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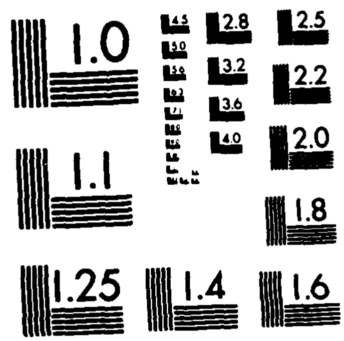

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MORSE TYPE INDEX THEORY FOR FLOWS AND PERIODIC SOLUTIONS FOR HAMILTONIAN EQUATIONS

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ABSTRACT

This paper has two aims. First, in an expository style an index theory for flows is presented, which extends the classical Morse-theory for gradient flows on manifolds. Secondly this theory is applied in the study of the forced oscillation problem of time dependent (periodic in time) and asymptotically linear Hamiltonian equations. Using the classical variational principle for periodic solutions of Hamiltonian systems a Morse-theory for periodic solutions of such systems is established. In particular a winding number, similar to the Maslov index of a periodic solution is introduced, which is related to the Morse-index of the corresponding critical point. This added structure is useful in the interpretation of the periodic solutions found.

AMS (MOS) Subject Classifications: 58E05, 70H05, 58F22, 70H25

Key Words: Index theory for flows, winding number of a periodic solution, forced oscillations of Hamiltonian equations, Morse theory for periodic solutions.

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SIGNIFICANCE AND EXPLANATION

The basic laws of Physics are governed by action principles. Equilibrium states are critical points of an 'action functional.' In most cases (in particular that of this report) these functionals are "infinitely indefinite" and classical Morse theory does not apply. In this report a modified theory is described and is applied to find periodic solutions of Hamiltonian systems of equations. In particular a theorem is proved which is analogous to Morse's theorem relating the index of a closed geodesic (as a critical point of an Energy functional) to the number of conjugate points on the geodesic.

The report is one of many steps in the development of a "Morse Theory" for infinitely indefinite functionals.

The responsibility for the wording and views expressed in this descriptive summary lies with MRC, and not with the authors of this report.

MORSE TYPE INDEX THEORY FOR FLOWS AND PERIODIC SOLUTIONS FOR HAMILTONIAN EQUATIONS

Charles Conley and Eduard Zehnder

Introduction

Let $h = h(t,x) \in C^2(R \times R^{2n})$, $n \geq 2$. We consider the time dependent Hamiltonian vectorfield

$$(1) \qquad \dot{x} = Jh'(t,x), \qquad (t,x) \in R \times R^{2n} ,$$

where J is the standard symplectic structure in R^{2n}, and where h' denotes the gradient of h with respect to x. Assuming the Hamiltonian function h to depend periodically on time:

$$h(t+T,x) = h(t,x) \quad ,$$

for some $T > o$, we are looking for periodic solutions of (1) having period T, $x(t) = x(t+T)$. Such solutions correspond in a one-to-one way to the critical points of the following functional f defined on the loop space which is simply the space of periodic functions having period T:

$$(2) \qquad f(x) := \int_0^T \{ \tfrac{1}{2} < \dot{x}, Jx > - h(t,x(t)) \} \, dt.$$

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In fact, the equation (1) is the Euler equation of the variational problem: extr $f(x)$, and in order to have periodic solutions one has to impose periodic boundary conditions: $x(o) = x(T)$. The first variation of f is then given by

$$\delta f(x)y = \int_o^T < -J\dot{x} - h'(t,x), y > dt.$$

In the following this variational approach will be used in order to find periodic solutions. We observe that the functional f is neither bounded from below nor from above. (If h is convex a different functional could be used which is bounded from below, however we do not make such a requirement.) It turns out, that the critical points of f are saddle points having infinite dimensional stable and unstable invariant manifolds. In fact for the second variation of f at a critical point x_o we find the expression:

$$\delta^2 f(x_o)(y_1,y_2) = \int_o^T < -J\dot{y}_1 - h''(t,x_o)y_1, y_2 > dt.$$

The selfadjoint operator of this bilinear form (defined on the dense subspace $\{x \in H^1(o,T; R^{2n}) \mid x(o) = x(T)\}$ of L_2) can be seen to have a purely discrete spectrum which is unbounded from below and from above (see section 2). In order to set up a Morse theory for periodic solutions we need a relation between the particular periodic solution of (1) and its corresponding critical point of (2). For this purpose we introduce for a periodic solution of (1) an index, which will turn out to be roughly the signature of the Hessian of f at the corresponding critical point. To do so we pick any periodic solution $x_0(t) = x_0(t+T)$ of (1) and look at the linearized equation along this solution, i.e. at the linear equation

$$(3) \qquad \dot{y} = Jh''(t,x_0(t))y .$$

Setting $A(t) := h''(t,x_0(t))$ we can rewrite this equation as

$$(4) \qquad \dot{y} = JA(t)y ,$$

where $A(t)$ is symmetric, $t \to A(t)$ continuous, and $A(t+T) = A(t)$, i.e. is periodic of period $T > 0$. If now $X(t)$ is the fundamental solution of (4) which satisfies $\dot{X}(t) = JA(t)\,X(t)$ and $X(o) = 1$, then $X(t)$, $o \leq t \leq T$ is an arc in the group of symplectic matrices starting at the identity. The eigenvalues of the symplectic matrix $X(T)$ are called the Floquet multipliers associated to the periodic solution $x_0(t)$. We shall single out as nondegenerate periodic solutions those which do not have 1 as Floquet multiplier and hence define:

<u>Definition:</u> A periodic solution $x_0(t)$ of (1) is called nondegenerate, if it has no Floquet multiplier equal to 1.

This definition requires that the linear system (4), with periodic coefficients, admits no nontrivial periodic solutions with period T, as is well known from Floquet theory.

We now consider the set of continuous loops of symmetric matrices, $A(t) = A(t+T)$. which have the additional property that the corresponding equation (4) has no Floquet multiplier equal to 1. We call this set P. In P an equivalence relation is introduced as follows: two loops $A_0(t)$ and $A_1(t)$ are called equivalent, if one loop can continuously be deformed into the other one without leaving the set P of loops under consideration. In other words, there exists a continuous family $A_\sigma(t)$, $o \leq \sigma \leq 1$ of loops, such that $A_\sigma(t) = A_0(t)$ for $\sigma = o$ and $A_\sigma(t) = A_1(t)$ for $\sigma = 1$, and such that 1 is not an eigenvalue of $X_\sigma(T)$ for all $o \leq \sigma \leq 1$, where $X_\sigma(t)$ is the fundamental solution satisfying $\dot{X}_\sigma(t) = JA_\sigma(t) \cdot X_\sigma(t)$ and $X_\sigma(o) = 1$. It turns out that the set P decomposes into countably many equivalence classes, which are characterized by an integer, which will first be defined for a special constant loop.

Let $A(t) = S$ be a constant loop in P, the corresponding fundamental solution is then $\exp(tJS)$, and 1 is not a Floquet multiplier if $\exp(TJS)$ has no eigenvalue equal to 1. For an eigenvalue λ of JS we therefore have $\lambda \notin i\tau Z$, where $\tau = \frac{2\pi}{T}$. We now consider the purely imaginary eigenvalues of JS, and assume them to be distinct from each other. They occur in pairs. If $(\lambda, \overline{\lambda})$ is a pair of purely imaginary eigenvalues with corresponding complex eigenvectors $e, \overline{e}$, then $< \overline{e}, Je > \neq o$ is purely imaginary, and we set $\alpha(\lambda) := \text{sign}(- i < \overline{e}, Je >) \operatorname{Im} \lambda$. Observe that $\alpha(\lambda) = \alpha(\overline{\lambda}) = \alpha(-\lambda)$. Since, by assumption, $\alpha(\lambda) \notin \tau Z$, there is an integer m such that $m\tau < \alpha(\lambda) < (m+1)\tau$. In this case we set $[\alpha(\lambda)] = m + \frac{1}{2}$, and define

(5) $$j(S) = \sum_\lambda [\alpha(\lambda)] \in Z ,$$

where in the above summation λ runs over all purely imaginary eigenvalues of JS. If there are no such eigenvalues the sum is understood to be zero. Observe that $j(S)$ is an integer, since there are an even number of purely imaginary eigenvalues. As index of this special constant loop $A(t) = S$ in P we set

$$\text{(6)} \qquad \operatorname{ind}(A(t)) = j(S).$$

With this notation we can formulate:

Theorem 1

Each equivalence class of the set P of loops contains constant loops $A(t) = S$ for which $\operatorname{ind}(A(t))$ is defined as above. All such constant loops in the same equivalence class have the same index, and constant loops in different components of P have different indices. To every integer $j \in Z$ their is exactly one equivalence class having the index j.

In view of this theorem it is only necessary to define the index for the special class of constant loops chosen above. The theorem states that the index is well defined on components.

By means of theorem 1 we shall associate to every nondegenerate periodic solution $x_0(t)$ of the equation (1) the index j of the corresponding linearized equation (3). After these explanations we can formulate an existence statement for periodic solutions of an asymptotically linear Hamiltonian system.

Theorem 2.

Let $h = h(t,x) \in C^2(R \times R^{2n})$, $n \geq 2$, *be periodic in time of period* $T > o$, $h(t+T,x) = h(t,x)$. *Assume* (i) *the Hessian of* h *is bounded:* $-\beta \leq h''(t,x) \leq \beta$ *for all* $(t,x) \in R \times R^{2n}$ *and for some constant* $\beta > o$. *Assume* (ii) *the Hamiltonian vectorfield to be asymptotically linear*

$$Jh'(t,x) = JA_\infty(t)x + o(|x|), \text{ as } |x| \to \infty .$$

uniformly in t, *where* $A_\infty(t) = A_\infty(t+T)$ *is a continuous loop of symmetric matrices. Assume* (iii) *that the trivial solution of the equation* $\dot{x} = JA_\infty(t)x$ *is nondegenerate and denote its index by* j_∞. *Then the following statements hold:*

(1) *There exists a periodic solution of period* T *for* (1). *If this periodic solution is nondegenerate with index* j_0, *then there is a second T-periodic solution, provided* $j_0 \neq j_\infty$. *Moreover if there are two nondegenerate periodic solutions there is also a third periodic solution.*

(2) *Assume all the periodic solutions are nondegenerate, then there are only finitely many of them and their number is odd. If* j_k, $1 \leq k \leq m$, *denote their indices we have the following identity:*

$$\sum_{k=1}^{m} t^{-j_k} = t^{-j_\infty} + t^{-d}(1+t)\, Q_d(t) ,$$

where $d > o$ *is an integer, and where* $Q_d(t)$ *is a polynomial having nonnegative integer coefficients.*

The theorem extends earlier results in [3] and [14]. We point out an interesting special case of the above statement, which can be viewed as a generalization to higher dimensions of the Poincaré-Birkhoff fixed point theorem for mappings in the plane. This well known theorem states that a measure preserving homeomorphism of an annulus, which twists the two boundaries in opposite directions has at least two fixed points, see G.D. Birkhoff [5] and, more recently, M. Brown and W.D. Neumann [6].

Corollary.

Let $h = h(t,x) \in C^2(R \times R^{2n})$, $n \geq 2$ *be periodic,* $h(t+T),x) = h(t,x)$ *and let the Hessian of* h *to be bounded. Assume*

$$Jh'(t,x) = JA_\infty(t)x + o(|x|) \quad \text{as} \quad |x| \to \infty$$

$$Jh'(t,x) = JA_o(t)x + o(|x|) \quad \text{as} \quad |x| \to o$$

uniformly in t, *for two continuous loops* $A_o(t+T) = A_o(t)$ *and* $A_\infty(t+T) = A_\infty(t)$. *Assume that the two linear systems* $\dot{x} = JA_\infty(t)x$ *and* $\dot{x} = JA_o(t)x$ *do not admit any nontrivial* T-*periodic solutions, and denote by* j_∞ *and* j_o *the indices of these two linear systems. If* $j_\infty \neq j_o$ *then there exists a nontrivial* T-*periodic solution of* (1). *Moreover, if this periodic solution is also nondegenerate then there is a second* T-*periodic solution.*

In other words, if the two linear systems with $A_0(t)$ and $A_\infty(t)$ cannot be continuously deformed into each other within the set P, then we conclude the existence of a T-periodic orbit. The corollary only claims the existence of one T-periodic solution except if the non-degeneracy condition is satisfied. This is in contrast to the Poincaré-Birkhoff fixed point theorem which always guarantees two fixed points. Birkhoff's original proof in [5] also suggests, that the integer $|j_0-j_\infty|$ is a measure for the lower bound of the number of periodic solutions of (1). Our proof of the above statement being based on a Morse-type index theory does not allow such a conclusion. However, our statements given here may allow improvements similar to those allowed by using Ljusternik's category theory, when it is added to the classical Morse theory. As a sideremark we recall, however, that under additional assumptions the following result has been proved by means of mini-max techniques:

Theorem [3]

Let h *be as in the corollary and assume, in addition,* $h(t,x) = h(t,-x)$ *for all* $(t,x) \in R \times R^{2n}$. *Moreover, let* $A_0(t) = A_0$ *and* $A_\infty(t) = A_\infty$ *be independent of* t. *Then* (1) *has at least* $|j_0-j_\infty|$ *nontrivial pairs* $(x(t), -x(t))$ *of* T-*periodic solutions.*

As for the proof of theorem 2. we are looking for critical points of the functional f defined on the loop space. The assumption (i) allows the application of an analytical device due to H. Amann [1], which in this context was already used in [2] and which reduces the study of critical points of f to the study of critical points of a related functional, a, defined on a finite dimensional space Z, namely the trigonometrical polynomials of a fixed finite order. There exists an injective map u from Z into the whole loop space such that the critical points of the functional $a(z) = f(u(z))$ correspond in a one to one way to the critical points $u = u(z)$ of f in the loop space. To the gradient flow $\dot{z} = a'(z)$ we then apply a Morse-type index theory for flows, which is represented in section 3.

In order to briefly outline this index theory for flows we consider a flow on a topological space which is not necessarily a gradient flow on a manifold. To an isolated invariant set S an index pair (N_1,N_0) can be associated, where $N_0 \subset N_1$ is roughly the "exit set" of N_1, and where $S \subset \operatorname{int}(N_1 \setminus N_0)$, see section 3. The homotopy type of the pointed space N_1/N_0 then does not depend on the particular choice of index pairs for S and is called the index of S, and denoted by $h(S) := [N_1/N_0]$. We therefore can associate to an isolated invariant set S the algebraic invariant $p(t,h(S))$, which is the series in t whose coefficients are the ranks of Čech cohomology of an index-pair (N_1,N_0) for S. The index theory for flows then relates the algebraic invariants of S to the algebraic invariants of a Morse decomposition of S. The result is as follows (for a precise formulation of the statement we refer to section 3)

Theorem 3

Let S *be an isolated invariant set, and let* $(M_1,\ldots,M_m)$ *be an ordered Morse decomposition of* S, *where* $M_k \subset S$ *are isolated and invariant. Then there is a filtration* $N_0 \subset N_1 \subset \ldots \subset N_m$ *for this Morse decomposition, such that* (N_m,N_0) *is an index pair for* S *and such that* (N_j,N_{j-1}) *is an index pair for* M_j. *If we set* $h(M_j) = [N_j/N_{j-1}]$ *and* $h(S) = [N_m/N_0]$, *then the following identity holds:*

$$\sum_{j=1}^{m} p(t,h(M_j)) = p(t,h(S)) + (1+t)\,Q(t) ,$$

where $Q(t)$ *is a series in* t *having only nonnegative integer coefficients. This identity can be viewed as a generalization of the Morse inequalities.*

The development outlined here extends some of the results in [4]. It can be viewed as a generalization of Morse theory for flows other than gradient flows on spaces other than manifolds. An index is associated not only to critical points but to any isolated invariant set of a local flow. In addition to the classical Morse theory it includes Smale's generalization for periodic orbits [8]. More cogently, an analogue of the "Homotopy Axiom" of Leray-Schauder degree theory is possible in this generalized Morse theory. In this connection we observe that even in applications to gradient flows it is necessary to have an index for sets other than critical points in order to have this analogue, since a critical point may under deformation of the flow be continued to a set which does not consist just of critical points. With this addition, the generalized Morse theory becomes a useful tool in bifurcation theory.

The application of this index theory to the problem of periodic solutions is as follows. We first observe that due to the assumption (ii) and (iii) in theorem 2 the set S of bounded solutions of the gradient flow a' is compact, hence has an index. Using the invariance of the index under deformations crucially, this index is computed to be the homotopy type of a pointed sphere:

$$h(S) = [\dot{S}^{m_\infty}], \quad m_\infty = \frac{1}{2}\dim Z - j_\infty ,$$

hence $p(t,h(S)) = t^{m_\infty}$. Here $\dot{S}^{m_\infty}$ denotes a sphere of dimension m_∞ with a distinguishes point, $*$, that is a pair $(S^{m_\infty}, *)$. The critical points of the functional a for which we are looking comprise a Morse decomposition of the isolated invariant set. It turns out that if a periodic solution is nondegenerate with index j, then the corresponding critical point, $z \in Z$, of a is an isolated invariant set with index

$$h(\{z\}) = [\dot{S}^{m}], \; m = \frac{1}{2}\dim Z - j,$$

hence $p(t,h(\{z\})) = t^{m}$. The statements in theorem 2 are then an immediate consequence of theorem 3.

The statement of the above Corollary generalizes a corresponding result of H. Amann and E. Zehnder [3] improved by Kung-Ching Chang [14], where the linear systems "at o" and "at ∞" are assumed to be independent of time. It should be said that there are many recent existence results of periodic solutions of timedependent Hamiltonian systems, which however postulate strong asymptotic nonlinearities of the Hamiltonian vectorfield. For example, assuming a superquadratic behaviour of the Hamiltonian function at ∞, and an elliptic timeindependent equilibrium point at o, P. Rabinowitz [15] finds not only a T-periodic solution, but also subharmonic solutions, i.e. solutions of period kT, $k \in \mathbb{N}$. In such a situation it is not obvious how to isolate a suitable set of bounded solutions of the gradient system, to which the above outlined index-theory can be applied. The proof of Rabinowitz follows different lines and is based on mini-max arguments. We point out that the number of T-periodic solutions for such highly nonlinear systems is expected to be large. In the special case of dimension $n = 1$ this is in fact known, see H. Jacobowitz [25] and P. Hartmann [26], special results in higher dimensions are due to A. Bahri and H. Beresticki [24].

The organization of the paper is as follows. In the first section we describe the index for periodic solutions of timedependent Hamiltonian systems. In the second section theorem 2 will be proved. In section 3 the index theory for flows is represented. It makes use of the concepts and tools developed in [4], however for the readers convenience it will be developed from the beginning. We point out that the setting in which this theory is developed is more general than might the first be noticed. For example, it readily adapts to diffusion reaction equations, to functional delay equations and (as will be seen in a later paper) to the treatment of indefinite functionals in infinite dimensions, for which it is not immediately clear there even is a flow. To bring this out some otherwise irrelevant propositions are added. The organization of the paper is seen from the following table of contents.

Finally we would like to thank J. Moser and H. Amann for helpful discussions and suggestions. The paper was done while the first autor was at the Ruhr-University Bochum as recipient of a Humboldt award.

