (2.1a,b) we obtain the differential equation

$$\chi(g + \chi g') = g'^2 \qquad \text{(D3)}$$

for $g(\chi)$, where without loss of generality we have replaced d_1, κ by 4 for convenience. The conditions on $g(\chi)$ are then

$$g(1-) = 0-, \quad \text{Real}\ (g) < 0 \quad \text{for } 0 < \chi < 1, \qquad \text{(D4,D5)}$$

the former to match with (D1). The required solution of (D3) can be written in the parametric form

$$g = \chi^3(v^2 - v) \text{ with } \chi^{12} = v^{-3}(4 - 3v)^{-5} . \qquad \text{(D6,D7)}$$

Hence g remains real and negative for the range $1 > \chi > \chi_0 = 5^{-5/12}\,2^{2/3}$ ($\doteq$ 0.41516), or $1 > v > 1/2$, with (D4) satisfied as $v \to 1-$, $\chi \to 1-$. As $v \to 1/2^+$, $d\chi/dv \to 0$ and so then $d^2g/d\chi^2$ has an inverse square root singularity although $g(< 0)$, $dg/d\chi$ (> 0) remain finite. The singularity in g is accompanied by one in $A(\chi)$ but these are smoothed out in a thinner layer near $\chi = \chi_0$ by means of Airy functions. Consequently g becomes complex in $\chi < \chi_0$. However, (D6), (D7) can continue to hold then, with v now complex, and the constraint (D5) is maintained for $0 < \chi < \chi_0$. As $\chi \to 0^+$ for instance we obtain

$$g(\chi) \sim 3^{-5/4}e^{-5\pi i/8} + O(\chi^{3/2}) \qquad \text{(D8)}$$

from (D6), since $v \sim 3^{-5/8}\chi^{-3/2}\exp(-5\pi i/16)$ from (D7), and (D8) satisfies (D5).

It follows that the decay or upstream influence for small positive x is purely exponential in $1 > \chi > \chi_0$ but oscillatory-exponential in $\chi_0 > \chi > 0$. An extra viscous layer also arises nearer the surface but the Airy equation holds again there and accommodates the exponential growth required by (D8) at the upper extremes of the viscous layer.

