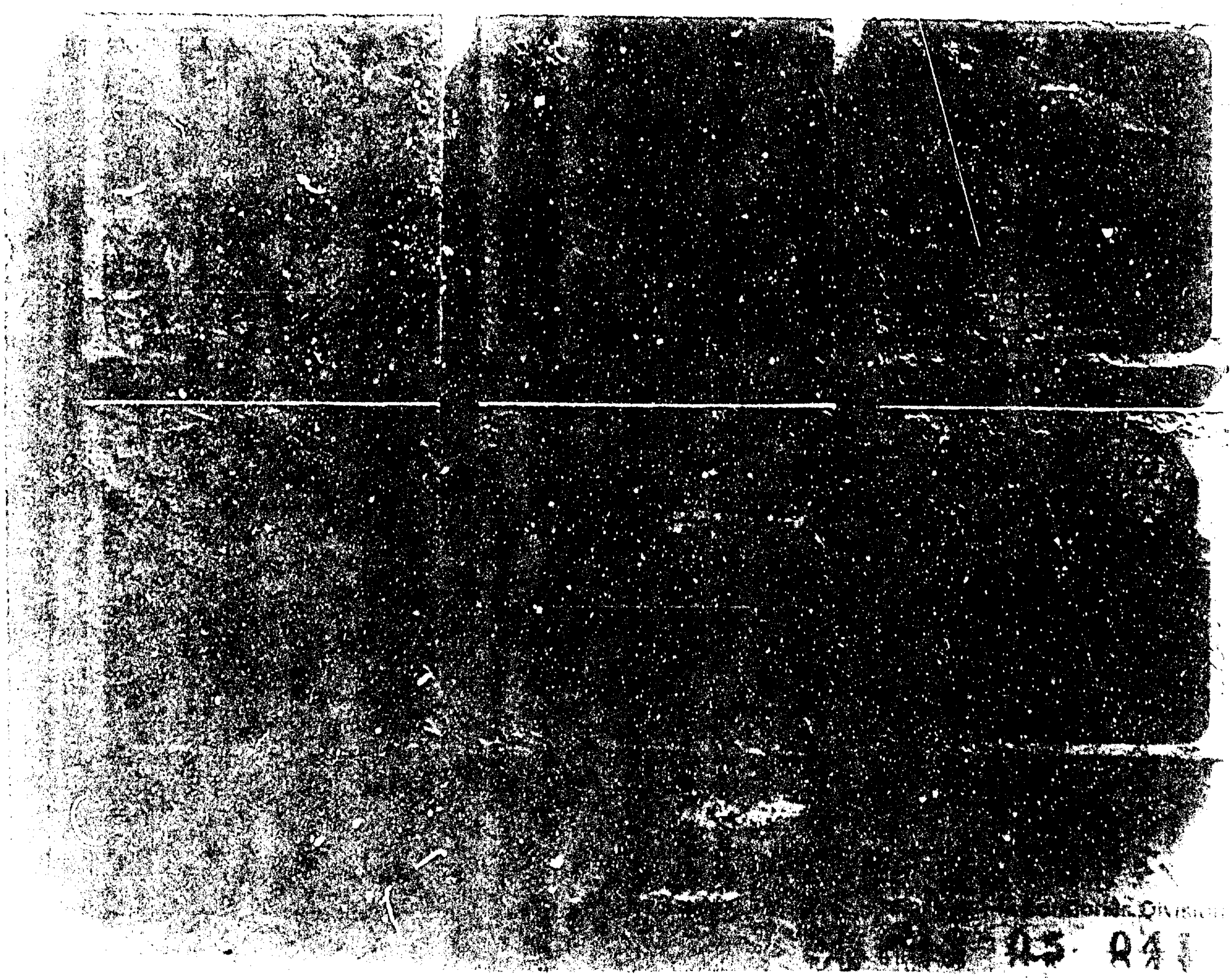
A STOCHASTIC NETWORK FORMULATION FOR PERFORMANCE ASSESSMENT AND LIFE-CYCLE MODELING IN COMPLEX SYSTEM DESIGN



UNCLASSIFIED

SECURITY CLASSIFICATION OF THIS PAGE (When Data Entered)

REPORT DOCUMENTATION PAGE		READ INSTRUCTIONS BEFORE COMPLETING FORM
1. REPORT NUMBER 18898.3-MA	2. GOVT ACCESSION NO. AD-A135337	3. RECIPIENT'S CATALOG NUMBER N/A
4. TITLE (and Subtitle) A Stochastic Network Formulation for Performance Assessment and Life-Cycle Modeling in Complex System Design		5. TYPE OF REPORT & PERIOD COVERED Reprint
		6. PERFORMING ORG. REPORT NUMBER N/A
7. AUTHOR(s) Austin J. Lemoine		8. CONTRACT OR GRANT NUMBER(s) ~~DAAG29 82 K 0151~~ N00014 82 C 0620
9. PERFORMING ORGANIZATION NAME AND ADDRESS North Carolina State University Raleigh, NC		10. PROGRAM ELEMENT, PROJECT, TASK AREA & WORK UNIT NUMBERS N/A
11. CONTROLLING OFFICE NAME AND ADDRESS U. S. Army Research Office P. O. Box 12211 Research Triangle Park, NC 27709		12. REPORT DATE May 83
		13. NUMBER OF PAGES 19
14. MONITORING AGENCY NAME & ADDRESS(if different from Controlling Office)		15. SECURITY CLASS. (of this report) Unclassified
		15a. DECLASSIFICATION/DOWNGRADING SCHEDULE

16. DISTRIBUTION STATEMENT (of this Report)

Submitted for announcement only.