1. Arcs in $Sp(n,R)$, $n \geq 2$

The aim of this section is to prove theorem 1 in the introduction. With $J \in \mathcal{L}(R^{2n})$ we denote the standard symplectic structure in R^{2n}:

$$J = \begin{pmatrix} o & -I \\ I & o \end{pmatrix} ,$$

where I is the identity matrix in R^n. We recall, that the group of symplectic matrices in R^{2n} is defined as $Sp(n,R) = \{M \in \mathcal{L}(R^{2n}) \mid M^T JM = J\}$, and we abbreviate in the following $W = Sp(n,R)$. With $W^* \subset W$ we denote the subset $W^* = \{M \in W \mid 1 \text{ is not an eigenvalue of } M\}$. Now consider the linear differential equation

$$\dot{x} = JA(t)x \tag{1.1}$$

in R^{2n}, where $A(t+1) = A(t)$ is continuous and periodic with period 1. If $X(t)$ is the fundamental solution:

$$\dot{X}(t) = JA(t)\, X(t),\ X(o) = 1 , \tag{1.2}$$

then $X(t)$, $o \leq t \leq 1$ is an arc in W. We consider loops $A(t)$ with the property that $X(1) \in W^*$. There is a one to one correspondence between the set of such equations and the set of continuously differentiable curves $X(t)$, $o \leq t \leq 1$, in W satisfying

$$(1.3) \qquad \begin{aligned} &X(o) = 1, \quad X(1) \in W^* \\ &X'(1) = X'(o)\, X(1) \,, \end{aligned}$$

the correspondence being given by $JA(t) = X'(t)\, X(t)^{-1}$. In order to prove theorem 1 we thus are led to investigate when two such paths in W can be continuously be deformed into each other without leaving that class. We introduce as P the set of paths $\gamma\colon [o,1] \to W$ such that $\gamma(o) = 1$ and $\gamma(1) \in W^*$:

$$(1.4) \qquad P = \{\gamma\colon [o,1] \to W \mid \gamma \text{ continuous, } \gamma(o) = 1 \text{ and } \gamma(1) \in W^*\} \,.$$

We give the set P the compact open topology and consider the equivalence classes defined as follows: We call γ_1 and $\gamma_2 \in P$ equivalent, $\gamma_1 \sim \gamma_2$, if there exist a continuous deformation $\delta\colon [o,1] \times [o,1] \to W$ satisfying

$$(1.5) \qquad \begin{aligned} &\delta(t,o) = \gamma_1(t) \quad \text{and} \quad \delta(t,1) = \gamma_2(t) \\ &\delta(1,\sigma) \in W^* \,, \qquad o \leq \sigma \leq 1 \\ &\delta(o,\sigma) = 1 \qquad\quad o \leq \sigma \leq 1 \end{aligned}$$

A simple example of a path in P is an exponential path, defined as follows. Pick a symmetric matrix $S \in \mathcal{L}(R^{2n})$, then $\gamma(t) = \exp(t\, JS)$, $o \leq t \leq 1$ is a path in W. This path is in P if and only if $\exp JS \in W^*$, or equivalently if and only if $2\pi i n$ is not an eigenvalue of JS for every integer $n \in Z$. Such a path will be called an exponential path. An exponential path corresponds to a constant loop $A(t) = S$, $o \leq t \leq 1$ in (1.1).

In order to formulate our first result, we define an index for an exponential path as we did in the introduction. We assume $\exp JS \in W^*$ and, in addition, we assume the purely imaginary eigenvalues of JS to be distinct. As index of the exponential path $\gamma \in P$, $\gamma(t) = \exp(tJS)$ we then define

$$(1.6) \qquad \operatorname{ind}(\gamma) = j(S) \in Z,$$

where the righthand side is defined by formula (5) of the introduction.

Theorem 1.1.

Each equivalence class of P *contains an exponential path,* $\gamma(t) = \exp(tJS)$ *for which* $\operatorname{ind}(\gamma)$ *is defined as above. All such exponential paths in the same equivalence class have the same index, and exponential paths in different components have different indices. To every integer* $j \in Z$ *there is exactly one equivalence class having the index* j.

In view of this theorem it is only necessary to define the index for the special class of paths in P chosen above. The theorem says, that the index actually depends only on the component of P containing the path. But at the end of the proof of theorem $|\cdot|$ we will be able to define an index for every $\gamma \in P$ intrinsically.

Theorem 1 of the introduction is an immediate consequence of theorem 1.1. In fact, the deformations (1.5) can be chosen to be differentiable and to satisfy (1.3) for every $0 \leq \sigma \leq 1$. This can effectively be proved using the local representation of canonical maps by means of "generating functions"; for details we refer to [13] . The proof of theorem 1.1 proceeds in several steps.

1.1 Contractible loops in Sp(n,R)

Every real symplectic matrix M can be represented in polar form as

$$(1.7) \qquad M = P \cdot O \ ,$$

where $P = (MM^T)^{1/2}$ is a positive definite symmetric und symplectic matrix, and where $O = P^{-1}M$ is an orthogonal symplectic matrix. This representation (1.7) is unique. The set of the above matrices P has the unique representation:

$$(1.8) \quad P = \exp A, \qquad A = \begin{pmatrix} a_1 & a_2 \\ a_2 & -a_1 \end{pmatrix} \ , \qquad a_1 = a_1^T, \quad a_2 = a_2^T \ ,$$

where $a_1, a_2 \in \mathcal{L}(R^n)$. In particular, the set of positive definite symplectic and symmetric matrices is contractible, so are then all the loops in this set.

Each matrix O, which is orthogonal and symplectic has the form

$$(1.9) \qquad O = \begin{pmatrix} u_1 & -u_2 \\ u_2 & u_1 \end{pmatrix} , \quad \overline{u} := u_1 + iu_2 \ ,$$

with $\bar{u} = u_1 + iu_2$ being a unitary matrix in $\mathcal{L}(C^n)$. This correspondence is one to one. These simple facts are well known, see for example M. Levi [16] and Gelfand-Lidskii [17].

Let now $\gamma: [0,1] \to Sp(n,R)$ be any continuous arc of symplectic matrices and let $\bar{u}(t)$ be the associated arc of unitary matrices. Let $\Delta(t)$ be a continuous function such that $\det \bar{u}(t) = \exp(i\Delta(t))$. Then $\Delta(1) - \Delta(0)$ depends only on γ. This number will be denoted by $\Delta(\gamma)$. If γ is a loop, i.e. $\gamma(0) = \gamma(1)$, then $\Delta(\gamma)$ is an integer multiple of 2π.

Lemma 1.1. The loop γ is contractible in $Sp(n,R)$ if and only if $\Delta(\gamma) = 0$.

Proof: The statement is well known for the group $U(n)$ of unitary matrices, to which we shall reduce the Lemma. According to (1.7) we have $\gamma(t) = P(t)\, O(t)$, where $P(t)$ is a loop of positive definite symmetric and symplectic matrices hence contractible, while $O(t)$ corresponds by (1.9) to a loop of unitary matrices, which is contractible if and only if $\Delta(\gamma) = 0$. ∎

1.2. Change of symplectic basis

A symplectic basis in R^{2n} is a basis $(e_1,\dots,e_n, f_1,\dots,f_n) =: (e,f)$ such that for the matrix $M := (e,f) \in \mathcal{L}(R^{2n})$ we have $M^TJM = J$ and $M^TM = 1$. Thus M is symplectic and orthogonal. Since the unitary group is connected, the set of symplectic basis is also connected.

Let $\overline{u}$ be the unitary matrix associated to an $M \in Sp(n,R)$, and let 0_o be a symplectic orthogonal matrix, then the unitary matrix associated to $0_o^{-1} \cdot M \cdot 0_o \in Sp(n,R)$ is $\overline{u}_o^{\,-1} \cdot \overline{u} \cdot \overline{u}_o$, where $\overline{u}_o$ corresponds to 0_o. Since $\det(\overline{u}^{-1} \cdot \overline{u} \cdot \overline{u}_o) = \det \overline{u}$ we conclude:

<u>*Lemma 1.2.*</u>

Let $0(t)$ *be an arc of symplectic orthogonal matrices, and let* $\gamma(t) = 0(t)^{-1} \cdot M \cdot 0(t)$, *for some* $M \in Sp(n,R)$. *Then* $\Delta(\gamma) = o$. *Thus if* γ *is an arc ending at* M, *and if* 0 *is symplectic orthogonal, then* γ *can be extended to an arc* $\tilde{\gamma}$ *ending at* $0^{-1} \cdot M \cdot 0$ *in such a way that* $\Delta(\gamma) = \Delta(\tilde{\gamma})$.

<u>1.3. Changing the spectrum</u>

Let $M \in Sp(n,R)$ then the eigenvalues of M occur in groups: if λ is an eigenvalue, then also λ^{-1}, $\overline{\lambda}$ and $\overline{\lambda}^{-1}$ are eigenvalues. Let $E(\alpha) = E_\alpha$ be the generalized eigenspace for the eigenvalue α of M, i.e. the nullspace of $(M-\alpha)^{2n}$. The following statement is well known:

<u>*Lemma 1.3.*</u>

If $\alpha\beta \neq 1$ *then* $< JE_\alpha, E_\beta > = o$.

<u>*Proof:*</u> Let E_α^k be the nullspace of $(M-\alpha)^k$, so that $o = E_\alpha^o \subset E_\alpha^1 \subset \ldots \subset E_\alpha^{2n} = E_\alpha$. It suffices to prove $< JE_\alpha^k, E_\beta^\ell > = o$ if $\alpha\beta \neq 1$ for all $k, \ell \geq o$, which will be done by induction with respect to $k + \ell$. For $k + \ell = o$ the statement is trivial and we shall assume $< JE_\alpha^\kappa, E_\beta^\lambda > = o$

for $\kappa+\lambda < k+\ell$. Let $s_\alpha \in E_\alpha^k$ and $s_\beta \in E_\beta^\ell$ and set $s_\alpha^1 = (M-\alpha)s_\alpha \in E_\alpha^{k-1}$ and $s_\beta^1 = (M-\beta)s_\beta \in E_\beta^{\ell-1}$. Thus $\alpha s_\alpha = Ms_\alpha - s_\alpha^1$ and $\beta s_\beta = Ms_\beta - s_\beta^1$, and therefore, M being symplectic, we conclude $\alpha\beta < Js_\alpha, s_\beta > =$ $= < Js_\alpha, s_\beta > - < JMs_\alpha, s_\beta^1 > - < Js_\alpha^1, Ms_\beta >$. Since $Ms_\alpha \in E_\alpha^k$ and $Ms_\beta \in E_\beta^\ell$, the last two terms vanish by the induction hypothesis and hence $(\alpha\beta - 1) < Js_\alpha, s_\beta > = o$, which proves the Lemma. •

We next describe how to change the eigenvalues of an eigenvalue group of a symplectic matrix in such a way that the eigenspaces remain unchanged. We pick an eigenvalue λ of $M \in Sp(n,\mathbb{R})$. For every complex number $\nu \in C$, $\nu \neq o$ we define a new matrix M_ν by:

$$M_\nu = M \text{ on } E(\mu) \text{ if } \mu \notin (\lambda,\lambda^{-1}, \bar{\lambda},\bar{\lambda}^{-1})$$

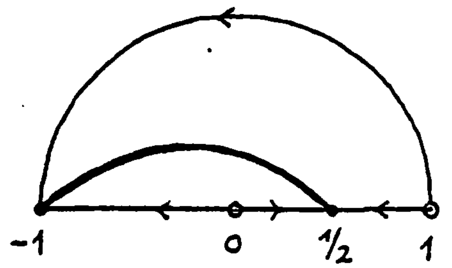
$$M_\nu \mid E(\phi(\lambda)) = \phi(\nu) \quad M \mid E(\phi(\lambda)) ,$$

where $\phi(z) = z, z^{-1}, \bar{z}$ or $\bar{z}^{-1}$. Observe the definition requires that if λ is real, then ν is real and if $\bar{\lambda} = \lambda^{-1}$, then $\bar{\nu} = \nu^{-1}$. The matrix M_ν is clearly real, it is also symplectic. Namely, if $x \in E(\alpha)$ and $y \in E(\beta)$, then by Lemma 1.3, $< JM_\nu x, M_\nu y > = < Jx,y > = o$ if $\alpha\beta \neq 1$. Assume $\alpha\beta = 1$ and assume $\alpha \notin (\lambda,\lambda^{-1}, \bar{\lambda},\bar{\lambda}^{-1})$, then $< JM_\nu x, M_\nu y > =$ $= < JMx, My > = < Jx,y >$. If on the other hand $\alpha \in (\lambda,\lambda^{-1}, \bar{\lambda},\bar{\lambda}^{-1})$ then by construction we again have $< JM_\nu x, M_\nu y > = < JMx, My > = < Jx,y >$, hence $M_\nu^T JM_\nu = J$ proving the claim. We now shall use the above construction in order to prove

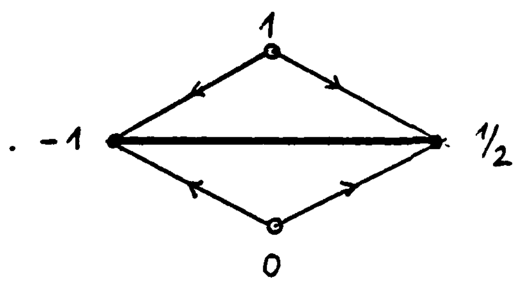
Lemma 1.4.

Let $W^{**} \subset W^{*}$ be the subset of matrices whose eigenvalues with unit modulus are equal to -1. Then W^{**} is a strong deformation retract of W^{*}.

Proof: The closed upper half disk in the complex plane minus the two points $\{o\}$ and $\{1\}$ admits a strong deformation retraction $r(z,t)$, $o \leq t \leq 1$ to an arc which is interior to the half disk except at the points $\{-1\}$ and $\{1/2\}$, and connecting these two points. We choose this deformation retraction to preserve reality and unit modulus. In order to construct $r(z,t)$ just observe that



is homeomorphic to



We extend r to the complex plane minus the two points $\{o\}$ and $\{1\}$ by setting $r(\overline{z},t) = \overline{r(z,t)}$ and $r(z^{-1},t) = r(z,t)^{-1}$. The deformation $\delta = \delta(M.t)$ of W^{*} is then carried out by simply deforming the spectra of $M \in W^{*}$ by means of r leaving the eigenspace alone: $\delta(M,t) = \prod_{\lambda} M_{\nu(\lambda,t)}$, where $\nu(\lambda,t) = r(\lambda,t)\lambda^{-1}$, and where the product runs over all the eigenvalue groups of M. •

1.4 Change to distinct eigenvalues

Lemma 1.5.

Any neighborhood of $M \in W$ *contains an arc in* W *connecting* M *to a matrix with distinct eigenvalues, none of which is equal to* -1.

Proof: Assume λ is an eigenvalue in the eigenvalue group $(\lambda,\lambda^{-1},\bar{\lambda},\bar{\lambda}^{-1})$ of M. By Lemma 1.3, if $\xi \in E(\lambda)$ and if $\eta \in E(\mu)$ then either $<\xi,J\bar{\eta}> = 0$. or $\mu = \bar{\lambda}^{-1}$. Choose an orthogonal basis $\xi_1,\ldots,\xi_k$ for $E(\lambda)$ such that $M\xi_1 = \lambda\xi_1$, and a dual basis $\eta_1,\ldots,\eta_k$ for $E(\bar{\lambda}^{-1})$ such that $<\xi_j, J\bar{\eta}_\ell> = \delta_{j\ell}$. Given a complex $\nu \in C$, define M_ν as follows:

$$M_\nu \xi_1 = \nu\lambda^{-1} \xi_1, \quad M_\nu \bar{\xi}_1 = \bar{\nu}\, \bar{\lambda}^{-1} \bar{\xi}_1$$

$$M_\nu \eta_1 = \bar{\nu}^{-1} \bar{\lambda} \eta_1, \; M_\nu \bar{\eta}_1 = \nu^{-1} \lambda \bar{\eta}_1 \quad ,$$

if ζ is any of the remaining $\xi_2,\ldots,\xi_k$, or $\eta_2,\ldots,\eta_k$ or if $\zeta \in E(\mu)$, $\mu \notin (\lambda,\lambda^{-1}, \bar{\lambda},\bar{\lambda}^{-1})$, we then set $M_\nu \zeta = \zeta$. One checks easily that M_ν is real and symplectic. Now set $B_\nu = M_\nu M$, then ν is an eigenvalue of B_ν, and if $\mu \notin (\lambda,\lambda^{-1}, \bar{\lambda},\bar{\lambda}^{-1})$ is an eigenvalue of M, it is also an eigenvalue of B_ν with the same eigenspace. One can verify by a calculation which we forego, that indeed the dimension of the generalized eigenspace of B_ν corresponding to λ is one less than that of M. A similar construction can be carried out in the cases that λ is real and that λ is on the unit circle. Finally, multiplying A by $e^{\varepsilon J}$ with a small ε, if necessary, it can be arranged that -1 is not an eigenvalue. Using induction the proof of the lemma follows. •

1.5 Normalforms for distinct eigenvalues

Assume $M \in Sp(n,R)$ has distinct eigenvalues none of which is equal to -1. Then, as it is easily verified, there is a symplectic base in which the matrix has a block diagonal form. Every block corresponds to an eigenvalue group and has one of the following three normalforms, where we abbreviate

$$R(\alpha) := \begin{pmatrix} \cos\alpha & -\sin\alpha \\ \sin\alpha & \cos\alpha \end{pmatrix},$$

(1) Hyperbolic plane (eigenvalue group (β,β^{-1}), β real)

$$M = \begin{pmatrix} \beta & 0 \\ 0 & \beta^{-1} \end{pmatrix}. \quad P = M, \quad O = 1, \quad J = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$$

(2) Elliptic plane $(\lambda,\bar{\lambda})$, $\lambda = e^{i\alpha}$, α real.

$$M = R(\alpha), \quad P = 1, \quad O = M, \quad J = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$$

(3) Complex eigenvalue group $(\lambda,\lambda^{-1},\bar{\lambda},\bar{\lambda}^{-1})$, $\lambda = \rho e^{i\theta}$

$$M = \begin{pmatrix} \rho R(\theta) & 0 \\ 0 & \rho^{-1}R(\theta) \end{pmatrix}$$

$$P = \begin{pmatrix} \rho & 0 \\ 0 & \rho^{-1} \end{pmatrix}, \quad O = \begin{pmatrix} R(\theta) & 0 \\ 0 & R(\theta) \end{pmatrix}, \quad J = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}.$$

After the normal form of the block, the corresponding block for P,O according to the polarform, $M = PO$ and the corresponding symplectic structure are indicated. For the corresponding $\bar{u}$ one reads off from these normalforms

$$(1.10) \qquad \det \bar{u} = \begin{cases} 1 & \text{case (1)} \\ e^{i\alpha} & \text{case (2)} \\ 1 & \text{case (3)} \end{cases} .$$

Note also, that in case (2), $e = (1,-i)$ is the eigenvector for the eigenvalue $e^{i\alpha}$ and $< \bar{e}, Je > = 2i$. If O is the orthogonal symplectic matrix which puts M into the block diagonal form $M_1 = O^{-1}MO$, we can connect the identity and the matrix O by an arc $O(t)$ in the set of symplectic orthogonal matrices and find by Lemma 1.2

Lemma 1.6.

Suppose $M \in Sp(n,R)$ *has distinct eigenvalues none of which is equal to* -1, *then* M *is connected by an arc* $\gamma(t)$ *to a matrix* M_1 *in the above block diagonal form, such that, in addition,* $\Delta(\gamma) = o$. •

Using this result we shall prove

Lemma 1.7.

W^* *has two components, each of which is simply connected relative to* W. *One component,* W^*_+, *contains the matrix* $W_+ = -\,\mathrm{id}$, *and the degree of the fixed point* o *of the map* $x \to Mx$ *is* $+1$ *if* $M \in W^*_+$. *The other component,* W^*_- *contains the matrix*

$$W_- = \begin{pmatrix} 2 & & & o \\ & -I & & \\ & & 1/2 & \\ o & & & -I \end{pmatrix}$$

where I *is the identity in* (n-1) *dimensions. The degree of* $M \in W^*_-$ *is* -1.