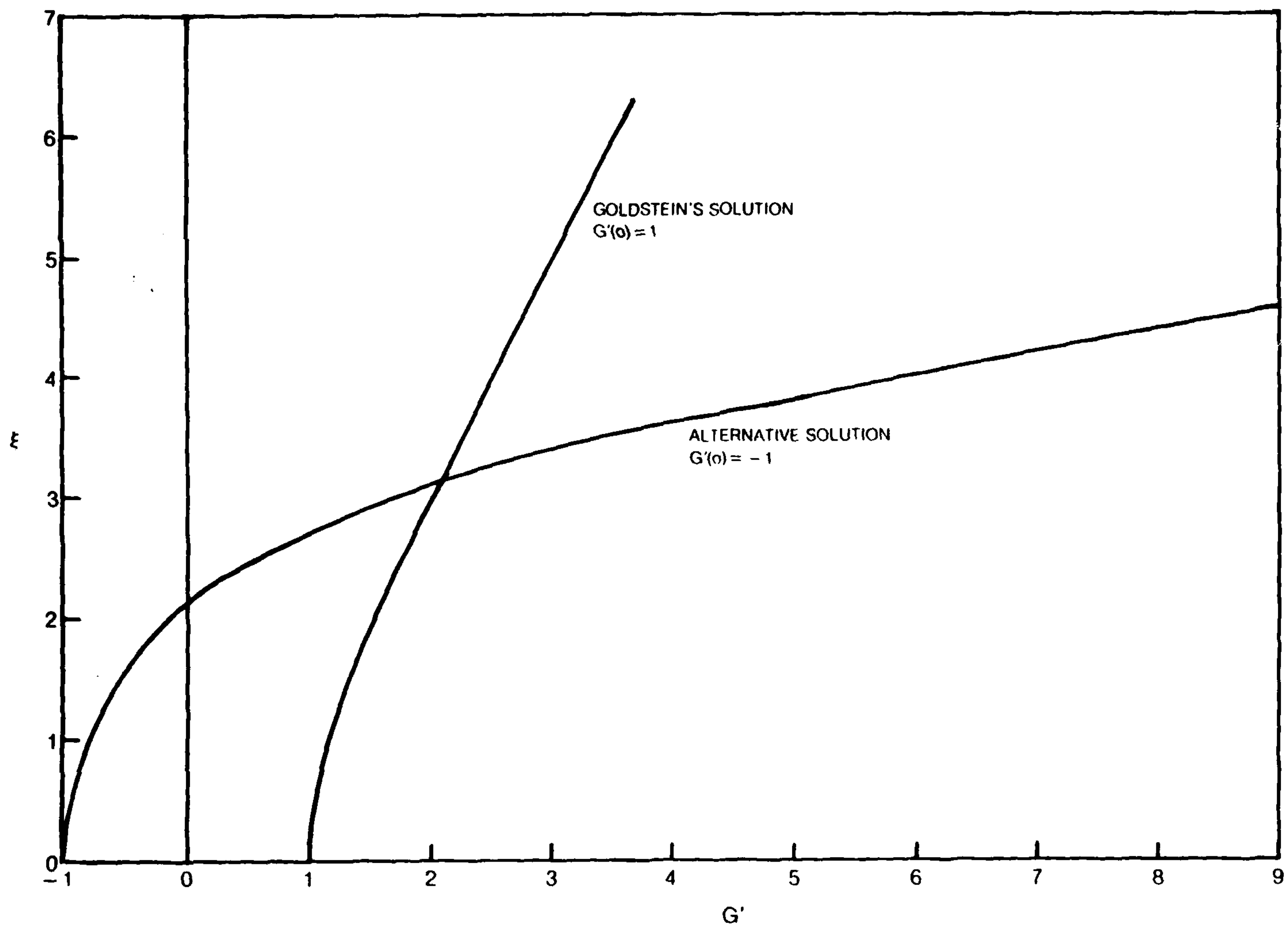

Figure 1 Nonuniqueness of Near-Wake Solutions
(a) Normalized Velocity Function

83–3–61–1

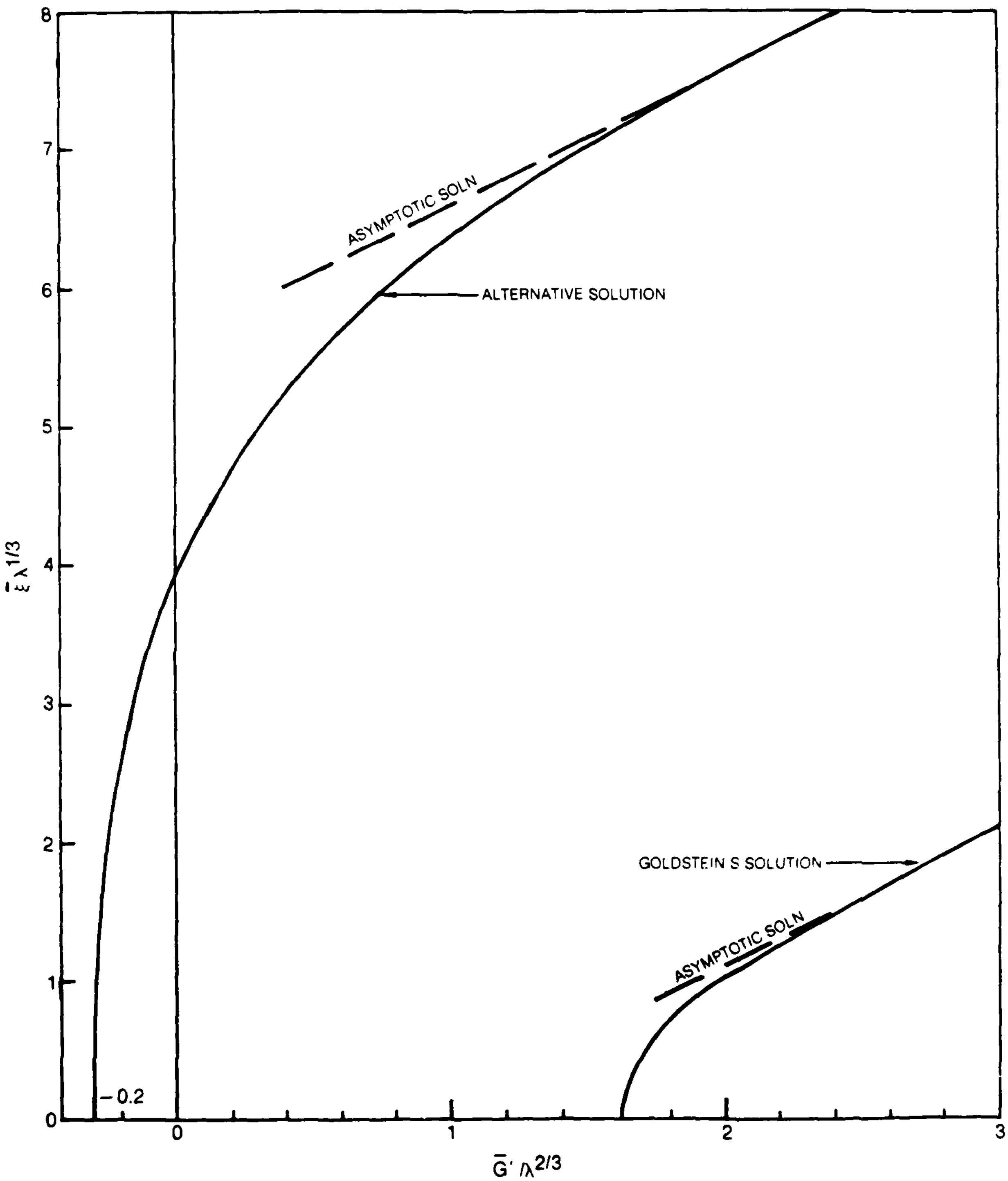

Figure 1 Nonuniqueness of Near-Wake Solutions (b) Velocity Function

83–3–61–2