17. DISTRIBUTION STATEMENT (of the abstract entered in Block 20, if different from Report)

18. SUPPLEMENTARY NOTES

19. KEY WORDS (Continue on reverse side if necessary and identify by block number)

20. ABSTRACT (Continue on reverse side if necessary and identify by block number)

DD FORM 1 JAN 73 1473 EDITION OF 1 NOV 65 IS OBSOLETE

UNCLASSIFIED

SECURITY CLASSIFICATION OF THIS PAGE (When Data Entered)

WDL-TR9954
May 25, 1983

A STOCHASTIC NETWORK FORMULATION FOR PERFORMANCE ASSESSMENT AND LIFE-CYCLE MODELING IN COMPLEX SYSTEM DESIGN

By:

Austin J. Lemoine

Software Technology Department

ABSTRACT

A network model incorporating various stochastic features is considered. The model seeks to represent a complex sequential process wherein an object or "system" moves through a succession of states (nodes) and operating modes (classes) in the course of carrying out or fulfilling its function, and with the transitions among states and operating modes occurring in a possibly random manner, and requiring (consuming) some resource in randomly varying amounts. Model structure is reminiscent of the BCMP [1] setup for an open network of queues with sufficient servers, but the viewpoint adopted here is motivated by our project experience and also by Harrison and Lemoine [4]. We discuss the routing behavior and resource requirements of typical objects moving through the network, the evolution of the network over time when external input follows a possibly nonhomogenous Poisson pattern, the "distance" between the time-dependent distribution of state and the limiting distribution when the Poisson input is homogenous, and some potential applications of the formulation and results.

This report is based in part on research supported by Army Research Office Contract DAAG29-82-K-0151 with North Carolina State University, entitled, "Networks of Queues and Queues with Periodic Poisson Input"; and by Office of Naval Research Contract N00014-82-C-0620 with Ford Aerospace & Communications Corporation, entitled, "Performance, Acquisition, and Ownership of Complex Systems."

CONTENTS

SECTION 1

INTRODUCTION

SECTION 2

MODEL DESCRIPTION

SECTION 3

ROUTING AND RESOURCE REQUIREMENTS

SECTION 4

POISSON INPUT

SECTION 5

MODEL DISCUSSION

REFERENCES

SECTION 1

INTRODUCTION

Stochastic networks, or vector random processes, provide an appealing framework for modeling complex sequential processes evolving under uncertainty. Stochastic networks find application in such diverse areas as queueing systems, computer and communication networks, logistics/repair systems, manufacturing processes, inventory systems, and population studies, as well as planning and management of research and development projects. (Cf. Kelly [6], Lemoine [8], [9], and Whitehouse [12]).

This paper discusses a simple but rather flexible network model that incorporates certain stochastic effects. We consider a network of stations or processors and associated functions, with uncontrolled input of objects or jobs (requests or demands for execution of functions), random movement (routes) of objects through the system, randomly varying resource requirements to execute the functions, and time horizons of shorter duration then necessary to achieve steady-state for situations where it might be important to respond to demands promptly and with minimal competition for available resources. The model structure is reminiscent of Baskett et al [1], but the approach taken in analyzing the model is prompted by Harrison and Lemoine [4].

This paper is organized as follows. Section 2 gives the basic features of the model. Section 3 considers movement of a typical object through the network, with particular emphasis on the resources required, global as well as nodal, to process the object until it leaves the system. Section 4 considers the evolution of the network over time when the pattern of demand input is a Poisson process and the processing capacity available at each node in the network is effectively unlimited (ie, there are sufficient servers at each node so that objects never wait in queue). An explicit representation is given for the time-dependent distribution of network state (defined here as the number of objects of each type at each node in the system). When the Poisson input process is homogenous, this time-dependent distribution converges to a limit, and the "distance" between the time-dependent and limiting distributions is estimated. Section 5 describes a few situations where the model formulation and results may be appropriate.

In Sections 3 and 4 it is necessary to distinguish between moment generating functions and Laplace transforms. In particular, let X be a nonnegative random variable, θ a real parameter, and $\phi(\theta) = E\{\exp(\theta X)\}$. Then $\phi(\theta)$ exists but could have value $+\infty$ for some $\theta > 0$. If $\phi(\theta)$ is finite for some $\theta > 0$, then X (more properly, the distribution of X) is said to have a moment generating function. The function $\phi(\cdot)$ restricted to θ values in $(-\infty, 0]$ is of course the Laplace transform (of the distribution) of X.

Acknowledgement. A preliminary version of this paper in outline form, coauthored with Dr. Lewis Meier, appeared in the *Proceedings of the Third International Conference on Distributed Computer Systems,* Miami, Florida, 1982. The conference paper was oriented to a specific application area in distributed systems and was prepared for U.S. Army Contract No.

