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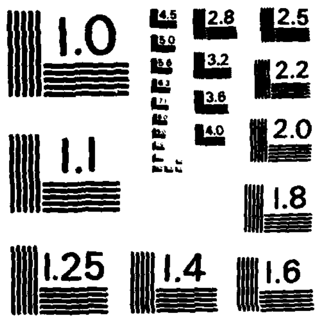

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The Gauss-Tchebyshev Inequality for Unimodal Distributions

by

S. W. Dharmadhikari and Kumar Joag-dev[1]

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and Florida State University

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Key Words and Phrases. Gauss' inequality, Tchebyshev's inequality, Markov's inequality, unimodal distributions, convex structure.

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Summary

Let X be a random variable whose distribution is unimodal with mean μ. For $r > 0$, let $\lambda_r = \{E|X - \mu|^r\}^{1/r}$. In this paper, we determine a value k_r such that

$$P(|X - \mu| \geq k\lambda_r) \leq [r/(r + 1)]^r . k^{-r},$$

for all $k \geq k_r$. This improves and extends a recent result of Vysochanskii and Petunin (1979) who have only considered the case $r = 2$ with a higher value for k_2. Our proof is also considerably simpler because it uses the convex structure of the class of unimodal distributions.

AMS 1980 Subject Classification: 60 E 15

Key Words and Phrases. Gauss' inequality, Tchebyshev's inequality, Markov's inequality, unimodal distributions, convex structure.

1. Introduction.

Let X be a real random variable with mean μ and let $r > 0$. Markov's inequality states that, for every given a and every $k > 0$,

$$(1.1) \qquad P(|X - a| \geq k) \leq E(|X - a|^r)/k^r.$$

If $a = \mu$ and $r = 2$, (1.1) reduces to the usual Tchebyshev inequality. Suppose now that the distribution of X is unimodal with a mode M. A result attributed to Gauss (1821) states that

$$(1.2) \qquad P(|X - M| \geq k) \leq (4/9)\, E\,(|X - M|^2)/k^2,$$

for all $k > 0$. In other words, if $a = M$, the bound on the right side of (1.1) can be reduced by a factor (4/9) when $r = 2$. As a consequence, if the distribution of X is both symmetric and unimodal, then $M = \mu$ and (1.2) gives

$$(1.3) \qquad P(|X - \mu| \geq k) \leq 4\sigma^2/(9k^2),$$

where σ^2 = Var (X). Recently, Vysochanskii and Petunin (1979) showed that (1.3) is valid without the assumption of symmetry as long as $k \geq \sqrt{8/3}$. In this paper, we first obtain the factor by which the bound in (1.1) can be improved if the distribution is unimodal and $a = M$. We then show that the improved bound is valid even if $a = \mu$ as long as k is suitably large. For $r = 2$, we need $k \geq \sqrt{19/3}$, which is better than the value $\sqrt{8/3}$ obtained by Vysocanskii and Petunin.

2. Preliminaries.

In this section we give some results on certain convex sets of distributions.

DEFINITION 2.1. A distribution function F is said to be unimodal about a mode M if F is convex on $(-\infty, M)$ and concave on (M, ∞).

Let $\mathcal{C}_M$ denote the set of all distributions on R that are unimodal about M. Then $\mathcal{C}_M$ is clearly convex (under mixtures). It is also closed under weak convergence; see Gnedenko and Kolmogorov (1968), Section 32. Let $\mathcal{U}_M$ denote the set of all uniform distributions on intervals with M as one end point. Then C_M is the closed convex hull of U_M. Another equivalent statement of this result is as follows; [see Feller (1971), p. 158].

THEOREM 2.1. A random variable X has a unimodal distribution with mode M if, and only if, X is distributed as M + UZ, where U is uniform on (0, 1) and U, Z are independent.

This theorem enables one to reduce many problems involving unimodal distributions to those involving uniform distributions.

Let $\mathcal{D}_\mu$ denote the set of all distributions on R which have mean μ and finite support. The following lemma is possibly known.

LEMMA. 2.1. Every distribution in D_μ is a finite convex mixture of one or two point distributions with mean μ.

