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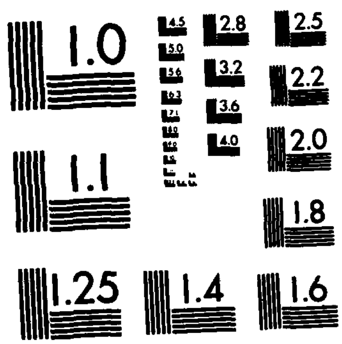

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AN ACCURATE THEORY AND SIMPLE FOURTH ORDER GOVERNING EQUATIONS FOR ORTHOTROPIC AND COMPOSITE CYLINDRICAL SHELLS

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October 1983

(Received May 26, 1983)

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ABSTRACT

A pair of complex conjugate fourth-order differential equations that govern the deformation of orthotropic circular cylindrical shells is presented. As shown in the paper, this pair of equations is as accurate as equations can be within the scope of the Kirchhoff assumptions. Also presented for the first time are several pairs of accurate and simple fourth order equations which can be systematically and explicitly deduced from the previously mentioned pair of equations. Because of their accuracy and simplicity, these simple equations are of practical importance. The advantage in applying those equations presented herein is that their solutions can be easily found in simple closed forms. This considerably simplifies calculations for solving problems of orthotropic and laminated composite cylindrical shells. Unlike other known equations in the literature, their general solutions remain unknown because of the algebraic complexities involved.

AMS(MOS) Subject Classification: 73L10, 73C30
Key Words: simple and accurate governing equations, closed form solutions, orthotropic and composite circular cylindrical shells, Shell theory, fourth order differential equations
Work Unit Number 2 - Physical Mathematics

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Sponsored by the United States Army under Contract No. DAAG29-80-C-0041.

AN ACCURATE THEORY AND SIMPLE FOURTH ORDER GOVERNING EQUATIONS FOR ORTHOTROPIC ANC COMPOSITE CYLINDRICAL SHELLS

Shun Cheng* and F. B. He*

Introduction

Considerable attention has been devoted to the study of isotropic cylindrical shells. Literature on this subject is quite extensive. In contrast, relatively little work has been done on the formulation of the basic equations for orthotropic cylindrical shells, although they are frequently employed as structural elements in industry [1-7]. Examples of orthotropic shells include laminated composite, perforated and stiffened cylindrical shells whose material behavior can be considered as orthotropic. Composite shells (5,6,7) constitute an example of great practical importance.

As is known in the literature, the classical shell theory is based on the same basic assumptions employed in the theory of thin plates, known as Kirchhoff assumptions. Since the inception of Love's first approximation, further simplifications or approximations beyond these basic assumptions have been introduced in developing the theory of thin shells. As the abundance of literature indicates, many versions of shell theories have been formulated, each depending on different versions of the various approximations. This has confronted engineers as well as researchers with a controversial problem with regard to the consistency of the theory, shortcomings of the derivations and accuracy of the resulting equations. Many sets of resulting equations have been proposed for isotropic shells [8-12] and especially, due to their importance in application and the fact that they display nearly every type of behavior found in general shell theory, for cylindrical shells [8-16]. Several publications are listed in the reference. Others may be found by consulting these references. As for orthotropic cylindrical shells,

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two types of basic equations, corresponding either to Flügges's or Donnell's equations for isotropic shells, have been formulated in the literature [1-3,6]. In either case, a resulting single eighth-order differential equation maybe deduced. However, the eighth-order equation for orthotropic shells is more complicated than the corresponding ones for isotropic shells. A common difficulty with these eighth-order equations in isotropic or orthotropic shell theory is that their general solutions remain unknown because of the algebraic complexities involved. For orthotropic cylindrical shells, even the simpler eighth-order equation based on Donnell approximations as seen in [2] suffers from the same complexity. Although a fourth-order equation is presented in [3], this equation does not yield accurate and dependable solutions as is illustrated through the comparison made between numerical results from analytical solutions and from experimental data presented in the same paper [3]. In computing the characteristic roots arising from solving these eighth-order equations by means of eigenfunctions, it is found that the two large roots and the two small roots in the same set of solutions for the characteristic equation are far apart and of different orders of magnitude. This makes the computation more tedious and time-consuming, even with the present day numerical techniques.

Recently a general theory for thin isotropic shells was developed by Markov [17]. It is a consistent theory, since it makes no simplifications or approximations beyond a clear set of fundamental hypotheses. Other advantages of the method of derivation as applied to shells of general curvature have also been illustrated in [17].

In the present paper, a pair of complex conjugate fourth-order partial differential equations that govern the deformation of orthotropic circular cylindrical shells is proposed. This pair of equations is deduced from a set of basic equations which is based on the following Kirchhoff hypotheses:

(a) The transverse normal stress is negligibly small and

(b) Normals to the middle surface of the shell remain normal to it and undergo no change in length during deformation.

The set of basic equations is exact in the sense that in deriving these equations all terms have been retained without introducing further simplifications or approximations beyond these fundamental hypotheses. Even those terms which are of higher-order are kept since they can be summed in closed form.

Because the pair of equations deduced herein is complex conjugates, only one of the equations needs to be considered. Further, closed form solutions of the characteristic equations that arise from solving the pair of governing equations by means of eigenfunctions can be easily obtained. The technique used is an extension of the one for isotropic shells presented in [15,16]. From the pair of equations, a number of simplified fourth-order governing equations can be systematically and explicitly deduced, as shown in the paper. These fourth-order equations for orthotropic cylindrical shells are new in the literature and of definite technical importance because these equations can be easily solved in closed forms and yet retain practically the same accuracy as the original eighth-order equation.

Basic Equations

In accordance with the fundamental hypotheses stated previously, the following basic equations can be deduced for orthotropic circular cylindrical shells. Let a be the radius of the midsurface of the shell, x, y, z the axial, circumferential and radial coordinates and α, β the dimensionless midsurface coordinates along lines of curvatures ($\alpha = \frac{x}{a}$, $\beta = \frac{y}{a}$). The three displacement components u_α, u_β, and u_z of an arbitrary point of the shell can be expressed in terms of midsurfae displacements u, v, and w as follows [8,16]

$$u_\alpha = u + z\omega_\beta \; , \quad u_\beta = v - z\omega_\alpha \; , \; u_z = w \qquad (1)$$

where

$$\omega_\alpha = \frac{1}{a}\left(\frac{\partial w}{\partial \beta} - v\right), \quad \omega_\beta = -\frac{1}{a}\frac{\partial w}{\partial \alpha}$$