DASG-60-78-L-0079 with Systems Control Technology, Inc., Palo Alto, California. The present paper represents a substantial generalization of its author's contribution to the conference paper. We are very grateful to Dr. Lewis Meier for illuminating discussions on distributed computing systems.

SECTION 2

MODEL DESCRIPTION

Consider a network (Figure 1) in which objects (jobs, items, messages, etc) in different classes (types, operating modes, etc) are moving through the system and also changing class, in a possibly random manner, and requiring a randomly varying amount of resource (service/processing/repair/replacement, eg, capacity-machine instructions per second, words of memory, buffer space, time, dollars, equipment, material, manpower and skill levels, etc) at each node visited. There are N nodes (stations, service/processing/repair/supply centers, tasks, milestones, etc) indexed by i, and there are M object classes indexed by m. Inputs (customers, demands, requests) at any node may originate directly from outside the system or by internal transfer within the network. On any visit to node i by an object in class m the object is dropped from the network (routed to sink) after processing at i with probability q_{mi}, independently of previous visits, classes, and other objects present in the system. In this event, the amount of resource required at node i to process the object is distributed as a random variable S_{mi}. On any visit to node i by an object in class m the object remains in the system after processing at node i and transfers to node j as an object in class r with probability p_{ij}^{mr}, independently of prior visits, classes, and other objects (if i = j then the object is sent back to node i), where

$$q_{mi} = 1 - \sum_{j=1}^{N} \sum_{r=1}^{M} p_{ij}^{mr}$$

for each m and i; and, in this event, the amount of resource required at node i to process the object is distributed as a random variable R_{ij}^{mr}. All objects eventually exit the system.

The preceding formulation easily accomodates multiple resource types by specifying, for each type of resource, appropriate probability distributions for R_{ij}^{mr} and S_{mi}. Subsequent sections of this paper do not distinguish between multiple resource types, but the results of Section 3 are valid for each resource type.

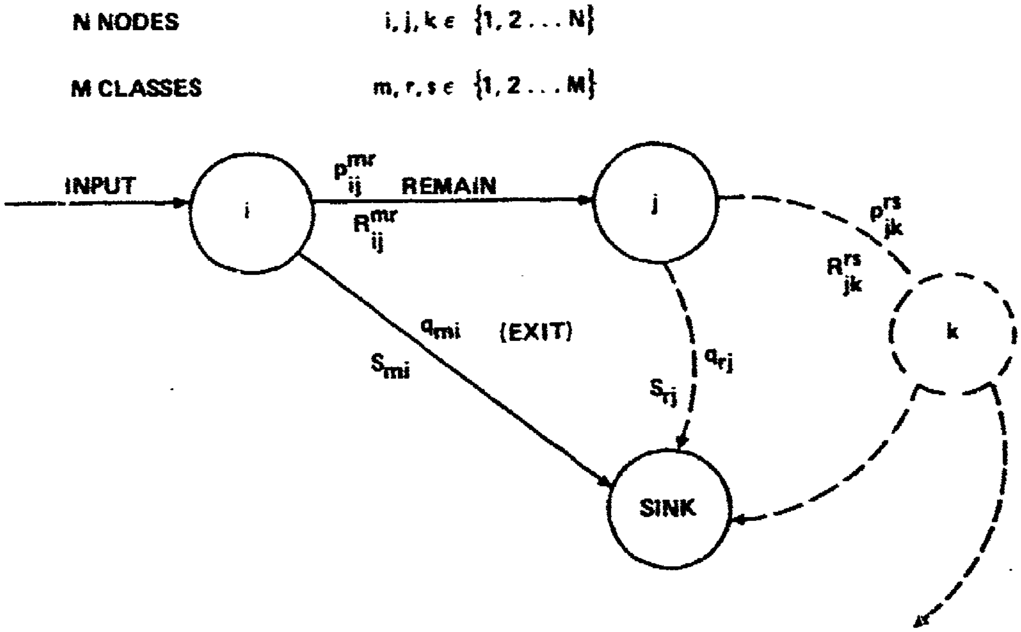


Figure 1. Network Model

SECTION 3

ROUTING AND RESOURCE REQUIREMENTS

Consider now the route of a generic object through the network. Let b_{mi} be the probability that an object enters the system through node i in class m. For i,j=1,...,N let $P_{ij} = [p_{ij}^{mr}]$ where m,r=1,...,M so that P_{ij} is an M by M matrix, and then put $P = [P_{ij}]$ so that P is an N by N block matrix. The routing-and-class history of a typical object moving through the system corresponds to the evolution of a finite absorbing Markov chain with a single absorbing state (the network sink) and with transitions among nonabsorbing states governed by P. Let I be the NM by NM identify matrix. Since all objects eventually exit the system the matrix I-P is invertible, and

$$(I - P)^{-1} = \sum_{x=0}^{\infty} P^x$$

If we put $(I - P)^{-1} = [A_{ij}]$ and $A_{ij} = [a_{ij}^{mr}]$ then a_{ij}^{mr} is the expected number of requests for processing at node j by an object in class r that originally enters the network through node i in class m. Let ρ_{ij}^{mr} be the probability an object entering via node i in class m reaches node j in class r at some later step. Then $\rho_{ii}^{mm} = 1 - (a_{ii}^{mm})^{-1}$ and $\rho_{ij}^{mr} = a_{ij}^{mr}/a_{jj}^{rr}$ otherwise. Moreover, if q_{ij}^{mr} is the probability that an object entering the network via node i in class m exits the system from node j in class r, then $q_{jj}^{rr} = q_{rj}a_{jj}^{rr}$ and $q_{ij}^{mr} = \rho_{ij}^{mr} q_{jj}^{rr}$ otherwise. Finally, if q_{mi}^{*} is the unconditional probability that an object exits the system from node i in class m then