Proof:

Pick $M \in W^*$ and connect it by an arc in W^* to an element which is an block diagonal form, using Lemma 1.5 and Lemma 1.6. Now, if $\beta < o$ in case (1), as well in the cases (2) and (3), the blocks are obviously connected to blocks -1 by an arc in W^*. Also the block in case (3) is connected to two blocks of type (1) with $\beta > o$ by connecting θ to zero with $\rho \neq 1$. One sees that conversely two positive hyperbolic planes can be brought together and connected to -1. (Observe that the block in (3) can also be connected to two elliptic planes, but the corresponding α's will have opposite signs as is clear since $\det \bar{u} = 1$; this is known from the study of strong stability classes, see [16]). Thus depending on the parity of the number of positive hyperbolic planes, $M \in W^*$ can be connected either to W_- or to W_+. But these two matrices cannot lie in the same component of W^* since they have different degrees for the fixed point o. That W^* is simply connected relative to W follows now from Lemma 1.4. In fact, if γ is any loop, then as one sees from the above forms, $\Delta(\gamma)$ depends only on the variation of the arguments of the eigenvalues in the elliptic planes (case (2)), since by Lemma 1.2 changes of basis do not contribute. By Lemma 1.4 any loop in W^* can be deformed to one on which the eigenvalues of modulus one are all equal to -1, hence to one for which $\Delta(\gamma)=o$. By Lemma 1.1, the loop is contractible in W. •

1.6. Proof of theorem 1.1.

Let $\gamma \in P$ be given. Extend γ by an arc in W^* using Lemma 1.7, to a path $\overline{\gamma}$ connecting 1 to either W_+ or W_-. We treat the case W_-, the other case is similar.

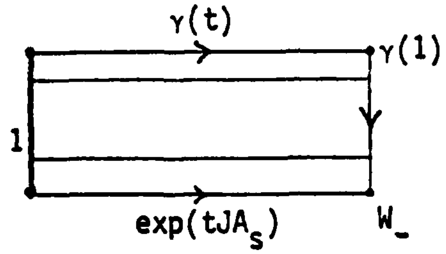
Using now the assumption $n \geq 2$ we observe that the matrix W_- has countably many real logarithms. Namely, if we define for an integer $\ell \in Z$ the symmetric matrix $A_\ell \in \mathcal{L}(R^{2n})$ by

$$A_\ell = \left(\begin{array}{c|c} \begin{matrix} (2\ell+1)\pi & & & \\ & \pi & & \\ & & \ddots & \\ & & & \pi \end{matrix} & \begin{matrix} \ln 2 & \\ & 0 \end{matrix} \\ \hline \begin{matrix} \ln 2 & \\ & 0 \end{matrix} & \begin{matrix} (2\ell+1)\pi & & & \\ & \pi & & \\ & & \ddots & \\ & & & \pi \end{matrix} \end{array}\right)$$

then

$$W_- = e^{JA_\ell}, \quad \ell \in Z.$$

For the special exponential arcs $\hat{\gamma}_\ell \in P$, defined by $\hat{\gamma}_\ell(t) = \exp(tJA_\ell)$, $o \leq t \leq 1$, which connect 1 with W_- we find by (1.10) that $\Delta(\hat{\gamma}_\ell) = 2\pi\ell + \pi(n-1)$. Pick some $\ell \in Z$ and define the loop γ_1 by first following $\overline{\gamma}$ from 1 to W_- and then following $\hat{\gamma}_\ell$ backwards from W_- to 1. Then $\Delta(\gamma_1) = \Delta(\overline{\gamma}) - \Delta(\hat{\gamma}_\ell) = 2\pi m$ for some integer m. Therefore, if we set $s = \ell+m$, we find for the new loop γ_2, defined by following $\overline{\gamma}$ to W_- but then following $\hat{\gamma}_s$ backwards to 1, that $\Delta(\gamma_2) = o$. Hence, by Lemma 1.1, the loop γ_2 is contractible in W. This shows that the path $\gamma \in P$ and the exponential path $\bar{\gamma}_s$, where $\bar{\gamma}_s(t) = \exp(tJA_s)$, are in the same component of P.



Now consider any exponential path $\gamma = \gamma(t) = e^{JSt}$, $o \leq t \leq 1$ where $JS \in \mathcal{L}(R^{2n})$ has distinct eigenvalues. If the purely imanigary eigenvalues λ of JS are $\pm i\alpha_1, \dots, \pm i\alpha_k$, normalized so that $e^{JS}\xi = i\alpha\xi$ implies $< \bar{\xi}, J\xi > i^{-1} > o$, then by means of the normalform (2) in section 1.5 one sees that

$$(1.11) \qquad \Delta(e^{JSt}) = \sum_{j=1}^{k} \alpha_j .$$

Extend now this path γ to a path $\bar{\gamma}$ connecting 1 to either W_+ or W_-, say W_-, in such a way that the eigenvalues of non unit modulus stay that way up to the last point, so that they do non contribute to the Δ of the extended path. If α_j, $1 \leq j \leq k$ lies in the open interval between $2n\pi$ and $2(n+1)\pi$ for some integer n, it is during the deformation changed to $(2n+1)\pi$, i.e. moved to the closest odd multiple of π. Therefore, if $\lambda = i\alpha$ denote the purely imaginary eigenvalues of JS we find

$$(1.12) \qquad \Delta(\bar{\gamma}) = \pi \sum_{\lambda} [\alpha(\lambda)] = \pi j(S).$$

Of course, since W^* is simply connected relative to W, any way of extending $\gamma(t) = e^{JSt}$ to W_- with an arc in W^* gives a $\bar{\gamma}_1$ with $\Delta(\bar{\gamma}_1) = \pi j(S)$. It follows that all exponential paths in the same component of P have the same index, defined by $\Delta(\bar{\gamma})$, and if two exponential paths have the same index, they lie in the same component of P. In fact, let e^{JAt} be the arc connecting 1 with W_- to which e^{JSt} is deformed, then $\Delta(e^{JAt}) = \pi j(S)$. If now $\gamma_1(t) = e^{JS_1 t}$ lies in the same component as $\gamma(t) = e^{JSt}$, it can also be deformed to the same path e^{JAt}, hence $\Delta(e^{JAt}) = \pi j(S_1)$, and therefore $j(S_1) = j(S)$. Conversely, if $j(S_1) = j(S)$ for two exponential path's, then they can be deformed to the same path e^{JAt} and lie therefore in the same component of P. This finishes the proof of theorem 1. •

We now can define an index for any path $\gamma \in P$, not just for an exponential path as follows. We extend γ by an arc in W^* to a path $\bar{\gamma}$ connecting 1 to either W_+ or W_- and put

$$j(\gamma) := \frac{1}{\pi} \Delta(\bar{\gamma}). \tag{1.13}$$

In view of Lemma 1.1. and Lemma 1.7. the right hand side does not depend on the extension. It moreover is an integer, which characterizes the component to which $\gamma(t)$ belongs.

If $\gamma \in P$, then there is the following relation between the index, $j(\gamma)$, and the fixed point degree $\sigma = \deg(\gamma(1))$ of the fixed point o of the symplectic map $\gamma(1) \in W^* : x \to \gamma(1)x$, namely:

$$\deg(\gamma(1)) = (-1)^{j(\gamma)+n}, \tag{1.14}$$

where R^{2n} is the space under consideration. We only have to prove this for an exponential path $\gamma(t) = \exp(tJS)$. By definition of the index we have $j(\gamma) = m+2L$, for some integer $L \in Z$, where m is the number of purely imaginary eigenvalue pairs of JS. Moreover, if ℓ is the number of hyperbolic planes of JS, then by definition of the fixed point degree of e^{JS}, $\sigma = (-1)^{\ell}$. But $2n = 2m + 2\ell + 4k$, where k is the number of complex eigenvalue groups of JS, hence $\ell = n-m-2k$ and the equality (1.14) follows.

1.7 Interpretation of the index as an intersection number

The integer $j(\gamma)$, $\gamma \in P$, can be related to the number of oriented intersections of a curve of Lagrange planes with a fixed Lagrange plane. If ω denotes the symplectic structure in R^{2n} given by the matrix J, we can introduce the symplectic structure ω_1 in $R^{2n} \times R^{2n}$ by setting $\omega_1 = \omega + (-\omega)$. A map $M \in \mathcal{L}(R^{2n})$ is then symplectic if the 2-form ω_1 vanishes on the (2n)-dimensional subspace graph $(M) := \{(x,Mx) | x \in R^{2n}\}$, that is, if graph (M) is a Lagrange subspace. Hence an arc $X(t) \in Sp(n,R)$ gives rise to an arc graph $(X(t))$ in the space of Lagrange planes. The diagonal $\Delta = \{(x,x) | x \in R^{2n}\}$ is a Lagrange plane, and for any $M \in Sp(n,R)$ we have graph $(M) \cap \Delta = \{o\}$ if and only if 1 is an eigenvalue of M. We now relate the integer $j(\gamma)$, $\gamma \in P$, to the number of intersections of graph$(\gamma(t))$ with Δ, $o < t < o$. Let γ be the special exponential arc $\gamma(t) = \exp(tJA_s)$ defined previously. Then one verifies easily that

$$(1.15) \qquad j(\gamma) - n + \frac{1}{2}(1 - \deg(\gamma(1))) = \pm \sum_{o < t < 1} \dim(\text{graph } \gamma(t) \cap \Delta),$$

where the signs $\pm$ correspond to $s \gtrless o$, i.e. they correspond to the orientation of the rotation in the distinguished elliptic plane which gives rise to a nontrivial intersection. The right hand side is understood to be zero in case $s = o$. We remark that $\dim(\text{graph } \gamma(\tau) \cap \Delta)$ for some $o < \tau < 1$ is the dimension of the solution space of the periodic boundary value problem $\dot{x} = JA(t)x$, $x(o) = x(\tau)$, where $JA(t) := \dot{\gamma}(t) \cdot \gamma(t)^{-1}$; in fact $\gamma(t)$ is the fundamental solution of this equation. As the left hand side of (1.15) depends only on the component of P, we can use formula (1.15) in order to associate to every element of a component of P a normalized oriented intersection number even if the intersections of the particular arc chosen are not "transversal". As for the intersection theory for curves of Lagrange spaces we refer to J. Duistermaats paper [18] "On the Morse Index in Variational Calculus", in which also the relation to the Maslov-index of a periodic solution is described. As for the latter index we refer to V. Arnol'd [19].

2. Periodic solutions of Hamiltonian equations.

In this section we shall prove theorem 2 of the introduction which guarantees T-periodic solutions of the equation

$$\text{(2.1)} \qquad \dot{x} = Jh'(t,x), \quad x(o) = x(T)$$

where $h(t+T,x) = h(t,x)$, and $h \in C^2(R \times R^{2n})$, $n \geq 2$. We shall first reformulate the problem (2.1) as an abstract variational problem for a functional in the loop space.

2.1 The variational problem

Let H be the real Hilbertspace $H = L_2(o,T;\ R^{2n})$. Define in H the linear operator $A : \mathrm{dom}(A) \subset H \to H$ by setting $\mathrm{dom}(A) = \{u \in H^1(o,T;R^{2n}) \mid u(o) = u(T)\}$ and $Au : = -J\dot{u}$, $u \in \mathrm{dom}(A)$. The continuous operator $F: H \to H$ is defined by $F(u)\,(t): = h'(t,u(t))$, $u \in H$. Its potential $\Phi(u)$ is given by

$$\Phi(u): = \int_o^T h(t,u(t))\, dt.$$

F is the gradient of Φ, that is $\Phi'(u) = F(u)$. Writing the equation (2.1) in the form $-J\dot{x} = h'(t,x)$ one sees that every solution $u \in \mathrm{dom}(A)$ of the equation

$$Au = F(u) \tag{2.2}$$

defines (by T-periodic continuation) a classical T-periodic solution of (2.1). Conversely, every T-periodic solution of (2.1) defines (by restriction) a solution u of the equation (2.2). The equation (2.2) is the Euler equation of the variational problem $\mathrm{extr}\ \{f(u) \mid u \in \mathrm{dom}(A)\}$, where

$$f(u) = \frac{1}{2} < Au,u > - \Phi(u), \tag{2.3}$$

which in classical notation is simply given by (2) of the introduction, with periodic boundary conditions $x(o) = x(T)$. Hence in order to find the required solutions of the equation (2.2) we can just as well look for critical points of f. We first summarize some information about the operator A :

Lemma 2.1.

The operator A *is selfadjoint,* $A = A^*$. *It has closed range and a compact resolvent. The spectrum of* $A, \sigma(A)$, *is a pure point spectrum and* $\sigma(A) = \tau Z$, $\tau = \frac{2\pi}{T}$. *Every eigenvalue* $\lambda \in \sigma(A)$ *has multiplicity* $2n$ *and the eigenspace* $E(\lambda) = \ker(\lambda - A)$ *is spanned by the orthogonal basis given by the following loops:*

$$t \to e^{t\lambda J} e_k = (\cos \lambda t)\, e_k + (\sin \lambda t) J e_k ,$$

$k = 1,2,\ldots,2n$; *where* $\{e_k \mid 1 \le k \le 2n\}$ *is the standard basis in* R^{2n}. *In particular* $\ker(A) = R^{2n}$, *that is consists of the constant loops.*

The proof is easy, see [2]. If $b = b(t)$ is a symmetric matrix $b(t) \in \mathcal{L}(R^{2n})$ and if b depends continuously and periodically on t with period $T > 0$, i.e. $b(t) = b(t+T)$, we define the selfadjoint operator $B \in \mathcal{L}(H)$ by

$$(2.4) \qquad (Bu)(t) = b(t) \cdot u(t), \qquad u \in H.$$

Lemma 2.2.

(i) *The operator* $A-B$ *defined on* $\mathrm{dom}(A)$ *is selfadjoint and has compact resolvent. Thus it has a pure point spectrum* $\sigma(A-B) = \sigma_p(A-B)$.

(ii) $0 \in \sigma(A-B)$ *if and only if* 1 *is a Floquetmultiplier for the linear Hamiltonian system* $\dot{x} = Jb(t)x$.

(iii) *If* $b(t) = b$ *does not depend on* t, *then the operator* B *commutes with the projections* $P := \int_{-\alpha}^{\alpha} dE_\lambda$ *for every* $\alpha > 0$, *where* (E_λ) *is the spectral resolution of* A.

Proof: (i): Standard arguments (see [2]) and Lemma 2.1 imply that $A-B$ is selfadjoint and has compact resolvent, since A has compact resolvent. (ii): $o \in \sigma(A-B)$ if the equation $(A-B)u = o$ has a nontrivial solution $u \in \mathrm{dom}(A)$, that is $u \in H^1(o,T;\ R^{2n})$ and $u(o) = u(T)$. Since u satisfies the equation $\dot{u} = Jb(t)u$, this is the case if and only if 1 is a Floquet multiplier of the above equation, as is well known from Floquet theory. As for (iii) we refer to ([2] , Lemma 12.3). •

2.2. Reduction to a finite dimensional variational problem

We shall assume from now on the Hessian of h to be bounded:

$$(2.5) \qquad -\beta \leq h''(t,x) \leq \beta$$

for all $(t,x) \in R \times R^{2n}$ and for some $\beta > o$. From (2.5) we conclude by the mean value theorem, that the potential operator F satisfies

$$(2.6) \qquad -\beta\ |u-v|^2 \leq < F(u)-F(v),\ u-v > \leq \beta\ |u-v|^2 \quad ,$$

for every $u,v \in H$. As observed in [1] , see also [2] , this estimate allows to reduce the problem on finding critical points of f to the problem of finding critical points of a function $a = a(z)$ defined on the finite dimensional space $Z: = PH \subset H$, where

$$P = \int_{-\beta}^{\beta} dE_\lambda$$

is the projection onto the eigenspace of A belonging to the eigenvalues in $(-\beta,\beta)$, E_λ being the spectral resolution of A. We assume $\beta \notin \sigma(A)$ and have the freedom to pick $\beta > 0$ as large as we need. We summarize this reduction to a finite dimensional variational problem in the following

Lemma 2.3.

There are a function $a \in C^2(Z,R)$ *and an injective* C^1*-map* $u: Z \to H$ *having its range in the domain of the operator* $A, u(Z) \subset \mathrm{dom}(A)$, *and with* $\mathrm{im}(u'(z)) \subset \mathrm{dom}(A)$ *for every* $z \in Z$, *with the following properties:*

(i) $z \in Z$ *is a critical point of the function* a, i.e. $a'(z) = 0$, *if and only if* $u(z)$ *is a solution of the equation* $Au = F(u)$, *i.e. a T-periodic solution of the Hamiltonian equation (2.1). If* u *is a solution of* $Au = F(u)$, *then* $u = u(z)$ *for a critical point* z *of* a.

(ii) u *has the form* $u(z) = z + v(z)$ *with* $Pv(z) = 0$.

(iii) *The function* a *is given by* $a(z) = f(u(z)) = \frac{1}{2} < Au(z), u(z) > - \Phi(u(z))$, *its derivative* a' *is globally Lipschitz continuous and*

$$a'(z) = Az - PF(u(z)) = Au(z) - F(u(z))$$

$$a''(z) = (A - F'(u(z))) \cdot u'(z) = A|Z - PF'(u(z)) \cdot u'(z).$$

(iv) *If* F *is linear,* $F(u) = Bu$, *and if* $Bu = bu$, b *a time independent symmetric matrix, then* $a(z) = (A-B)z$ *(Here* $BP = PB$ *is used).*

(v) *If* Σ *is a topological space, and if* $F: \Sigma \times H \to H$ *is a continuous map, such that, for every* $\sigma \in \Sigma$, *the function* $F(\sigma,\cdot): H \to H$ *is a continuous potential operator satisfying the estimate* (2.6) *with the constant independent of* σ, *then the corresponding* $u = u(\sigma,z)$ *is continuous.*

Proof: The proof of this crucial Lemma is contained in ([2], Lemma 12.2, Lemma 3.1. Prososition 4.5 and Remark 2.2.). •

In view of this Lemma, the required periodic solutions of (2.1) are in one to one correspondence to the critical points of this function a, which is defined on the finite dimensional space Z. It remains to determine the critical points of a.

2.3 Morse theory for the reduced problem

In order to find the critical points of a we shall apply the Morse theory discribed in section 3 below to the gradient flow defined by

$$\dot{z} = a'(z) \,, \tag{2.7}$$

which, according to Lemma 2.3 (ii) does exist. We shall first show, that the set S of bounded solutions of (2.7) is compact, provided the assumptions of theorem 1 in the introduction are met. We therefore shall assume, in addition to (2.5), that our Hamiltonian vectorfield is asymptotically linear, requiring that

$$Jh'(t,x) = JA_{\infty}(t)x + o(|x|), \qquad |x| \to \infty \tag{2.8}$$

uniformly in t, where $A_{\infty}(t+T) = A_{\infty}(t)$ is a continuous loop of symmetric matrices.

Lemma 2.4.

Assume (2.8), and assume the linear Hamiltonian equation $\dot{x} = JA_\infty(t)x$ *to be nondegenerate. Denote its index by* j_∞. *Then the set* S *of bounded solutions of (2.7) is compact, hence has an index, which is the homotopy type of a pointed sphere* S^{m_∞} *of dimension* m_∞ :

$$h(S) = [\dot{S}^{m_\infty}] , \qquad m_\infty = \frac{1}{2} \dim Z - j_\infty .$$

Therefore $p(t,h(S)) = t^{m_\infty}$.

