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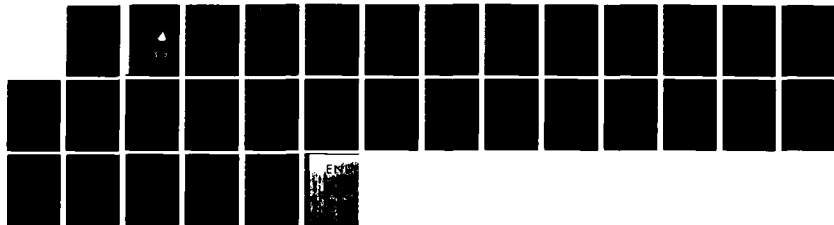
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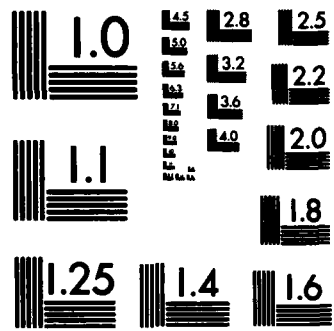
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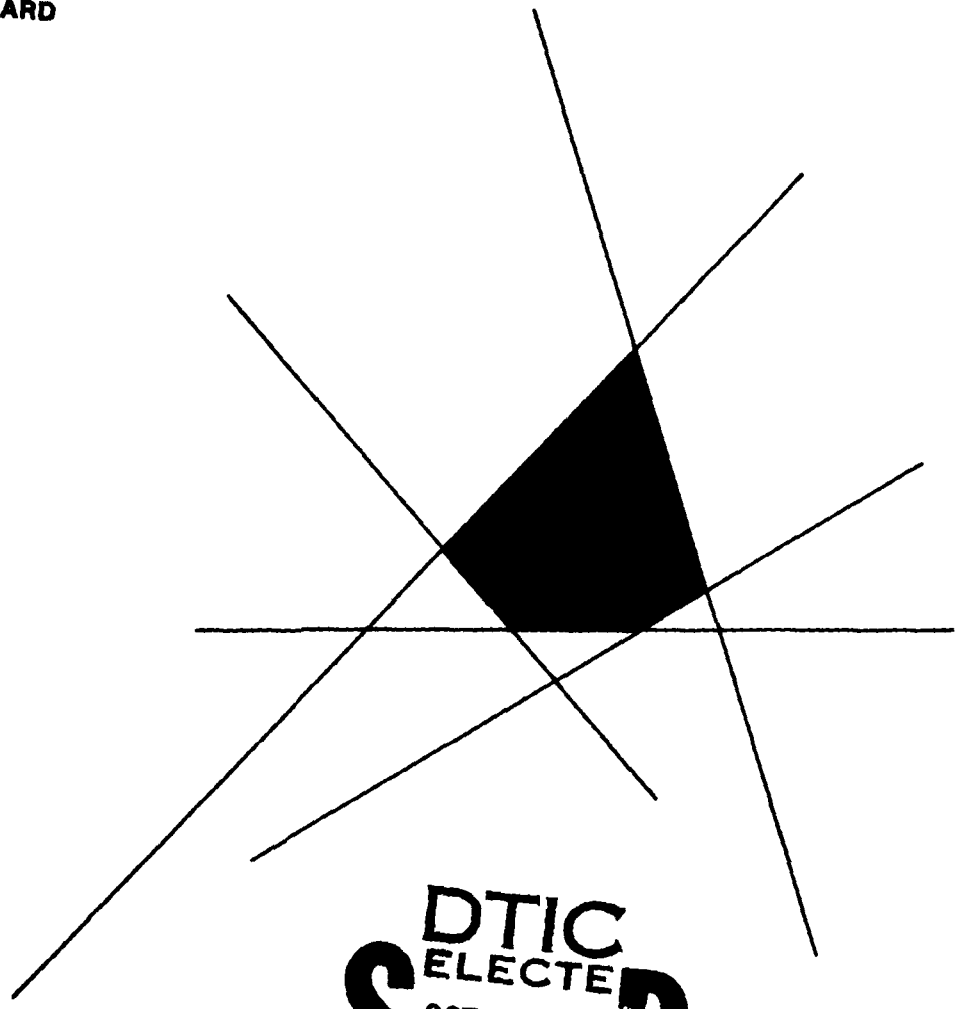
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PRODUCTION NETWORKS: A DYNAMIC MODEL OF PRODUCTION

by
STEVEN T. HACKMAN
and
RONALD W. SHEPHARD

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PRODUCTION NETWORKS: A DYNAMIC MODEL OF PRODUCTION

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DEDICATION

I would like to dedicate this paper to its most influential contributor, Professor Ronald W. Shephard, who died July 23, 1982.