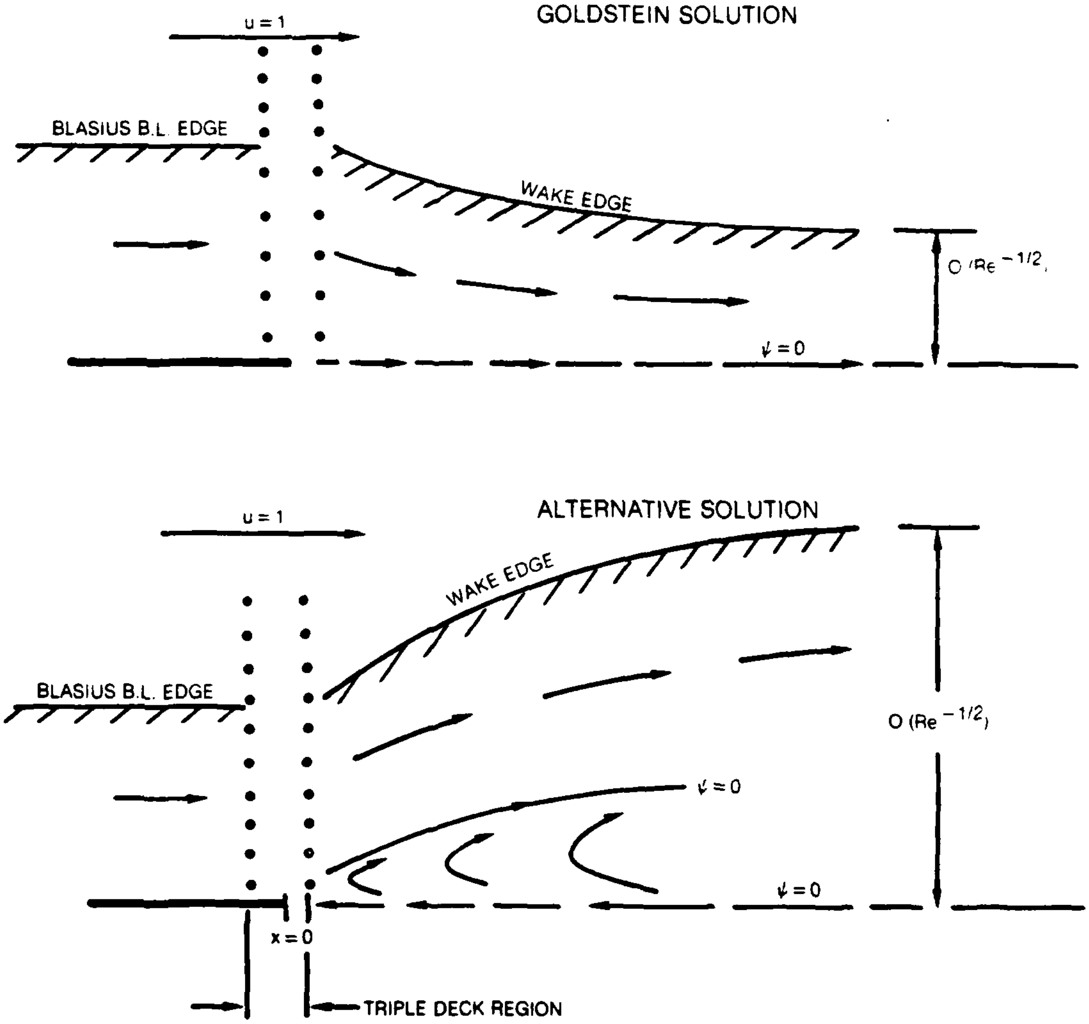

Figure 1 Nonuniqueness of Near-Wake Solutions
(c) Implied Flow Patterns

83–3–61–3

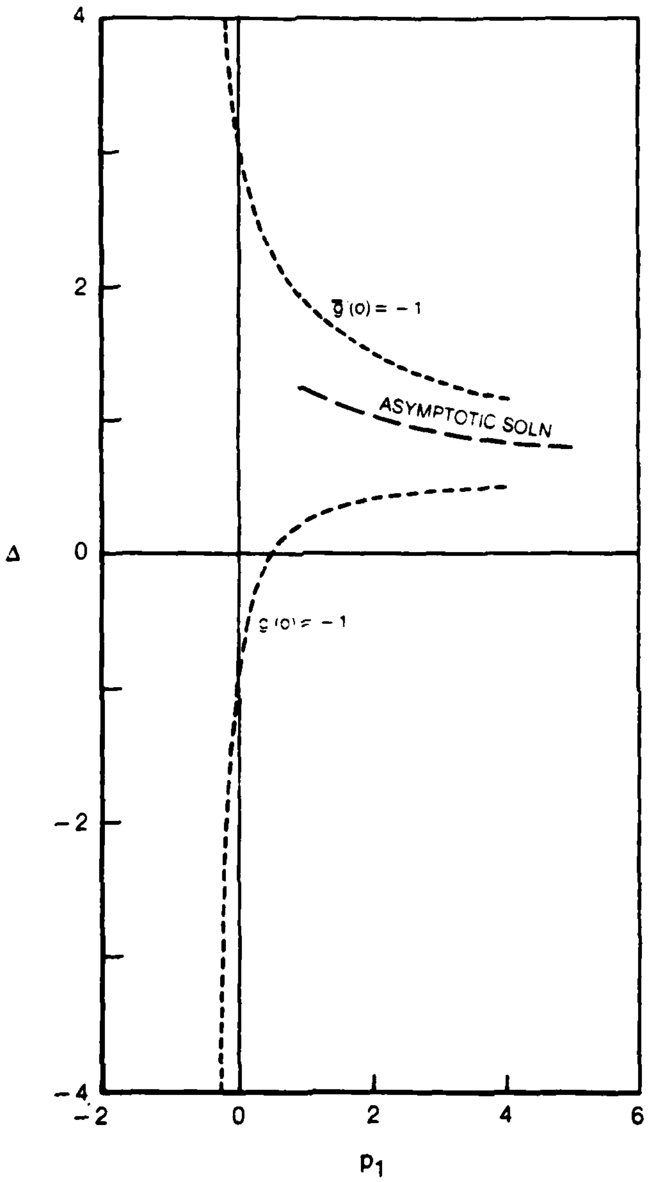

Figure 1 Nonuniqueness of Near-Wake Solutions
(d) Very-Near-Wake Displacement Distribution