Proof:

By theorem 1 in the introduction there exists a continuous family $B_\sigma(t)$, $o \leq \sigma \leq 1$ of loops $B_\sigma(t+T) = B_\sigma(t)$ having the properties that 1 is not a Floquet multiplier of $\dot{x} = JB_\sigma(t)x$ for all $o \leq \sigma \leq 1$, and that for $\sigma = 1$, $B_1(t) = A_\infty(t)$ and that for $\sigma = o$, $B_0(t) = A_0$ is a constant loop having the index $j_\infty = j(A_0)$ as defined in the introduction. Define the continuous family F_σ of potential operators

$$(2.9) \qquad F_\sigma(u) = B_\sigma u + \sigma(F(u) - A_\infty u),$$

$o \leq \sigma \leq 1$ and $u \in H$. It has the properties that for $\sigma = 1$, $F_1(u) = F(u)$, and for $\sigma = o$, $F_0(u) = A_0 u$. Moreover F_σ satisfies the estimate (2.6) for some $\beta > o$ which is independent of σ and therefore gives rise by Lemma 2.3 (v) to a continuous family of gradient systems.

$$(2.10) \qquad \dot{z} = a_\sigma'(z), \qquad z \in Z.$$

With $u = u(\sigma,z)$ we have by Lemma 2.3 (iii) and by (2.9):

$$(2.11) \qquad a_\sigma'(z) = Au - F_\sigma(u) = (A-B_\sigma)u-\sigma(F(u) - A_\infty(u)).$$

We shall prove, that there are constants $\nu > o$ and $\delta > o$ independent of σ, such that for all $z \in Z$

$$(2.12) \qquad |a_\sigma'(z)| \geq \frac{\nu}{2} |z| - \delta .$$

First observe that by Lemma 2.2 (i) and (ii) $o \notin \sigma(A-B_\sigma)$, and, since $\sigma \to B_\sigma \in \mathcal{L}(H)$ is continuous and the resolvent of $(A-B_\sigma)$ is compact, there is a constant $\nu > o$ independent of σ, such that $(A-B_\sigma)^{-1} \in \mathcal{L}(H)$ and $|(A-B_\sigma)^{-1}| \leq \nu^{-1}$, hence for every $u \in \mathrm{dom}(A)$

$$(2.13) \qquad |(A-B_\sigma)u| \geq \nu|u|, \qquad o \leq \sigma \leq 1.$$

On the other hand, from (2.8), we conclude, that

$$(2.14) \qquad \lim_{|u| \to \infty} \frac{1}{|u|} |F(u) - A_\infty u| = o.$$

Since by Lemma 2.3 (ii), $|u(\sigma,z)|^2 = |z|^2 + |v(\sigma,z)|^2$, hence $|u(\sigma,z)| \geq |z|$, the claimed estimate (2.12) follows from (2.11) together with the estimates (2.13) and (2.14).

Let S_σ denote the set of bounded solutions of the equation (2.10), that is $S_\sigma = \{z \in Z \mid \text{there is a bounded orbit containing } z\}$. Then the estimate (2.12) implies the existence of a compact set $K \subset Z$ containing S_σ in its interior for all $o \leq \sigma \leq 1$. Thus K is an isolating neighborhood for S_σ, $\sigma \in [o,1]$, which are therefore related by continuation

([4] section IV.I Theorem 3.1). Thus by the invariance of the homotopy index ([4] section IV.I Theorem 1.4), the homotopy index of S_σ is independent of $\sigma \in [o,1]$, i.e. $h(S_\sigma) = h(S)$. For $\sigma = o$, the vectorfield a_σ' is, in view of Lemma 2.3 (iv), given by

$$(2.15) \qquad a_o'(z) = (A-A_o)\, z.$$

Since by assumption $o \notin \sigma(A-A_o)$, it follows that for $\sigma = o$, the isolated invariant set S_o of (2.15) consists just of the hyperbolic rest point $z = o$, hence $S_o = \{o\}$. It is shown in section 3 that the homotopy index of a hyperbolic rest point is the homotopy type of a pointed shere $\dot{S}^m$ whose dimension, m, equals the dimension of the stable invariant manifold of the rest point. Hence it remains to compute the dimension m of a maximal subspace Z_+ of Z such that $(A-A_o)|Z_+ > o$. Here it is important to recall, that A_o is the bounded linear operator, which is defined as in (2.4) by a symmetric matrix, also denoted by A_o, which moreover does not depend on t. For this special case the dimension $m = \dim Z_+$ has been computet in ([3], Lemma 1). In fact, denoting by $j(A_o)$ the integer introduced in section 1, it is proved in that paper that $m = \frac{1}{2} \dim Z - j(A_o)$. Since $j_\infty = j(A_o)$, the Lemma is proved. •

We next consider a special periodic solution of (2.1), namely an equilibrium point x_o of the Hamiltonian vectorfield, which we assume to be the origin, such that $Jh'(t,o) = o$ for all $t \in R$.

Lemma 2.5.

Assume o *is an equilibrium point of the Hamiltonian equation, and assume the trivial periodic solution,* $x_0(t) = o$, $t \in R$ *to be nondegenerate and denote its index by* $j_0 \in Z$. *Then the corresponding critical point* $z_0 = o \in Z$ *is an isolated invariant set, whose index is given by:*

$$h(\{z_0\}) = [\dot{S}^{m_0}] , \quad m_0 = \frac{1}{2} \dim Z - j_0 .$$

Therefore $p(t,h\{z_0\}) = t^{m_0}$.

Proof: Since o is an equilibrium point, $Jh'(t,x) = JA_0(t)x + o(|x|)$ as $|x| \to o$, with $A_0(t) = h''(t,o)$. By assumption $x_0(t) \equiv o$ is nondegenerate and therefore by Theorem 1 there is a deformation $B_\sigma(t) = B_\sigma(t+T)$ connecting the loop $A_0(t) = B_1(t)$ for $\sigma = 1$ with a constant loop $B_0(t) = A_1$ for which the index is given by $j(A_1) = j_0$. By definition, 1 is not a Floquet multiplier for the linear systems $\dot{x} = JB_\sigma(t)x$, $\sigma \in [o,1]$. Define the family F_σ of potential operators as

$$F_\sigma(u) = B_\sigma u + \sigma(F(u) - A_0 u) ,$$

then $F_1(u) = F(u)$ and $F_0(u) = A_1 u$, where the operator $B_\sigma, A_1, A_2 \in \mathcal{L}(H)$ are defined as in (2.4) by means of the corresponding matrices. F_σ satisfies the estimate (2.6) for $\sigma \in [o,1]$ with a constant β independent of σ and, by Lemma 2.3, gives rise to a family a'_σ of gradient systems on Z with a corresponding family $u(\sigma,z)$, such that $a'_\sigma(o) = o$ and $u(\sigma,o)=o$ for $\sigma \in [o,1]$. Explicitely we have with $u = u(\sigma,z)$: $a'_\sigma(z) = (A-B_\sigma)u - \sigma(F(u) - A_0 u)$. As in the proof of the previous Lemma there is a $\nu > o$, such that for $\sigma \in [o,1]$ we have the estimate $|(A-B_\sigma)u| \geq \nu|u|$, for $u \in \mathrm{dom}(A)$. Moreover, since $F(o) = o$ and $F'(o) = A_0 \in \mathcal{L}(H)$ we have $F(u) - A_0 u =$

$o(|u|)$ as $|u| \to o$ in H. With $|u(\sigma,z)| \geq |z|$ we conclude that there is an $\varepsilon > o$ independent of σ, such that if $|z| < \varepsilon$ then $|a'_\sigma(z)| \geq \frac{\nu}{2} |z|$. Hence $z = o$ is an isolated critical point for every $\sigma \in [o,1]$ and therefore an isolated invariant set of the corresponding gradient flow. As in the proof of the previous Lemma we conclude that the index of this isolated invariant set does not depend on $\sigma \in [o,1]$ and so is the index of the critical point of $a'_\sigma(z)$ for $\sigma = o$, which by Lemma 2.3 (iv) is given by $a'_o(z) = (A-A_1)z$. Since $o \notin \sigma(A-A_1)$ the critical point $z = o$ is hyperbolic and hence the index is the homotopy type of a pointed sphere of dimension m_o, which as in the previous Lemma is computed to be equal to $\frac{1}{2}\dim Z - j_o$, with $j_o = j(A_o)$. This finishes the proof of the Lemma. •

We shall use this Lemma in order to establish a relation between the index of a nondegenerate periodic solution of the equation (2.1) and the index of the corresponding critical point of the gradient flow on the loopspace.

Lemma 2.6.

Let $x_o(t)$ be a nondegenerate T-periodic solution of the Hamiltonian equation (2.1) with index j. Then the corresponding critical point, z_o, of the functional a on the loopspace Z is an isolated invariant set with index given by

$$h(\{z_o\}) = [\dot{S}^m], \quad m = \frac{1}{2}\dim Z - j.$$

Therefore $p(t,h(\{z_o\})) = t^m$. Moreover, the signature of the Hessian of *a at z_0 is equal to $2j$. The local degree of a' in a neighborhood of z_0 is equal to $(-1)^{n+j}$.*

Proof:

Let z_0 be the critical point of a corresponding to the given periodic solution $x_0(t)$, such that $u(z_0)(t) = x_0(t)$, and set $u_0 = u(z_0)$. By Lemma 2.3 (iii) we find $a''(z_0) = (A-F'(u_0))\, u'(z_0)$, where $F'(u_0) \in \mathcal{L}(H)$ is defined by the matrix $h''(t,x_0(t))$. By the nondegeneracy of the periodic solution we have by Lemma 2.2 (ii) the estimate $|(A-F'(u_0))u| \geq \geq \nu\, |u|$ for $u \in \mathrm{dom}(A)$. Moreover, since $u(z) = z + v(z)$ with $Pv(z) = o$ we conclude, that $|u'(z_0)\zeta|^2 = |\zeta|^2 + |v'(z_0)\zeta|^2 \geq |\zeta|^2$ for every $\zeta \in Z$ and therefore $|a''(z_0)\zeta| \geq \nu|\zeta|$ for every $\zeta \in Z$. Therefore z_0 is an isolated critical point. To compute its index, we shall reduce the problem to the situation of the previous Lemma and define the following family of potential operators satisfying (2.6):

$$(2.16) \qquad F_\sigma(u) = F(u + \sigma u_0) - \sigma F(u_0), \quad \sigma \in [o,1] .$$

Clearly $F_0(u) = F(u)$ and $F_1(u) = F(u + u_0) - F(u_0)$ and so $F_1(o) = o$ and $F'(o) = F'(u_0)$. Put $v_\sigma := (1-\sigma)u_0$, we claim

$$(2.17) \qquad Av_\sigma = F_\sigma(v_\sigma).$$

In fact, since u_0 is a periodic solution, we know $Au_0 = F(u_0)$ and therefore $Av_\sigma = (1-\sigma)Au_\sigma = (1-\sigma)\, F(u_0)$. On the other hand $F_\sigma(v_\sigma) = F(v_\sigma + \sigma u_0) - \sigma F(u_0) = F(u_0) - \sigma F(u_0)$ and hence the claim follows. Denote by a_σ and u_σ the family of functionals and maps belonging to

(2.16). By Lemma 2.3 there is to every v_σ in (2.17) a unique critical point z_σ of a_σ such that

$$v_\sigma = u(\sigma, z_\sigma) .$$

The critical points z_σ are isolated. In fact, since $F_\sigma'(v_\sigma) = F'(u_0)$ we have for the Hessian of a_σ at z_σ:

$$a_\sigma''(z_\sigma) = (A-F'(u_0)) \cdot u_\sigma'(z_\sigma)$$

and therefore we have for $\sigma \in [0,1]$ the estimate $|a_\sigma''(z_\sigma)\zeta| \geq \nu|\zeta|$ for all $\zeta \in Z$ and for some $\nu > 0$ independent of σ. Hence the isolated invariant sets $S_\sigma = \{z_\sigma\}$ of a_σ' are related by continuation and therefore $h(S_\sigma)$ is independent of σ and hence is equal to the index of the critical point $z_1 = 0$ of the flow $\sigma = 1$. This flow is defined by $a_1'(z)$ belonging to the problem $Au = G(u)$, where $g(u) = F(u + u_0) - F(u_0)$. Since $G(0) = 0$ and $G'(0) = F'(u_0)$ the problem is reduced to the proof of the previous Lemma, where this time $A_0(t)$ is replaced by $h''(t,x_0(t))$, $x_0(t)$ being the periodic solution. The first statement now follows by a further deformation of the gradient a_1' to the linear system $(A-A_1)(z)$ for some constant loop A_1 with $j(A_1) = j$. To prove the second part of Lemma we simply observe that by definition of the space Z and by Lemma 2.1, $\dim Z = 2n + 4\ell$ for some positive integer ℓ. Hence for some open neighborhood U of z_0 we have $\deg(U,a'(z),0) = \text{signum}(\det a''(z_0)) = (-1)^{n+j}$, as claimed. •

Proof of theorem 2.

Set $d = \frac{1}{2} \dim Z$, d is an integer. Let S be the set of bounded orbits of the gradient flow $\dot{z} = \nabla a(z)$. It consists of critical points of a and of connections between them. The critical points corresponds by Lemma 2.3 in a one to one way to the periodic solutions of the Hamiltonian equations (2.1) we are looking for.

By Lemma 2.4 the invariant set $S \subset Z$ is compact and of homotopy type $h(S) = [\dot{S}^{m_\infty}]$ with $m_\infty = d - j_\infty$. This is not the index of the empty set which is a pointed one point space hence has the homotopy type $[(\{p\},p)]$ for an arbitrary point p. Therefore $S \neq \emptyset$ and because the limit set of a bounded orbit of a gradients system consists of critical points, the function a possesses at least one critical point and consequently the Hamiltonian equation admists at least one T-periodic solution.

Remark: As a sideremark we observe that the existence of one critical point could also be established by a degree argument. In fact, it follows from Lemma 2.4 for a large ball $\Omega \subset Z$, that $\deg(\Omega, a', o) = (-1)^{n+j_\infty} = (-1)^{m_\infty} \neq o$.

If the periodic orbit found above is nondegenerate, it has by Theorem 1 an index denoted by $j \in \mathbb{Z}$. The corresponding critical point z of a is then, by Lemma 2.6, an isolated invariant set with index $h(\{z\}) = [\dot{S}^m]$, where $m = d - j$. Assume z is the only critical point of a, then $S = \{z\}$, since we are dealing with a gradient system and therefore $h(S) = [\dot{S}^m]$ which, on the other hand is equal to $[\dot{S}^{m_\infty}]$ and consequently $m = m_\infty$. Therefore if $j \neq j_\infty$ and hence $m \neq m_\infty$ there must be more than one critical point of a.

Assume now that the Hamiltonian system possesses two nondegenerate periodic orbits having indices j_1 and j_2. We claim that there is at least a third periodic orbit. In fact, if this is not the case, then the isolated invariant set S contains precisely two isolated critical points z_1 and z_2 with indices $h(\{z_1\}) = [\dot{S}^{m_1}]$, $m_1 = d - j_1$ and $h(\{z_2\}) = [\dot{S}^{m_2}]$, $m_2 = d - j_2$. If we label them such that $a(z_1) \leq a(z_2)$, then (z_1, z_2) is an admissible Morse-decomposition of S. From theorem 3.3 we conclude the identity $p(t,h(\{z_1\})) + p(t,h(\{z_2\})) = p(t,h(S)) +$ $+ (1+t)\, Q(t)$, which, by Lemma 2.6, leads to the identity

$$t^{m_1} + t^{m_2} = t^{m_\infty} + (1+t)\, Q(t).$$

Setting $t = 1$ we find the equation $2 = 1 + 2Q(1)$ with a non-negative integer $Q(1)$. This is nonsense, hence we must have at least three critical points of a.

Assume finally all the periodic solutions to be nondegenerate and denote their indes by j_k, $k = 1,2,\ldots$ They correspond to the critical points of a, which are isolated. Since S is compact there are only finitely many of them, say $(z_1,\ldots,z_n)$. We order them such that $a(z_i) \leq a(z_j)$ if $i < j$. Then $(z_1,\ldots,z_n)$ is an admissible ordering of a Morse decomposition of S, and by Theorem 3.3 and Lemma 2.4 we have

$$\sum_{k=1}^{n} p(t,h(z_k)) = t^{m_\infty} + (1+t)\, Q(t),$$

with $m_\infty = d - j_\infty$. By assumption the periodic solutions are nondegenerate, hence by Lemma 2.6 we know $p(t,h(z_k)) = t^{m_k}$, $m_k = d - j_k$, so that

$$\sum_{k=1}^{n} t^{m_k} = t^{m_\infty} + (1+t)\, Q(t),$$

which after multiplication by t^{-d}, $d = \frac{1}{2} \dim Z$ becomes the advertized identity in Theorem 2. We conclude that there is at least one periodic solution having index j_∞. Also, setting $t = 1$ we find $n = 1 + 2\, Q(t)$, hence the number of periodic solutions is odd as claimed in Theorem 2. This finishes the proof of Theorem 2.

As an illustration, we assume there are precicely 3 nondegenerate periodic solutions with indices j_k, $1 \leq k \leq 3$. Then, by (2.19), $t^{m_1} + t^{m_2} + t^{m_3} = t^{m_\infty} + (1+t)\, Q(t)$, hence $Q(t) = 1$ and therefore $Q(t) = t^{\ell}$ for some integer ℓ. We conclude that one of the j_k's agrees with j_∞, say $j_3 = j_\infty$. The remaining indices are therefore bound to satisfy $|j_1 - j_2| = 1$. It would be interesting to have an example of a Hamiltonian system realizing this rather special situation.

3. Morse theory for flows.

3.1. Set up, Morse decompositions, isolated invariant sets.

Let Γ be a topological space. A flow on Γ is a continuous map from $\Gamma \times R$ onto Γ, $(\gamma,t) \to \gamma\cdot t$ satisfying for all $\gamma \in \Gamma$ and all $s,t \in R$ the two conditions $\gamma\cdot o = \gamma$ and $(\gamma\cdot s)\cdot t = \gamma\cdot(s+t)$. For two subsets $\Gamma' \subset \Gamma$ and $R' \subset R$ we set $\Gamma'\cdot R' = \{\gamma\cdot t \in \Gamma \mid \gamma \in \Gamma'$ and $t \in R'\}$. A subset $I \subset \Gamma$ is then called invariant, if $I\cdot R = I$. If $N \subset \Gamma$ is a subset we denote the invariant set contained in N by $I(N)$:

$$(3.1) \qquad I(N) := \{\gamma \in N \mid \gamma\cdot R \subset N\} .$$

Clearly $I(N)$ is invariant, it is closed if N is closed. For a subset $Y \subset \Gamma$ we define its ω-limit sets by

$$(3.2) \qquad \omega(Y) = I(\mathrm{cl}\ \{Y\cdot[o,\infty)\}) \text{ and } \omega^*(Y) = I(\mathrm{cl}\ \{Y\cdot(-\infty,o]\}).$$

It follows from the definitions, that if I is a closed and invariant subset of Γ, and if $Y \subset I$, then $\omega(Y)$ and $\omega^*(Y)$ are closed and invariant subsets contained in I. If, in addition, I is compact and Hausdorff relative to I and if Y is connected, then $\omega(Y)$ is connected too.

Definition (3.1) (*Morse decomposition*)

Assume I is a compact, Hausdorff, invariant set in Γ. A Morse decomposition of I is a finite collection $\{M_\pi\}_{\pi \in P}$ of subsets $M_\pi \subset I$, which are disjoint, compact and invariant, and which can be ordered $(M_1, M_2, \ldots, M_n)$ so that for every $\gamma \in I \setminus \bigcup_{1 \leq j \leq n} M_j$ there are indices $i < j$ such that

$$\omega(\gamma) \subset M_i \quad \text{and} \quad \omega^*(\gamma) \subset M_j .$$

Such an ordering will then be called an admissible ordering. There may be several admissible orderings of the same decomposition. The elements M_j of a Morse decomposition of I will be called Morse sets of I.