The stress-strain relations for orthotropic materials [18,19] are

$$\sigma_\alpha = \frac{E_1}{1 - \nu_1\nu_2}(e_\alpha + \nu_1 e_\beta), \quad \sigma_\beta = \frac{E_2}{1 - \nu_1\nu_2}(\nu_2 e_\alpha + e_\beta),$$

$$\tau_{\alpha\beta} = G e_{\alpha\beta} \tag{2}$$

where E_1, E_2 are the moduli of elasticity along the principal directions α and β, respectively, G is the shear modulus which characterizes the change of angles between principal directions α and β, $\nu_1 = \nu_{\beta\alpha}$ is the Poisson's ratio which characterizes the decrease in α-direction due to tension applied in β-direction, and $\nu_2 = \nu_{\alpha\beta}$ is the Poisson's ratio which characterizes the decrease in β-direction due to tension applied in α-direction. Among these material constants there exists the relation [18,19]:

$$E_1\nu_1 = E_2\nu_2 \tag{3}$$

The components of strain at an arbitrary point of the shell are related to the midsurface displacements by [8,15,16]

$$e_\alpha = \frac{1}{a}\left(\frac{\partial u}{\partial \alpha} - \frac{z}{a}\frac{\partial^2 w}{\partial \alpha^2}\right), \quad e_\beta = \frac{1}{a}\left(\frac{\partial v}{\partial \beta} + w\right) - \frac{z}{a(a+z)}\left(\frac{\partial^2 w}{\partial \beta^2} + w\right)$$

$$e_{\alpha\beta} = \frac{1}{a+z}\left[\frac{\partial u}{\partial \beta} + \frac{\partial v}{\partial \alpha} + 2\frac{z}{a}\left(\frac{\partial v}{\partial \alpha} - \frac{\partial^2 w}{\partial \alpha \partial \beta}\right) + \left(\frac{z}{a}\right)^2\left(\frac{\partial v}{\partial \alpha} - \frac{\partial^2 w}{\partial \alpha \partial \beta}\right)\right] \tag{4}$$

The bending $(\eta_\alpha, \eta_\beta)$ and twisting (τ) strains are

$$\eta_\alpha = -\frac{1}{a^2}\frac{\partial^2 w}{\partial \alpha^2}, \quad \eta_\beta = -\frac{1}{a^2}\left(\frac{\partial^2 w}{\partial \beta^2} + w\right), \quad \tau = -\frac{1}{2a^2}\left(\frac{\partial u}{\partial \beta} - \frac{\partial v}{\partial \alpha} + 2\frac{\partial^2 w}{\partial \alpha \partial \beta}\right) \tag{5}$$

Let h be the wall thickness, K_1, K_2 the extensional rigidity, D_1, D_2 the flexural rigidity

$$K_1 = \frac{E_1 h}{1-\nu_1\nu_2}, \quad K_2 = \frac{E_2 h}{1-\nu_1\nu_2}, \quad D_1 = \frac{h^2}{12} K_1, \quad D_2 = \frac{h^2}{12} K_2 \tag{6}$$

and define

$$k = \frac{E_2}{E_1}, \quad k_1 = \frac{G(1-\nu_1\nu_2)}{E_1} \tag{7}$$

Let N_α, N_β be the normal stress resultants, S_α, S_β the shear stress resultants, M_α, M_β the bending moments, $M_{\alpha\beta}$, $M_{\beta\alpha}$ the twisting moments, and Q_α, Q_β the transverse stress resultants [15]. These are stress resultants (N, S, Q) and couples (M) per unit length of the middle surface and are related to the midsurface displacements through the stress-strain relations as

$$N_\alpha = \frac{K_1}{a}\left[\frac{\partial u}{\partial \alpha} + \nu_1\left(\frac{\partial v}{\partial \beta} + w\right) - c^2 \frac{\partial^2 w}{\partial \alpha^2}\right],$$

$$N_\beta = \frac{K_2}{a}\left[\frac{\partial v}{\partial \beta} + \nu_2 \frac{\partial u}{\partial \alpha} + w + c^2 \left(\frac{\partial^2 w}{\partial \beta^2} + w\right)(1+\delta)\right]$$

$$S_\alpha = \frac{Gh}{a}\left[\frac{\partial u}{\partial \beta} + \frac{\partial v}{\partial \alpha} - c^2 \left(\frac{\partial^2 w}{\partial \alpha \partial \beta} - \frac{\partial v}{\partial \alpha}\right)\right]$$

$$S_\beta = \frac{Gh}{a}\left[\frac{\partial u}{\partial \beta} + \frac{\partial v}{\partial \alpha} + c^2 \left(\frac{\partial^2 w}{\partial \alpha \partial \beta} + \frac{\partial u}{\partial \beta}\right)(1+\delta)\right] \tag{8}$$

$$M_\alpha = -\frac{D_1}{a^2}\left[\frac{\partial u}{\partial \alpha} + \nu_1 \frac{\partial v}{\partial \beta} - \left(\frac{\partial^2 w}{\partial \alpha^2} + \nu_1 \frac{\partial^2 w}{\partial \beta^2}\right)\right], \quad M_\beta = \frac{D_2}{a^2}\left[\left(\frac{\partial^2 w}{\partial \beta^2} + w\right)(1+\delta) + \nu_2 \frac{\partial^2 w}{\partial \alpha^2}\right]$$

$$M_{\alpha\beta} = \frac{Gh^3}{6a^2}\left(\frac{\partial v}{\partial \alpha} - \frac{\partial^2 w}{\partial \alpha \partial \beta}\right), \quad M_{\beta\alpha} = -\frac{Gh^3}{12a^2}\left[\frac{\partial u}{\partial \beta} - \frac{\partial v}{\partial \alpha} + 2\frac{\partial^2 w}{\partial \alpha \partial \beta} + \delta\left(\frac{\partial u}{\partial \beta} + \frac{\partial^2 w}{\partial \alpha \partial \beta}\right)\right]$$

$$Q_\alpha = \frac{D_1}{a^3}\left[\frac{\partial^2 u}{\partial\alpha^2} - k_1(1+\delta)\frac{\partial^2 u}{\partial\beta^2} + (k_1+\nu_1)\frac{\partial^2 v}{\partial\alpha\partial\beta} - \frac{\partial^3 w}{\partial\alpha^3} - (2k_1+\nu_1+\delta k_1)\frac{\partial^3 w}{\partial\alpha\partial\beta^2}\right]$$

$$Q_\beta = \frac{D_2}{a^3}\left[2\frac{k_1}{k}\frac{\partial^2 v}{\partial\alpha^2} - (1+\delta)\left(\frac{\partial^3 w}{\partial\beta^3} + \frac{\partial w}{\partial\beta}\right) - \left(2\frac{k_1}{k} + \nu_2\right)\frac{\partial^3 w}{\partial\alpha^2\partial\beta}\right]$$

in which

$$c^2 = \frac{h^2}{12a^2}$$

and

$$\delta = \frac{(\tanh^{-1}\sqrt{3}\,c - \sqrt{3}\,c)}{\sqrt{3}\cdot c^3} - 1 = 9\,c^2\left[\frac{1}{5} + \frac{1}{7}(\sqrt{3}\,c)^2 + \frac{1}{9}(\sqrt{3}c)^4 + \ldots\ldots\right].$$