83–3–61–4

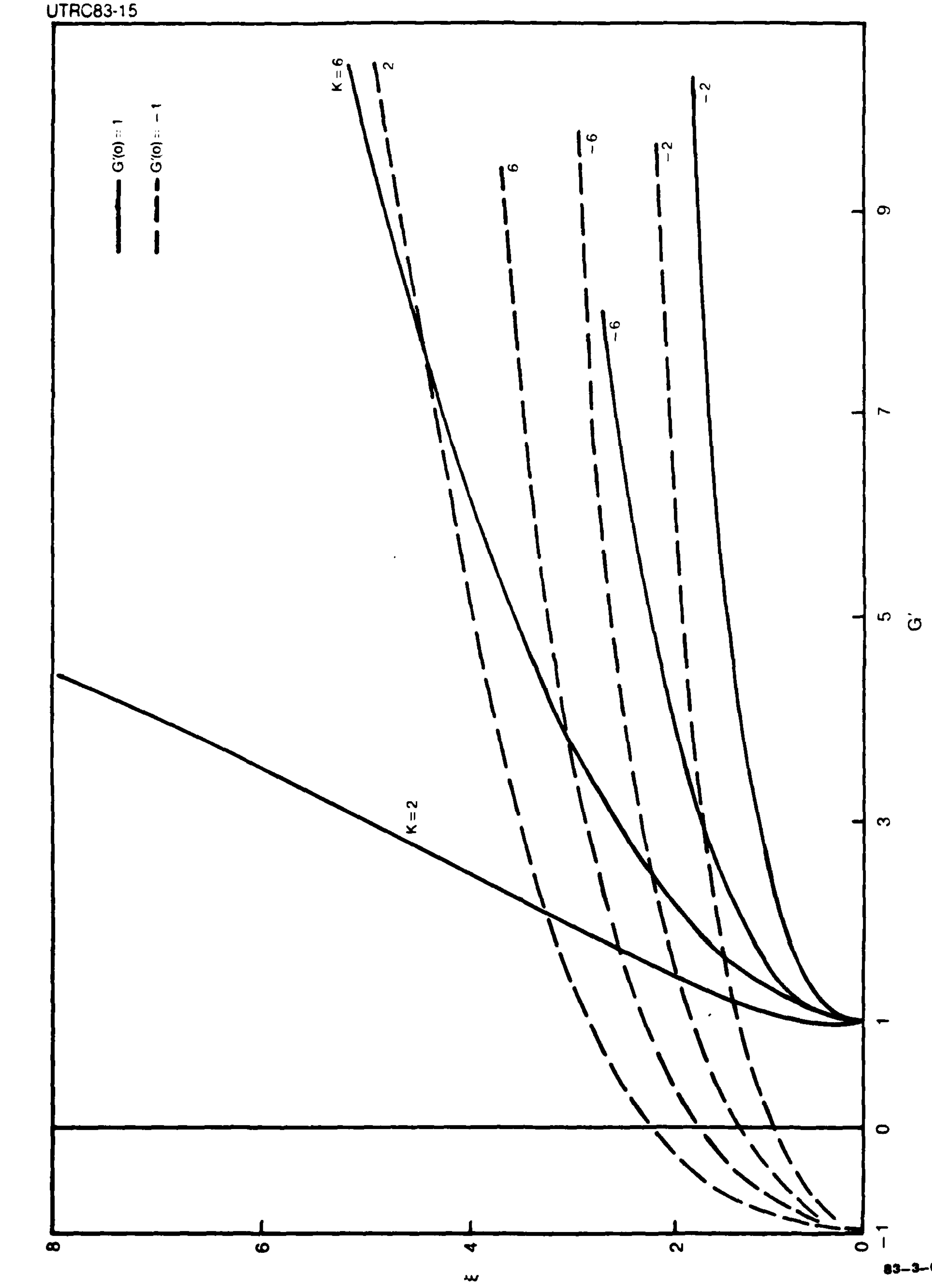

Figure 2 More General Near Wake
(a) Velocity Function Distribution

83–3–61–5

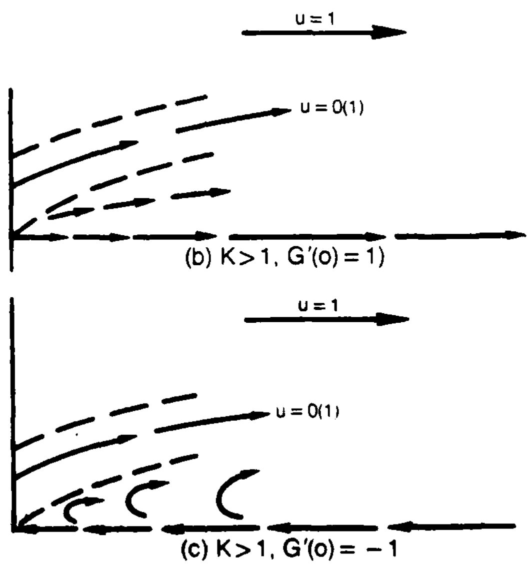

Figure 2 More General Near Wakes