I hope that each of us has the opportunity to care for someone as much as he cared for his students. I will judge myself truly successful if I can match the inspiration that he gave to his students, and the respect and admiration he received in return.

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ACKNOWLEDGEMENT

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ABSTRACT

This paper introduces the dynamic production network as a dynamic model of production which exhibits the real complexity underlying the simple treatments commonly used in economic analyses. The model reflects the truly dynamic character of production by illustrating directly the time substitutions of both input and output and modeling the intermediate product transfers *explicitly*. The abstract dynamic production network correspondence which relates input to output for this model is defined. From this, the concept of the technically efficient subset is introduced for which a proof of its non-emptiness is provided. The optimization problem that is used to obtain a technically efficient point can be used for resource allocation; a preliminary treatment is provided. Some limitations of the present model are also discussed.

PRODUCTION NETWORKS: A DYNAMIC MODEL OF PRODUCTION

by

Steven T. Hackman and Ronald W. Shephard

1. INTRODUCTION

The typical production function of economic theory is a steady state (static) relationship between inputs of goods and services and outputs of the same kind, abstracting from the intermediate products involved in the system so represented. Intermediate products have been made explicit in the so-called 'activity analysis' models of production, on a macroeconomic basis, by Leontief and his successors. These models were steady state until recently, but the dynamic treatments involved have been treated on an ad hoc basis.

Dynamic models of production systems which abstract from intermediate products are not very convincing. On this account one is directed to dynamic production networks, and to seek to develop a more or less general theory for the same. The objective is to better understand economic dynamics with the ultimate goal of applying the theory to real production processes.

To accomplish this task, the production system is modeled as a directed network, the nodes of which represent primitive production activities. These primitive production activities are those within which the intermediate product transfers need not be considered for the purposes at hand. The nodes are connected by directed arcs to define transfers of intermediate and final products. System exogenous inputs are treated as transfers from an initial node A_0 . For a system with N producing activities, final outputs are taken as delivered to node A_{N+1} . Thus, a production system is regarded as

a jointly operating, finite number of interrelated primitive production activities A_1, A_2, \dots, A_N sharing a common source of system exogenous inputs of goods and services used in production.

The primitive elements for the flows of goods and services should have a truly dynamic character, instead of indexing steady state relationships with a time variable t . These flows will be called *time rate histories* and will be taken as elements of a subset of the nonnegative functions defined on the nonnegative part of the real line. If x represents the flow of a good or service, then $x(t)$ represents the rate of flow of the good or service at time t .

To define the dynamic production network, one needs to know precisely the dynamic relationship between the input and the output time rate histories for the primitive activities. This purely technological relationship is modeled by a dynamic production correspondence (introduced in [7]) and defined in Section 2, thus describing the dynamic production network in full.

The dynamic production network correspondence is a purely technological relationship between the input and the output time rate histories at the network level and will be defined in Section 3. Related to this correspondence is the concept of technical efficiency. Its introduction and proof of its existence will be given in Section 4 as well as a preliminary treatment of resource allocation. Section 5 discusses the limitations of the present model and the appendix contains the proof of two simple propositions used in the proof of the non-emptiness of the technically efficient subset.

2. THE DYNAMIC PRODUCTION NETWORK

2.1 Time Rate Histories

As mentioned in the introduction, the primitive elements for the flows of goods and services (time rate histories) should have a truly dynamic character. If x represents such a flow, then $x(t)$ represents the rate of the flow at time t .

Let L represent the space of "acceptable" flows of goods and services. The model described so far implies that, at a minimum, flows in L should be defined on the nonnegative part of the real line, nonnegative, and measurable with respect to the usual Borel σ -field \mathcal{B} on \mathbb{R}_+ . To integrate these flows, it is assumed that the underlying measure μ is Lebesgue restricted, however, to some interval $[0, T]$ (T positive) which can be thought of as the planning horizon. Furthermore, such flows in L should not be able to become arbitrarily "high" and hence will be assumed bounded in the usual L^∞ norm. Finally, one sees in practice that the amount of time over which a flow is sufficiently large cannot be made arbitrarily small. Thus, positive constants k and δ , $\delta < T$, are assumed known for which one only considers those functions x satisfying the following property:

P.1 For all $c > k$, if $\mu\{x > c\} \neq 0$, then $\mu\{x > c\} \geq \delta$.

In other words, L , the space of acceptable flows of goods and services, will be taken as the subset of $L^\infty_+(\mathbb{R}_+, \mathcal{B}, \mu)$ satisfying P.1.

2.2 The Activity's Dynamic Production Correspondence

The production model should display explicitly the intermediate product transfers. This display is essential for dynamic models of production

since final output evolves as the evolutionary flow of intermediate products to final products, the organization of which determines final output flows.

