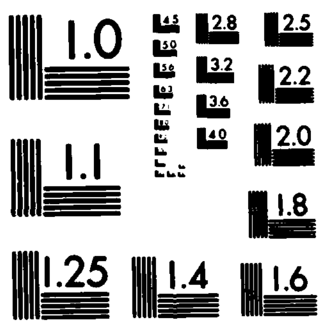
AD-A136 353 FINITE DIFFERENCE METHODS FOR THE INCOMPRESSIBLE NAVIER-STOKES EQUATIONS -..(U) WISCONSIN UNIV-MADISON MATHEMATICS RESEARCH CENTER J C STRIKWERDA OCT 83

UNCLASSIFIED MRC-TSR-2584 DAAG29-80-C-0041 F/G 12/1 NL

1/1

END
DATE
FILMED
2-84
DTIC



MICROCOPY RESOLUTION TEST CHART
NATIONAL BUREAU OF STANDARDS-1963-A

AD A136353

MRC Technical Summary Report #2584

FINITE DIFFERENCE METHODS FOR THE INCOMPRESSIBLE NAVIER-STOKES EQUATIONS - A SURVEY

John C. Strikwerda

Mathematics Research Center
University of Wisconsin—Madison
610 Walnut Street
Madison, Wisconsin 53705

October 1983

(Received August 4, 1983)

Sponsored by

U. S. Army Research Office
P. O. Box 12211
Research Triangle Park
North Carolina 27709

UNIVERSITY OF WISCONSIN - MADISON
MATHEMATICS RESEARCH CENTER

FINITE DIFFERENCE METHODS FOR THE INCOMPRESSIBLE NAVIER-STOKES EQUATIONS - A SURVEY

John C. Strikwerda

Technical Summary Report #2584

October 1983

ABSTRACT

This paper is a survey of finite difference techniques for solving the incompressible Navier-Stokes equations. We consider what features are important for schemes in order to be useful in scientific and engineering applications. In so far as possible we try to point out the strengths and weaknesses of various methods.

AMS (MOS) Subject Classifications: 65N05, 76D05

Key Words: Navier-Stokes Equations, Finite Difference Methods

Work Unit Number 3 - Numerical Analysis and Scientific Computing

---

Sponsored by the United States Army under Contract No. DAAG29-80-C-0041.

SIGNIFICANCE AND EXPLANATION

The incompressible Navier-Stokes equations describe the flow of fluids such as water, air at low speeds, and many other common liquids and gases. Efficient and accurate numerical methods for their solution are therefore of great importance for a wide variety of engineering and scientific applications. In this paper, several existing finite difference methods are examined and suggestions for further research are given.

FINITE DIFFERENCE METHODS FOR THE INCOMPRESSIBLE NAVIER-STOKES EQUATIONS - A SURVEY

John C. Strikwerda

1. Introduction

The incompressible Navier-Stokes equations describe the motion of fluids such as water, air at low speeds, and many other common liquids and gases. As such they are of widespread scientific and engineering significance and numerical solution procedures equations are of great importance. In this paper we will examine common numerical methods for solving the incompressible Navier-Stokes equations and discuss various strengths and weaknesses of the methods. We will particularly concern ourselves with finite difference methods although much of this paper will apply to general numerical methods.

The incompressible Navier-Stokes equations are

$$(1.1) \qquad \vec{u}_t + (\vec{u} \cdot \vec{\nabla})\vec{u}) + \vec{\nabla} p - \frac{1}{R} \nabla^2 \vec{u} = \vec{f}(\vec{x},t)$$

$$\vec{\nabla} \cdot \vec{u} = 0$$

where $\vec{u}(\vec{x},t)$ is the velocity $p(\vec{x},t)$ is the pressure, and $\vec{f}(\vec{x},t)$ represents forces on the fluid. The parameter R is the Reynolds number which is a non-dimensional measure of the ratio of inertial forces to viscous forces. The equations (1.1) hold for t in $(0,T)$ and $\vec{x}$ in a domain Ω in $\mathbb{R}^n$, and on the boundary of Ω there are n scalar boundary conditions. For definiteness we will consider the boundary conditions to be Dirichlet conditions on the velocity, i.e.

$$(1.2) \qquad \vec{u}(\vec{x},t) = \vec{b}(\vec{x},t) \quad \text{on} \quad \partial\Omega x [0,T]$$

Much of what we say will not depend on the form of the boundary condition.