Proof. Without loss of generality, let $\mu = 0$. Let $P \in D_0$ and let ν be the size of the support of P. The lemma holds if $\nu \le 2$. Suppose the lemma holds for $\nu \le n$, where $n \ge 2$. Let Y be a random variable with distribution P and suppose Y takes exactly (n + 1) values. Since Y is not degenerate and $E(Y) = 0$, we can find $a > 0$ such that

$$\xi = P(Y = -a) > 0 \text{ and } \eta = P(Y = b) > 0.$$

Without loss of generality, assume that $a\xi \ge b\eta$. Consider the two-point distribution P_0 which puts mass $a/(a + b)$ at the point b mass $b/(a + b)$ at the point (-a). Then P_0 has zero mean and

$$(2.1) \qquad P = \alpha P_0 + (1 - \alpha)P_1,$$

where $\alpha = \eta(a + b)/a$. Note that αP_0 accounts for all the mass at b. It is clear that $\alpha > 0$. On the other hand, since Y takes at least 3 values, we must have $\xi + \eta < 1$. Therefore

$$\eta(a + b) = a\eta + b\eta \leq a\eta + a\xi = a(\xi + \eta) < a. \text{ Thus } \alpha < 1.$$

The quantity P_1 in (2.1) is a distribution which puts positive mass at $\leq n$ points, since the mass at b is accounted for by αP_0. By the induction hypothesis, P_1 is expressible as a mixture of one or two point distributions with zero mean. Therefore, by (2.1), P also can be expressed as a mixture of the required type. The proof of the lemma is now complete.

The following lemma is standard.

LEMMA 2.2. _Let_ $r > 0$ _and let_ X _be a real random variable with_ $E(|X|^r) < \infty$. _Then we find a sequence of random variables_ X_n _such that each_ X_n _takes only a finite number of values and_ $E(|X_n - X|^r) \to 0$. _Moreover, if_ $r \geq 1$, _then we can shoose the_ X_n _in such a way that_ $E(X_n) = E(X)$ _for all_ n.

3. The Gauss-Tchebyshev inequality.

The Markov inequality states that

$$(3.1) \qquad P(|X - a| \geq k) \leq E(|X - a|^r)/k^r,$$

where X is a real random variable, $a \in R$, $r > 0$ and $k > 0$. If $a = E(X)$ and $r = 2$, (3.1) gives the usual Tchebyshev inequality. If X has a distribution which is unimodal about M, then the bound on the right side of (3.1) can be reduced by a factor which depends on r. This is made precise by Theorem 3.1. below. For the special case $r = 2$, Theorem 3.1 goes back to Gauss (1821).

THEOREM 3.1. Let X have a distribution which is unimodal about M. Then for every $r > 0$ and every $k > 0$,

$$P(|X - M| \geq k) \leq \left(\frac{r}{r+1}\right)^r \frac{(E(|X - M|^r)}{k^r}. \tag{3.2}$$

Moreover, this bound is sharp.

Proof. Without loss of generality, let $M = 0$. Since (3.2) is trivially true if $E|X|^r = \infty$, we assume that $E|X|^r < \infty$. Since X is unimodal about zero, by Theorem 2.1, X has the same distribution as UZ, where U is uniform on (0, 1) and U, Z are independent. Now $E|X|^r = E(|Z|^r)/(r + 1)$. Therefore $E|Z|^r < \infty$. Lemma 2.2 shows that it is sufficient to establish (3.2) in the case where Z takes only a finite number of values. Now the set of distributions of Z, for which (3.2) is valid, is clearly convex. Therefore we need only consider the case where Z is degenerate. Finally, (3.2) is clearly unaffected by a change of scale. Therefore we may and do assume that Z is degenerate at 1, so that X has the uniform distribution on (0, 1). In this case, $E|X|^r = 1/(r + 1)$ and

$$P(|X| \geq k) = \begin{cases} (1 - k), & \text{if } 0 < k \leq 1 \\ 0, & \text{if } k \geq 1. \end{cases}$$

Therefore

$$k^r P(|X| \geq k) = \begin{cases} k^r(1 - k) & \text{if } 0 < k \leq 1 \\ 0, & \text{if } k \geq 1. \end{cases}$$

For fixed r, the last quantity becomes maximum when $k = r/(r + 1)$. The maximum value is $r^r/(r + 1)^{r+1}$. Therefore

$$k^r P(|X| \geq k \leq \left(\frac{r}{r+1}\right)^r \cdot \frac{1}{(r+1)} = \left(\frac{r}{r+1}\right)^r E|X|^r,$$

which proves (3.2). Further the above calculation shows that the bound is sharp.