Accordingly, the production model presented here assumes that the primitive structures of the production network are the activities. The inputs to the activities are the system exogenous inputs and the intermediate products relevant to the particular activity's production process. The outputs may be intermediate products used as inputs by other activities, or final products, or mixtures of both as in the case of spare parts.

As notation, let

$x_{0i} = (x_{0i}^1, \dots, x_{0i}^n) \in (L)^n$ be a vector of n system exogenous input rate histories allocated to the i^{th} activity, $i = 1, 2, \dots, N$

$v_{ij} = (v_{ij}^1, \dots, v_{ij}^m) \in (L)^m$ be a vector of m transfer rate histories of net outputs of the i^{th} activity to the j^{th} activity, $i \neq j$, $i = 1, 2, \dots, N$ and $j = 1, 2, \dots, N, N+1$

$v_i = (v_i^1, \dots, v_i^m) \in (L)^m$ be a vector of m net output rate histories of the i^{th} activity, $i = 1, 2, \dots, N$.

Note that in the foregoing representation, common function spaces $(L)^n$ and $(L)^m$ are taken for convenience so that some of the components of x_{0i} , v_{ji} , and v_i may be null functions depending on the production process of the i^{th} activity.

Abstractly, the *activity's dynamic production correspondence* is taken as a correspondence from $v_i \in (L)^m \rightarrow L_1(v_i) \subset (L)^n \times (L)^m$ where one interprets (loosely) the statement that $(x_{0i}, \sum_{j \neq i} v_{ji}) \in L_1(v_i)$ to mean that the

activity may produce V_i if allocated x_{0i} as system exogenous input over time and $\sum_{j \neq i} V_{ji}$ as intermediate product transfers from other activities over time. This includes the possibilities of (free) disposal and storage of both inputs and outputs.

To be more precise, one needs to differentiate between the concepts of allocation and application. Since disposal and storage are not excluded, what is *allocated* to the activity as input $\left(x_{0i}, \sum_{j \neq i} V_{ji}\right)$ may not be what is *applied* as input to the production process over time. Let $(y_i, W_i) \in (L)^n \times (L)^m$ denote a vector of input applied to the production process and let $T_i \in (L)^{n+m}$ denote a vector of disposal rate histories for both input and output. For an activity A_i , $i = 1, 2, \dots, N$, the histories (y_i, W_i) and T_i must satisfy an inventory balance equation in order to be feasible for a particular allocation $\left(x_{0i}, \sum_{j \neq i} V_{ji}\right)$:

if the j^{th} system exogenous input is activity storable, $1 \leq j \leq n$,

$$(2.1) \quad 0 \leq s_i^j + \int_0^t (x_{0i}^j - T_i^j - y_i^j) d\mu \leq C_i^j, \quad \text{for all } t \in [0, T]$$

if the j^{th} system exogenous input is not activity storable, $1 \leq j \leq n$,

$$(2.2) \quad 0 = x_{0i}^j - T_i^j - y_i^j$$

where $s_i^j \in \mathbb{R}_+$ represents initial stock, if any, and $C_i^j \in \mathbb{R}_+ \cup \{\infty\}$ represents the stock capacity level constant over time, $s_i^j \leq C_i^j$.

(The capacity level could be modeled as a function over time, if

desired for more generality.) It is understood that $T_i^j = 0$ if the j^{th} system exogenous input is not activity disposable.

Constraint (2.1) simply states that the inventory at time t must be non-negative and cannot exceed capacity. Constraint (2.2) states that if inventory is not allowed, then after disposal one must apply the rest to production.

To complete the discussion of the correspondence L_i , the structure of the model at the activity level assumes the existence of a *production function* F_i which takes vectors of inputs applied to the production process $(y_i, W_i) \in (L)^n \times (L)^m$ into the realized vector of final outputs of the activity $F_i(y_i, W_i) \in (L)^m$ obtained through production. Whereas there exists a correspondence between any given output vector V_i and feasible input vectors of allocations $(x_{0i}, \sum_{j \neq i} V_{ji})$, there is only a functional relationship between the input vector applied to the production process (y_i, W_i) and the realized output vector $F_i(y_i, W_i)$.

Two restrictions on the production function are assumed. The system exogenous inputs are usually different types of labor or capital services; if a service input is "bounded" in the usual sense, then the outputs produced by the activity should not be able to become arbitrarily "high" (regardless of the magnitude of intermediate product input). Furthermore, an activity must be able to produce some product(s), and the set of products which it can produce are not needed themselves as intermediate product input from other activities. These two restrictions on the production function F_i do not seem to be serious; most production systems one can think of satisfy them.

Mathematically, one can describe these two restrictions as follows:

A.1 For each activity $i = 1, 2, \dots, N$, for all positive constants C , there exists a constant $B_i(C)$ such that:

$$\sup_{\|y_i\| < C} \|F_i(y_i, W_i)\| < B_i(C)$$

where, for $x \in (L^\infty)^k$, $\|x\| = \max_{1 \leq i \leq k} \|x_i\|_\infty$.