There are a number of systems similar to the Navier-Stokes equations (1.1) which are also of significant interest. Among these are the Bossinesq equations describing fluids with small temperature gradients and buoyancy effects, equations of non-Newtonian fluid

Sponsored by the United States Army under Contract No. DAAG29-80-C-0041.

flow, and the Stokes equations describing slow viscous flow. The Stokes equations may be written as

$$(1.3) \qquad \begin{aligned} \vec{u}_t - \vec{\nabla} p &= \nabla^2 \vec{u} \\ \vec{\nabla}\cdot\vec{u} &= 0 \end{aligned} \qquad \text{in } \Omega \times [0,T].$$

For the mathematical theory of the systems (1.1) and (1.3) we refer to Ladyzhenskaya (1963) and Temam (1979).

We begin our discussion of numerical methods by considering what features are required in a solution algorithm. First, numerical solutions are often required for non-simple gemetries which involve curvilinear grids arising from boundary-fitted coordinate systems. Also, the grids may be stretched to resolve boundary layers or other flow phenomena of interest. Secondly, both the velocity and pressure should be accurately determined. The pressure can be mathematically eliminated from (1.1) and (1.3) by projecting the first equation in each system onto the space of divergence free vector fields. However the determination of the pressure is very important for physical applications, in fact, it is often the primary objective. One is frequently interested in the pressure force induced on an object by a given velocity distribution.

These considerations show that what is desired for the incompressible Navier-Stokes equations are schemes which give accurate determination of the velocity and pressure for non-rectilinear grids. It is also desirable to have methods that easily extend to more general systems such as the Bossinesq equations.

A great many schemes have been proposed for solving the incompressible Navier-Stokes equations and, in a review such as this, it is impossible to acknowledge more than a small fraction of them. We have tried to consider only questions of a more general nature without dwelling on particular features of the schemes. Thus we refer to only those papers which were the first to introduce new methods or ideas. In particular, the many fascinating papers dealing with applications of schemes to problems of scientific interest are not mentioned at all.

We apologise to those whose work we have overlooked. Any notification of such oversight will be appreciated and will be corrected in a more complete review which the author intends to write later. We hope nonetheless that this review will stimulate further work on this important topic.

2. Modifications of the Navier-Stokes Equations

The Navier-Stokes equations (1.1) are frequently modified with the intent of obtaining a form which is more amenable to numerical solution. In this section we examine several of these modifications.

One common way of modifying the Navier-Stokes equations (1.1) is to replace it by the system

$$\vec{u}_t + (\vec{u} \cdot \vec{\nabla})\vec{u} + \vec{\nabla}p - \frac{1}{R}\nabla^2\vec{u} = \vec{f}(\vec{x},t)$$

(2.1)

$$\nabla^2 p = \vec{\nabla}\cdot\vec{f} - \sum_{i,j} u^i_{x_j} u^j_{x_i}$$

The last equation of (2.1) is obtained by taking the divergence of the first equation of (1.1) and using the last equation of (1.1) to eliminate the divergence of the velocity. The motivation for using the system (2.1) is the hope that, when discretized, the pressure can be obtained using standard methods for inverting the discrete Laplacian. However the system (2.1) has a grave disadvantage in that it requires an additional boundary condition. Several suggestions have been made for this boundary condition.