The special case $r = 2$ gives the Gauss inequality.

COROLLARY 3.1. (Gauss). *If* X *has a distribution which is unimodal about* M, *then, for all* $k > 0$,

$$P(|X - M| \geq k) \leq 4\, E(|X - M|^2)/(9k^2)$$

COROLLARY 3.2. *Let* X *have a symmetric and unimodal distribution. Let* $\mu = E(X)$ *and* $\sigma^2 = \mathrm{Var}(X)$. *Then, for all* $k > 0$,

$$P(|X - \mu| \geq k\sigma) \leq 4/(9k^2). \tag{3.3}$$

Proof. Immediate from corollary 3.1, because $M = \mu$.

Recently, Vysochanskii and Petunin (1979) showed that (3.3) holds for unimodal random variables without the assumption of symmetry provided that $k \geq \sqrt{(8/3)}$. We improve and generalize their results below (Theorem 3.2). Our proof is also considerably simpler because we use the convex structures introduced in Section 2.

THEOREM 3.2. *Let* X *have a unimodal distribution with mean* μ. *Let* $\tau_r = E(|X - \mu|^r)$. *Then, for every* $k > 0$,

$$P(|X - \mu| \geq k) \leq \max\left[\frac{(r+1)\tau_r - k^r}{rk^r}, \left(\frac{r}{r+1}\right)^r \frac{\tau_r}{k^r}\right].$$

Proof. Without loss of generality assume that $\mu = 0$. Suppose X is unimodal about M. If 0 is also a mode of X, then the theorem follows from Theorem 3.1. So, suppose that X is not unimodal about 0. Again, we may assume that $M > 0$. By Theorem 2.1, X has the same distribution as $M + UZ$, where U is uniform on (0, 1) and U, Z are independent. Now $0 = E(X) = M + \frac{1}{2}E(Z)$.

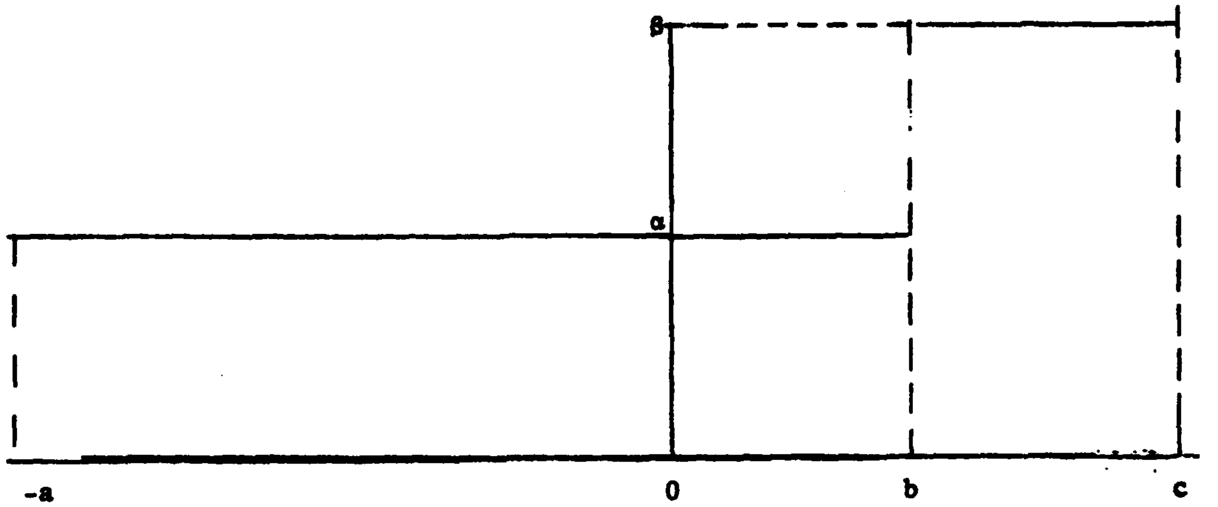

Fig. 1. Graph of the density f in the proof of Theorem 3.2.