A.2 Let P_i denote the set of products which the i^{th} activity is capable of producing, i.e.,

$$P_i = \left\{ k : \exists (y_i, W_i) \in (L)^n \times (L)^m \text{ such that } \|F_i^k(y_i, W_i)\|_\infty > 0 \right\}.$$

Then it is assumed that P_i is non-empty for each i and that for $k \in P_i$ the following property on F_i holds:

$$F_i^k(y_i, W_i) \leq F_i^k\left(y_i, \left(W_i, \dots, W_i^{k-1}, 0, W_i^{k+1}, \dots, W_i^m\right)\right)$$

for all $(y_i, W_i) \in (L)^n \times (L)^m$.

In view of A.2, the V_i 's and the W_i 's will be restricted in the following manner:

$$(2.3) \quad V_i^k = 0 \quad \text{if } k \notin P_i$$

$$(2.4) \quad W_i^k = 0 \quad \text{if } k \in P_i.$$

To be feasible, the output functions $F_i(y_i, W_i)$, $\sum_{j \neq i} V_{ji}$, V_i , and T_i (restricted to the last m coordinates) must satisfy their own, similar set of inventory balance equations, the explanation of which is clear:

if the k^{th} product is activity storable,

$$(2.5) \quad 0 \leq s_i^k + \int_0^t \left(\sum_{j \neq i} v_{ji}^k - T_i^k - W_i^k \right) d\mu \leq C_i^k ,$$

for all $t \in [0, T]$, $k \notin P_i$

$$(2.6) \quad 0 \leq s_i^k + \int_0^t \left(F_i^k(y_i, W_i) - T_i^k - V_i^k \right) d\mu \leq C_i^k ,$$

for all $t \in [0, T]$, $k \in P_i$

if the k^{th} product is not activity storable,

$$(2.7) \quad 0 = \sum_{j \neq i} v_{ji}^k - T_i^k - W_i^k , \quad k \notin P_i$$

$$(2.8) \quad 0 = F_i^k(y_i, W_i) - T_i^k - V_i^k , \quad k \in P_i$$

where, as before, $s_i^k \in \mathbb{R}_+$ represents the initial stock, if any, and $C_i^k \in \mathbb{R}_+ \cup \{\infty\}$ represents the (constant) capacity, so that $s_i^k \leq C_i^k$. It is understood that $T_i^k = 0$ if the k^{th} product is not activity disposable.

Thus, in summation, to say that $(x_{0i}, \sum_{j \neq i} v_{ji}) \in L_i(V_i)$ means that there exist functions $(y_i, W_i) \in (L)^n \times (L)^m$, $T_i \in (L)^{n+m}$ such that constraints (2.1) - (2.8) are met. This completes the discussion of the activity correspondences and so the discussion of the primitive elements of the model. In the next section, the dynamic production network correspondence is developed.

3. THE DYNAMIC PRODUCTION NETWORK CORRESPONDENCE

The *dynamic production network correspondence*, similar in vein to the activity correspondence, is a correspondence from $u \in (L)^m \rightarrow \mathbb{LN}(u) \subset (L)^n$ which, loosely described, is the set of all system exogenous input rate histories $x \in (L)^n$ that when allocated to the system as a whole may produce u , possibly through disposal and storage. Again, free disposal for inputs and outputs is not excluded nor is storage for either system exogenous input or final products. However, system intermediate product storage or disposal is *not* allowed, thus requiring that $v_i = \sum_{j \neq i} v_{ij}$. This last restriction, made for convenience, does not appear to be a restriction for the following two reasons:

- (1) Storage and disposal of intermediate products are already allowed at the activity level.
- (2) If the production system is such that it is more appropriate to dispose of or store intermediate product at the system level--for example, a warehouse--then the acts of disposal and storage may be modeled as an activity where one "duplicates" the products involved so as to not violate A.2 and takes the production function to be the projection map onto the last m coordinates. This would make sense if the act of disposing costs something; then, one would probably want to define an activity for this act of disposing.

To describe when an $x \in \mathbb{LN}(u)$, one needs system inventory balance equations similar to those at the activity level. Below, $T_0 \in (L)^n$ and $T_{N+1} \in (L)^m$ represent the disposal rate functions for activities A_0 and A_{N+1} .