One suggestion is that the additional boundary condition be given by the normal derivative of the pressure as determined by the first equation of (1.1) or (2.1) evaluated on the boundary, e.g. Roache (1972,p 194). The formula is

$$(2.2) \qquad \frac{\partial p}{\partial n} = -\vec{n}\cdot(\vec{u}_t + (\vec{u}\cdot\vec{\nabla})\vec{u} - \frac{1}{R}\nabla^2\vec{u} - \vec{f}(\vec{x},t)).$$

This however is not satisfactory as a boundary condition since it is not independent of the system of differential equations (2.1), and thus leaves the overall system underdetermined.

Another boundary condition which is used along boundaries corresponding to physical surfaces is to set the normal derivative of the pressure equal to zero, which is valid in the limit for high Reynolds number flows. This gives a well-determined system, however, its solutions are not in general solutions of (1.1).

A good way to illustrate these ideas is to consider steady two-dimensional flow in a channel. In the region $0 \leq x \leq L$, $0 \leq y \leq 1$ we have the exact solution to (1.1)

$$u = y(1-y)$$
$$v = 0 \qquad (2.3)$$
$$p = -\frac{2}{R}x$$

where $\vec{u} = (u,v)$ is the velocity. We regard this solution as determined by the specification of the velocity components on the boundary, i.e. (1.2). Using the pressure-equation system (2.1), if one specifies the normal derivative of pressure then one needs to know the exact value of dp/dx or one obtains a solution to (2.1) which is different from (2.3) and is not a solution of (1.1).

On the other hand, the boundary condition (2.2) underspecifies the solution since any solution of (2.1) with

$$\frac{\partial p}{\partial n} = \phi(x,y)$$

for arbitrary ϕ satisfies (2.1) and (2.2).

This simple example shows that solving (1.1) via (2.1) is unlikely to give correct solutions. Indeed the methods using (2.1) or similar systems have difficulty with accurate pressure fields and with satisfying the incompressibility condition on the velocities (see for example the work by Boney, Hefner, Hirsh, and Zoby reported in Rubin and Harris (1975)). Roache (1972) has a discussion of the difficulties of obtaining a zero divergence for the velocity field when using the above approach (see also Harlow and Welch (1964)).

The only boundary condition of which this author knows which will make (2.1) equivalent to (1.1) is to specify the divergence of the velocity on the boundary. This will make (2.1) equivalent to (1.1) for low Reynolds numbers. However, this destroys the possibility of iteratively solving (2.1) by solving the Poisson equation for pressure since there is no boundary condition on pressure. Because of these difficulties it seems best to avoid the use of (2.1) and similar "pressure-equation" approaches to solve the Navier-Stokes equations (1.1).

A number of researchers make transformations similar to those that give rise to (2.1) but use the discrete system of equations obtained from either a finite difference or finite element approximation of (1.1). It is usually difficult to discern enough details from their writings to know if their modified equations are equivalent to their original discrete equations. It would seem that most of these discrete pressure-equation methods would have the same difficulties as the differential pressure-equation approach. That is, the number of possible solutions is increased and an additional boundary condition is introduced to guarantee uniqueness. Difficulties would arise from having either inaccurate boundary conditions, e.g. normal derivative of pressure set to zero, or a boundary condition consistent with (2.2) which could cause ill-conditioning. Chorin (1967) discusses other shortcomings of this approach. In any case, this matter is more important than the cursory treatment it usually receives in the algorithm descriptions.

Another approach to solving the steady incompressible Navier-Stokes equations is the artificial compressibility method. The basic idea is to solve a time-dependent system of equations, whose steady-state solutions are the same as the steady-state solutions of (1.1), until a steady state is reached. Methods have been proposed by Chorin (1967), Yanenko (1967), and Aubert and Deville (1983). The convergence rate of these methods is a dependent on the choice of time-discretization used to solve the system. A difficulty which arises is that the finite difference equations may not have a steady-state solution. The reason for this is discussed in section 4.