Therefore $E(Z) = -2M$. It is clear from Lemma 2.2 that it is sufficient to prove the theorem in the case where Z takes only a finite number of values. Moreover, since the mean of X is fixed at 0, the class of distributions of X for which the theorem holds is convex. Therefore the second assertion of Lemma 2.2 and Lemma 2.1 show that it is sufficient to prove the theorem in the case where Z takes exactly two values. We have thus reduced our problem to the case where X has the density f given by

$$f(x) = \begin{cases} \alpha & , \text{ if } -a < x < b, \\ \beta & , \text{ if } b < x < c, \\ 0 & , \text{ elsewhere} \end{cases}$$

Here a, b, c are suitable positive constants. A graph of f is given in Fig. 1. Since f is not to be unimodal about 0, we must have $\alpha < \beta$. Further the condition $E(X) = 0$ requires that $b < c < a$. Three cases arise.

Case 1. Suppose $0 < k < b$. Here $P[|X| < k] = 2\alpha k$ and so

$$\int_{|t|<k} |t|^r f(t)dt = \frac{2\alpha k^{r+1}}{(r+1)} = \frac{k^r P[|X| < k]}{(r+1)} . \tag{3.4}$$

Case 2. Suppose $b < k < c$. Here

$$P[|X| < k] = \alpha(b + k) + \beta(k - b),$$

and

$$\int_{|t|<k} |t|^r f(t)dt = \frac{\alpha(b^{r+1}+k^{r+1}) + \beta(k^{r+1}-b^{r+1})}{(r + 1)} .$$

Simple algebraic manipulations yield

$$(3.5)\qquad \int_{|t|<k} |t|^r f(t)\,dt - \frac{k^r P[|X|<k]}{(r+1)} = \frac{b(\beta-\alpha)\,(k^r-b^r)}{(r+1)}\,.$$

Since $\alpha < \beta$ and $0 < b < k$, the right side of (3.5) is positive.

Consider the two cases together. That is, let $0 < k < c$. Then (3.4) and (3.5) show that

$$(3.6)\qquad \int_{|t|<k} |t|^r f(t)\,dt \geq \frac{k^r P[|X|<k]}{(r+1)}\,.$$

Now

$$\tau_r = E|X|^r = \int_{|t|\geq k} |t|^r f(t)\,dt + \int_{|t|<k} |t|^r f(t)\,dt$$

$$\geq k^r P[|X|\geq k] + \frac{k^r P[|X|<k]}{(r+1)}, \quad \text{[using (3.6)]}.$$

Writing $P[|X| < k] = 1 - P[|X| \geq k]$, we get

$$\tau_r \geq k^r \left[\left(\frac{r}{r+1}\right) P[|X| \geq k] + \frac{1}{(r+1)}\right],$$

Therefore

$$(3.7)\qquad P[|X| \geq k] \leq \frac{(r+1)\tau_r - k^r}{rk^r}$$

<u>Case 3.</u> Suppose that $c < k$. Define a new density g as follows.

$$g(x) = \begin{cases} \gamma\,, & \text{if } 0 < x < c, \\ f(x)\,, & \text{elsewhere} \end{cases}$$

Since g agrees with f outside the interval (0, c), the constant γ must satisfy

$$(3.8)\qquad \gamma c = \int_0^c f(t)dt = \alpha b + \beta(c-b).$$

Now let $\delta_r = \int_{-\infty}^{\infty} |t|^r g(t)dt$. Then

$$(r+1)\ (\tau_r-\delta_r) = (r+1)\ [\int_0^c t^r f(t)dt - \int_0^c t^r g(t)dt]$$

$$= \alpha b^{r+1} + \beta(c^{r+1}-b^{r+1}) - \gamma c^{r+1}$$

$$= \alpha b^{r+1} + \beta(c^{r+1}-b^{r+1}) - c^r[\alpha b+\beta(c-b)],\ [\text{using } (3.8)]$$

$$= b(\beta-\alpha)\ (c^r - b^r).$$

Since $\alpha < \beta$ and $0 < b < c$, we see that $\delta_r \le \tau_r$. Let Y be a random variable with density g. Since g is unimodal about 0, Theorem 3.1 shows that

$$P(|Y| \ge k) \le (\frac{r}{r+1})^r \frac{\delta_r}{k^r} \le (\frac{r}{r+1})^r \frac{\tau_r}{k^r}\ .$$

But since $k > c$, the densities g and f agree on the set $(-\infty, -k] \cup [k, \infty)$. Therefore

$$(3.9)\qquad P[|X| \ge k] = P[|Y| \ge k] \le (\frac{r}{r+1})^r \frac{\tau_r}{k^r}\ .$$

The theorem now follows from (3.7) and (3.9).