For an admissible ordering $(M_1, \ldots, M_n)$ of a Morse decomposition of I we define the subsets $M_{ji} \subset I$ as follows:

$$(3.3) \qquad M_{ji} := \{\gamma \in I \mid \omega(\gamma) \text{ and } \omega^*(\gamma) \subset M_i \cup M_{i+1} \cup \ldots \cup M_j\}.$$

In particular $M_{jj} = M_j$. The following statement then follows immediately from the definitions.

Proposition 3.1.

Assume $(M_1, \ldots, M_n)$ *is an admissible ordering of a Morse decomposition of* I. *If* $i \leq j$, *then* $M_1, \ldots, M_{i-1}, M_{ji}, M_{j+1}, \ldots, M_n)$ *is an admissible ordering of a Morse decomposition of* I.

Moreover, $(M_i, M_{i+1}, \ldots, M_{j-1}, M_j)$ *is an admissible ordering of a Morse decomposition of* M_{ji}. •

In the classical Morse theory the topological space $\Gamma = M$ is a manifold and the flow under consideration is the gradient flow of a function defined on M, which is assumed to have finitely many critical points. These critical points serve as the sets of a Morse decomposition of the invariant set $I = M$ which in this case is the whole manifold. The statement of Morse theory then relates the dimensions of the unstable invariant manifolds of these critical points to algebraic invariants of the whole manifold. In our more general setting, the invariant set I is just a subset of Γ, and the flow is not necessarily a gradient flow. The aim is to relate algebraic invariants of the Morse sets of a Morse-decomposition of I to algebraic invariants of all of I. The invariants will depend on the behavior of the flow in a neighborhood of I. In order to be flexible in the applications we shall introduce the notion of a local flow.

Definition (3.2). (*Local flow*)

Assume $X \subset \Gamma$ is a locally compact and Hausdorff subset of Γ. For simplicity assume X to be a *metric* space. X is called a local flow, if for every $\gamma \in X$ there are a neighborhood $U \subset \Gamma$ of γ and an $\varepsilon > 0$ such that

$$(X \cap U) \cdot [0,\varepsilon) \subset X .$$

To illustrate the purpose of this notion in the applications we mention some examples: 1. Consider on $\Gamma = R^{2n}$ the flow of a time independent Hamiltonian vectorfield given by a function h. Then h is an integral of the flow. If the invariant set I is contained in the set $\{x \in R^{2n} \mid h(x) = c\}$, this energy surface is a local flow containing I. 2. We point out that there are many ways meeting local flows if one studies partial differential equations, which are not defined on locally compact spaces ab initio. To be more precise we describe in the Appendix a system of parabolic equations, which leads in a natural way to a local flow.

Definition 3.3. *(Isolated invariant set)*

Let $N \subset X$ be a *compact* subset of a local flow X. If

$$I(N) \subset \text{int } N \text{ (relative to } X)$$

then N is called an *isolating neighborhood* (in X) and $I(N)$ is called an *isolated* invariant set. (Note that the interior of N may be empty).

Proposition 3.2.

Assume S to be an isolated invariant set in the local flow X and let $\{M_\pi\}_{\pi \in P}$ be a Morse decomposition of S. Then the sets M_π are also isolated invariant sets in X.

Proof: By assumption there is a compact $N \supset S$ with $I(N) = S \subset \text{int } N$ (relative to X). Pick any compact X-neighborhood N_π of a set M_π which is disjoint from the remaining Morse sets and contained in N. This set N_π is an isolating neighborhood of M_π. Let $\gamma \in I(N_\pi)$, so that $\gamma\cdot R \subset N_\pi$ hence $\gamma\cdot R \subset N$ and consequently $\gamma \in S$. Since both $\omega(\gamma)$ and $\omega^*(\gamma)$ are contained in N_π they cannot be in any other Morse set other than M_π. From the definition of a Morse decomposition it now follows that $\gamma \in M_\pi$ and thus $I(N_\pi) = M_\pi \subset \text{int } N_\pi$ relative to X. •

2. Index pairs for Morse decompositions

If $Z \subset Y \subset \Gamma$ are subsets, we call Z positively invariant relative to Y, if $\gamma \in Z$ and $\gamma\cdot[o,t] \subset Y$ together imply that $\gamma\cdot[o,t] \subset Z$. Under a compact pair (Z_2,Z_1) we mean an ordered pair of compact spaces with $Z_1 \subset Z_2$. The following concept is crucial.

Definition 3.4. (*Index Pair*)

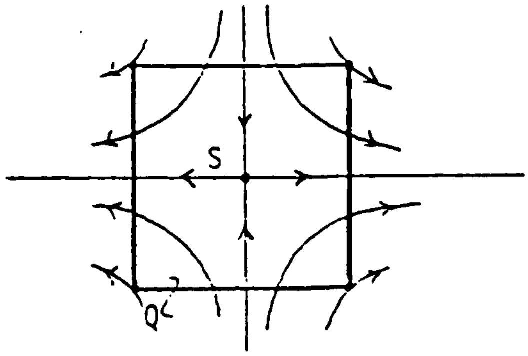
Let S be an isolated invariant set in the local flow X. A compact pair (N_1,N_0) in X is called an index pair for S, if

(i) $\quad cl(N_1 \setminus N_0)$ is an isolating neighborhood for S.

(ii) $\quad N_0$ is positively invariant relative to N_1

(iii) $\quad$ if $\gamma \in N_1$, and $\gamma\cdot R^+ \not\subset N_1$ then there is a $t \geq o$ such that $\gamma\cdot[o,t] \subset N_1$ and $\gamma\cdot t \in N_0$.

Observe that positive orbits can leave N_1 only through the "face" N_0. We illustrate this concept by an example. We consider $\Gamma = X = R^2$ and the flow defined by $\dot{x} = x$, $\dot{y} = -y$. The set $S = \{o\}$ is an isolated invariant set.



Any closed square $Q = N_1$ centered at o with N_0 being the closed faces left and right can be taken as index pair (N_1, N_0) for the set S.

The algebraic invariants for an isolated invariant set S referred to in the previous section will actually be invariants of an index pair for S. It will turn out however that these invariants do not depend on the particular choice of an index pair for S. In this sence they will depend only on the way S sits in the local flow X. The first step is to construct an index pair for an isolated invariant set, which is done in the next theorem.

<u>*Theorem 3.1.*</u> *(Existence of a filtration of index pairs)*
Let S be an isolated invariant set und let $(M_1, \ldots, M_n)$ be an admissible ordering of a Morse decomposition of S. Then there exists an increasing sequence of compact sets

$$(3.4) \qquad N_0 \subset N_1 \subset \ldots \subset N_n ,$$

such that for any $i \leq j$, the pair (N_j, N_{i-1}) is an index pair for M_{ji}. In particular (N_n, N_0) is an index pair for S, and (N_j, N_{j-1}) is an indexpair for M_j. Moreover, given any isolating neighborhood N of S and given any neighborhood U of S, then the sets N_j can be chosen *so that* $cl(N_n \setminus N_0) \subset U$ *and such that the sets* N_j *are positively invariant relative to* N.

The rest of this paragraph is devoted to the proof of this theorem (Lemma 3.1. - Lemma 3.4.). We first choose any isolating neighborhood N of S, hence $I(N) = S$, and define for $j = 1,2,\dots,n$ the following subsets of N, which stay in N in forward respectively backward time:

$$(3.5) \qquad \begin{aligned} I_j^+ &= \{\gamma \in N \mid \gamma\cdot R^+ \subset N \text{ and } \omega(\gamma) \subset M_j \cup \dots \cup M_n\} \\ I_j^- &= \{\gamma \in N \mid \gamma\cdot R^- \subset N \text{ and } \omega^*(\gamma) \subset M_1 \cup \dots \cup M_j\}. \end{aligned}$$

We claim that $I_i^+ \cap I_j^- = M_{ji}$. In fact, if $\gamma \in I_i^+ \cap I_j^-$, then $\gamma\cdot R \subset N$ and hence $\gamma \in S$. Furthermore $\omega(\gamma) \subset M_i \cup \dots \cup M_n$ and $\omega^*(\gamma) \subset M_1 \cup \dots \cup M_j$, and the claim follows from the definition of a Morse decomposition.

<u>*Lemma 3.1.*</u> *The sets* $I_j^\pm$ *are compact*

<u>*Proof:*</u> a) The sets I_1^+ and I_n^- are compact: Observe $I_1^+ = \{\gamma \in N \mid \gamma\cdot R^+ \subset N\}$. Therefore if $\gamma \notin I_1^+$ then $\gamma\cdot t^* \notin N$ for some $t^* > o$. By the compactnes of N and by the continuity of the flow, there exists an open neighborhood $U \subset \Gamma$ of γ such that $U\cdot t^* \cap N = \phi$. Consequently if $\gamma \in U \cap N$ then $\gamma \notin I_1^+$ and $N \setminus I_1^+$ is open relative to N and hence I_1^+ is compact. The proof that I_n^- is compact is similar.

b) The special case $n = 2$: Let (M_1,M_2) be an admissible ordering of a Morse decomposition of S. By definition $I_2^+ \subset I_1^+$ and by

a) the set I_1^+ is compact; it remains to show that I_2^+ is closed. Let $\gamma = \lim_{n\to\infty} \gamma_n$, $\gamma_n \in I_2^+$, then $\gamma \in I_1^+$, hence $\omega(\gamma) \subset M_1 \cup M_2$ and we have to show $\omega(\gamma) \subset M_2$. Assume by contradiction $\omega(\gamma) \subset M_1$. Since M_1, M_2 are disjoint and compact we can choose open neighborhoods U_1 and U_2 of M_1 and M_2 with $\mathrm{cl}(U_1) \cap \mathrm{cl}(U_2) = \emptyset$. Since $\omega(\gamma_n) \subset M_2$ and $\omega(\gamma) \subset M_1$ there exists a sequence t_n' such that $\gamma_n \cdot [t_n', \infty) \subset U_2$ and there exists a sequence t_n'' such that $\gamma_n \cdot t_n'' \in U_1$. Therefore we find a sequence t_n such that $\gamma_n \cdot [t_n, \infty) \subset N \setminus U_1$ and $\gamma_n \cdot t_n \in N \setminus (U_1 \cup U_2)$. Take a subsequence such that $\gamma^* = \lim_{n\to\infty} (\gamma_n \cdot t_n)$ exists, then $\gamma^* \notin M_1 \cup M_2$ and $\gamma^* [o, \infty) \subset N \setminus U_1$ and therefore $\omega(\gamma^*) \subset M_2$. If the sequence (t_n) is bounded, then $\gamma^* \in \gamma \cdot R$ hence $\omega(\gamma) = \omega(\gamma^*) \subset M_2$ contradicting $\omega(\gamma) \subset M_1$. If the sequence (t_n) is unbounded then given any $t > o$, $\gamma^* \cdot [-t, o]$ is a limit of segments $\gamma_n \cdot t_n \cdot [-t, o]$ $= \gamma_n \cdot [t_n - t, t_n]$. For n large these segments are contained in $\gamma_n \cdot R^+ \subset N$ and it follows that $\gamma^* \cdot [-t, o] \subset N$. Since this holds true for every $t > o$, the set $\gamma^* \cdot R^-$ and hence $\gamma^* \cdot R$ is contained in N and therefore $\gamma^* \in S$. Since (M_1, M_2) is an admissible ordering of a Morse decomposition of S we conclude from $\omega(\gamma^*) \subset M_2$ that $\gamma^* \in M_2$ contradicting $\gamma^* \notin M_1 \cup M_2$.

c) The general case: We observe that if $j > 1$, then I_j^+ is simply the set $\overline{I}_2^+$ where $\overline{I}_2^+$ corresponds to the Morse set $\overline{M}_2 = M_{nj}$ of the two decomposition $(\overline{M}_1 = M_{(j-1)1}, \overline{M}_2 = M_{nj})$ of S. A similar remark applies to I_{j-1}^- for $j \leq n$, hence the Lemma follows from b). •

For a subset $Z \subset N$ we define the set $Z \subset P(Z) \subset N$ as the "swept out set" of Z by the flow in positive time as follows:

(3.7) $$P(Z) := \{\gamma \in N \mid \text{there exist } \gamma' \in Z \text{ and } t' \geq o \text{ such that } \gamma'\cdot[o,t'] \subset N \text{ and } \gamma'\cdot t' = \gamma\}$$

The set $P(Z)$ is positively invariant relative to N.

<u>Lemma 3.2.</u> *Let* V *be any* Γ*-neighborhood of* I_j^-*. Then there is a compact* N*-neighborhood* Z *of* I_j^- *such that* $P(Z)$ *is compact and* $P(Z) \subset V$.

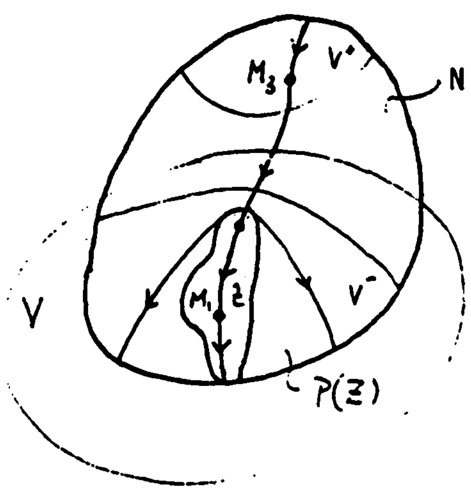
<u>*Proof:*</u> Since I_j^- and I_{j+1}^+ are disjoint and (by Lemma 3.1.) compact we can pick open X-neighborhoods V^+ of I_{j+1}^+ and V^- of I_j^- such that $V^- \subset V$ and $cl(V^+) \cap cl(V^-) = \phi$. We first claim, that there is a $t^* > o$ with the property, that for every $\gamma \in N \setminus V^-$ the arc $\gamma\cdot[-t^*,o]$ contains a point in V^+ or in $\Gamma \setminus N$:

(*) if $\gamma \in N \setminus V^-$ then $\gamma\cdot[-t^*,o] \not\subset N \setminus V^+$.

In fact, if $\gamma \in N \setminus V^-$ and $\gamma\cdot R^- \not\subset N$ there is a $t = t(\gamma)$ such that $\gamma\cdot(-t) \notin N$. If $\gamma\cdot R^- \subset N$, it follows from $\gamma \notin I_j^-$ that $\omega^*(\gamma) \subset M_{j+1} \cup \ldots \cup M_n \subset I_{j+1}^+ \subset V^+$, and there is a $t = t(\gamma)$ so that $\gamma\cdot(-t) \in V^+$. In either case there is a neighborhood W of γ such that $W\cdot t(\gamma)$ is contained in the complement of $(N \setminus V^+)$. The claim now follows since $N \setminus V^-$ is compact. In order to define Z let $\gamma \in I_j^-$, then $\gamma\cdot R^- \subset I_j^- \subset V^-$ and we can pick a compact neighborhood C_γ of γ such that $C_\gamma\cdot[-t^*,o] \subset V^-$. Since by Lemma 3.1. the set I_j^- is compact, a finite collection of such C_γ's cover I_j^-, and we let Z be their union. Z is a compact neighborhood of I_j^- and we claim that $P(Z) \subset V^-$. Assume not, then there is a $\gamma \in P(Z)$ with $\gamma \notin V^-$. By definition of

$P(Z)$, $\gamma = \gamma' \cdot t'$ for some $\gamma' \in Z$ and some $t' \geq o$ and $\gamma' \cdot [o,t'] \subset N$. Pick τ such that $\gamma'[o,\tau) \subset V^-$ and $\gamma^* := \gamma' \cdot \tau \in N \setminus V^-$. Then $\gamma^* \cdot [-\tau, o) \subset V^-$ and $\gamma^*(-\tau) = \gamma' \in Z$. By definition of Z, $\gamma' \cdot [-t^*, o] \subset V^-$, hence $\gamma^* \cdot [-(t^* + \tau), o] \subset cl(V^-) \subset N \setminus V^+$ in contradiction to (*), hence $P(Z) \subset V^-$. It remains to show that $P(Z)$ is compact. We shall show that the complement of $P(Z)$ in N is open and assume $\gamma \notin P(Z)$. Then $\omega^*(\gamma) \not\subset M_1 \cup \ldots \cup M_j$ and therefore there exists a $t = t(\gamma)$ such that $\gamma(-t) \notin N \setminus V^+$. Let $t_1 = \sup \{t \geq o | \gamma \cdot [-t, o] \subset N \setminus V^+\}$, then $\gamma \cdot [-t_1, o] \subset N \setminus V^+$ since $N \setminus V^+$ is closed. Moreover $\gamma \cdot [-t_1, o] \cap Z = \phi$ since $\gamma \notin P(Z)$. Because Z is compact there is a $t_2 > t_1$ such that $\gamma \cdot [-t_2, o] \cap Z = \phi$ and $\gamma \cdot (-t_2) \notin N \setminus V^+$. By the continuity of the flow and by the compactnes of $N \setminus V^+$ there is a neighborhood W of γ such that $W \cdot [-t_2, o] \cap Z = \phi$ and $W \cdot (-t_2) \cap (N \setminus V^+) = \phi$. Since $P(Z) \subset V^-$, we conclude, that if $\gamma' \in W$ then there is no orbit segment from Z to γ' which is contained in $N \setminus V^+$, hence $\gamma' \notin P(Z)$ and the complement of $P(Z)$ in N is open, hence $P(Z)$ compact. This finishes the proof of Lemma 2. •

The construction of Lemma 3.2 is schematically illustrated by the following Figure:



We next construct the index pair (N_n, N_o) for $S = I(N)$. We know $I_1^+ \cap I_n^- = S \subset \text{int}\,(N)$. Therefore, since by Lemma 3.1 the sets I_1^+ and I_n^- are compact, we can choose open neighborhoods U^+ of I_1^+ in N and U^- of I_n^- in N such that $\text{cl}(U^+ \cap U^-) \subset U \cap \text{int}\,N$, for a given neighborhood U of S. Define:

$$N_o = P(N \setminus U^+). \tag{3.7}$$

Then, by definition, N_o is positively invariant relative to N. We shall prove that N_o is compact. Since $N \setminus U^+$ is compact and disjoint from $I_1^+ = \{\gamma \in N \mid \omega(\gamma) \subset M_1 \cup \ldots \cup M_n\}$ there is a $t^* > o$ such that $\gamma \in N \setminus U^+$ implies $\gamma\cdot[o,t^*] \not\subset N$. Let $\gamma^* = \lim_{n\to\infty} \gamma_n$, $\gamma_n \in N_o$. By definition, $\gamma_n = \gamma_n' \cdot t_n'$ with $\gamma_n' \in N \setminus U^+$ and $\gamma_n'\cdot[o,t_n] \subset N$. Therefore $o \leq t_n \leq t^*$ and since $N \setminus U^+$ is compact, there are a $\gamma \in N \setminus U^+$ and $t \geq o$ such that $\gamma^* = \gamma\cdot t$ with $\gamma \in N \setminus U^+$ and $\gamma\cdot[o,t] \subset N$. Consequently $\gamma^* \in N_o$, hence N_o is compact.

In order to define N_n we apply Lemma 3.2 and take a compact neighborhood $N_n' \subset U^-$ of the set I_n^-, which is positively invariant relative to N and set

$$N_n = N_n' \cup N_o\,. \tag{3.8}$$

By construction N_n is positively invariant relative to N and (N_n, N_o) is a compact pair.

<u>Lemma 3.3.</u> (N_n, N_o) *is an index pair of* S. *Moreover* $\text{cl}(N_n \setminus N_o) \subset U$.