83-3-61-6

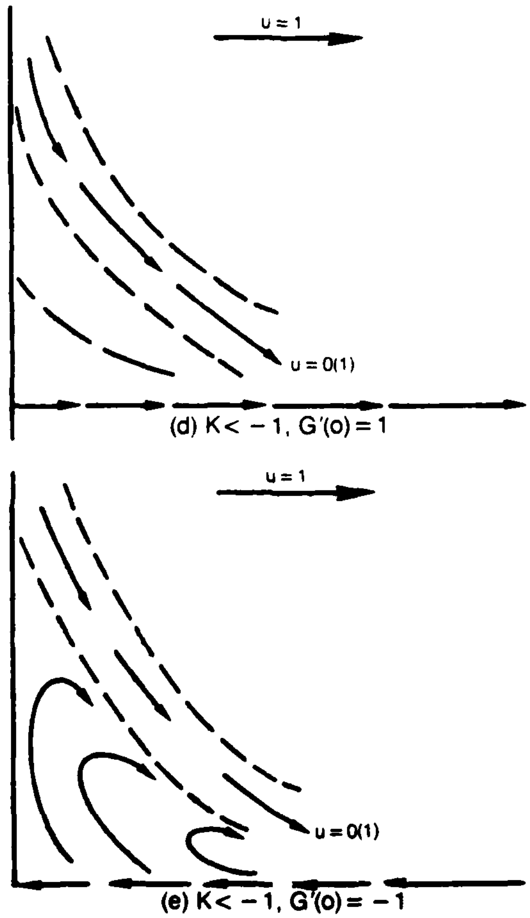

Figure 2 More General Near Wakes

83-3-61-7

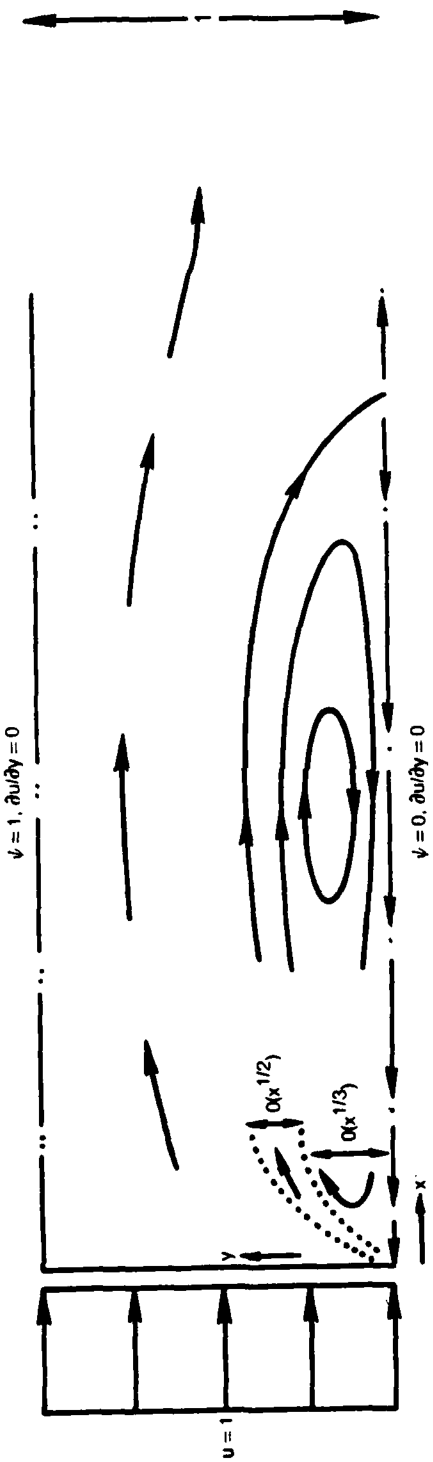

Figure 3 Cascade Flow Branching Solution