The equations of static equilibrium are

$$\frac{\partial N_\alpha}{\partial\alpha} + \frac{\partial S_\beta}{\partial\beta} + aX = 0, \quad \frac{\partial N_\beta}{\partial\beta} + \frac{\partial S_\alpha}{\partial\alpha} + Q_\beta + aY = 0;$$

$$N_\beta - \frac{\partial Q_\alpha}{\partial\alpha} - \frac{\partial Q_\beta}{\partial\beta} - aZ = 0 \tag{9}$$

$$\frac{\partial M_{\alpha\beta}}{\partial\alpha} - \frac{\partial M_\beta}{\partial\beta} - aQ_\beta = 0, \quad \frac{\partial M_{\beta\alpha}}{\partial\beta} - \frac{\partial M_\alpha}{\partial\alpha} - aQ_\alpha = 0$$

in which X, Y and Z are surface loads per unit area in x, y and z directions, respectively.

Pair of Accurate Complex Conjugate Fourth-Order Equations for Normal Deflection

Substituting equation (8) into equation (9), a system of three differential equations is obtained for the three basic functions. This system is presented in Table 1 and possesses a symmetrical structure. The three linear partial differential equations with constant coefficients can be reduced to a single differential equation of higher order that is more convenient to solve and/or analyze with regard to the present problem. These three equations presented in Table 1 shall be considered as algebraic equations in u, v, and w having coefficients which are constants (elastic constants and c^2) and the symbols of differentiations. Let D_o be the 3x3 determinant of Table 1 and calculate its cofactors D_{11}, D_{12}, ... D_{33}. Let

$$u = D_{i1}\phi_i \;,\quad v = D_{i2}\phi_i \;,\quad w = D_{i3}\phi_i \qquad (\text{sum on } i,\; i = 1,2,3) \tag{10}$$

and substitute these expressions in the three equations in Table 1. Then, in accordance with the theory of linear algebra

$$D_o\phi_i + \frac{1 - \nu_1\nu_2}{E_1 h} a^2 X_i = 0 \;, \qquad (i = 1,2,3) \tag{11}$$

are obtained, in which $X_1 = X$, $X_2 = Y$, $X_3 = Z$. *If only a normal surface* load Z is applied on the shell, ϕ_1 and ϕ_2 can be set equal to zero in equations (10) and (11). Calculating cofactors D_{31}, D_{32}, D_{33} and D_o from Table 1 and replacing ϕ_3 by $\frac{\phi}{k_1}$, the following are obtained from equations (10) and (11)

$$u = \frac{\partial}{\partial\alpha}\{k \frac{\partial^2}{\partial\beta^2} - \nu_1 \frac{\partial^2}{\partial\alpha^2} + c^2[\frac{\partial^4}{\partial\alpha^4} - k \frac{\partial^4}{\partial\beta^4} + (2K-8k_1-4\nu_1) \frac{\partial^4}{\partial\alpha^2\partial\beta^2}]\}\,\phi$$

$$v = \frac{\partial}{\partial\beta}\{-k \frac{\partial^2}{\partial\beta^2} - \frac{1}{k_1}(k-\nu_1 k_1-\nu_1^2) \frac{\partial^2}{\partial\alpha^2} + 2c^2[\frac{\partial^4}{\partial\alpha^4} + (2k_1+\nu_1) \frac{\partial^4}{\partial\alpha^2\partial\beta^2}]\}\,\phi \tag{12}$$

$$w = [\frac{\partial^4}{\partial\alpha^4} + \frac{1}{k_1}(k-2\nu_1 k_1-\nu_1^2) \frac{\partial^4}{\partial\alpha^2\partial\beta^2} + k \frac{\partial^4}{\partial\beta^4} + 4c^2 k_1 \frac{\partial^4}{\partial\alpha^2\partial\beta^2}]\,\phi \tag{13}$$

$$\frac{1}{c^2} D_o\phi = \{\frac{\partial^8}{\partial\alpha^8} + 2K \frac{\partial^8}{\partial\alpha^6\partial\beta^2} + k_2 \frac{\partial^8}{\partial\alpha^4\partial\beta^4} + 2kK \frac{\partial^8}{\partial\alpha^2\partial\beta^6} + k^2 \frac{\partial^8}{\partial\beta^8} + 2\nu_1 \frac{\partial^6}{\partial\alpha^6}$$

$$+ k_2 \frac{\partial^6}{\partial\alpha^4\partial\beta^2} + 2k(2K-\nu_1) \frac{\partial^6}{\partial\alpha^2\partial\beta^4} + 2k^2 \frac{\partial^6}{\partial\beta^6} + [\frac{(k-\nu_1^2)}{c^2} + 4k - 3\nu_1^2] \frac{\partial^4}{\partial\alpha^4}$$

$$+ 2k(K-\nu_1) \frac{\partial^4}{\partial\alpha^2\partial\beta^2} + k^2 \frac{\partial^4}{\partial\beta^4}\}\,\phi = \frac{a^4}{D_1} Z \tag{14}$$

in which $K = \frac{k}{2k_1}(1-\nu_1\nu_2) + 2k_1 \;,\quad k_2 = 6k + 4\nu_1(K-4k_1) - 8\nu_1^2$ (15)

Table 1

u	v	w	Load Terms	
$\frac{\partial^2}{\partial\alpha^2} + k_1[1+c^2(1+\delta)]\,\frac{\partial^2}{\partial\beta^2}$	$(k_1+\nu_1)\,\frac{\partial^2}{\partial\alpha\partial\beta}$	$\nu_1\,\frac{\partial}{\partial\alpha} - c^2[\frac{\partial^3}{\partial\alpha^3} - k_1(1+\delta)\,\frac{\partial^3}{\partial\alpha\partial\beta^2}]$	$+\,\frac{1-\nu_1\nu_2}{E_1h}\,a^2X$	$= 0$
$(k_1+\nu_1)\,\frac{\partial^2}{\partial\alpha\partial\beta}$	$k\,\frac{\partial^2}{\partial\beta^2} + k_1(1+3c^2)\,\frac{\partial^2}{\partial\alpha^2}$	$k\,\frac{\partial}{\partial\beta} - c^2(3k_1+\nu_2k)\,\frac{\partial^3}{\partial\alpha^2\partial\beta}$	$+\,\frac{1-\nu_1\nu_2}{E_1h}\,a^2Y$	$= 0$
$\nu_1\,\frac{\partial}{\partial\alpha} - c^2[\frac{\partial^3}{\partial\alpha^3} - k_1(1+\delta)\,\frac{\partial^3}{\partial\alpha\partial\beta^2}]$	$k\,\frac{\partial}{\partial\beta} - c^2(3k_1+\nu_2k)\,\frac{\partial^3}{\partial\alpha^2\partial\beta}$	$k\{1+c^2[\frac{1}{k}\,\frac{\partial^4}{\partial\alpha^4} + (4\,\frac{k_1}{k} + 2\nu_2+\delta\frac{k_1}{k})$ $\frac{\partial^4}{\partial\alpha^2\partial\beta^2} +(1+\delta)(\frac{\partial^2}{\partial\beta^2}+1)^2]\}$	$-\,\frac{1-\nu_1\nu_2}{E_1h}\,a^2Z$	$= 0$