Proof: We verify conditions (i-iii) of definition 3.4. ad (i): S and N_o are compact and disjoint, hence $N \setminus N_o$ is a neighborhood of S, also N'_n hence N_n is a neighborhood of S by construction, hence $N_n \setminus N_o$ is a neighborhood of S. Furthermore, since $N \setminus U^+ \subset N_o$ and $N'_n \subset U^-$ we conclude that $N_n \setminus N_o \subset U^- \cap U^+$ and hence $cl(N_n \setminus N_o) \subset cl(U^+ \cap U^-) \subset U \cap \text{int } N$. In particular $cl(N_n \setminus N_o)$ is an isolating neighborhood of S. ad(ii): If $\gamma \in N_o$ and $\gamma\cdot[o,t] \subset N_n$ then $\gamma\cdot[o,t] \subset N$ and so $\gamma\cdot[o,t] \subset N_o$, since N_o is invariant relative to N. ad (iii): If $\gamma \in N_o$ there is nothing to prove. Assume $\gamma \in N_n \setminus N_o$ and $\gamma\cdot R^+ \not\subset N_n$. Put $t^* = \sup\{t \geq o \mid \gamma\cdot[o,t] \subset N_n \setminus N_o\}$, then $\gamma\cdot t^* \in cl(N_n \setminus N_o) \subset \text{int } N$ (relative to X). We now use the fact, that X is a local flow: since $\gamma\cdot t^* \in X$, there is a Γ-neighborhood W of $\gamma\cdot t^*$ and an $\varepsilon > o$ such that $W \cap X\cdot[o,\varepsilon) \subset X$. Since $\gamma\cdot t^* \in \text{int } N$ (rel X) there is therefore an $\varepsilon > o$ such that $\gamma\cdot[t^*, t^* + \varepsilon] \subset N$. But N_n is positively invariant relative to N and hence $\gamma\cdot[t^*, t^* + \varepsilon] \subset N_n$. By definition of t^* we conclude for a τ in $t^* < \tau < t^* + \varepsilon$ that $\gamma\cdot\tau \in N_o$. Since $\gamma\cdot[o,\tau] \subset N_n$, the crucial third condition of the definition of an index pair is verified. •

We finally construct the advertized filtration $N_o \subset N_1 \subset \ldots \subset N_n$. Applying Lemma 3.2, the definitions (3.5) and (3.6) to $N_n \subset N$ instead of N, we find for every $1 \leq j \leq n-1$ a compact neighborhood N'_j of I^-_j such that

1) $I^-_j \subset N'_j \subset N_n$

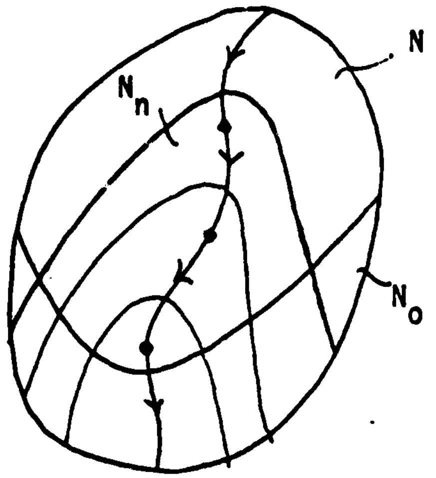
2) $N'_j \cap I^+_{j+1} = \phi$

3) N'_j is positively invariant relative to N_n.

Recall that $I_j^- \cap I_{j+1}^+ = \emptyset$. Now define iteratively:

(3.9) $$N_j = N'_j \cup N_{j-1}, \ 1 \leq j \leq n-1.$$

Schematically:



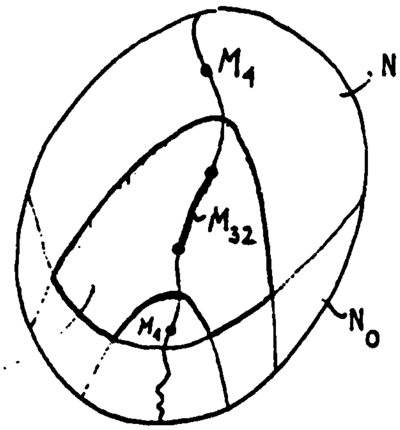
The following Lemma then finishes the proof of Theorem 3.1.

Lemma 3.4: (N_j, N_{i-1}), $i \leq j$, is an index pair for M_{ji}.

Proof. ad (i): To show that $cl(N_j \setminus N_{i-1})$ is an isolating neighborhood of M_{ji}, assume $\gamma \cdot R \subset cl(N_j \setminus N_{i-1})$. Then $\gamma \in S$ and since $\gamma \notin I_{i-1}^-$ we conclude $\omega^*(\gamma) \subset M_i \cup \ldots \cup M_n$. On the other hand $\gamma \notin I_{j+1}^+$ hence $\omega(\gamma) \subset M_1 \cup \ldots \cup M_j$ and therefore $\gamma \in M_{ji}$ by definition of this set. Clearly $M_{ji} \subset I_j^- \subset N_j$ and $M_{ji} \subset I_i^+$ and $I_i^+ \cap N_{i-1} = \emptyset$. Therefore $M_{ji} \subset N_j \setminus N_{i-1}$ proving our claim. ad (ii): By construction, N_{i-1} is positively invariant relative to N_n, also $N_j \subset N_n$ and therefore N_{i-1} is positively invariant relative to N_j. ad (iii): Assume $\gamma \in N_j \setminus N_{i-1}$ and $\gamma \cdot R^+ \not\subset N_j$. Then $\gamma \cdot R^+ \not\subset N_n$, since N_j is positively invariant relative to N_n. Therefore, by Lemma 3.3

there exists a τ such that $\gamma\cdot[o,\tau] \subset N_n$ and $\gamma\cdot\tau \in N_o$. From $\gamma \in N_j$, we deduce $\gamma\cdot[o,\tau] \subset N_j$. Also, by construction, $N_o \subset N_{j-1}$. Summarizing we have shown: if $\gamma\cdot R^+ \not\subset N_j$ there exists a $\tau \geq o$, such that $\gamma\cdot[o,\tau] \subset N_j$ and $\gamma\cdot\tau \in N_{j-1}$, hence also the third condition of an index pair is verified. •

Schematically:



$cl(N_3 \setminus N_1)$

3.3. The Morse "Inequalities" for a filtration.

The statement of the Morse inequalities for a filtration $N_o \subset N_1 \subset \ldots \subset N_n$ of any compact spaces is an immediate consequence of the axioms of elementary cohomology theory. If (Y,Z) is a compact pair, we denote by $H(Y,Z)$ the Čech-cohomology with coefficients in some fixed ring. This particular cohomology is chosen because it is defined for compact spaces and has the continuity property which does not hold for singular cohomology, for example. (The continuity property states: if $X = \cap X_n$ then $H(X) = \varinjlim H(X_n)$). For the cohomology theory we need we refer to E.H. Spanier [6] and [7]. If $A \supset B \supset C$ are compact spaces, then there is an exact sequence

$$o \to H^o(A,B) \to H^o(A,C) \to H^o(B,C) \xrightarrow{\delta^o}$$

$$(3.10) \qquad \xrightarrow{\delta^o} H^1(A,B) \to H^1(A,C) \to H^1(B,C) \xrightarrow{\delta^1}$$

$$\xrightarrow{\delta^1} H^2(A,B) \to \ldots \qquad .$$

Assuming the modules $H^P(X,Y)$ to be of finite rank we denote by $r^P(X,Y)$ the rank of $H^P(X,Y)$ and with $d^P(A,B,C)$ we denote the rank of the image of δ^P. If (X,Y) is a compact pair and if $A \supset B \supset C$ are compact spaces we can define the following formal power series

$$p(t,X,Y) := \sum_{n \geq o} r^n(X,Y)\, t^n$$

$$(3.11)$$

$$q(t,A,B,C) := \sum_{n \geq o} d^n(A,B,C)\, t^n \, .$$

The coefficients of these formal series are nonnegative integers.

Proposition 3.3.

Assume $N_o \subset N_1 \subset N_2 \subset \ldots \subset N_n$ *are compact spaces. Then*

$$\sum_{j=1}^{n} p(t,N_j,N_{j-1}) = p(t,N_n,N_o) + (1+t)\, Q(t).$$

where $Q(t) = \sum_{j=2}^{n} q(t,N_j,N_{j-1},N_o)$.

Proof: For the compact spaces $A \supset B \supset C$ we conclude from the exactnes of the sequence (3.10) for every $m \geq o$:

$$r^o(A,B) - r^o(A,C) + r^o(B,C)$$
$$- r^1(A,B) + r^1(A,C) - r^1(B,C) + \ldots$$
$$+ (-1)^m r^m(A,B) - (-1)^m r^m(A,C) + (-1)^m r^m(B,C)$$
$$- (-1)^m d^m(A,B,C) = o \quad .$$

From this we deduce:

$$(-1)^m d^m(A,B.C) = (-1)^{m-1} d^{m-1}(A,B,C)$$
$$+ (-1)^m r^m(A,B) - (-1)^m r^m(A,C) + (-1)^m r^m(B.C).$$

Multiplication of this equation by $(-1)^m t^m$ and addition over m yields

$$q(t,A,B,C) = - t\, q(t,A,B,C)$$
$$+ p(t,A,B) - p(t,A,C) + p(t,B,C),$$

or equivalently:

$$p(t,A,B) + p(t,B.C) = p(t,A,C) + (1 + t)\, q(t,A,B,C).$$

Application of this equality to the triples $N_j \supset N_{j-1} \supset N_o$, $j \geq 2$ gives

$$p(t,N_j,N_{j-1}) + p(t,N_{j-1},N_o) = p(t,N_j,N_o) + (1+t)\, q(t,N_j,N_{j-1},N_o).$$

Adding these equations over $j \geq 2$ and setting $Q(t) := \sum_{j \geq 2} q(t,N_j,N_{j-1},N_o)$

one finds

$$\sum_{j \geq 1}^{n} p(t,N_j,N_{j-1}) = p(t,N_n,N_o) + (1+t)\, Q(t) ,$$

as claimed. •

3.4. The Morse Index and the Morse inequalities for an isolated invariant set.

Proposition 3.3 is in particular applicable to the filtration $N_o \subset N_1 \subset \ldots \subset N_n$ found by Theorem 3.1 for a Morse decomposition of an isolated invariant set. There is however not a unique filtration for a given Morse decomposition, in fact there is also no unique index pair (N_1,N_o) for an isolated invariant set. But we shall prove that $H(N_1,N_o) \cong H(\overline{N}_1, \overline{N}_o)$ for any two index pairs (N_1,N_o) for the same isolated invariant set S. To do so we first recall the notion of a pointed space. For any pair (X,A), the pointed space X/A is the pair

$$X/A = ((X \setminus A) \cup [A], [A]). \tag{3.12}$$

The points of X/A consist therefore of the points $x \in X \setminus A$ and an additional distinguished point $[A]$. The topology of X/A is defined as follows: a set $U \subset X/A$ is open if either $U \subset X \setminus A$ und U is open in X, or if $[A] \in U$ and $\{U \cap (X \setminus A)\} \cup A$ is open in X. In particular, if $A = \phi$, then $[A]$ is open and closed in X/A. Another way to define the pair (3.12) is $X/A = (X/\sim, [A])$, with $\sim$ being the equivalence relation in X defined by: $x \sim y$ if either $x = y$ in case $x,y \notin A$ or if $x,y \in A$. This equivalence relation simply identifies the points in A. Clearly if (X,A) is a compact Hausdorff pair, then X/A is also.

After these preliminaries we can formulate the crucial fact, that the homotopy type of N_1/N_2 depends only on S.

Theorem 3.2.

Let (N_1,N_0) *and* $(\bar{N}_1,\bar{N}_0)$ *be two index pairs for the isolated invariant set* S. *Then the pointed topological spaces* N_1/N_0 *and* $\bar{N}_1/\bar{N}_0$ *are homotopically equivalent:* $[N_1/N_0] = [\bar{N}_1/\bar{N}_0]$, *if we denote by* [ ] *the equivalence class of pointed spaces. We therefore can associate to* S *the unique equivalence class*

$$h(S) = [N_1/N_0] \ , \tag{3.13}$$

where (N_1,N_0) *is any index pair for the isolated invariant set* S. *We call* h(S) *the (homotopy) index of* S.

Postponing the proof of this theorem to the next section we first state and prove the Morse theorem. Observe that if (X,Y) is any pair, then for the Čech-cohomology (see [7]) $H(X,A) = H(X/A)$. Using then the fact, that the cohomologies of two homotopically equivalent pairs are isomorphic we therefore conclude from Theorem 3.2. the

Corollary:

Let (N_1,N_0) *and* $(\bar{N}_1,\bar{N}_0)$ *be two index pairs for the isolated invariant set* S. *Then* $H(N_1,N_0) \cong H(\bar{N}_1,\bar{N}_0)$.

With $H(N_1,N_0)$ we have found, up to isomorphisms, the algebraic invariant of the isolated invariant set S we are looking for. It is independent of the particular index pair for S chosen. We can therefore define:

$$(3.14) \qquad p(t,h(S)) = p(t,N_1,N_0) ,$$

where (N_1,N_0) is any index pair for S. With this notation, we formulate the main result of this section.

Theorem 3.3.

Assume S *is an isolated set in the local flow* X. *Let* $(M_1,\ldots,M_n)$ *be an admissible ordering of a Morse-decomposition of* S. *Then*

$$\sum_{j=1}^{n} p(t,h(M_j)) = p(t,h(S)) + (1+t)\,Q(t) ,$$

where the series $Q(t)$ *is defined as in Proposition 3.3. In particular the coefficients of* Q *are nonnegative integers.*

Proof: By theorem 3.1. there is a filtration $N_0 \subset N_1 \subset \ldots \subset N_n$ for the Morse-decomposition, such that (N_n,N_0) is an index pair for S and (N_j,N_{j-1}) is an index pair for M_j, $1 \le j \le n$. In view of (3.14) the statement is an immediate consequence of Proposition 3.3. •

We point out that the term $q(t,N_j,N_{j-1},N_0)$ in theorem 3.3 gives some measure of the number of algebraic connections from M_j to $M_{j-1,1}$. In fact we shall prove

Proposition 3.4.

If $q(t,N_j,N_{j-1},N_o) \neq o$, *then* $M_{j,1} \subset M_{j-1,1} \cup M_j$ *but* $M_{j,1} \neq M_{j-1,1} \cup M_j$.

Proof: With the notation as in the proof of the previous theorem we consider the compact sets $N_j \supset N_{j-1} \supset N_o$, where (N_j,N_o) is an index pair for $M_{j,1}$, the pair (N_{j-1},N_o) is an index pair for $M_{j-1,1}$ and (N_j,N_{j-1}) is an index pair for M_j. If (N_1,N_o) is an index pair for S we can write $H(h(S)) \cong H(N_1,N_o)$. With this notation we have by (3.10) the exact sequence: $\xrightarrow{\delta} H(h(M_j)) \to H(h(M_{j,1})) \to H(h(M_{j-1,1})) \xrightarrow{\delta} \ldots$ Assuming $M_{j,1} = M_{j-1,1} \cup M_j$ we shall conclude $\delta = o$, hence $q(t,N_j,N_{j-1},N_o) = o$ in contradiction to the assumption. Rewording, we set $S = M_{j1}$ and $S_1 = M_{j-1,1}$ and $S_2 = M_j$, and have $S = S_2 \cup S_1$ with $S_2 \cap S_1 = \phi$. The above exact sequence then reads

$$(3.15) \qquad \xrightarrow{\delta} H(h(S_2)) \to H(h(S_1 \cup S_2)) \to H(h(S_1)) \xrightarrow{\delta} .$$

We need a Lemma. We first recall that the sum $\vee$ of two pointed spaces (A,a) and (B,b) is defined to be $A \cup B / \{a,b\}$ or, in other words, it is the pointed space obtained taking the union and identifying the two distinguished points a and b. This sum is denoted by $(A,a) \vee (B,b)$. It is easily seen to be well defined on homotopy classes, so that $[(A,a)] \vee [(B,b)]$ can be defined to be $[(A,a) \vee (B,b)]$.

Lemma 3.5. (Sum formula for the index) If S_1 *and* S_2 *are isolated invariant sets with* $S_1 \cap S_2 = \emptyset$, *then* $S_1 \cup S_2$ *is an isolated invariant set and* $h(S_1 \cup S_2) = h(S_1) \vee h(S_2)$.

Proof of the Lemma: Choose disjoint index pairs (N_1,N_0) for S, and $(\overline{N}_1,\overline{N}_0)$ for S_2, i.e. $N_1 \cap \overline{N}_1 = \phi$, then $(N_1 \cup \overline{N}_1, N_0 \cup \overline{N}_0)$ is an index pair for $S_1 \cup S_2$ and it is easy to see that $[N_1 \cup \overline{N}_1/N_0 \cup \overline{N}_0] = [N_1/N_0] \vee [\overline{N}_1/\overline{N}_0]$. Thus $h(S_1 \cup S_2) = h(S_1) \vee h(S_2)$ as claimed. •

By the Lemma, we have $H(h(S_1 \cup S_2)) = H(h(S_1) \vee h(S_2))$ which is isomorphic to $H(h(S_1)) \oplus H(h(S_2))$. Therefore the exact sequence (3.15) must break up into a collection of short exact sequences $o \to H^r(h(S_2)) \to H^r(h(S_1 \cup S_2)) \to H^r(h(S_1)) \to o$, and so, in particular, the maps δ are all trivial. This finishes the proof of Proposition 3.4. •

It is clear that by breaking up the Morse decomposition in different ways, Q can be written as sum of terms measering connections between different Morse sets; of course the sum of these terms would be the same.

3.5. Proof of theorem 3.2.

Let (X,A) be a pair, then the quotient map $X \to X/A$, $x \to [x]$, defined by $[x] = x$ if $x \in X \setminus A$ and $[x] = [A]$ if $x \in A$, is continuous. It is surjective expect if $A = \emptyset$ in which case it just misses the distinguished point $[A]$. The following statement is obvious:

Proposition 3.5.

Let $f: (X,A) \to (Y,B)$ *be a continuous map between the two pairs* (i.e. $f(A) \subset B$). *Then the induced map of pointed spaces* $\tilde{f}: X/A \to Y/B$ *defined* *by* $\hat{f}([x]) = [f(x)]$ *is also continuous.* •

If (N_1, N_0) is an index pair for the isolated invariant set S, we define for $t \geq o$ the following subsets of N_1:

$$(3.16)\qquad \begin{aligned} N_1^t &:= \{\gamma \in N_1 \mid \gamma\cdot[-t,o] \subset N_1\} \\ N_0^{-t} &= \{\gamma \in N_1 \mid \gamma\cdot[o,t] \cap N_0 \neq \phi\} \quad \supset N_0 \end{aligned}$$

These sets are compact, and positively invariant relative to N_1. Roughly N_1^t is N_1 "pushed forward" in time t and N_0^{-t} is N_0 "pulled backward" in time t. One can readily verify that (N_1, N_0^{-t}) is also an index pair for S. In general, however, $(N_0 \cap N_1^t)$ is not an index pair for S anymore, although it is, if X is a two sided local flow.

Now let $i: (N_1^t, N_1^t \cap N_0) \to (N_1, N_0)$ be the inclusion map and denote by $\hat{i}$ the induced map between the corresponding pointed spaces as defined in proposition 3.5, then:

Lemma 3.6. *Let* $t \geq o$. *Then the induced map*

$$(3.17)\qquad \hat{i} : N_1^t/(N_0 \cap N_1^t) \to N_1/N_0$$

is a homotopy equivalence.