$$q_{mi}^{*} = q_{mi} \sum_{j=1}^{N} \sum_{r=1}^{M} b_{rj} a_{ji}^{rm}$$

The above expressions for the probabilities ρ_{ij}^{mr}, q_{ij}^{mr} and q_{mi}^{*} follow immediately from well known results for finite absorbing Markov chains; cf. Kemeny and Snell [7].

Observe that the crucial factor in determining these various routing and class parameters is computation of the matrix $(I - P)^{-1}$. In many applications of interest this matrix I - P will be relatively sparse, indeed even upper triangular.

Turning to global network resource requirements, let L_{mi} be the total level of resources necessary to process an object until it reaches the network sink, given that the object enters the system at node i in class m. For $\theta \leq 0$ let $f_{mi}(\theta)$, $g_{mi}(\theta)$, and $h_{mi}(\theta)$ be the Laplace transforms for the distributions of S_{mi}, R_{ij}^{mr} and L_{mi}, respectively. By virtue of the independence assumptions regarding routing and class and resource requirements on visits to nodes, the transforms $\{L_{mi}(\theta)$: m = 1,...,M, and i = 1,...,N$\}$ satisfy the network flow equations

$$h_{mi}(\theta) = q_{mi} f_{mi}(\theta) + \sum_{j=1}^{N} \sum_{r=1}^{M} p_{ij}^{mr} [g_{ij}^{mr}(\theta) h_{rj}(\theta)] \qquad (1)$$

These network flow equations have a unique solution: Let $f(\theta)$ and $h(\theta)$ be MN column vectors consisting of N blocks of length M where entry m of block i in $f(\theta)$ is $q_{mi}f_{mi}(\theta)$ and entry r of block j in $h(\theta)$ is $h_{rj}(\theta)$. Let $D(\theta)$ be an N by N block matrix where the (ij)th block, say $D_{ij}(\theta)$, is the M by M matrix $[p_{ij}\, g_{ij}^{mr}(\theta)]$. Then equation (1) is equivalent to

$$h(\theta) = f(\theta) + D(\theta)h(\theta) \tag{2}$$

or

$$[I - D(\theta)]\, h(\theta) = f(\theta)$$

Since $0 \leq D(\theta) \leq P$ componentwise, it follows that the matrix $[I - D(\theta)]$ is invertible for all $\theta \leq 0$; indeed

$$[I - D(\theta)]^{-1} = \sum_{x=0}^{\infty} [D(\theta)]^x$$

Thus

$$h(\theta) = [I - D(\theta)]^{-1} f(\theta) \tag{3}$$

Computation of the transforms $\{h_{mi}(\theta)\}$ from (3) is not entirely straightforward, but we will provide an explicit representation for those transforms using a slightly different approach.

Repeated differentiation of the network flow equations with respect to θ provides a recursive procedure for computing the moments of the cumulative resource variables $\{ L_{mi} \}$. Let $h_{min} = E\{(L_{mi})^n\}$, $g_{ijn}^{mr} = E\{(R_{ij}^{mr})^n\}$ and $f_{min} = E\{(S_{mi})^n\}$. We first observe that if the variables $\{S_{mi}, R_{ij}^{mr}\}$ have finite moments of order n then the variables $\{L_{mi}\}$ also have this property. Differentiating both sides of (2) n times with respect to θ and using Leibnitz's Rule leads to

$$h^{(n)}(\theta) = f^{(n)}(\theta) + \sum_{x=0}^{n} \binom{n}{x} D^{(x)}(\theta) h^{(n-x)}(\theta)$$

or

$$h^{(n)}(\theta) = [I - D(\theta)]^{-1}\Big[f^{(n)}(\theta) + \sum_{x=1}^{n} \binom{n}{x} D^{(x)}(\theta) h^{(n-x)}(\theta)\Big] \tag{4}$$

where the superscripts correspond to componentwise differentiation in (2). Suppose that $L_{mi})^{n-1}$ has a finite mean (this certainly holds for n = 1). Then each component of $h^{(n-x)}(\theta)$ has a finite limit as $\theta \to 0$ for x=1,...,n. Since $[I - D(\theta)]^{-1} \to [I - P]^{-1}$ as $\theta \to 0$, it follows from

(4) that each component of $h^{(n)}(\theta)$ has a finite limit as $\theta \to 0$, and so $(L_{mi})^n$ has finite mean. Thus by mathematical induction L_{mi} has a finite moment of order n. Moreover, (4) yields a compact recursive formula for computing these moments. Let δ_x, h_x, and h_o be NM column vectors where the entries corresponding to (m,i) are $q_{mi}f_{mix}$, h_{mix}, and 1, respectively. Also, let D_x be an NM by NM matrix where the entry corresponding to (m,i), (r,j) is $p_{ij}^{mr}g_{ijx}^{mr}$. Then

$$h_n = [I - P]^{-1}\left[\delta_n + \sum_{x=1}^{n} \binom{n}{x} D_x h_{n-x}\right] \quad (5)$$

Recursion relation (5) requires only the given network parameters (ie, the matrix P, the moments of the variables $\{R_{ij}^{mr}, S_{mi}\}$ and computation of the matrix $(I - P)^{-1}$.