Another common method is to use the "parabolized" Navier-Stokes equations in which the second-derivatives in the stream-wise direction are removed. Because of its limited applicability and uncertain justification we will not discuss this method here except to note that often an analogue of (2.1) is derived and thus some of our observations on (2.1) also apply to the parabolized equations. Raithby and Schneider (1979) discuss these difficulties for three-dimensional flow problems.

A popular method for two-dimensional and axisymmetric flows is the stream-function and vorticity formulation. For two dimensions, the stream function $\psi(x,y,t)$ is defined by

$$u = \psi_y \ , \ v = -\psi_x$$

and the vorticity $\zeta(x,y,t)$ is given by

$$\zeta = v_x - u_y .$$

The resulting system is then

$$\text{(2.4)} \qquad \zeta_t + \psi_y \zeta_x - \psi_x \zeta_y = \frac{1}{R} \nabla^2 \zeta$$

$$\nabla^2 \psi = -\zeta \ .$$

Two boundary conditions are required by this system and corresponding to (1.2) they are given by specifying both components of the gradient of ψ. Note however that the system (2.4) would be much easier to solve if one had one boundary condition on ψ and one on ζ. In fact, what is usually done in finite difference calculations is that (2.4b) is used to supply values of ζ on the boundary, and the second normal derivative in $\nabla^2 \psi$ is approximated using the boundary conditions for the gradient. The nonlinear system is then solved iteratively, (2.4a) determing ζ given ψ and (2.4b) determining ψ given ζ. This approach works, but the boundary condition on ζ causes great difficulty. That this must be so is easy to see. Suppose at some iteration there are errors in ψ of magnitude ϵ. Then using a discrete version of (2.4b) as the boundary condition for ζ, the next iterate of ζ will have errors of magnitude ϵh^{-2} where h is the normal grid spacing. (The h^{-2} comes from the second normal derivative in the Laplacian.) Thus the errors in ψ become magnified in ζ. The usual way to treat this is to under-relax the boundary values of ζ on the boundary. That is

$$\zeta^{n+1} = (1-\mu)\zeta^n + \mu \ (-\nabla_h^2 \psi^{n+1}) \ ,$$

where as we have seen μ must be proportional to h^2, giving a very slow rate of convergence. In spite of being so slow this procedure is often used and can give accurate solutions.

If one eliminates the vorticity from the system (2.4) one obtains a single equation which for steady flow is a fourth order elliptic equation in ψ,

$$(2.5) \qquad \frac{1}{R} \nabla^4\psi - \psi_y \nabla^2\psi_x + \psi_x \nabla^2\psi_y = 0.$$

This equation requires two boundary conditions and, equivalent to (1.2), they are given by specifying ψ and its normal derivative. This gives a very well-conditioned numerical problem. Recent work using (2.5) has been done by Schreiber and Keller (1983a). Once a solution is obtained for (2.5), the pressure can be obtained by integrating the equation for the gradient of the pressure from (1.1). If the equation (2.5) is used with non-uniform grids and coordinate transformations, the discretized system can be quite formidable.

Based on the above observations it would seem that of the methods we've discussed only the original version of the Navier-Stokes equations (i.e. (1.1)) and the stream function formulation (2.5) can lead to well-conditioned numerical methods.

3. Finite Difference Schemes

In this section we discuss finite difference schemes for the Navier-Stokes equations (1.1) in the form given. We will concentrate on schemes which are formally second-order accurate, although there are a number of schemes which are first-order accurate (e.g. Kzivickii and Ladyzhenskaya (1966), Temam (1979)).