Proof: Let $t \geq o$ and define the map

(3.18) $$F : (N_1/N_0) \times [0,1] \to N_1/N_0$$

by setting $F([\gamma],\sigma) = [\gamma\cdot\sigma t]$ in case $\gamma\cdot[0,\sigma t] \subset N_1 \setminus N_0$ and $F([\gamma],\sigma) = [N_0]$ otherwise. By definition 3.4 of an index pair, this map is well defined as a map between pointed spaces. We shall prove that it is continuous. Suppose $F([\gamma],\sigma) \neq [N_0]$, then, by Def. 3.4 (ii), $\gamma\cdot[0,\sigma t] \subset N_1 \setminus N_0$. Let U be any neighborhood of $\gamma\cdot\sigma t$ disjoint from N_0, and let V be any neighborhood of $\gamma\cdot[0,\sigma t]$ disjoint from N_0. By the continuity of the flow there are neighborhoods W of γ and W' of σ such that if $(\gamma',\sigma') \subset W \times W'$ then $\gamma'\cdot\sigma' t \in U$ and $\gamma'\cdot[0,\sigma' t] \subset V$. It now follows from definition 3.4 (iii) and $V \cap N_0 = \phi$ that $\gamma'\cdot[0,\sigma' t] \subset N_1 \setminus N_0$, hence $F([\gamma'], \sigma') = [\gamma'\cdot\sigma' t] = \gamma'\cdot\sigma' t \in U$; hence F is continuous at $([\gamma],\sigma)$. Suppose $F([\gamma],\sigma) = [N_0]$. If $\gamma\cdot[0,\sigma t] \not\subset N_1$, then for γ' close to γ and σ' close to σ we have $\gamma'\cdot[0,\sigma' t] \not\subset N_1$, hence $F([\gamma'],\sigma') = [N_0]$, and F is continuous at $([\gamma],\sigma)$. Suppose finally that $F([\gamma],\sigma) = [N_0]$ and $\gamma\cdot[0,\sigma t] \subset N_1$, then $\gamma\cdot\sigma t \in N_0$. Let $\mathcal{U}$ be a neighborhood of $[N_0]$ in N_1/N_0. Then there is a neighborhood U of N_0 such that $[N \cap U] = \mathcal{U}$ Now if (γ',σ') is close to (γ,σ), then $\gamma'\cdot\sigma' t \in U$ by the continuity of the flow. If $\gamma'\cdot[0,\sigma' t] \subset N_1$, then $F(\gamma',\sigma') = [\gamma'\cdot\sigma' t] \in \mathcal{U}$. If $\gamma'\cdot[0,\sigma' t] \not\subset N_1$, then $F(\gamma',\sigma') = [N_0] \in \mathcal{U}$. In either case, if (γ',σ') is close to (γ,σ), $F(\gamma',\sigma') \in \mathcal{U}$, so F is continuous at $([\gamma],\sigma)$ and having exhausted all the possibilities, F is continuous.

If $\sigma = 1$, the map $F(\cdot,1)$: N_1/N_0 has its range in $(N_1^t \cup N_0)/N_0 = N_1^t/(N_0 \cap N_1^t)$. Let f be the map $F(\cdot,1)$ but considered as a map from N_1/N_0 into $N_1^t/(N_0 \cap N_1^t)$. Then, $\sim$ meaning homotopic to,

$$(3.19) \qquad \hat{i} \circ f = F(\cdot,1) \sim \mathrm{id} \quad \text{on} \quad N_1/N_0,$$

by definition of F. On the other hand, since N_1^t is positively invariant relative to N_1, the restriction of F to $(N_1^t \cup N_0)/N_0 \times [o,1]$ has range in $(N_1^t \cup N_0)/N_0$. Let F_r denote this restricted map as a map into $(N_1^t \cup N_0)/N_0$. Then

$$(3.20) \qquad f \circ \hat{i} = F_r(\cdot,1) \sim \mathrm{id} \quad \text{on} \quad N_1^t/(N_0 \cap N_1^t).$$

From (3.19) and (3.20) the Lemma follows. •

Next define for $t \geq o$ the map

$$(3.21) \qquad g\colon N_1/N_0^{-t} \to N_1^t \,/\, (N_0 \cap N_1^t)$$

by setting $g([\gamma]) = [\gamma\cdot t]$ if $\gamma\cdot[o,t] \subset N_1 \setminus N_0$ and $g([\gamma]) = [N_0 \cap N_1^t]$ otherwise. This map is in fact well defined as a map between pointed spaces. Indeed, if $\gamma \in N_0^{-t}$, then, by definition, $\gamma\cdot[o,t] \not\subset N_1 \setminus N_0$, hence $g([\gamma]) = [N_0 \cap N_1^t]$. Also, if $\gamma\cdot[o,t] \subset (N_1 \setminus N_0)$ then $\gamma\cdot t \in N_1^t$ and so $g([\gamma]) = [\gamma\cdot t] \in N_1^t/(N_0 \cap N_1^t)$ as required.

__Lemma 3.7.__ The map g *defined by* (3.21) *is a homeomorphism.*

__Proof:__ Assume $g([\gamma]) = [N_0 \cap N_1^t]$, then $\gamma\cdot[o,t] \not\subset N_1 \setminus N_0$ and therefore, by Definition 3.4, $\gamma\cdot[o,t] \cap N_0 \neq \emptyset$, hence $\gamma \in N_0^{-t}$ and thus $g^{-1}([N_0 \cap N_1^t]) = [N_0^{-t}]$. Moreover, if $[\gamma_1]$ and $[\gamma_2]$ are not equal to

$[N_0^{-t}]$ in N_1/N_0^{-t}, then $\gamma_1\cdot[0,t]$ and $\gamma_2\cdot[0,t]$ are contained in $N_1 \setminus N_0$ and $\gamma_1\cdot t \neq \gamma_2\cdot t$. Therefore $[\gamma_1\cdot t] \neq [\gamma_2\cdot t]$ in $N_1^t/(N \cap N_1^t)$ and g is injective. We claim that g is surjective. Let $\gamma \in N_1^t \setminus N_0$, then there is a $\gamma' \in N_1$ with $\gamma'\cdot[0,t] \subset N_1 \setminus N_0$ and $\gamma'\cdot t = \gamma$ proving the claim. Proceeding as in Lemma 3.6 one sees that the map g is continuous, since g is a map between compact Hausdorff spaces it must therefore be a homeomorphism. •

Let now $j : (N_1,N_0) \to (N_1,N_0^{-t})$ be the inclusion map. For its induced map $\hat{j}$ we shall prove

<u>*Lemma 3.8.*</u> *The map* $\hat{j} : N_1/N_0 \to N_1/N_0^{-t}$ *is a homotopy equivalence.*

<u>*Proof:*</u> Consider the sequence of maps

$$N_1/N_0 \xrightarrow{\hat{j}} N_1/N_0^{-t} \xrightarrow{g} N_1^t/(N_0 \cap N_1^t) \xrightarrow{\hat{i}} N_1/N_0 .$$

By the definitions, $g \circ \hat{j} = f$, where f is defined in Lemma 3.6, hence $(\hat{i} \circ g) \circ \hat{j} = \hat{i} \circ f \sim \mathrm{id}$ on N_1/N_0, by (3.19). Also, by (3.20), $(g \circ \hat{j}) \circ \hat{i} = f \circ \hat{i} \sim \mathrm{id}$ on $N_1^t/(N_0 \cap N_1^t)$. Therefore, since, by Lemma 3.7, the map g is a homeomorphism, $\hat{j} \circ (\hat{i} \circ g) \sim \mathrm{id}$ on N_1/N_0^{-t}, which proves the Lemma. •

It was shown in theorem 3.1, that if N is any isolating neighborhood of $S = I(N)$, then there is an index pair (N_1,N_0) for S such that $N_0 \subset N_1 \subset N$, and such that, moreover, N_1 and N_0 are positively invariant relative to N. Such an index pair will be called an index pair contained in N. We observe that for an index pair (N_1,N_0) contained in

N we have $I^-(N) \subset N_1$ and $I^+(N) \cap N_0 = \phi$, where $I^-(N) = \{\gamma \in N \mid \gamma \cdot R^- \subset N\}$ and $I^+(N) = \{\gamma \in N \mid \gamma \cdot R^+ \subset N\}$.

<u>*Lemma 3.9.*</u> *Let* (N_1, N_0) *and* $(\overline{N}_1, \overline{N}_0)$ *be two index pairs for* $S = I(N)$ *contained in* N. *Then there exists a* $t > o$ *such that:*

$$(N_1^t,\ N_0 \cap N_1^t) \subset (\overline{N}_1, \overline{N}_0^{-t})$$

$$(\overline{N}_1^t,\ \overline{N}_0 \cap \overline{N}_1^t) \subset (N_1, N_0^{-t}) \ .$$

<u>*Proof:*</u> Since $I^-(N) \subset N_1$, if $\gamma \in cl(N \setminus N_1)$ then $\gamma \cdot R^- \not\subset N$. By compactnes of $cl(N \setminus N_1)$ there is a $t_1 \geq o$ such that $\gamma \in cl(N \setminus N_1)$ implies $\gamma \cdot [-t_1, o] \not\subset N$. Similarly, if $\gamma \in N_0$, then $\gamma \cdot R^+ \not\subset N$ and there is a t_0 such that $\gamma \in N_0$ implies $\gamma \cdot [o, t_0] \not\subset N$. Let $\overline{t}_1$ and $\overline{t}_0$ be the corresponding numbers for the pair $(\overline{N}_1, \overline{N}_0)$, and put $\tau = \max \{t_1, \overline{t}_1, t_0, \overline{t}_0\}$. Suppose $\gamma \in N_1^\tau$. Then $\gamma \cdot [-\tau, o] \subset N_1 \subset N$ and hence $\gamma \notin cl(N \setminus \overline{N}_1)$ and therefore $\gamma \in \overline{N}_1$ and $N_1^\tau \in \overline{N}_1$. If $\gamma \in (N_0 \cap N_1^\tau) \subset \overline{N}_1$, then $\gamma \cdot [o, \tau] \not\subset N$ and so by definition (3.4) (iii) there is a $t \leq \tau$, such that $\gamma \cdot [o, t] \subset \overline{N}_1$ and $\gamma \cdot t \in \overline{N}_0$, therefore $\gamma \in \overline{N}_0^{-t}$ and so $\gamma \in \overline{N}_0^{-\tau}$. Consequently $(N_1^\tau,\ N_0 \cap N_1^\tau) \subset (\overline{N}_1,\ \overline{N}_0^{-\tau})$. The other inclusion follows the same way. •

<u>*Lemma 3.10.*</u> *Let* (N_1, N_0) *and* $(\overline{N}_1, \overline{N}_0)$ *be two index pairs for* S *contained in the isolating neighborhood* N *of* $S = I(N)$. *Then* $[N_1/N_0] = [\overline{N}_1/\overline{N}_0]$.

<u>*Proof:*</u> Let $t > o$ be as in Lemma 3.9, and let i_1 and i_2 be the

inclusion maps of the corresponding pairs in Lemma 3.9. Denote by $\hat{i}_1$ and $\hat{i}_2$ the corresponding induced maps between the pointed spaces. Consider the sequence of maps:

$$N_1^t/(N_0 \cap N_1^t) \xrightarrow{\hat{i}_1} \bar{N}_1/\bar{N}_0^{-t} \xrightarrow{\bar{g}} \bar{N}_1^t/(\bar{N}_0 \cap \bar{N}_1^t) \xrightarrow{\hat{i}_2}$$

$$N_1/N_0^{-t} \xrightarrow{g} N_1^t/(N_0 \cap N_1^t) \xrightarrow{\hat{i}_1} \bar{N}_1/\bar{N}_0^{-t} \; ,$$

where the map g respectively $\bar{g}$ is defined by means of the flow in (3.21). Observe now that by definition $\hat{i}_2 \circ \bar{g} \circ \hat{i}_1 = \hat{j} \circ \hat{i} \circ \bar{g} \circ \hat{j} \circ \hat{i}$, since both mappings map $[\gamma]$ onto $[\gamma \cdot t]$ or onto $[N_0^{-t}]$. Hence, by the Lemmata 3.6 - 3.8, $\hat{i}_2 \circ \bar{g} \circ \hat{i}_1$ is a homotopy equivalence. Similarly $\hat{i}_1 \circ g \circ \hat{i}_2$ is a homotopy equivalence. Now, if one has any sequence $X_0 \xrightarrow{\phi} X_1 \xrightarrow{\chi} X_2 \xrightarrow{\psi} X_3$ of maps such that $\chi \circ \phi$ and $\psi \circ \chi$ are homotopy equivalences, then all the maps ϕ, χ and ψ are homotopy equivalences. Applying this observation we conclude, that $\bar{g} \circ \hat{i}_1$, $\hat{i}_2$ and $\hat{i}_1 \circ g$ are homotopy equivalences and hence, by Lemma 3.7 also the map $\hat{i}_1$ is a homotopy equivalence. Now the sequence of maps

$$N_1/N_0 \xrightarrow{\hat{j}} N_1/N_0^{-t} \xrightarrow{g} N_1^t/(N_0 \cap N_1^t) \xrightarrow{\hat{i}_1}$$

$$\bar{N}_1/\bar{N}_0^{-t} \xrightarrow{\bar{g}} \bar{N}_1^t/(\bar{N}_0 \cap \bar{N}_1^t) \xrightarrow{\hat{i}} \bar{N}_1/\bar{N}_0$$

shows that $[N_1/N_0] = [\bar{N}_1/\bar{N}_0]$. •

In order to prove theorem 3.2 we have to show that N_1/N_0 is equivalent to $\bar{N}_1/\bar{N}_0$ for two arbitrary index pairs (N_1, N_0) and

$(\bar{N}_1,\bar{N}_0)$ for S. For this purpose we shall show that they are equivalent to two index pairs both contained in a common isolating neighborhood of S. In view of Lemma 3.10 this will then finish off the proof of theorem 3.2.

Let (N_1,N_0) be an index pair for S. Pick an isolating neighborhood N' of S whose interior contains $cl(N_1 \setminus N_0)$, and define the pair $(\tilde{N}_1,\tilde{N}_0) = (N' \cap N_1, N' \cap N_0)$ which is an index pair for S contained in the isolating neighborhood $N: = N' \cap N_1$ of S. In fact, from $\tilde{N}_1 \setminus \tilde{N}_0 = N_1 \setminus N_0$ we conclude that $cl(\tilde{N}_1 \setminus \tilde{N}_0)$ is an isolating neighborhood for S. Moreover, $\tilde{N}_0$ is positively invariant relative to $\tilde{N}_1$. In fact, if $\gamma \in \tilde{N}_1 \setminus \tilde{N}_0$ and if $\gamma\cdot[o,t] \not\subset \tilde{N}_1$ define $t^* = \sup \{s \mid \gamma\cdot[o,s] \subset \tilde{N}_1 \setminus \tilde{N}_0\}$. Then $\gamma\cdot t^* \in cl(\tilde{N}_1 \setminus \tilde{N}_0) = cl(N_1 \setminus N_0) \subset N'$. But $\gamma\cdot t^*$ is not in the interior of $cl(N_1 \setminus N_0)$, therefore $\gamma\cdot t^* \in N_0$ and so $\gamma\cdot t^* \in N_0 \cap N' = \tilde{N}_0$.

<u>*Lemma 3.11.*</u> N_1/N_0 *is homeomorphic to* $\tilde{N}_1/\tilde{N}_0$.

<u>*Proof:*</u> Clearly $N_1 \setminus N_0 = \tilde{N}_1 \setminus \tilde{N}_0$ and $\tilde{N}_1 \subset N_1, N_0 \subset \tilde{N}_0$. The inclusion map $i: = (\tilde{N}_1,\tilde{N}_0) \to (N_1,N_0)$ therefore induces the required homeomorphism $\bar{i}$. •

Let now $\hat{N}$ be any isolating neighborhood of S contained in $N_1 \setminus N_0 = \tilde{N}_1 \setminus \tilde{N}_0$. By theorem 3.1 there is an index pair $(\hat{N}_1,\hat{N}_0)$ in N_1 such that $\hat{N}_1$ and $\hat{N}_0$ are positively invariant relative to N and such that $cl(\hat{N}_1 \setminus \hat{N}_0) \subset \text{int } \hat{N}$. As above, the pair $(\hat{N}_1 \cap \hat{N}, \hat{N}_0 \cap \hat{N})$ is an index pair for S in $\hat{N}$ such that $(\hat{N}_1 \cap \hat{N}) / (\hat{N}_0 \cap \hat{N})$ is homeomorphic to $\hat{N}_1/\hat{N}_0$ (Lemma 3.11). But also $(\tilde{N}_1,\tilde{N}_0)$ is an index pair in N and therefore, by Lemma 3.10, we conclude that N_1/N_0 is homotopically equivalent to $(\tilde{N}_1 \cap \tilde{N})/(\tilde{N}_0 \cap \tilde{N}$.

Summarizing we have proved that N_1/N_0 has the homotopy type of an index pair in $\tilde{N}$, namely of that given by $(\tilde{N}_1 \cap \tilde{N})/(\tilde{N}_0 \cap \tilde{N})$. Thus if (N_1,N_0) is any index pair and $\tilde{N} \subset \text{int}\,(N_1 \setminus N_0)$ is any compact neighborhood of S, then N_1/N_0 has the same homotopy type as the index pairs of S in $\tilde{N}$. If $(\tilde{N}_1,\tilde{N}_0)$ is another index pair of S, we now simply choose $\tilde{N}$ interior to $\text{cl}(\overline{N}_1 \setminus \overline{N}_0)$ and with Lemma 3.10 the proof of theorem 3.2 follows. •

3.6. Comparison with classical Morse-theory

The present approach is a generalization of Morse's theory which had as its original aim to find lower bounds for the number of critical points of a smooth function on a manifold.

Suppose $f(x)$ is a smooth function on a compact closed manifold M of dimension d. Then the equation $\dot{x} = \nabla f(x)$ defines a flow on M. This flow will be taken as the flow on $\Gamma = M$, also the local flow X will be taken to be all of M as well as the isolated invariant set S. Hence $\Gamma = M = X = S$. Suppose now that f has only finitely many critical points, say $\{x_\pi\}_{\pi \in P}$. We claim that the sets $M_\pi := \{x_\pi\}$ form a Morse decomposition of $S = M$. In fact, if $x \in M$ and $t > o$ then either $x \cdot t = x$ or $f(x \cdot t) > f(x)$. This implies that f is constant on the limit sets $\omega(x)$ and $\omega^*(x)$, so both these sets must be rest points and, unless x is a rest point, $f(\omega(x)) > f(\omega^*(x))$. Now let $(x_0,x_1,\ldots,x_n)$ be any ordering of the rest points such that if $j > i$ then $f(x_j) \geq f(x_i)$. Then the condition (in Definition 3.1) that

$(x_0, x_1, \dots, x_n)$ is an admissible ordering of a Morse decomposition is satisfied.