Consider now local resource requirements. Let L_{ij}^{mr} be the resources required at node j to process an object in class r that enters the network at node i in class m. L_{ij}^{mr} is the total loading induced at node j by an object in class r entering at node i in class m. Observe that

$$L_{mi} = \sum_{j=1}^{N} \sum_{r=1}^{M} L_{ij}^{mr}$$

Let $\phi_{ij}^{mr}(\theta)$ be the Laplace transform of L_{ij}^{mr}. For an arbitrary node k and object class s let $q_{sk} + u_{sk} = 1 - \rho_{kk}^{ss}$ and

$$\phi_{sk}(\theta) = \frac{q_{sk} f_{sk}(\theta) + u_{sk} g_{sk}^{sk}(\theta)}{1 - \rho_{kk}^{ss} g_{sk}^{sk}(\theta)} \quad (6)$$

The quantity u_{sk} is the probability that an object at node k in class s departs the network either without returning to node k or from node k but in a different class. We then have

$$\phi_{ij}^{mr}(\theta) = \begin{cases} \phi_{mi}(\theta) \text{ if } i = j \text{ and } m = r, \text{ and} \\ 1 - \rho_{ij}^{mr} + \rho_{ij}^{mr} \phi_{rj}(\theta) \text{ otherwise} \end{cases} \quad (7)$$

Moreover, in reference to the comment following (3) above,

$$h_{mi}(\theta) = \prod_{j=1}^{N} \prod_{m=1}^{M} \phi_{ij}^{mr}(\theta) \quad (8)$$

Next, we observe that, if the resource variables $\{ S_{mi}, R_{ij}^{mr} \}$ have moment generating functions, then so do the cumulative resource variables $\{L_{mi}\}$. In particular, suppose the transforms $\{f_{mi}(\theta), g_{ij}^{mr}(\theta)\}$ converge for all θ-values in the interval $(-\infty,\theta_1)$ where $\theta_1 > 0$. Select a node k and object class s. Since $\rho_{kk}^{ss} < 1$ there is a θ_0 in $(0,\theta_1)$ for which $\rho_{kk}^{ss} g_{kk}^{ss}(\theta_0) < 1$.

If $t > 0$ then

$$E\{\exp[\min(\theta_o L_{kk}^{ss}, t)]\} = q_{sk} E\{\exp[\min(\theta_o L_{sk}, t)]\} + u_{sk} E\{\exp[\min(\theta_o R_{kk}^{ss}, t)\}$$

$$+ \rho_{kk}^{ss} E\{\exp[\min(\theta_o R_{kk}^{ss} + \theta_o \hat{L}_{kk}^{ss}, t)]\}$$

where $\hat{L}_{kk}^{ss}$ is independent of R_{kk}^{ss} and distributed as L_{kk}^{ss}. Noting that $\min(a+b,c) \leq \min(a,c) + \min(b,c)$ when a, b, and c are nonnegative, it then follows that

$$E\left\{\exp[\min(\theta_o L_{kk}^{ss}, t)]\right\} \leq \frac{q_{sk} f_{sk}(\theta_o) + u_{sk} g_{kk}^{ss}(\theta_o)}{1 - \rho_{kk}^{ss} g_{kk}^{ss}(\theta_o)}$$

Letting $t \to \infty$, we see that $\phi_{sk}(\theta_0)$ is finite and that (6) is valid for $\theta \leq \theta_0$. Thus (1), (6), (7), and (8) are also valid for all θ-values in some interval $(-\infty, \theta_2)$ where $\theta_2 > 0$.

Finally, let L_{mi}^{*} be the total resources required at node i in class m by a typical job entering the network. Then L_{mi}^{*} has Laplace transform (or moment generating function).

$$\sum_{j=1}^{N} \sum_{r=1}^{M} b_{rj} \phi_{ji}^{rm}(\theta)$$

SECTION 4

POISSON INPUT

Consider now the evolution of the network over time when objects enter the system according to a Poisson process $A = \{A(t), t \geq 0\}$ with intensity function $\{\lambda(t), t \geq 0\}$ and the processing capacity available to each node in the network is unlimited. The variables R_{ij}^{mr} and S_{mi} are then interpreted as service or processing times at node i. The discussion we give follows Section 3 in Harrison and Lemoine [4].

The assumption of Poisson input as stated above means that the numbers of objects arriving in nonoverlapping time intervals are independent random variables and that

$$P\{A(t) = n\} = e^{-\Lambda(t)} \frac{[\Lambda(t)]^n}{n!}, t \geq 0$$

where

$$\Lambda(t) = \int_0^t \lambda(y)dy$$

In particular, if $A_{mi}(t)$ is the number of objects entering the network through node i in class m up to time t and $A_{mi} = \{A_{mi}(t), t \geq 0\}$, then A_{mi} are independent Poisson processes and A_{mi} has intensity function $\{b_{mi}\,\lambda(t), t \geq 0\}$.