We consider first the so-called staggered mesh schemes. The second-order accurate staggered mesh scheme for a uniform cartesian grid assigns the values of each of the velocity components and the pressure to different interlaced grids. In two dimensions with velocity components u and v, one may assign values of u to grid locations $((i+\frac{1}{2})h, jh)$, values of v to $(ih, (j+\frac{1}{2})h)$, and values of p to (ih, jh), e.g. Harlow and Welch (1965), Patankar and Spalding (1972), Raithby and Schneider (1979), Brandt and Dinar (1979). This method works very well as long as the geometry is rectangular and the grid is uniform. Non-uniform grids and grid mapping techniques cannot be conveniently handled, although Liu and Krause (1979) have developed a staggered mesh scheme for use with general geometries. The staggered mesh schemes also have some difficulty at boundaries. For example, when both velocity components are specified at a boundary then that velocity component whose mesh lines do not lie on the boundary requires some special treatment. A method for treating curved boundaries has been developed by Viecelli (1971).

The central difference scheme on a uniform rectangular mesh assigns values of all the variables to each grid point. The divergence and gradient operators are approximated using central differences and the Laplacian is approximated by the standard five-point discrete Laplacian. Central difference schemes have been introduced by Chorin (1967, 1968) for time-dependent calculations, and Aubert and Deville (1983) have developed a fourth order accurate central difference scheme for steady flows.

An important concept for finite difference schemes for elliptic systems such as (1.1) and (1.2) is that of regularity (see Bube and Strikwerda (1983), and also Frank (1968), Brandt and Dinar (1979)). Regular schemes give rise to regularity estimates analogous to those in the theory of elliptic systems of differential equations. Solutions to regular difference schemes will in general be smoother than solutions to non-regular schemes and also will be more accurate approximations to the solutions of the differential equations. The central difference scheme is non-regular (Bube and Strikwerda (1983)),. which results in non-smooth solutions. The lack of smoothness is most noticeable in the pressure. The staggered mesh scheme however is regular. The advantage of the central difference scheme is that it is easily implemented with non-uniform grids as introduced by coordinate changes.

It should be emphasized that none of the difficulties mentioned above are insurmountable. Both the staggered mesh and central differencing schemes have been used and often quite successfully.

A scheme which incorporates both the ease of use which central differences possess and regularity is the regularized central difference scheme which has been introduced by the author (Strikwerda (1983)). This scheme has been shown to be second-order accurate for both the velocity and pressure when applied to the Stokes equations (1.3) (Strikwerda (1983)). In this scheme the derivatives of pressure are approximated as

$$\frac{\partial p}{\partial x_k} = \delta_{k0}p - \alpha h_k^2 \delta_{k-}\delta_{k+}^2 p \tag{3.1}$$

and the first derivatives of the velocity in the divergence equation are approximated as

$$\frac{\partial u^k}{\partial x_k} = \delta_{k0}u^k - \alpha h_k^2 \delta_{k+}\delta_{k-}^2 u^k , \tag{3.2}$$

where α is a non-zero constant and δ_{k0}, δ_{k+} and δ_{k-} are the centered, forward, and backward divided differences, respectively. The Laplacian is approximated with the usual five-point stencil. Note that for α equal to one-sixth the approximations (3.1) and (3.2) are third-order accurate.

Since the regularized central difference scheme is a variant of the central difference scheme it is easy to implement with coordinate changes and boundary-fitted grids.

4. The Integrability Condition

In order for a solution to exist to the system (1.1) with boundary conditions (1.2) it is necessary that the boundary data satisfy the integrability condition. This condition is obtained by the divergence theorem as

$$0 = \int\int_\Omega \vec{\nabla}\cdot\vec{u} = \int_{\partial\Omega} \vec{n}\cdot\vec{u} = \int_{\partial\Omega} \vec{n}\cdot\vec{b} . \tag{4.1}$$

Thus if the integral of the normal component of the boundary data is non-zero, there is no solution.

Now when one discretizes the equations (1.1) one obtains a discrete analog of the incompressibility condition

$$\vec{D}_n\cdot\vec{u}_h = 0 \quad \text{in } \Omega_h \tag{4.2}$$

and the boundary condition

$$\vec{u}_h = \vec{b}_n \quad \text{on } \partial\Omega_h . \tag{4.3}$$

There is also a discrete integrability condition analogous to (4.1) which must be satisfied.