The constant coefficients in equations (12), (13) and (14) may contain coefficients higher than those shown (of the order c^2 or higher). These coefficients have been omitted since, in thin shell theory, $\frac{h}{a} \leq \frac{1}{20}$, thus $c^2 \leq 2\times10^{-4}$ is a very small number. The complete expression of D_0 is given in the Appendix. Comparison of the magnitude of the coefficients of the terms which were omitted with the coefficients of the terms (having the same partial differentiations) which were retained in equation (14) reveals that these omitted terms are truly of smaller orders of magnitude. This fact has been further verified through the actual computation of thes coefficients using available numerical data drawn from elastic constants o many orthotropic materials. In Table 2, the elastic constants of a few materials are presented. Thus, equation (14) is an accurate governing equation for orthotropic cylindrical shells because this equation is derived from the basic hypotheses without introducing further approximations in its derivation except that, as just stated, some negligibly small terms have been omitted. These small terms have been totally dropped in all the known equations of orthotropic shells. In some publications, even certain terms in equation (14) are neglected. In the following analysis, some of these negligibly small terms will be retained so that equation (14) can be reduced to a pair of fourth order complex conjugate equations. This not only tremendously simplifies calculation of the roots of the characteristic equation which arise from solving the equation by separation of variables but it also facilitates obtaining solutions in simple explicit forms. As stated previously, finding solutions of equation (14) in explicit forms is almost prohibitively difficult due to the algebraic complexity involved. In addition to keeping some small terms, the following approximate relation as given in [20,21] is employed:

$$G = \frac{\sqrt{E_1 E_2}}{2(1+\sqrt{\nu_1 \nu_2})} \tag{16}$$

Table 2 Mechanical Properties of Materials

Species	$k = E_2/E_1$	G/E_1	ν_1	ν_2
Glass/epoxy	0.3333	0.1666	0.0833	0.2500
Boron/epoxy	0.1000	0.0333	0.0300	0.3000
Graphite/epoxy	0.0250	0.0125	0.0063	0.2500
Douglas-fir	0.0500	0.0780	0.0220	0.4490

Table 3 Characteristic Roots ($w = e^{p\beta}\cos n\alpha$)
(Boron/Epoxy)

1/c	n	Equation	p_1, p_2	p_3, p_4
50	0.001	(14)	0.009297 + 0.008429i	0.000079 + 1.000000i
		(18)	0.009278 + 0.008414i	0.000079 + 1.000000i
		(24)	0.009278 + 0.008414i	0.000079 + 1.000000i
		(27)	0.008900 + 0.008843i	0.000079 + 0.999997i
		(25)	0.008872 + 0.008871i	0.000079 + 0.999992i
		(28)	0.008872 + 0.008871i	0.000079 + 0.999997i
		(26)	0.103515 + 0.042850i	0.042877 + 0.103450i
		(29)	0.103496 + 0.042858i	0.042869 + 0.103469i
50	0.010	(14)	0.093577 + 0.083514i	0.007852 + 1.000139i
		(18)	0.093411 + 0.083407i	0.007857 + 1.000138i
		(24)	0.093414 + 0.083402i	0.007857 + 1.000138i
		(27)	0.089696 + 0.087736i	0.007870 + 0.999889i
		(25)	0.089463 + 0.088064i	0.007883 + 0.999372i
		(28)	0.089420 + 0.088024i	0.007872 + 0.999839i
		(26)	0.328262 + 0.135123i	0.135969 + 0.326220i
		(29)	0.327653 + 0.135377i	0.135719 + 0.326828i
5000	0.100	(14)	3.217938 + 1.376923i	1.331418 + 3.327954i
		(18)	3.217917 + 1.376970i	1.331469 + 3.327934i
		(24)	3.217926 + 1.376965i	1.331472 + 3.327925i
		(27)	3.211572 + 1.380148i	1.329567 + 3.333750i
		(25)	3.217595 + 1.377790i	1.332245 + 3.327594i
		(28)	3.211551 + 1.380201i	1.329618 + 3.333728i
		(26)	3.282622 + 1.351234i	1.359694 + 3.262197i
		(29)	3.276528 + 1.353769i	1.357185 + 3.268280i

in which $\sqrt{E_1E_2}$ and $\sqrt{\nu_1\nu_2}$ are geometric mean values for the modulus E and Poisson's ratio ν, respectively. From equations (3), (15) and (16), we obtain

$$2k_1 = \sqrt{k}(1-\sqrt{\nu_1\nu_2}), \quad K = 2(2k_1+\nu_1)=2\sqrt{k}, \; k_2 = 6k \tag{17}$$

In equation (14), replacing the coefficient k_2 of the middle term $\frac{\partial^8}{\partial\alpha^4\partial\beta^4}$ (only in this term) by 6k and keeping some of these omitted small terms, from equations (13) and (14), the governing differential equation for normal deflection w may be written as

$$L\bar{L}\, w = \frac{a^4}{D_1}\left[\frac{\partial^4}{\partial\alpha^4} + \frac{1}{k_1}(k-2\nu_1k_1-\nu_1^2)\frac{\partial^4}{\partial\alpha^2\partial\beta^2} + k\frac{\partial^4}{\partial\beta^4}\right] Z \tag{18}$$

in which

$$L = \frac{\partial^4}{\partial\alpha^4} + K\frac{\partial^4}{\partial\alpha^2\partial\beta^2} + k\left(\frac{\partial^4}{\partial\beta^4} + \frac{\partial^2}{\partial\alpha^2} + \frac{\partial^2}{\partial\beta^2}\right)$$
$$+ i\left[\frac{1}{c_1}\frac{\partial^2}{\partial\alpha^2} + c_1\left(k_3\frac{\partial^4}{\partial\alpha^4} + k_4\frac{\partial^4}{\partial\alpha^2\partial\beta^2} + k_5\frac{\partial^4}{\partial\beta^4} + k_5\frac{\partial^2}{\partial\beta^2} + k_6\frac{\partial^2}{\partial\alpha^2}\right)\right] \tag{19}$$

and $\bar{L}$ is the complex conjugate linear differential operator of L, $i = \sqrt{-1}$ and

$$c_1 = \frac{c}{\sqrt{k(1-\nu_1\nu_2)}}, \quad k_3 = \nu_1 - k, \quad k_4 = \frac{1}{2}(k_2-2kK-2k),$$
$$k_5 = k(K-k-\nu_1), \quad k_6 = \frac{1}{2}(4k-k^2-3\nu_1^2) \tag{20}$$