On the input side,

if the j^{th} system exogenous input is system storable, $1 \leq j \leq n$,

$$(3.1) \quad 0 \leq s^j + \int_0^t \left(x^j - T_0^j - \sum_i x_{0i}^j \right) d\mu \leq C^j, \quad \text{for all } t \in [0, T]$$

if the j^{th} system exogenous input is not system storable, $1 \leq j \leq n$,

$$(3.2) \quad 0 = x^j - T_0^j - \sum_i x_{0i}^j$$

where $s^j \in \mathbb{R}_+$ is the initial stock, if any, and $C^j \in \mathbb{R}_+ \cup \{\infty\}$ represents the (constant) capacity, $s^j \leq C^j$. It is understood that $T_0^j = 0$ if the j^{th} system exogenous input is not system disposable.

On the output side,

if the k^{th} final product is system storable, $1 \leq k \leq m$,

$$(3.3) \quad 0 \leq s_{N+1}^k + \int_0^t \left(\sum_i v_{iN+1}^k - T_{N+1}^k - u^k \right) d\mu \leq C_{N+1}^k, \quad \text{for all } t \in [0, T]$$

if the k^{th} final product is not system storable, $1 \leq k \leq m$,

$$(3.4) \quad 0 = \sum_i v_{iN+1}^k - T_{N+1}^k - u^k$$

where $s_{N+1}^k \in \mathbb{R}_+$ is the initial stock, if any, and $C_{N+1}^k \in \mathbb{R}_+ \cup \{\infty\}$ represents the (constant) capacity, $s_{N+1}^k \leq C_{N+1}^k$. It is understood that $T_{N+1}^k = 0$ if the k^{th} final product is not system disposable.

Finally, one needs to make sure that the individual activities can produce what is required of them, so one adds

$$(3.5) \quad \left(x_{0i}, \sum_{j \neq i} v_{ji} \right) \in L_i(v_i) \quad \text{for all } i = 1, 2, \dots, N$$

and, in view of A.2 (what is made as output is not needed as input), one adds

$$(3.6) \quad \sum_{j \neq i} v_{ji}^k = 0, \quad k \in P_i.$$

In summation, to say that $x \in (L)^n$ is in $LN(u)$ means that there exist functions

$$x_{0i} \in (L)^n, \quad i = 1, 2, \dots, N,$$

$$v_{ij} \in (L)^m, \quad i = 1, 2, \dots, N \\ j = 1, 2, \dots, N, N+1,$$

$$v_i \in (L)^m, \quad i = 1, 2, \dots, N,$$

$$T_0 \in (L)^n,$$

$$T_{N+1} \in (L)^m$$

such that constraints (3.1) - (3.6) are met. One additional comment is in order here. By the nature of the constraint (3.3), it is implied that the output level u is obtained if the cumulative amount of output obtained through production, at all times, exceeds the cumulative amount of output required--after possible disposal.

In the next section, the technically efficient subset of $LN(u)$ is defined and a proof of its non-emptiness is provided (assuming, of course, that $LN(u)$ is non-empty). The proof is an application of some of the basic theorems of functional analysis; references for some of the mathematical content may be found in Halmos [2], Kelley [3], and Rudin [4].

4. THE TECHNICALLY EFFICIENT SUBSET

The technically efficient subset $EN(u)$ of $LN(u)$ is defined to be:

$$EN(u) = \{x \in LN(u) : \text{if } y \underset{\neq}{\leq} x, \text{ then } y \notin LN(u)\}$$

where one defines for

- (i) $f, g \in L^\infty$, $f \leq g$ to mean that $f(t) \leq g(t)$ except on a μ -null subset.
- (ii) $f, g \in (L^\infty)^n$, $f \leq g$ to mean that $f_i \leq g_i$ for $i = 1, 2, \dots, n$.
- (iii) $f, g \in L^\infty$, $f \underset{\neq}{\leq} g$ to mean that $f \leq g$ and there exists a subset of positive measure on which $f(t) < g(t)$.
- (iv) $f, g \in (L^\infty)^n$, $f \underset{\neq}{\leq} g$ to mean that $f \leq g$ and for some i , $f_i \underset{\neq}{\leq} g_i$.

Assuming one is "charged" a positive amount for system exogenous input and the only objective is to obtain output level u , then as a minimum requirement the production planner would want to choose an efficient point. Thus, it is important to know that the model presented here always has the property that the efficient subset is non-empty when the set $LN(u)$ is non-empty.

The proof of its non-emptiness utilizes 3 simple propositions of which the first two are verified in the appendix so as to not disturb the flow of the main proof with unnecessary details.

Theorem:

$LN(u) \neq \emptyset$ implies that $EN(u) \neq \emptyset$.

Proof:

Since $LN(u)$ is non-empty, pick a z in $LN(u)$. Consider the following optimization problem:

$$\text{Inf } \int \left(\sum_j x^j \right) d\mu$$

subject to

$$(1) \quad x \in LN(u)$$

$$(2) \quad x \leq z$$

where $x = (x^1, \dots, x^n)$.