The histories of objects moving through the system are independent and distributed as a process $Y = \{Y(t), 0 \leq t \leq L\}$ where L is the total length of time a generic object is in the network. In particular, with probability b_{mi}, $Y(0) = (m,i)$ and L has the same distribution as the variable L_{mi}. Now define

$$\gamma_{rj}(t) \equiv P\{L > t, Y(t) = (r,j)\} = \sum_{i=1}^{N} \sum_{m=1}^{M} b_{mi} P\{L_{mi} > t, Y(t) = (r,j) \mid Y(0) = (m,i)\}$$

$$\xi_{rj}(t) \equiv \int_0^t \lambda(y)\gamma_{rj}(t-y)dy,$$

and

$$\xi(t) \equiv \sum_{j=1}^{N} \sum_{r=1}^{M} \xi_{rj}(t) = \int_0^t \lambda(y)P\{L > t-y\}dy$$

Assuming the network is empty at time 0, the quantity $\xi_{rj}(t)$ is the expected number of objects in class r occupying node j at time t, $\xi(t)$ is the expected number of objects in the system, and $\Lambda(t) - \xi(t)$ is the expected number of objects that have arrived and departed from the system by time t.

Now let $C_{rj}(t)$ be the number of objects in class r occupying node j at time t, and let

$$\mathcal{C}(t) = \left\{ C_{rj}(t),\ r = 1, \ldots, M \text{ and } j = 1, \ldots, N \right\}$$

be the state of the network at time t where $C = \{c_{rj}\}$ is a generic state. Then the following remarkable result holds (cf [4]):

$$P\left\{ \mathcal{C}(t) = C \right\} = e^{-\xi(t)} \prod_{j=1}^{N} \prod_{r=1}^{M} [\xi_{rj}(t)]^{c_{rj}}/(c_{rj}!) \tag{9}$$

That is, the number of objects in class r occupying node j at time t has a Poisson distribution with mean $\xi_{rj}(t)$, and the numbers occupying the various nodes and classes are independent random variables. Moreover, the total number of departures from the system by time t has a Poisson distribution with mean $\Lambda(t) - \xi(t)$.

Implementation of (9) requires computation of the mean values $\{\xi_{rj}(t)\}$, which is difficult for nonhomogenous input. Suppose, however, that A is a homogenous Poisson process with $\lambda(t) \equiv \lambda$ for $t \geqslant 0$. Let

$$\xi_{rj} = \lambda \int_0^{\infty} \gamma_{rj}(y)dy = \lambda \sum_{i=1}^{N} \sum_{m=1}^{M} b_{mi} E\left\{ L_{ij}^{mr} \right\}$$

and

$$\xi = \sum_{j=1}^{N} \sum_{r=1}^{M} \xi_{rj}$$

In the homogenous case we have $\xi_{rj}(t) \uparrow \xi_{rj}$ as $t \uparrow \infty$. It then follows from (9) that

$$\lim_{t \to \infty} P\left\{ \mathcal{C}(t) = C \right\} = e^{-\xi} \sum_{j=1}^{N} \sum_{r=1}^{M} (\xi_{rj})^{c_{rj}}/(c_{rj}!) \tag{10}$$

This limiting or asymptotic distribution depends only on the arrival rate λ and the *expected* amount of time ξ_{rj} that an object spends at node j in class r while in the network, and *not* upon the forms of the distributions for service times at the various nodes. Moreover the numbers of objects occupying the various nodes and classes are independent, Poisson distributed, random variables. For the case of homogenous input, the process of departures from the system is Poisson with intensity function $\{\lambda H(t), t \geqslant 0)\}$ where $H(t) = P\{L < t\}$. Thus, as $t \uparrow \infty$, the process of departures is asymptotically Poisson with rate λ. Indeed, the process of departures from node j by objects in class r is asymptotically Poisson with rate λq_{rj}^{*}, and these node/class departure processes are asymptotically independent across the various nodes and classes; cf Kelly [6].

In the case of homogenous input, the time-dependent distribution of state given by (9) can be approximated by the asymptotic distribution given in (10) when t is large enough. The closeness of this approximation can be gauged as follows: Let $\pi_t(C)$ denote the right side of (9) and $\pi(C)$ the right side of (10) and

$$d(\pi_t,\pi) \equiv \sum_{C} |\pi_t(C) - \pi(C)|$$

We can think of $d(\pi_t,\pi)$ as the *distance* between the distribution of state at time t, namely π_t, and the asymptotic distribution of state π. (In measure theory, $d(\pi_t, \pi)$ is the total variation of the signed measure $\pi - \pi_t$.) For the multidimensional Poisson distributions π_t and π (cf Cinlar [3], pp. 564-565)

$$d(\pi_t,\pi) \leq 2 \sum_{j=1}^{N} \sum_{r=1}^{M} [\xi_{rj} - \xi_{rj}(t)]$$

But the summation on the right side of this inequality is

$$\xi - \xi(t) = \lambda \int_t^{\infty} P\{L > y\}\, dy$$

Assume henceforth that the resource variables $\{S_{mi}, R_{ij}^{mr}\}$ have moment generating functions. This is not a severe restriction as the distributions customarily used in the modeling of service systems and queueing networks have this property. Take $\theta > 0$ for which $\{h_{mi}(\theta)\}$ are finite. Put $\phi(\theta) = E\{\exp(\theta L)\}$, where, of course,

$$\phi(\theta) = \sum_{j=1}^{N} \sum_{r=1}^{M} b_{rj} h_{rj}(\theta)$$

Now $P\{L > y\} = P\{\exp(\theta L) > \exp(\theta y)\}$ and by Chebyshev's Inequality (cf Chung [2], p. 48), $P\{\exp(\theta L) > \exp(\theta y)\} \leq \phi(\theta) \exp(-\theta y)$.