83-3-61-8

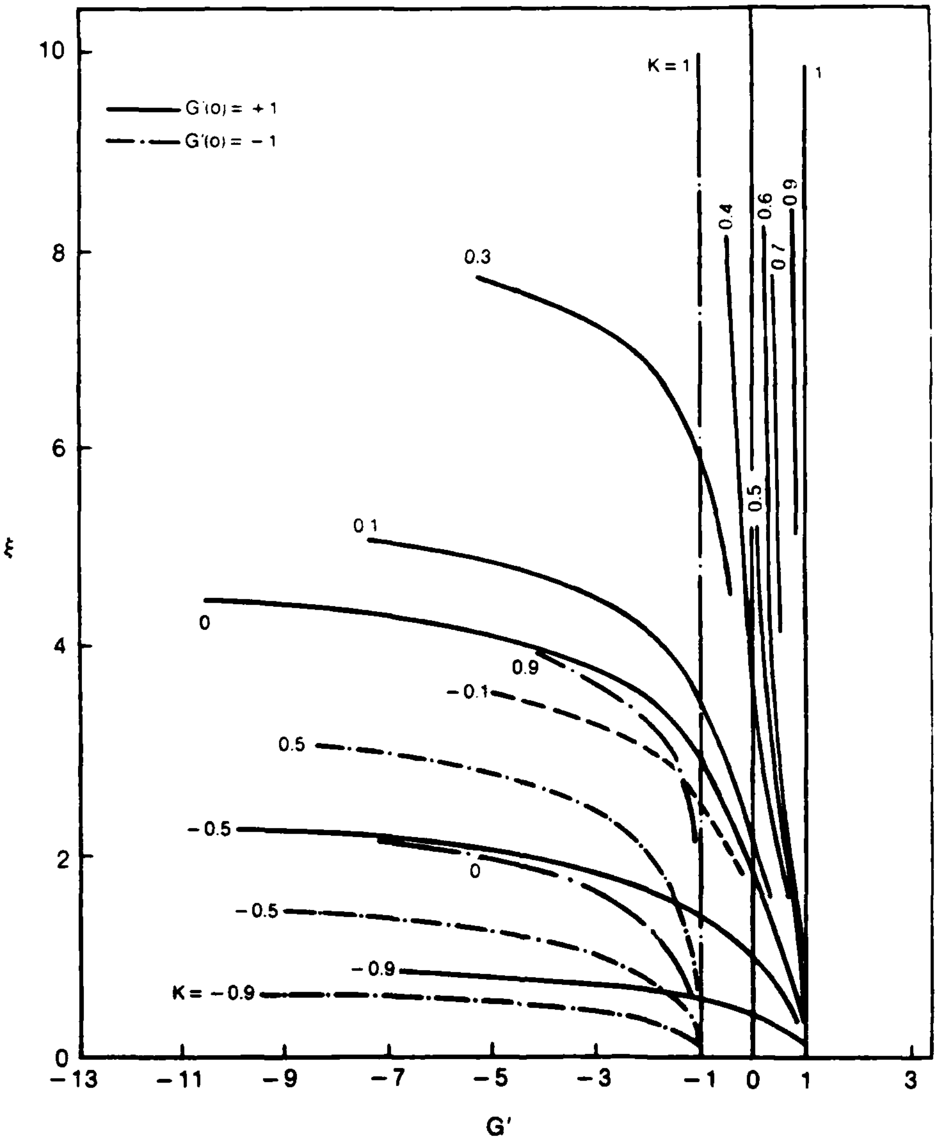

Figure 4

83–3–61–9

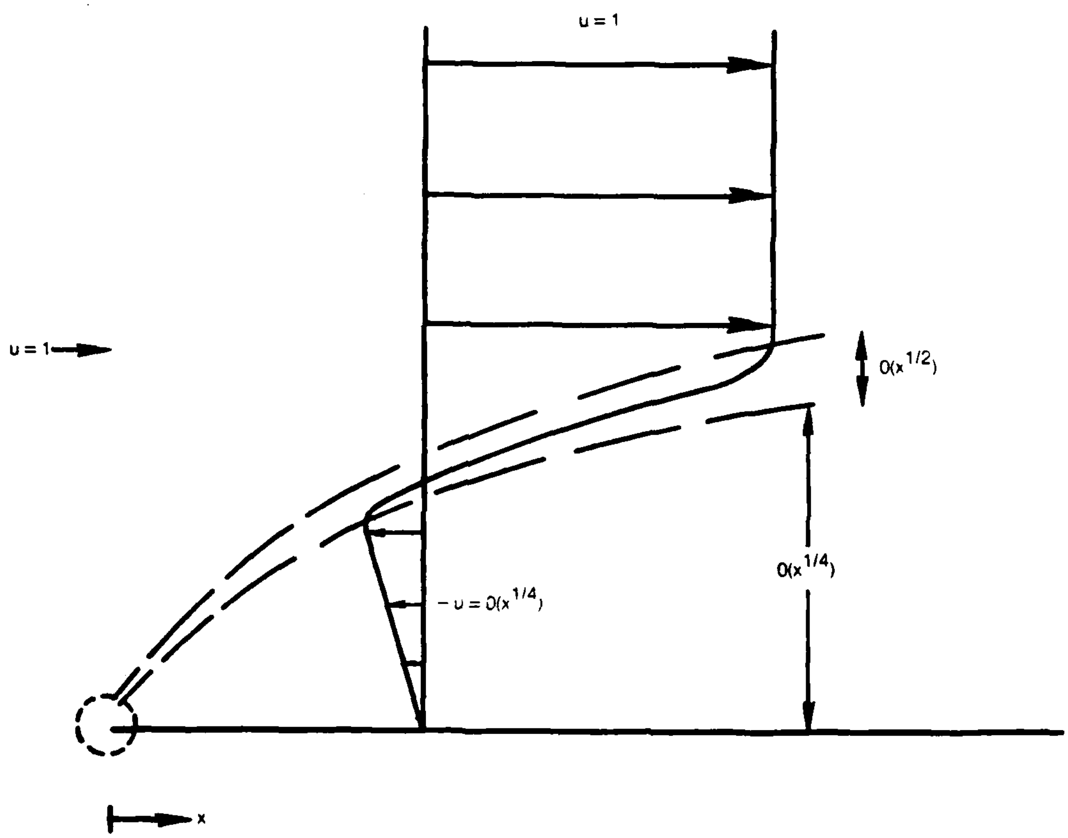

Figure 5 Blasius Solution Alternative

83-3-61-10

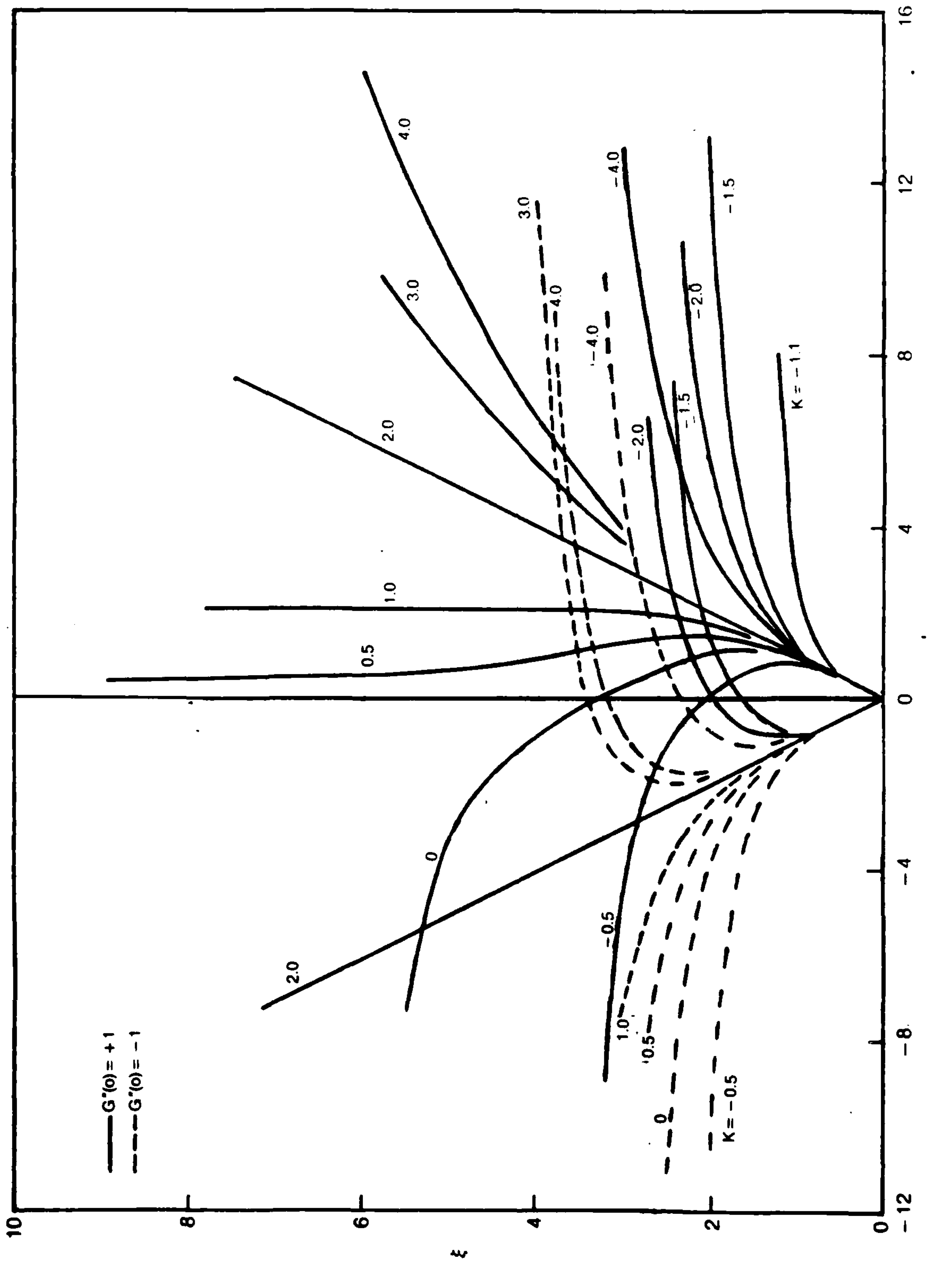