Endow L with the relative weak-star topology (viewing it as a subset of L^∞) and $(L)^n$ with the product topology. The objective function, viewed as a map from $(L)^n$ into \mathbb{R}_+ , is continuous since addition is a continuous operation and L^1 is the pre-dual of L^∞ . (The integrable function here is $p(t) = 1$ for all $t \in \mathbb{R}_+$.)

As an application of the theorems of Banach-Alaoglu (closed unit ball in L^∞ is weak-star compact) and Tychonoff (product of an arbitrary number of compact sets is compact in the product topology), the set of x in $(L)^n$ satisfying constraint (2) is contained in a compact subset of $(L^\infty)^n$.

Since L is weak-star closed in L^∞ (verified in Proposition 2, which utilizes a simple technical result verified in Proposition 1), the set of x in $(L)^n$ satisfying constraint (2) is compact in $(L)^n$. Since a continuous (real-valued) function on a compact set achieves its infimum, then in order to show that $EN(u)$ is non-empty, it suffices to show that $LN(u)$ is a weak-star closed subset of $(L)^n$. This is because any point achieving the minimum in the optimization problem is clearly an efficient point.

To show that $LN(u)$ is a weak-star closed subset of $(L)^n$, let $\{x_\alpha\}$ be a net in $LN(u)$ converging to x . (One argues in terms of nets because L^∞ under the weak-star topology does not have a countable neighborhood base.) If x_α is in $LN(u)$, one can find all of those functions necessary to satisfy constraints (3.1) - (3.6). (To satisfy constraint (3.5), one can find all of those functions necessary to satisfy constraints (2.1) - (2.8).) To show that x is in $LN(u)$, one needs to find "limit" functions which satisfy constraints (3.1) - (3.6). (To satisfy constraint (3.5), one needs to find those "limit" functions which satisfy constraints (2.1) - (2.8).)

To find the "limit" functions, one shows that for each net of functions, there exists a sub-net which is uniformly bounded in the usual norm. Then, by repeated application of the theorem of Banach-Alaoglu (and the fact that the weak-star topology on L^∞ makes L^∞ into a topological vector space which means that the operation of scaling is a homeomorphism), one can extract convergent sub-nets for each of these nets obtaining limit functions. By duality, the integral constraints will hold for the limit functions. The proof would then be complete.

All of the constraints have the following structural form:

$$0 \leq s + \int_0^t (x - T - y)d\mu \leq C, \text{ for all } t \in [0, T]$$

where s represents the initial stock, if any, C the stock capacity ($s \leq C$), and where x represents the flow of input into the system, T represents the disposal out of the system, and y the flow of input into the production process. This leads one to propose and prove Proposition 3. Once shown, it paves the way for a "verbal proof" of the theorem.

Proposition 3:

Let $\{x_\alpha\}$, $\{T_\alpha\}$, $\{y_\alpha\}$ be nets in L satisfying, for each α ,
 $0 \leq s + \int_0^t (x_\alpha - T_\alpha - y_\alpha) d\mu \leq C$, for all $t \in [0, T]$. Assume that the net
 $\{x_\alpha\}$ converges weak-star to x . Then there exist sub-nets of the original
 nets which are uniformly bounded.

Proof:

First, one shows that there exists some β such that for all $\gamma \geq \beta$
 $\{x_\gamma\}$ is uniformly bounded. If this were not true, this would imply the
 existence of a countable sub-net $\{x_n\}$ such that $\|x_n\|_\infty > k$ for each n
 and for any positive integer m , one can find an n such that $\|x_n\|_\infty > m$.

Fix $\epsilon > 0$. Since $x_n \rightarrow x$, there exists some N for which

$$\int x_n d\mu \leq \int x d\mu + \epsilon \quad \text{for all } n > N.$$

But because the net is in L , one has that for $0 < \gamma_n < \min \{\|x_n\|_\infty - k, \epsilon\}$

$$\int x_n d\mu \geq (\|x_n\|_\infty - \gamma_n) \mu\{x_n > \|x_n\|_\infty - \gamma_n\} > (\|x_n\|_\infty - \gamma_n) \delta.$$

Hence, $\|x_n\|_\infty < \frac{\int x d\mu + 2\epsilon}{\delta}$ for all n , which is a contradiction. Let
 B be a bound on the x_γ 's.

Now it is easy to see that the sub-net $\{y_\gamma\}$ (and $\{T_\gamma\}$) must have
 the same property as that stated for the original net $\{x_\alpha\}$; if not, ex-
 tract a further sub-net with the same properties as before. Then one has
 the following chain of inequalities for $0 < \gamma_n < \min \{\|y_n\|_\infty - k, \epsilon\}$:

$$\begin{aligned}
\|y_n\|_{\infty} \delta &< \int y_n d\mu + \delta \cdot \gamma_n \\
&\leq s + \int x_n d\mu + \delta \cdot \epsilon \\
&\leq s + T \cdot B + \delta \cdot \epsilon \quad \text{for all } n
\end{aligned}$$

and a contradiction would be reached. The result follows easily. \square

Now one may proceed using repeated applications of Proposition 3 and Banach-Alaoglu to obtain the desired result--as long as it is done in the proper order.