Hence

$$\int_t^{\infty} P\{L > y\}\, dy < \frac{\phi(\theta)}{\theta} e^{-\theta t}$$

Thus, if

$$\mu(t) = \frac{\lambda}{\theta} \phi(\theta) e^{-\theta t}$$

then

$$\xi - \mu(t) \leq \xi(t) \leq \xi \qquad (11)$$

and

$$d(\pi_t, \pi) \leq 2\mu(t)$$

Similar results hold for the marginal distributions of π_t and π. Indeed, if B is a subset of $\{(r,j)\}$, and π_{Bt} and π_B are the corresponding marginal distributions of π_t and π,

$$\sum_B \xi_{rj} - \mu_B(t) \leq \sum_B \xi_{rj}(t) \leq \sum_B \xi_{rj} \qquad (12)$$

and

$$d(\pi_{Bt}, \pi_B) \leq 2\mu_B(t)$$

where

$$\mu_B(t) = \frac{\lambda}{\theta} e^{-\theta t} \sum_B b_{rj} h_{rj}(\theta)$$

Moreover, if X has a Poisson distribution with mean β (cf Hoel et al [5], p. 107)

$$P\left\{X < \beta/2\right\} \leq (\sqrt{2/e})^{\beta}$$

and

$$P\left\{X > 2\beta\right\} < (e/4)^{\beta}$$

Let

$$\alpha_B(t) = \sum_B \xi_{rj} - \mu_B(t)$$

Then

$$P\left\{C_{rj}(t) < \xi_{rj}(t)/2 \text{ for all } (r,j) \text{ in } B\right\} = \prod_B P\left\{C_{rj}(t) < \xi_{rj}(t)/2\right\} \leq \prod_B (\sqrt{2/e})^{\xi_{rj}(t)} \leq (\sqrt{2/e})^{\alpha_B(t)} \qquad (13)$$

and

$$P\left\{C_{rj}(t) > 2\xi_{rj}(t) \text{ for all } (r,j) \text{ in } B\right\} < (e/4)^{\alpha_B(t)} \qquad (14)$$

Finally, if $\mathscr{C}^* = \{C^*_{rj}\}$ is a random vector having distribution π, then letting $t \to \infty$ in (13) and (14),

$$P\left\{C^*_{rj} \leq \xi_{rj}/2 \text{ for all } (r,j)\right\} \leq (\sqrt{2/e})^{\xi} \qquad (15)$$

and

$$P\left\{C^*_{rj} \geq 2\xi_{rj} \text{ for all } (r,j)\right\} \leq (e/4)^{\xi} \qquad (16)$$

SECTION 5
MODEL DISCUSSION

This section discusses briefly a few situations where the model considered in this paper may be relevant.

The informative book of Whitehouse [12] provides an extensive discussion in Chapters 8 through 11 of a methodology called GERT (Graphical Evaluation and Review Technique). GERT combines flowgraph theory, transform theory, and PERT (Program Evaluation and Review Technique) to obtain solutions to stochastic problems from a network oriented point of view. A large, varied, and impressive array of applications, theoretical and applied, of the GERT approach to problem solving is presented in Chapters 8, 10, and 11 of [12]. Topics include derivation of first-passage distributions, queueing systems, repairman problems and reliability systems, manufacturing processes, inventory systems, quality control sampling plans, planning and management of research and development programs, gaming, traffic problems, and more.

Typical output of the GERT solution procedure includes probability of traversal of a network path or loop, mean time of traversal of a path or loop, and second moment of the traversal time. Implementation of the GERT procedure as presented in [12] requires an appropriate network formulation followed by application of Mason's Rule, a result from flowgraph theory.

The model considered in this paper is sufficiently general to accommodate the plethora of stochastic network applications and scenarios presented in [12]; indeed, the model structure offers enhanced flexibility in problem formulation and solution. The results given in Section 3 should be simpler to compute with than the "pictorial flowgraph technique" based on Mason's Rule. In fact, a broader and richer range of results should be computationally tractable using the methodology presented in this paper.

The formulation of Section 2 is a plausible model for a complex sequential process wherein an object or "system" moves through a succession of states (nodes) and operating modes (classes) in the course of carrying out or fulfilling its function, with the transitions among states and operating modes occurring in a possibly random manner, and requiring (consuming) some resource in randomly varying amounts.