COROLLARY 3.3. Let X be a unimodal random variable with mean μ. Let $\lambda_r = \{E(|X-\mu|^r)\}^{1/r}$. Then, for every $k > 0$,

$$P[|X-\mu| \ge k\lambda_r] \le \max \left\{ \frac{(r+1) -k^r}{rk^r}\ ,\quad [\frac{r}{(r+1)k}]^r \right\}.$$

Proof. Immediate from Theorem 3.2, if we replace k by $k\lambda_r$ and note $\lambda_r^r = \tau_r$. Observe that

$$\frac{(r+1)-k^r}{r} \le \left(\frac{r}{r+1}\right)^r \text{ whenever } k \ge k_r, \text{ where}$$

$$k_r = \left[\frac{(r+1)^{r+1} - r^{r+1}}{(r+1)^r}\right]^{1/r} . \tag{3.10}$$

Therefore, the following corollary is immediate.

COROLLARY 3.4. *With the same notation as in Corollary* 3.3,

$$P(|X - \mu| \ge k\lambda_r) \le \left(\frac{r}{r+1}\right)^r k^{-r} ,$$

for all $k \ge k_r$, *where* k_r *is given by* (3.10).

For a comparison of our results with those given by Vysochanskii and Petunin, we write the special cases of the last two corollaries when $r = 2$.

COROLLARY 3.5. *Let* X *be a unimodal random variable with mean* μ *and variance* σ^2. *Then, for every* $k > 0$,

$$P(|X - \mu| \ge k\sigma) \le \max\left[\frac{3-k^2}{2k^2}, \frac{4}{9k^2}\right] . \tag{3.11}$$

Consequently, for every $k \ge \sqrt{19}/3$,

$$P(|X - \mu| \ge k\sigma) \le \frac{4}{9k^2} . \tag{3.12}$$

Proof. We only need to note that $k_2 = \sqrt{19}/3$.

REMARK. The inequality (3.11) is an improvement of the result of Vysochanskii and Petunin (1979). They have $(4-k^2)/3$ in place of our $3-k^2/2$. Consequently, they prove (3.12) for all $k \ge \sqrt{8/3}$.

It is to be noted that (3.12) does not hold for all $k > 0$, if the distribution is not symmetric. The following detailed analysis of the example considered by Vysochanskii and Petunin shows that (3.12) can fail if $k = 1.385$. We note that $1.385 < \sqrt{19/3}$.

EXAMPLE 3.1. Let $a \ge 1$ and consider a random variable X such that $P(X = 1) = (a - 1)/(a + 1)$ and

$$P(X \le x) = 2(x + 1)/(a + 1)^2 \ , \ -a < x < 1.$$

It is easy to check that $\mu = E(X) = 0$ and $\sigma^2 = \mathrm{Var}(X) = (2a - 1)/3$. Now

$$P(|X| \ge 1) = \frac{a - 1}{a + 1} + \frac{2(a - 1)}{(a + 1)^2} = \frac{(a - 1)\ (a + 3)}{(a + 1)^2}\ .$$

We now set $k\sigma = 1$. That is, $k = (1/\sigma)$. Then

$$\begin{aligned} k^2 P(|X \quad \mu| \ge k\sigma) &= \sigma^{-2}\ P(|X| \ge 1) \\ &= \frac{3(a - 1)\ (a + 3)}{(2a - 1)\ (a + 1)^2} = g(a),\ \text{say} \end{aligned}$$

The condition $g(a) > (4/9)$ reduces to,

$$8a^3 - 15a^2 - 54a + 77 < 0. \tag{3.13}$$

Numerical calculations show that (3.13) holds for $1.2816 \le a \le 3.05$. Since $k = \sigma^{-1}$, we see that (3.12) can fail if $.767 \le k \le 1.385$.

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