First treat the storage case. Since $x_{\alpha}^j \rightarrow x^j$, for each j , this implies by use of Proposition 3 and constraint (3.1) that there exist sub-nets of x_{0i}^j nets which are uniformly bounded. By Proposition 3 and constraint (2.1), there exist sub-nets of the y_i^j nets which are uniformly bounded too. Using assumption A.1, one has that there exist sub-nets of the $F_i^k(y_i, W_i)$ nets which are uniformly bounded. Constraint (2.6) and the usual argument gives us the uniform boundedness of the appropriate sub-nets associated with the V_i nets. Constraint (2.5) and the fact that $V_i = \sum_{j \neq i} V_{ij}$ in turn imply the existence of the appropriate sub-nets of the V_{ij} nets and the W_i nets which are uniformly bounded. Finally, in a similar vein, sub-nets exist associated with the disposal rate function nets which are similarly uniformly bounded. Thus, one can find sub-nets of the original nets which have, by Banach-Alaoglu, convergent sub-nets.

The limit functions satisfy all of the constraints necessary to make x in $LN(u)$ by duality where, in this case, the integrable function p is the function which is identically 1. Under non-storability, the arguments are even simpler and will be omitted. Constraints (2.3) and (2.4) will follow for the limit functions by Proposition 1. The proof is now complete. $\|\|$

The objective function in the optimization problem given above has the functional form:

$$(4.1) \quad \int \left(\sum_j p^j x^j \right) d\mu$$

where the p^j 's were all taken to be the function which is identically 1. For any $(p^1, \dots, p^n) \in (L^1)^n$, the functional expression given in (4.1), viewed as a map from $(L)^n$ into \mathbb{R} , is continuous when $(L)^n$ is endowed with the product topology (L endowed with the relative weak-star topology). Thus, a solution to the optimization problem posed in this section for general $p \in (L^1)^n$ exists. If the $p \in (L^1)^n$ were taken to be strictly positive (an element of $(L^1)_{++}$, where the $++$ denotes strictly positive functions), then any point achieving the infimum would clearly be efficient with respect to the particular p .

This leads one to define a correspondence from $(p,u) \in (L^1)_{++}^n \times (L)^m \rightarrow \psi(p,u) \in (L)^n$, where $\psi(p,u)$ denotes the set of all minimizers for the optimization problem given in this section with the objective functions given in (4.1). The p^j 's may be thought of as weighting factors for the resources which take into consideration their relative value to the production planner and the fact the future costs may need to be discounted. In fact, future costs were not discounted in the original problem; any point $x \in \psi(p,u)$, where the p^j 's are taken to be the function which is identically 1, minimizes the total amount of resource.

Each point in $\bigcup_{p \in (L^1)_{++}^n} \psi(p,u)$ is efficient and has the property that it can be obtained through minimization of an appropriate optimization problem.

(An interesting theoretical problem is under what conditions all efficient points can be obtained in this way, i.e., when is $\bigcup_{p \in (L^1)_{++}^n} \psi(p,u)$ actually equal to $EN(u)$?) A solution to the problem for general p , if found, still does not inform the production planner what is an efficient choice for the distribution of the resources among the activities, (the x_{0i} 's), the output requirements for the activities, (the V_i 's), and the intermediate product transfers, (the V_{ij} 's), since there may be many alternatives for a given $x \in \psi(p,u)$. Even if these latter functions were obtained in some way, the individual activities must still select the exogenous inputs to be applied to their production process, (the y_i 's), and the intermediate product transfer inputs to be applied to the production process, (the W_i 's), in order to obtain their designated output requirements, (the V_i 's).

This could be achieved by solving appropriate optimization problems in stages. First, one "selects" an appropriate $p \in (L^1)_{++}^n$ and "obtains" an $x \in \psi(p,u)$. From this, one defines a correspondence that maps an (x,u) pair into the set of all allocations, (the x_{0i} 's), output requirements, (the V_i 's), and intermediate product transfers, (the V_{ij} 's), which appear in a representation of a *feasible flow for x to support the output level u* . By the latter statement, we mean any collection of functions x_{0i} 's, V_i 's, V_{ij} 's, y_i 's, W_i 's, T_i 's satisfying the constraints (2.1) - (2.8) and (3.1) - (3.6) that make x an element of $LN(u)$. One then minimizes some sort of objective function, similar in spirit to (4.1), subject to the constraint that these functions are members of the correspondence.