As will be shown later, the replacement of coefficient k_2 of the middle term in equation (14) by 6k has only a negligibly small effect on the solutions of the problem, although the expression (16) is merely an approximate relation and may not be as accurate as is the corresponding relation for isotropic materials. In equation (19), it is seen that the last term $c_1k_6\frac{\partial^2}{\partial\alpha^2}$ is very small as compared with the term $\frac{1}{c_1}\frac{\partial^2}{\partial\alpha^2}$ and hence can be dropped. This should not yield any noticeable effect on the accuracy of the equation as will be further elaborated later. The homogeneous solutions of equation (18) are obtained from

$$Lw = 0, \quad \mathcal{L}w = 0 \tag{21}$$

From equations (12) and (13), we can express u and v in terms of w [15]. Equation (18) reduces to the same governing equation for isotropic cylindrical shells as deduced in [15]. Using the relation (16), equation (18) can also be reduced to the orthotropic plate equation [19] as the radius of the shell goes to infinity.

Solutions by Eigenfunctions

It may be shown that homogeneous equation (21) and suitable boundary conditions are satisfied by making use of the following solution when the eigenfunctions are trigonometric along a generator:

$$w = e^{p\beta}\cos n\alpha \tag{22}$$

in which $n = \frac{m\pi a}{\ell}$, m is an arbitrary integer, ℓ represents the length of the shell and e is the base of natural logarithms. When the eigenfunctions are trigonometric in the circumferential direction, w can be taken as

$$w = e^{p\alpha}\cos n\beta \tag{23}$$

in which n is a real number. It is an integer value when the cylinder is closed and a noninteger value when the shell is open. Substituting expressions (22) and (23) into the governing equation (21) yields characteristic equations for the determination of the roots p. Four complex roots are obtained and the other four roots are the complex conjugate numbers to these four roots. The characteristic equations are quadratic equations in p^2. Hence solutions of the present problem can be easily found in closed forms.

Simple Equations

The accurate fourth-order equation (21) can be used to obtain a number of simplified equations which are new in the literature and are of importance in practice. Considering the actual values of elastic constants of various orthotropic materials and the smallness of c^2, it can be easily shown that the last term in equation (19) $c_1 k_6 \frac{\partial^2}{\partial\alpha^2}$ is much smaller than the term $\frac{1}{c_1}\frac{\partial^2}{\partial\alpha^2}$, hence this term can be dropped in equation (19) as previously stated. When the same considerations are applied, terms with coefficients k_3, k_4 and k_5 in equation (19) can also be neglected because they are of a smaller order of magnitude in comparison with other terms which have the same partial differentiations in the equation. Dropping these terms in equation (19) yields the following simplified equation

$$Lw = [\frac{\partial^4}{\partial\alpha^4} + K\frac{\partial^4}{\partial\alpha^2\partial\beta^2} + k\frac{\partial^4}{\partial\beta^4} + k(\frac{\partial^2}{\partial\alpha^2} + \frac{\partial^2}{\partial\beta^2}) \pm \frac{i}{c_1}\frac{\partial^2}{\partial\alpha^2}]\, w = 0 \qquad (24)$$

If, in equation (24), new dimensionless coordinates ξ and ζ are introduced by stretching the variables α and β such that $\alpha = \sqrt{c_1}\,\xi$ and $\beta = \sqrt{c_1}\,\eta$, the fourth term and the fifth term in equation (24) become $kc_1(\frac{\partial^2}{\partial\xi^2} + \frac{\partial^2}{\partial\eta^2})$. The fourth term is small as compared with the last term $\frac{1}{c_1^2}\frac{\partial^2}{\partial\xi^2}$ and hence has little effect on the characteristic roots. Therefore this term can be dropped in equation (24) and another simple equation

$$Lw = (\frac{\partial^4}{\partial\alpha^4} + K\frac{\partial^4}{\partial\alpha^2\partial\beta^2} + k\frac{\partial^4}{\partial\beta^4} + k\frac{\partial^2}{\partial\beta^2} \pm \frac{i}{c_1}\frac{\partial^2}{\partial\alpha^2})\, w = 0 \qquad (25)$$

is obtained. If the fourth term in equation (25) is also dropped, one obtains

$$Lw = (\frac{\partial^4}{\partial\alpha^4} + K\frac{\partial^4}{\partial\alpha^2\partial\beta^2} + k\frac{\partial^4}{\partial\beta^4} \pm \frac{i}{c_1}\frac{\partial^2}{\partial\alpha^2})\, w = 0 \qquad (26)$$

Unlike others presented in this section, the preceding equation is obtained by dropping a term in equation (25) without the usual presence in that equation of a similar term having a much larger coefficient with which, as a justification for the resulting simplification, the dropped term can be compared. This procedure unavoidably affects the accuracy of the equation, as will be seen in the next section.

In deducing the fourth-order equation (21) from the original equation (14) only the coefficient k_2 of the term $\frac{\partial^8}{\partial\alpha^4\partial\beta^4}$ in equation (14) is replaced by its approximate value $6k$ (17). If the same approximation is also used for the coefficient K in equation (19), then equations (24), (25) and (26) become

$$(\nabla_o^4 + k\nabla^2 \pm \frac{i}{c_1}\frac{\partial^2}{\partial\alpha^2})\, w = 0 \tag{27}$$

$$(\nabla_o^4 + k\frac{\partial^2}{\partial\beta^2} \pm \frac{i}{c_1}\frac{\partial^2}{\partial\alpha^2})\, w = 0 \tag{28}$$

$$(\nabla_o^4 \pm \frac{i}{c_1}\frac{\partial^2}{\partial\alpha^2})\, w = 0 \tag{29}$$

in which $\nabla_o^2 = \frac{\partial^2}{\partial\alpha^2} + \sqrt{k}\,\frac{\partial^2}{\partial\beta^2}$, $\nabla^2 = \frac{\partial^2}{\partial\alpha^2} + \frac{\partial^2}{\partial\beta^2}$, $k = \frac{E_2}{E_1}$ and $c_1 = \frac{h}{2a\sqrt{3k(1-\nu_1\nu_2)}}$.