Consider a job (message) moving through a computer (communication) system. Performing the job entails execution of software modules embedded in a network of processors. The job, depending on its type, may require execution of a particular set of software modules with attendant variation in the machine instructions per second (or words of memory) required at processing centers. Sending a message or data packet entails routing through a network of transmitting stations. The message, depending on its type (priority), may require routing along a particular path with attendant variation in buffer space and frames required for transmission through each station along the path. A system in the course of its life cycle moves through stages of availability (mission readiness) and unavailability (failure, repair, maintenance). The system may be fully operational or degraded, component failures may have varying impact, repair or maintenance may restore the system to some previous level of readiness or be faulty and inadequate and possibly damaging to the system. Maintenance and repair may require varying skill levels, support equipment, and replacement of defective components (spares provisioning). Built-in test equipment or alarm mechanisms may trigger system shutdown based on erroneous signals, or fail to monitor a deteriorating situation and issue appropriate warnings or shutdown commands.

The model formulation is sufficiently general to capture, albeit imperfectly, the important factors and interactions in these examples. Performing the job, moving the message, or supporting the system through its life cycle induce loads or resource requirements at each step or stage along the way, and the results of Section 3 assess the extent of these resource requirements at the nodal and global levels.

The transforms of some important distributions in models for networks of queues satisfy the network flow equations (1). For open BCMP [1] networks, the transform (of the distribution) for the total service time of a job entering the system through node i in class m satisfies (1). In that case, the distributions of S_{mi} and R_{ij}^{m} are identical and depend only on i. Moreover, if the routing structure is acyclic and every two nodes are connected by at most one directed path, the transform of the steady-state total sojourn time (response time) distribution for a job entering through node i in class m satisfies (1); cf Walrand and Varaiya [11] and Lemoine [10]. The recursive formula (5) then provides a compact scheme for computing moments of these distributions.

Section 4 provides a fairly complete picture of the behavior of a system of infinite-server queues with Poisson input where the history of a typical individual or object moving through the network is as described in Section 2. In this setting, time is the resource required. The overall simplicity of the results in Section 4 can be traced to the order-statistic property of the Poisson process and the fact that objects do not compete with one another when sufficient service capacity is always available.

Viewed from a performance standpoint, infinite-server systems represent an upper bound on performance potential since measures such as response (sojourn) time or system throughput depend only upon routing, service times, and input rates. However, results for such systems are of interest for design and performance prediction.

The upper and lower confidence levels given in (15) and (16) indicate the extent to which the network population will remain within certain bounds, and hence provide a means of assessing the adequacy of alternate system designs, which, of course, will have finite capacity. Indeed, in systems which must gather and process a huge amount of data in real time and at phenomenally high rates, it is important to assess at the design stage the adequacy of the proposed processing resources to handle the anticipated workloads. Moreover, the formulation presented here provides additional flexibility in modeling diverse phenomena, and the relative simplicity of the results facilitates computation in studying alternative possibilities and configurations.

Finally, consider a logistics-repair system comprising many units of diverse types and a variety of support functions (maintenance, repair, replenishment, etc). Suppose the process that records the initial failures or epochs of unavailability of units (measured from when they are put into operation for the first time) is Poisson. In view of the comments above regarding life cycle modeling of an individual unit, the results of Section 4 provide a relatively simple means of approximating the status of the overall system as it evolves through time, while accounting for any number of important factors and considerations, and of assessing the adequacy of various support resources in relation to mission requirements and operational availability.

REFERENCES

[1] Baskett, F., Chandy, K.M., Muntz, R.R., and Palacios, F.G. "Open, Closed and Mixed Networks of Queues with Different Classes of Customers, *Journal of the Association for Computing Machinery 22*, pp. 248-260, 1975.

[2] Chung, K.L., *A Course in Probability Theory*, 2nd Ed., Academic Press, New York, 1974.

[3] Cinlar, E., "Superposition of Point Processes," *Stochastic Point Processes: Statistical Analysis, Theory, and Applications*, pp. 549-606, P.A.W. Lewis, Editor, Wiley, New York, 1981.

[4] Harrison, J.M. and Lemoine, A.J., "A Note on Networks of Infinite-Server Queues," *Journal of Applied Probability 18*, pp. 561-567, 1981.

[5] Hoel, P.G., Port, S.C., and Stone, C.J., *Introduction to Probability Theory*, Houghton Miffin, Boston, 1971.

[6] Kelly, F.P., *Reversibility and Stochastic Networks*, Wiley, New York.

[7] Kemeny, J.G. and Snell, J.L., *Finite Markov Chains*, Van Nostrand, Princeton, New Jersey, 1960.

[8] Lemoine, A.J., "Networks of Queues - A Survey of Equilibrium Analysis," *Management Science 24*, pp. 464-481, 1977.

[9] Lemoine, A.J., "Networks of Queues - A Survey of Weak Convergence Results," *Management Science 24*, pp. 1175-1193, 1978.

[10] Lemoine, A.J., "Total Sojourn Time in Networks of Queues," Tech Rep. No. 79-020-1, System Control, Inc., Palo Alto, CA, 1979.

[11] Walrand, J., and Varaiya, P., "Sojourn Times and the Overtaking Condition in Jacksonian Networks," *Advances in Applied Probability 12*, pp. 1000-1018, 1980.

[12] Whitehouse, G.E., *Systems Analysis and Design Using Network Techniques*, Prentice-Hall, Englewood Cliffs, NJ, 1973.