Figure 6 Eddy Closure on a Solid Surface

83-3-81-11

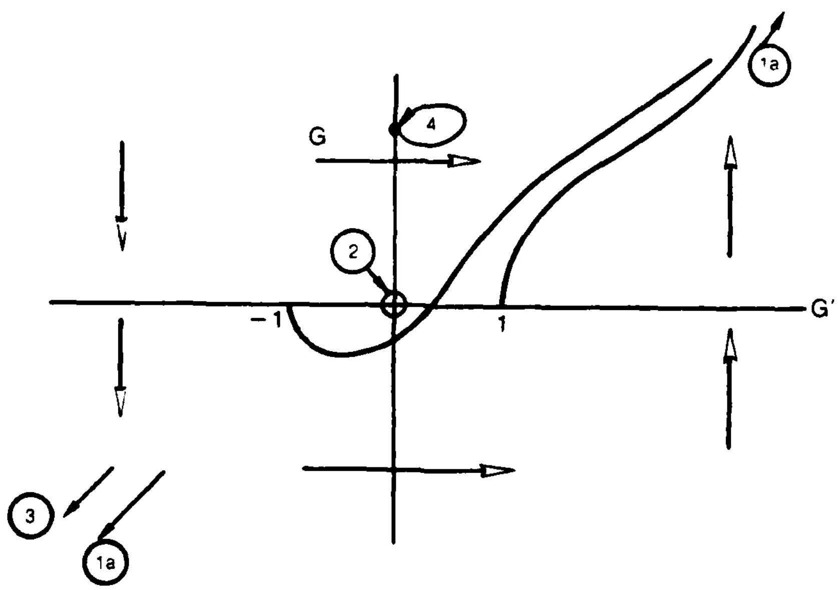

Figure B1 Phase Plane Trajectories |K| >1

83-3-81-12

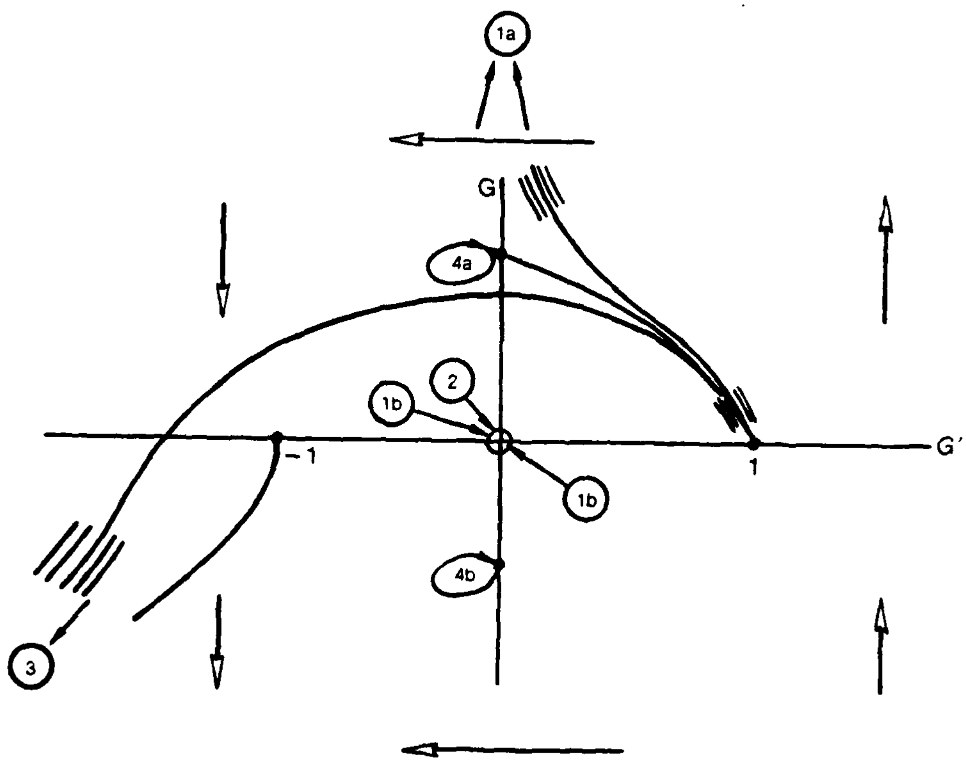