By substituting the differential operators L given by equations (24), (25) and (26) into the left-side of equation (18), the complete version of these equations including the load term can be written as

$$L\,\bar{L}w = \frac{a^4}{D_1}\left[\frac{\partial^4}{\partial\alpha^4} + 2(K-2k_1-\nu_1)\frac{\partial^4}{\partial\alpha^2\partial\beta^2} + k\frac{\partial^4}{\partial\beta^4}\right] Z \tag{30}$$

in which L represents any one of the three linear differential operators of equations (24), (25) and (26). Similarly the complete version of equations (27), (28) and (29) are

$$L\,\bar{L}w = \frac{a^4}{D_1}\nabla_o^4\, Z \tag{31}$$

in which L represents any one of the three linear differential operators of equations (27), (28) and (29). The Morley, Novozhilov and Donnell equations for isotropic shells [15] are special cases of the equations (27), (28) and (29).

Axially Symmetric Case

In this case the following equation is obtained from equations (14) and (13):

$$\left[\left(\frac{d^2}{d\alpha^2} + \nu_1\right)^2 + \frac{k(1-\nu_1\nu_2)}{c^2}\right] w = \frac{(1-\nu_1\nu_2)h}{12\, E_1 c^4} z \tag{32}$$

which can be further simplified to

$$\left[\frac{d^2}{d\alpha^2} + i\,\frac{\sqrt{k(1-\nu_1\nu_2)}}{c}\right]\left[\frac{d^2}{d\alpha^2} - i\,\frac{\sqrt{k(1-\nu_1\nu_2)}}{c}\right] w = \frac{(1-\nu_1\nu_2)h}{12\, E_1 c^4} z \tag{33}$$

Problems of Thin Rings and Long Tubes

When a ring is loaded by forces applied at the boundary, parallel to the plane of the ring, the stress components are zero on both faces of the ring. Such a state of stress is called plane stress. Following the procedures presented in [22]

$$\begin{aligned} &\frac{d^2w}{d\beta^2} + w = \frac{a^2M}{E_2I} \\ &\frac{dv}{d\beta} + w = 0 \\ &\omega = \frac{1}{a}\left(\frac{dw}{d\beta} - v\right) \end{aligned} \tag{34}$$

where $I = bh^3/12$, b = width of the ring, M is the bending moment ($M = bM_2$), and ω represents the rotation of radial cross sections of the ring. For an infinitely large radius a the preceding equations coincide with that for a straight beam.

When a long circular tube is under the action of lateral loads uniformly distributed along the axis of the cylinder, we have a state of plane strain.

In this case, displacement along the axis of the tube u is zero and v and w are functions of β only. Following the procedures similar to the deductions of the basic equations for thin rings, the following basic equations for the bending of long tubes of orthotropic materials can be obtained:

$$\frac{d^2w}{d\beta^2} + w = \frac{12(1-\nu_1\nu_2)a^2M_2}{E_2\ h^3}$$

$$\frac{dv}{d\beta} + w = 0 \tag{35}$$

$$\omega = \frac{1}{a}\left(\frac{dw}{d\beta} - v\right)$$

Thus the basic equations of the present theory contain both ring bending and bending of long circular tubes as special cases. However, as stated in [16], the equations for bending of thin rings and long tubes cannot be deduced from the Donnell equations.

<u>Comparisons and Conclusions</u>

Utilizing the computer, the relative accuracy of the differential equations presented previously can be further studied through numerical techniques. This can be done by calculating the numerical values of the characteristic roots of the equations and making a comparison of the closeness of these roots. The elastic constants of several typical orthotropic materials are collected in Table 2. Using these values, the roots calculated from homogeneous equations of (14) and (18) and equations (24-29) are obtained. Many roots for other orthotropic materials have also been calculated. Similarities in the properties of these roots for different materials can be observed. However, due to space limitations and the fact that the same conclusions can be drawn from different materials, only Boron-epoxy, Glass-epoxy and Graphite-epoxy are presented in Table 3 through Table 8 for a range of significant parameters. From all the numerical results, it may be concluded that the differences between the roots of equations (14), (18) and (24) are negligibly small for all values of n and c. Numerical results also show that simplified equations (25), (27) and (28) can yield accurate solutions as seen from the closeness of the characteristic roots of these equations to those of equations (14), (18) or (24). As is expected, the thinner the shell, the closer in value these roots will be. The simplified equations (26) or (29), which is only one term less than equation (25) or (28), is not always as accurate and dependable as other equations [16]. These two equations are apparently inaccurate in the case $w = e^{p\beta}\cos n\alpha$ when n is small. Hence, special care is needed when they are employed. All the preceding conclusions hold also for the case when E_1 and E_2, ν_1 and ν_2 are interchanged in the calculations and can be applied to laminated composite shells. In conclusion, equations (24), (25), (27) and (28) deduced herein have the two essential properties of accuracy and simplicity and hence are of importance in applications.

Table 4 Characteristic Roots ($w = e^{p\alpha}\cos n\beta$)
(Boron/Epoxy)

1/c	n	Equation	p_1, p_2	p_3, p_4
50	0	(14)	2.802685 + 2.808032i	0
		(18)	2.803801 + 2.809136i	0
		(24)	2.803801 + 2.809136i	0
		(27)	2.796462 + 2.814284i	0
		(25)	2.805359 + 2.805359i	0
		(28)	2.805359 + 2.805359i	0
		(26)	2.805359 + 2.805359i	0
		(29)	2.805359 + 2.805359i	0
50	1	(14)	2.939060 + 2.6[illegible]959i	0
		(18)	2.945199 + 2.671730i	0
		(24)	2.945332 + 2.671609i	0
		(27)	2.853203 + 2.758317i	0
		(25)	2.948208 + 2.669431i	0
		(28)	2.862275 + 2.749575i	0
		(26)	2.948680 + 2.670070i	0.058927 + 0.053359i
		(29)	2.862805 + 2.750173i	0.057446 + 0.055186i
5000	2	(14)	28.109116 + 27.998002i	0.019553 + 0.019495i
		(18)	28.109206 + 27.998089i	0.019553 + 0.019495i
		(24)	28.109254 + 27.998038i	0.019553 + 0.019495i
		(27)	28.075257 + 28.031950i	0.019539 + 0.019509i
		(25)	28.109475 + 27.997827i	0.019563 + 0.019485i
		(28)	28.076149 + 28.031060i	0.019540 + 0.019508i
		(26)	28.109477 + 27.997829i	0.022589 + 0.022500i
		(29)	28.076151 + 28.031062i	0.022563 + 0.022526i

Table 5 Characteristic Roots ($w = e^{p\beta}\cos n\alpha$)
(Glass/Epoxy)