Of course, any type of hierarchical production planning must be developed more precisely to see what kind of solutions one can obtain, if any,

and what properties they have. Correspondences could be analyzed as to what conditions on the production functions would be necessary and sufficient to guarantee continuity-like properties.

Finally, all decision-making developed here is deterministic; demand levels (the function u) are typically stochastic as are the outputs obtained through activity production (the F_1^k 's). An investigation into an appropriate stochastic framework should be undertaken.

5. ADDITIONAL COMMENTS

The scope of this introductory treatment was simply to illustrate the complexities of developing a general model of production. Therefore, some additional comments are in order.

First, (weak-star) continuity of the Production Function is necessary to complete the proof of the non-emptiness of the technically efficient subset. At this stage of development, this property will be assumed.

Second, the property defining the space of feasible flows of goods and services, while not unreasonable, does not address the issue of the structure of a feasible flow. Note, the property implies that if the maximum of a flow is larger than k then the set on which the maximum is attained has measure at least δ (which is somewhat restrictive).

Third, the correspondences $\psi(p,u)$, as defined, depend on the vector z given in the optimization problem. By property P.1, however, we may restrict attention to a bounded subset of $LN(u)$ anyway. (Observe that if x^* represents an optimal solution, assuming that it exists, then $\|x^*\|_\infty \leq \max \left\{ \frac{\int z^j d\mu}{\delta}, k \right\}$. Thus, we need only know that a z exists--this eliminates the need to restrict attention to the set $\{x : x \leq z\}$.)

Finally, by the use of Lebesgue measure, the model introduced here applies only to continuous flow production systems. Construction projects, for example, do not fit this framework. A more elaborate model is required.

It is noted that general model of production is being developed. The model which is presented *axiomatically* (1) allows for event-based flows (thus almost all production systems are modeled), (2) justifies weak-star continuity of the Production Function, (3) parametrically restricts the flows of goods and services (thus giving a rigorous explanation of their structure),

(4) shows that various laws of production hold, and (5) illustrates how planning models such as Material Requirements Planning and the ordinary Critical Path Method are special cases of the model introduced. In addition, the model is used to address some of the *metascientific queries* in the field of production. See Hackman [1] for details.

6. APPENDIX

Proposition 1:

Endow L_+^∞ with the relative weak-star topology. If $\{x_\alpha\}$ and $\{y_\alpha\}$ are nets in L_+^∞ such that $x_\alpha \leq y_\alpha$ for all α , and if $x_\alpha \rightarrow x$, $y_\alpha \rightarrow y$, then $x \leq y$.

Proof:

Suppose not. This implies that one can find a set A with positive (finite) measure and an $\epsilon > 0$ such that the function x exceeds y by ϵ on the set A . Let $p = 1_A$. By the finiteness of the measure, one has that $p \in L^1$. Let ϕ_p denote the continuous linear functional associated with p . Then one has that $\phi_p(x_\alpha) \leq \phi_p(y_\alpha)$, $\phi_p(x_\alpha) \rightarrow \phi_p(x)$, $\phi_p(y_\alpha) \rightarrow \phi_p(y)$, but $\phi_p(x) - \phi_p(y) > \epsilon\mu(A) > 0$. The contradiction is readily apparent. \square

Proposition 2:

L is weak-star closed in L^∞ .

Proof:

By Proposition 1, it is easy to see that L_+^∞ is weak-star closed in L^∞ , so it is sufficient to show that L is weak-star closed in L_+^∞ .

For this, one shows that the complement is open.

Let $z \in L_+^\infty$ but $z \notin L$. This implies the existence of some constant $c > k$ such that $0 < \mu\{z > c\} < \delta$. Let $p = 1_{\{z > c\}}$. Choose γ so that $0 < \gamma < c - k$ and let $\epsilon = (c - k - \gamma)\mu\{z > c\}$. Let $N(q, \beta)$ denote a basic open neighborhood about the origin in L^∞ , i.e.,

$$N(q, \beta) = \{h \in (L^\infty) : \left| \int (q \cdot h) d\mu \right| < \beta\} .$$

Let $h \in (z + N(p, \epsilon)) \cap L_+^\infty = \{h \in L_+^\infty : \left| \int p(z - h) d\mu \right| < \epsilon\} .$

If $h \leq k + \gamma$ on $\{z > c\}$, then one would have that

$$\left| \int_{\{z > c\}} (z - h) d\mu \right| \geq (c - k + \gamma) \mu\{z > c\} = \epsilon$$

which contradicts h being an element of $(z + N(p, \epsilon)) \cap L_+^\infty$. Therefore, there exists a subset A of $\{z > c\}$ of positive measure less than δ for which $h > k + \gamma$. Hence, $h \notin L$.

A neighborhood of z has been found which does not meet L , showing that the complement of L (in L_+^∞) is open. \square

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