Figure B2 Phase Plane Trajectories |K| <1

83-3-61-13

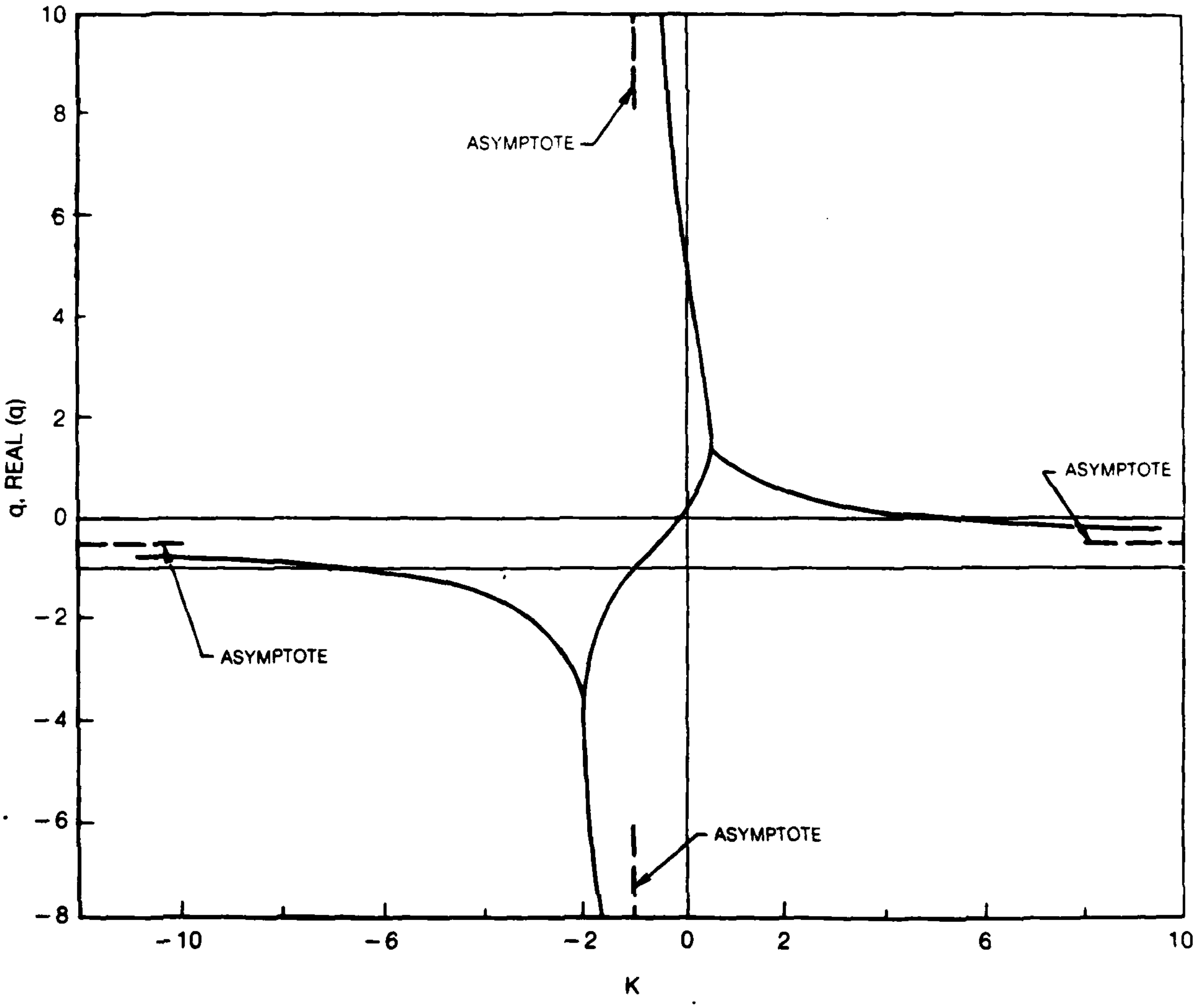

Figure B3 Limit Form Exponent

83–3–61–14

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