1/c	n	Equation	p_1, p_2	p_3, p_4
50	0.001	(14)	0.006679 + 0.006393i	0.000043 + 1.000000i
		(18)	0.006684 + 0.006406i	0.000043 + 1.000000i
		(24)	0.006688 + 0.006403i	0.000043 + 1.000000i
		(27)	0.006585 + 0.006508i	0.000043 + 0.999999i
		(25)	0.006546 + 0.006546i	0.000043 + 0.999998i
		(28)	0.006546 + 0.006546i	0.000043 + 0.999998i
		(26)	0.088900 + 0.036816i	0.036824 + 0.088881i
		(29)	0.088900 + 0.036817i	0.036824 + 0.088833i
50	0.010	(14)	0.067138 + 0.063729i	0.004278 + 1.000033i
		(18)	0.067147 + 0.063739i	0.004284 + 1.000033i
		(24)	0.067149 + 0.063741i	0.004285 + 1.000033i
		(27)	0.066126 + 0.064801i	0.004285 + 0.999923i
		(25)	0.065748 + 0.065187i	0.004287 + 0.999847i
		(28)	0.065748 + 0.065187i	0.004286 + 0.999873i
		(26)	0.281398 + 0.116309i	0.116559 + 0.280794i
		(29)	0.281363 + 0.116327i	0.116545 + 0.280837i
5000	0.100	(14)	2.738576 + 1.193250i	1.132930 + 2.884387i
		(18)	2.738575 + 1.193251i	1.132931 + 2.884387i
		(24)	2.738617 + 1.193267i	1.132948 + 2.884425i
		(27)	2.738139 + 1.193649i	1.132994 + 2.884726i
		(25)	2.738452 + 1.193565i	1.133228 + 2.884256i
		(28)	2.738106 + 1.193733i	1.133074 + 2.884691i
		(26)	2.813983 + 1.163089i	1.165591 + 2.807942i
		(29)	2.813632 + 1.163268i	1.165446 + 2.808373i

Table 6 Characteristic Roots ($w = e^{p\alpha}\cos n\beta$) (Glass/Epoxy)

1/c	n	Equation	p_1, p_2	p_3, p_4
50	0	(14)	3.773718 + 3.784743i	0
		(18)	3.775179 + 3.786174i	0
		(24)	3.775101 + 3.786093i	0
		(27)	3.757170 + 3.801267i	0
		(25)	3.779234 + 3.779234i	0
		(28)	3.779154 + 3.779154i	0
		(26)	3.779234 + 3.779234i	0
		(29)	3.779154 + 3.779154i	0
50	1	(14)	3.860588 + 3.696090i	0
		(18)	3.862294 + 3.697873i	0
		(24)	3.862230 + 3.697778i	0
		(27)	3.833873 + 3.725216i	0
		(25)	3.867971 + 3.692534i	0
		(28)	3.856292 + 3.703559i	0
		(26)	3.868686 + 3.693356i	0.078077 + 0.074538i
		(29)	3.857015 + 3.704375i	0.077861 + 0.074779i
5000	2	(14)	37.826903 + 37.757812i	0.026479 + 0.026442i
		(18)	37.826902 + 37.757811i	0.026479 + 0.026442i
		(24)	37.826113 + 37.757011i	0.026478 + 0.026441i
		(27)	37.819909 + 37.763213i	0.026479 + 0.026440i
		(25)	37.827467 + 37.757273i	0.026485 + 0.026436i
		(28)	37.822116 + 37.761010i	0.026481 + 0.026438i
		(26)	37.827470 + 37.757277i	0.030582 + 0.030525i
		(29)	37.822119 + 37.761013i	0.030578 + 0.030528i

Table 7 Characteristic Roots ($w = e^{p\beta}\cos n\alpha$) (Graphite/Epoxy)

1/c	n	Equation	p_1, p_2	p_3, p_4
50	0.001	(14)	0.013357 + 0.011729i	0.000158 + 1.000000i
		(18)	0.013303 + 0.011695i	0.000158 + 1.000000i
		(24)	0.013304 + 0.011694i	0.000158 + 1.000000i
		(27)	0.012591 + 0.012548i	0.000158 + 0.999994i
		(25)	0.012572 + 0.012568i	0.000158 + 0.999980i
		(28)	0.012571 + o.o12567i	0.000158 + 0.999994i
		(26)	0.123248 + 0.050992i	0.051051 + 0.123106i
		(29)	0.123199 + 0.051012i	0.051031 + 0.123155i
50	0.010	(14)	0.1352917 + 0.115102i	0.015698 + 1.000601i
		(18)	0.134745 + 0.114782i	0.015702 + 1.000595i
		(24)	0.134744 + 0.114782i	0.015702 + 1.000595i
		(27)	0.127835 + 0.123473i	0.015784 + 1.000040i
		(25)	0.127839 + 0.123844i	0.015855 + 0.998577i
		(28)	0.127646 + 0.123680i	0.015787 + 0.999991i
		(26)	0.391769 + 0.160409i	0.162270 + 0.387268i
		(29)	0.390212 + 0.161058i	0.161632 + 0.388826i
5000	0.100	(14)	3.863535 + 1.625197i	1.598486 + 3.928208i
		(18)	3.863438 + 1.625419i	1.598720 + 3.928115i
		(24)	3.863425 + 1.625414i	1.598715 + 3.928103i
		(27)	3.847469 + 1.632947i	1.593312 + 3.943178i
		(25)	3.862934 + 1.626713i	1.599938 + 3.927580i
		(28)	3.847456 + 1.632978i	1.593343 + 3.943165i
		(26)	3.917688 + 1.604092i	1.622698 + 3.872768i
		(29)	3.902121 + 1.161058i	1.616319 + 3.888262i

Table 8 Characteristic Roots ($w = e^{p\alpha}\cos n\beta$) (Graphite/Epoxy)

1/c	n	Equation	p_1, p_2	p_3, p_4
50	0	(14)	1.986613 + 1.988186i	0
		(18)	1.987408 + 1.988976i	0
		(24)	1.987389 + 1.988970i	0
		(27)	1.984245 + 1.990535i	0
		(25)	1.987400 + 1.987400i	0
		(28)	1.987387 + 1.987387i	0
		(26)	1.987400 + 1.987400i	0
		(29)	1.987387 + 1.987387i	0
50	1	(14)	2.111661 + 1.854832i	0
		(18)	2.119304 + 1.863529i	0
		(24)	2.119284 + 1.863523i	0
		(27)	2.024353 + 1.951096i	0
		(25)	2.120223 + 1.862897i	0
		(28)	2.027557 + 1.948014i	0
		(26)	2.120533 + 1.863358i	0.042075 + 0.036972i
		(29)	2.027929 + 1.948434i	0.040542 + 0.038953i
5000	2	(14)	19.925442 + 19.822453i	0.013807 + 0.013753i
		(18)	19.925560 + 19.822572i	0.013807 + 0.013753i
		(24)	19.925434 + 19.822448i	0.013807 + 0.013753i
		(27)	19.889481 + 19.858286i	0.013791 + 0.013769i
		(25)	19.925642 + 19.822496i	0.013815 + 0.013744i
		(28)	19.889795 + 19.857972i	0.013791 + 0.013769i
		(26)	19.925643 + 19.822498i	0.015953 + 0.015870i
		(29)	19.889797 + 19.857973i	0.015924 + 0.015899i