To arrive at Morse's statements it is necessary to find the $p(t, h(x_i))$ for the Morse sets $M_i = \{x_i\}$. In the case that x_i is a non-degenerate rest point, the gradient equation can be written in a local corrdinate system y centered at the rest point $y = o$, and after stretching the variables $y = \varepsilon x$, as

$$\dot{x}_- = A_- x_- + g_-(x)$$

$$\dot{x}_+ = A_+ x_+ + g_+(x) .$$

Here $x = (x_-, x_+) \in E_- \times E_+ = E = \mathbb{R}^d$, and $\langle x_-, A_- x_- \rangle \leq -\lambda |x_-|^2$ and $\langle x_+, A_+ x_+ \rangle \geq \lambda |x_+|^2$ for some $\lambda > o$. Moreover $g = (g_-, g_+)$ satisfies $g(o) = o$, $g'(o) = o$, and in $|x| \leq 2$ we have the estimate $|g| \leq \delta$, $|g'| \leq \delta$ with δ tending to zero as ε tends to zero. Let now Q be the unitsquare $Q = \{x \in E \mid |x_-| \leq 1$ and $|x_+| \leq 1\}$. If $x \in \partial Q$ then $|x_-| = 1$ or $|x_+| = 1$. If $x \in Q$ and $|x_-| \leq |x_+|$ then $\frac{d}{dt} |x_-|^2 =$ $= 2 \langle x_-, A_- x_- + g_-(x) \rangle \leq -\lambda |x_-|^2$ choosing $4\delta \leq \lambda$. Similarly, if $|x_+| \leq |x_-|$, then $\frac{d}{dt} |x_+|^2 \geq \lambda |x_+|^2$. This implies, that if $N_1 = Q$ and $N_0 =$ $\{x \in Q \mid |x_+| = 1\}$, then (N_1, N_0) is an index pair for the critical point x_i under consideration. We claim that

$$p(t, h(x_i)) = t^{d_i}, \quad d_i = \dim E_+. \tag{3.22}$$

This is seen as follows: the pair (N_1, N_0) is obviously homotopically equivalent to the pair $(D^m, \partial D^m)$, $m = d_i$, where $D^m = \{(x_+, o) \mid |x_+| \leq 1\}$ is the closed unit disc in E_+, and $\partial D^m = \{(x_+, o) \mid |x_+| = 1\}$ the unit sphere. Therefore N_1/N_0 is homotopically equivalent to $D^m/\partial D^m$ which is the same as the sphere of dimension $m = d_i$ with a distinguished point.

Therefore the cohomology has rank o in all dimensions except the dimension d_i, where it has rank one. Thus (3.22) is verified.

Also, $(M,\emptyset)$ is an index pair for $S = M$ and $p(t,h(S)) = \sum_{j=o}^{d} \beta_j t^j$ where β_j is the j'th Betti number of the manifold M. The statement $\sum_{i=o}^{n} p(t,h(x_i)) = p(t,h(S)) + (1+t)\,Q(t)$ then includes the Morse inequalities for gradient flows on compact closed manifolds. If all the critical points are nondegenerate we have:

$$\sum_{i=o}^{n} t^{d_i} = \sum_{j=o}^{d} \beta_j t^j + (1+t)\,Q(t),$$

where $Q(t) = q_o + tq_1 + \ldots$ is a polynomial with integer nonnegative coefficients $q_j \geq o$. In particular we read off that $\nu_j \geq \beta_j$, where ν_j is the number of critical points $\{x_i\}$ having $d_i = j$ as dimensions of its unstable invariant manifold.

In S. Smale's generalization of the above Morse theorem [8], some of the critical points are replaced by the periodic solutions of a flow, which together serve as Morse sets. He assumes that these periodic solutions and critical points are finite in number and comprise the non wandering set. That they form a Morse decomposition comes from his imposed "no cycle" condition. It can easily be shown that for a nondegenerate periodic orbit M_j(i.e. no Floquet-multiplier equal to 1) having an orientable unstable manifold, $p(t,h(M_j)) = t^d + t^{d+1}$, where d is the dimension of the unstable manifold of the Poincaré map. Also, when the unstable manifold is non-orientable, it is the same, if Z_2 coefficients are used, and is o otherwise. In fact, let us consider the orientable case. It is well known, that locally in a neighborhood of a nondegenerate periodic solution, the flow is topologically equivalent to a linear flow: $\dot{x}_- = -x_-$, $\dot{x}_+ = x_+$, $\dot{\theta} = 1$. Here $\theta(\text{mod } 2\pi) \in S^1$ and $S^1 \cong \{(o,o,\theta) \mid \theta \in S^1\}$ corresponds to the periodic solution. Now with B_ε and ∂B_ε as above for the critical point it is obvious that if $\overline{N}_1 := B_\varepsilon \times S^1$ and $\overline{N}_o := \partial B_\varepsilon \cap \{|x_+| = \varepsilon\} \times S^1$ then $(\overline{N}_1, \overline{N}_o)$ is an index pair for the periodic orbit. If $\dim x_+ = d^i > 2$ then the cohomology of $B_\varepsilon \times S^1$ has rank 1 in dimensions o and 1 and o otherwise, while that for $\partial B_\varepsilon \cap \{|x_+| = \varepsilon\}$ has rank 1 in dimensions $o, 1, d^i-1$ and d^i. Then the exact sequence for the pair $(\overline{N}_1, \overline{N}_o)$ implies that the cohomology of $(\overline{N}_1, \overline{N}_o)$ has rank 1 in dimensions d^i and d^{i+1}. If $d_i \leq 2$ the proof follows similar lines.

Remarks: We end this section with an informal description of some further properties of the index and with some remarks about their possible use in applications. For the precise statements and their proofs we refer, however, to [4] and to H.L. Kurland [23].

a) We have seen that the polynomial equality in Theorem 3.3 comes immediately from the filtration $N_o \subset \ldots \subset N_n$ and apriori does not depend on what Morse set is inside $N_j \setminus N_{j-1}$. That is, the homotopy type of N_j / N_{j-1} doesn't allow much to be concluded about M_j itself unless it is known to be a nondegenerate critical point or periodic orbit or say, some invariant manifold with a hyperbolic normal flow. We should mention however that there is a general relation between the cohomology of the index of S, that of S and that of the unstable set from S. Namely, let N be an isolating neighborhood of S and let $I^-(N) = \{\gamma \in N \mid \gamma \cdot R^- \subset N\}$. On $I^-(N) \setminus S$ define the equivalence relation: $\gamma \sim \gamma'$ if there is an orbit segment contained in N

and connecting γ and γ'. Let a^- be the unstable set from S defined as the quotient space, $a^- = (I^-(N) \setminus S)/\sim$. Then one can conclude by the continuity property of Čech cohomology that there is an exact sequence

$$\rightarrow H(a^-) \xrightarrow{\delta} H(h(S)) \rightarrow H(S) \rightarrow .$$

Thus the index has in it that part of the cohomology of S and of a^- that do not cancel each other. For a proof we refer to R. Churchill [10], see also [5]. For example, in the case where S an orientable periodic orbit other then an attractor, the first cohomology of S always cancels a corresponding class in a^-, and so does not show up in the index. (This is not generally true for isolated invariant sets that are circles).

b) The present approach to Morse theory shows that the part of Morse's work concerning existence of critical points makes no actual use of special properties of the spaces, for example they need not be manifolds, nor is it necessary to approximate infinite dimensional spaces by fin'te dimensional ones. The required compactness can be there even in infinite dimensions. Here it is particularly useful to have the flex'bility provided by the concept of (one-sided) local flow X. One of the main points of the present approach is that the sets which have an index need not be critical points, in fact an index is defined for any isolated invariant set. The value of this becomes particularly apparent when families of flows are treaded. In such problems it is seen that there is a natural way to each isolated invariant set of a given flow a corresponding isolated invariant set for each nearby flow. It can be shown that these "continued" sets have the same index as those from which they arise under perturbation. This is like the "homotopy axiom" of degree theory. Now even if one starts with a critical point, it may continue to an invariant set which is not comprised just of critical points. For example, consider the family $\dot{x} = \mu x - x^3$. When $\mu < o$, the set $\{x=o\}$ is an attracting point. However, it continues for $\mu > o$, to the full set of bounded solutions $S = \{x \mid -\mu \leq x \leq + \mu\}$. This set contains the three critical points together with the two "connecting" solutions. Therefore, to use Morse's theory to study critical points of parametrized families of gradient flows (for example to do "bifurcation theory") it is already necessary to have an index for sets other than sets of critical points. One other point should be made about the above sketched continuation theorem. In contrast to the situation in degree theory, the homotopy index has some "internal" structure, namely, that of a pointed space. The continuation theorem says that the isomorphism between the indices of two sets related by continuation is determined by the homotopy class of the arc along which the continuation takes place; and it may be different for arcs in different homotopy classes. This added structure can also be useful in studying families of flows.

c) As with degree theory, there are also "sum" and "product" formulas for the index theory. If S_1 and S_2 are disjoint isolated invariant sets of a local flow then $S_1 \equiv S_1 \cup S_2$ is isolated and its index is the "sum" of those of S_1 and S_2 (i.e. the pointed space obtained by identifying the distinguished point of $h(S_1)$ with that of $h(S_2)$). For example, one sometimes wants to prove the existence of a solution connecting two critical points. This can be done if the points

Appendix: An example of a local flow.

In order to illustrate the purpose of the concept of a local flow introduced in section 3.1 we consider the solutions of a special partial differential equation. More precisely we consider the following simple example of an initial boundary value problem for a weakly coupled semilinear parabolic system, which is not necessarily of variational structure. It is a special case of a general problem studied by H. Amann [20].

$$
\text{(A.1)}\qquad
\begin{aligned}
\frac{\partial u}{\partial t} - A(x,D)u &= f(x,u) && \text{in} \quad \Omega \times (o,\infty) \\
B(x,D)u &= o && \text{on} \quad \partial\Omega \times (o,\infty) \\
u(o,\cdot) &= u_o && \text{in} \quad \bar{\Omega} .
\end{aligned}
$$

Here $\Omega \subset R^n$ is a bounded domain whose boundary, $\partial\Omega$, is an $(n-1)$-dimensional $C^{2+\mu}$-manifold for some $\mu \in (o,1)$. The differential operator $A(x,D)u = (a(x,D)u_1,\ldots,a(x,D)u_N)$, $u = (u_1,\ldots,u_N): \Omega \times [o,\infty) \to R^N$ is a diagonal uniformly elliptic and positive second order differential operator with coefficients in $C^\mu(\bar{\Omega})$. The boundary operator $B(x,D)$ is a diagonal Dirichlet or Neumann boundary operator. Let $D \subset R^N$ be an arbitrary closed bounded convex set containing $o \in R^N$. For the non-linearity we impose the following smoothnes conditions: $f : \bar{\Omega} \times D \to R^N$ is continuous, $f(\cdot,\xi): \bar{\Omega} \to R^N$ is C^μ-Hölder continuous uniformly with respect to $\xi \in D$ and $f(x,\cdot): D \to R^N$ is locally Lipschitz-continuous uniformly with respect to $x \in \bar{\Omega}$. In addition, we impose on the boundary, ∂D, of the set D the following tangency condition for the vectorfields $f(x,\cdot)$: let $\xi_o \in \partial D$, then we denote by $N(\xi_o)$ the set of outer normals to ∂D at ξ_o, and we require that make up a Morse decomposition of an isolated invariant set S whose index is not the sum of those of the critical points (sée [9] for example). This brings out again the significance of Q in Theorem 3.3. Namely, the non-zero coefficients in $Q(t, N_j, N_{j-1}, N_0)$ correspond to connections from M_j to $M_{j-1,1}$. The product theorem applies when the flow breaks up into two flows e.q. $\dot{x} = f(x)$, $\dot{y} = g(y)$. If S_1 is an isolated invariant set for the first flow and S_2 for the second, then $S = S_1 \times S_2$ is an isolated invariant set for the full equations, and $h(S)$ is the "smash product" of $h(S_1)$ and $h(S_2)$. This theorem finds applications when a given set (not a product) can be continued to one which is a product.

$$(A.2) \qquad < p, f(x,\xi) > \leq o$$

for every $x \in \Omega$, every $\xi \in \partial D$ and every $p \in N(\xi)$. It is well known [20, Theorem 1], that in view of the restriction (A.2) the system (A.1) has a unique regular solution u for every given $u_o \in C^2(\overline{\Omega}, R^N)$ which satisfies $Bu_o = o$ and $u_o(\overline{\Omega}) \subset D$, moreover $u \in C^{2+\mu}(\overline{\Omega} \times (o,\infty), R^N)$. By a regular solution of (A.1) we mean a function $u \in C^{2.1}(\overline{\Omega} \times (o,\infty),R^N) \cap C^{1,o}(\overline{\Omega} \times [o,\infty), R^N)$ such that $u(x,t) \in D$ and $Lu(x,t) = f(x,u(x,t))$ for $(x,t) \in \overline{\Omega} \times (o,\infty)$ (where $L = \frac{\partial}{\partial t} + A(x,D)$), and $Bu(x,t) = o$ for $(x,t) \in \partial\Omega \times (o,\infty)$ and $u(o,x) = u_o(x)$ for $x \in \overline{\Omega}$.

For our purpose it is more conventient to formulate problem (A.1) as an abstract semilinear evolution equation in the Banach space $B = L_p(\Omega,R^N)$ for $p > 2n$, whose norm we denote by $| \;|$. Let $W : = \{u \in W_p^2 (\Omega,R^N) \mid B(x,D)u = o\}$, then the operator $A = A(x,D)$ with domain $D(A) : = W$ generates an analytic semigroup in $L_p(\Omega,R^N)$. Defining for $o \leq \alpha \leq 1$ the scale of Banachspaces $B_\alpha : = D(A^\alpha)$ with norms $|u|_\alpha := |A^\alpha u|$, and setting $B_o = B$, one knows that for $o \leq \alpha \leq \beta \leq 1$, the space B_β is continuously and densely embedded in B_α. Moreover, since the resolvent of the operator A is compact, this embedding is compact if $\alpha < \beta$. For more details and references we refer to [20] . Now set $M: = \{u \in B \mid u(x) \in D$ a.e. in $\Omega\}$ and put $F(u)(x) : = f(x,u(x))$. Then (A.1) is equivalent to the evol tion equation

$$(A.3) \qquad \dot{u} + Au = F(u), \qquad u(o) = u_o \in M \cap B_\beta$$

for some $\beta > o$ [20, Lemma 7.2] , and if u is a solution, then

$u(t) \in M \cap B_\beta$ for all $t \geq o$. Moreover u solves the integral equation

$$\text{(A.4)} \qquad u(t) = e^{At} u_o + \int_o^t e^{A(t-s)} F(u(s))\, ds .$$

Since $u(t) \in M \cap B_\beta$ for all $t \geq o$ we conclude from (A.4), in view of the well known properties of linear analytic semigroups, the following estimates for all $t, \tau \geq o$:

$$\text{(A.5)} \qquad \begin{aligned} &|u(t)|_\alpha \leq c(\alpha,\beta)\ (1 + |u_o|_\beta) \\ &|u(t) - u(\tau)|_\alpha \leq c(\alpha,\beta,\nu)\ (1 + |u_o|_\beta)\ |t-\tau|^\nu , \end{aligned}$$

where $o < \alpha < \beta$ and $o \leq \nu < \beta-\alpha$. Conversely, it is easily seen that a solution $u(t)$ of the integral equation satisfying the estimates (A.5) for some $o < \alpha < \beta$ is a solution of the evolution equation (A.3).

After these preliminaries we introduce a flow and a subflow promted by the estimates (A.5). Namely we let Γ be the Banach-space of continuous curves $\gamma : R \to B_\alpha$ with the sup-norm for some $\alpha > o$. On Γ a trivial continuous flow is in a natural way introduced by setting for every $\tau \in R$

$$\gamma \cdot \tau(t) := \gamma(\tau+t), \quad \text{all} \quad t \in R.$$

We now define a subset $X \subset \Gamma$ as follows. Let $o < \alpha < \beta < 1$, $o < \nu < \beta-\alpha$ and $C_1 > o$ be given constants and set $X = \{\gamma \in \Gamma \mid \gamma(t) \in B_\beta,\ \gamma(o) \in M \cap B_\beta,\ |\gamma(t)|_\beta \leq C_1$ and $|\gamma(t) - \gamma(\tau)|_\alpha \leq C_2\, |t-\tau|^\nu$ for all $t, \tau \in R$, moreover, for $t \geq o$ the curve γ is a solution of the equation (A.3) with initial condition $\gamma(o)\}$. We claim

Lemma A.1. $X \subset \Gamma$ *is compact.*

Proof. Since $B_\beta \subset B_\alpha$ is compactly embedded, we conclude by the Arzela-Ascoli theorem that the closure of X in Γ, $\overline{X}$, is a compact subset of Γ. It remains to show that $\overline{X} = X$. We first observe that the closed ball $K = \{u \in B_\beta \mid |u|_\beta \leq 1\}$ in B_β is closed in B. Indeed, let $x_n \in K$ with $x_n \to x$ in B, then $|x_n|_\beta = |A^\beta x_n| \leq 1$. By the reflexivity of B we conclude for a subsequence $A^\beta x_n \to y$ weakly in B and $|y| \leq 1$. Since $A^{-\beta} \in \mathcal{L}(B)$ is compact we conclude $x_n \to A^{-\beta}y$ in B and hence $x = A^{-\beta}y$. Therefore $x \in B_\beta$ and $|x|_\beta = |A^\beta x| = |y| \leq 1$ as claimed. Pick $\gamma \in \overline{X}$, then there is a sequence $\gamma_n \in X$ with $\gamma_n \to \gamma$ in Γ and by the above observation we conclude that for every $t \in R$, $|\gamma(t)|_\beta \leq C_1$ and moreover $|\gamma(t) - \gamma(\tau)|_\alpha \leq C_2|t-\tau|^\nu$ for $t, \tau \in R$, in addition $\gamma(o) \in M$, as M is closed in B. It remains to show that for $t \geq o$ the curve γ is a solution of (A.3), but this now follows since γ satisfies the integral equation (A.4). The Lemma is proved. •

From the local existence and uniqueness of the equation (A.3) we conclude that $(X \cap U) \cdot [o,\varepsilon) \subset X$ for every $U \subset \Gamma$ open and every $\varepsilon > o$. Since, by Lemma A.1, the subset $X \subset \Gamma$ is locally compact we conclude that X is a local flow of Γ. In this sense the index-theory desdribed in section 3 is applicable to the semiflow generated by the partial differential equation (A.1). Similar to the example described above a delay equation can give rise to a subflow of a trivial flow. Here the past history, properly restricted, determines curves which satisfy the relevant equation for positive times. For details and also for more examples of local flows we refer to [4, Chapter IV.6] .

$u(t) \in M \cap B_\beta$ for all $t \geq o$. Moreover u solves the integral equation

$$\text{(A.4)} \qquad u(t) = e^{At} u_0 + \int_0^t e^{A(t-s)} F(u(s))\, ds .$$

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From the local existence and uniqueness of the equation (A.3) we conclude that $(X \cap U) \cdot [o,\varepsilon) \subset X$ for every $U \subset \Gamma$ open and every $\varepsilon > o$. Since, by Lemma A.1, the subset $X \subset \Gamma$ is locally compact we conclude that X is a local flow of Γ. In this sense the index-theory desdribed in section 3 is applicable to the semiflow generated by the partial differential equation (A.1). Similar to the example described above a delay equation can give rise to a subflow of a trivial flow. Here the past history, properly restricted, determines curves which satisfy the relevant equation for positive times. For details and also for more examples of local flows we refer to [4, Chapter IV.6].

In view of possible applications to partial differential equations we point out that by relaxing the required compactnes assumptions, K.P. Rybakowski [21], [22], recently extended the concept of the homotopy index to one-sided semiflows which are not necessarily defined on locally compact metric spaces. His definition applies more directly to semilinear parabolic equations.

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This paper has two aims. First, in an expository style an index theory for flows is presented, which extends the classical Morse-theory for gradient flows on manifolds. Secondly this theory is applied in the study of the forced oscillation problem of time dependent (periodic in time) and asymptotically linear Hamiltonian equations. Using the classical variational principle for periodic solutions of Hamiltonian systems a Morse-theory for periodic solutions of such systems is established. In particular a winding number, similar to the Maslov index of a periodic solution is introduced, which is related to the Morse-index of the corresponding critical point. This added structure is useful in the interpretation of the periodic solutions found.

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