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Appendix

$$D_o = k_1 \{(k-\nu_1^2) \frac{\partial^4}{\partial\alpha^4}$$

$$+ c^2 \{\frac{\partial^8}{\partial\alpha^8} + \frac{1}{k_1} (k-\nu_1^2+4k_1^2+\delta k_1^2) \frac{\partial^8}{\partial\alpha^6\partial\beta^2}$$

$$+ \frac{1}{k_1} [4k_1k - 7k_1\nu_1^2 - 6\nu_1k_1^2 + 2\nu_1k - 2\nu_1^3 - (\nu_1^2k_1+2\nu_1k_1^2-2kk_1)(1+\delta)] \frac{\partial^8}{\partial\alpha^4\partial\beta^4}$$

$$+ \frac{1}{k_1} [3k_1^2k + 2\nu_1k_1k + (k^2-\nu_1^2k-2\nu_1kk_1+k_1^2k)(1+\delta)] \frac{\partial^8}{\partial\alpha^2\partial\beta^6} + k^2(1+\delta) \frac{\partial^8}{\partial\beta^8}$$

$$+ 2\nu_1 \frac{\partial^6}{\partial\alpha^6} + [4k + \frac{2\nu_1}{k_1} (k-\nu_1^2) - \nu_1(6k_1+8\nu_1) + (2k-2k_1\nu_1)(1+\delta)] \frac{\partial^6}{\partial\alpha^4\partial\beta^2}$$

$$+ [(6k_1+2\nu_1)k + (\frac{2k^2}{k_1} - \frac{2k\nu_1^2}{k_1} - 4\nu_1k + 2kk_1)(1+\delta)] \frac{\partial^6}{\partial\alpha^2\partial\beta^4}$$

$$+ 2k^2(1+\delta) \frac{\partial^6}{\partial\beta^6} + [3k - 3\nu_1^2 + k(1+\delta)] \frac{\partial^4}{\partial\alpha^4}$$

$$+ \frac{k}{k_1} [3k_1^2 + (k-\nu_1^2-2\nu_1k_1+k_1^2)(1+\delta)] \frac{\partial^4}{\partial\alpha^2\partial\beta^2} + k^2(1+\delta) \frac{\partial^4}{\partial\beta^4}\}$$

$$+ c^4 \{2 \frac{\partial^8}{\partial\alpha^8} + [9k_1 + 8\nu_1 - \frac{1}{k_1} (k-\nu_1^2) + 6k_1(1+\delta)] \frac{\partial^8}{\partial\alpha^6\partial\beta^2}$$

$$+ [-\nu_1^2 +(6k - 6k_1\nu_1-2\nu_1^2)(1+\delta)] \frac{\partial^8}{\partial\alpha^4\partial\beta^4}$$

$$+ [(6kk_1 + 2\nu_1k)(1+\delta) + kk_1(1+\delta)^2] \frac{\partial^8}{\partial\alpha^2\partial\beta^6} + k^2(1+\delta)^2 \frac{\partial^8}{\partial\beta^8}$$

$$+ 6\nu_1 \frac{\partial^6}{\partial\alpha^6} + 6(k-k_1\nu_1](1+\delta) \frac{\partial^6}{\partial\alpha^4\partial\beta^2}$$

$$+ [12kk_1 + 2k\nu_1 + 2kk_1(1+\delta)](1+\delta) \frac{\partial^6}{\partial\alpha^2\partial\beta^4}$$

$$+ 2k^2(1+\delta)^2 \frac{\partial^6}{\partial\beta^6} + 3k(1+\delta) \frac{\partial^4}{\partial\alpha^4}$$

$$+ [(6kk_1 + kk_1(1+\delta)](1+\delta) \frac{\partial^4}{\partial\alpha^2\partial\beta^2} + k^2(1+\delta)^2 \frac{\partial^4}{\partial\beta^4}\}\}$$

SECURITY CLASSIFICATION OF THIS PAGE (When Data Entered)

REPORT DOCUMENTATION PAGE		READ INSTRUCTIONS BEFORE COMPLETING FORM
1. REPORT NUMBER 2580	2. GOVT ACCESSION NO.	3. RECIPIENT'S CATALOG NUMBER
4. TITLE (and Subtitle) AN ACCURATE THEORY AND SIMPLE FOURTH ORDER GOVERNING EQUATIONS FOR ORTHOTROPIC CYLINDRICAL SHELLS		5. TYPE OF REPORT & PERIOD COVERED Summary Report - no specific reporting period 6. PERFORMING ORG. REPORT NUMBER
7. AUTHOR(s) Shun Cheng and F. B. He		8. CONTRACT OR GRANT NUMBER(s) DAAG29-80-C-0041
9. PERFORMING ORGANIZATION NAME AND ADDRESS Mathematics Research Center, University of Wisconsin 610 Walnut Street Madison, Wisconsin 53706		10. PROGRAM ELEMENT, PROJECT, TASK AREA & WORK UNIT NUMBERS Work Unit Number 2 - Physical Mathematics
11. CONTROLLING OFFICE NAME AND ADDRESS U. S. Army Research Office P.O. Box 12211 Research Triangle Park, North Carolina 27709		12. REPORT DATE October 1983 13. NUMBER OF PAGES 25
14. MONITORING AGENCY NAME & ADDRESS(if different from Controlling Office)		15. SECURITY CLASS. (of this report) UNCLASSIFIED 15a. DECLASSIFICATION/DOWNGRADING SCHEDULE

16. DISTRIBUTION STATEMENT (of this Report)

Approved for public release; distribution unlimited.

17. DISTRIBUTION STATEMENT (of the abstract entered in Block 20, if different from Report)

18. SUPPLEMENTARY NOTES

19. KEY WORDS (Continue on reverse side if necessary and identify by block number)

simple and accurate governing equations, closed form solutions, orthotropic and composite circular cylindrical shells, Shell theory, fourth order differential equations

20. ABSTRACT (Continue on reverse side if necessary and identify by block number)

A pair of complex conjugate fourth-order differential equations that govern the deformation of orthotropic circular cylindrical shells is presented. As shown in the paper, this pair of equations is as accurate as equations can be within the scope of the Kirchhoff assumptions. Also presented for the first time are several pairs of accurate and simple fourth order equations which can be systematically and explicitly deduced from the

20. Abstract (cont.)

previously mentioned pair of equations. Because of their accuracy and simplicity, these simple equations are of practical importance. The advantage in applying those equations presented herein is that their solutions can be easily found in simple closed forms. This considerably simplifies calculations for solving problems of orthotropic and laminated composite cylindrical shells. Unlike other known equations in the literature, their general solutions remain unknown because of the algebraic complexities involved.

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