There are at least two ways to satisfy the discrete integrability condition. The first method would be to analyze the matrix corresponding to (4.2) and determine the null space of the adjoint matrix. If the data is constrained to be orthogonal to this null space then a solution will exist. This approach is impractical for many situations, especially if coordinate changes have been employed since then the matrices are not easy to analyze. However, Amit, Hall, and Porshing(1982) have given a method which determines the null space of the discrete divergence operator for special rectilinear grids for two-dimensional problems.

A second approach, introduced by Strikwerda (1983), is to replace (4.2) by

(4.4')
$$\vec{D}_h \cdot \vec{u}_h = \delta_h$$

where δ_h is a constant chosen to guarantee a solution. The value of δ_h must be determined as part of the solution. As shown in the examples of Strikwerda (1983) δ_h is at least $O(h^2)$ for the regularized central scheme.

Another way of looking at the condition (4.4) is as follows. As shown by Temam (1979) and others, any discrete divergence operator $\vec{D}_h$, defined only on the interior of the grid, has a corresponding gradient operator $\vec{G}_h'$ defined by

(4.5)
$$(\vec{D}_h \cdot \vec{u}, \phi) + (\vec{u}, \vec{G}_h' \phi) = 0$$

for all grid vector function $\vec{u}$ and scalar functions ϕ which vanish on the boundary. If one wishes to satisfy (4.2) at each point of the interior then (4.5) with ϕ taken to be one at each interior point gives the requirement that

(4.6)
$$(\vec{D}_h \cdot \vec{u}, 1) + (\vec{u}, \vec{G}_h' 1) = 0$$

must be satisfied. This formula is the analog of the integrability condition (4.1), and the second term in (4.6) will usually involve only the values of $\vec{u}$ on and near the boundary. If the constraint (4.6) is not satisfied then the data must be modified so that (4.6) is satisfied. The use of (4.4) in place of (4.2) is one say by which (4.6) can be satisfied. An advantage of using (4.4) over approaches which would modify the boundary data of $\vec{u}$ is that (4.4) requires no explicit knowledge of $\vec{G}_h'$.

It is interesting to note that for the staggered mesh scheme on a uniform grid one can easily satisfy the discrete integrability condition since the calculus of finite differences mimics the differential calculus very closely, see e.g. Krivickii and Ladyzhenskaya (1966). Similarly, Chorin (1969) proves the convergence of a central difference scheme for the time-dependent Navier-Stokes equations on a periodic rectangular mesh. An essential element of these proofs is that one has a convenient form of the finite difference analogue of the divergence theorem of the differential calculus. Liu and Krause (1979) develop a staggered grid scheme for non-rectangular grids and the success of their scheme is due to their careful treatment of the integrability constraint.

5. Solving the Finite Difference Equations.

Having determined which form of the Navier-Stokes equations is to be used and given the finite difference equations to be employed, one is still left with finding an efficient method for solving the equations. We have already discussed solving the vorticity-streamfunction formulation in section 2. The formulation using just the streamfunction can be solved by iterative methods which employ direct solvers for the biharmonic equation or by Newton-like methods (Schreiber and Keller (1983a)).

For the primitive variable formulation the basic problem reduces to solving systems of the form

$$\begin{aligned} A(u)\,u + Gp &= b_1 \\ Du &= b_2 \end{aligned} \qquad (5.1)$$

where $A(u)u$ arises from discretizing the velocity portion of the first equation of (1.1) and G and D are the discrete gradient and divergence respectively. A common method to solve (5.1) is to use an iterative method each iteration consisting of one step of an iterative method appropriate for

$$A(u)\,w = c_1$$

together with a pressure update of the form

$$p^{\nu+1} = p^{\nu} - \gamma D u^{\nu+1}.$$

Several methods have been proposed. The first seems to have been given by Chorin (1968) for the time-dependent Navier-Stokes. Other methods have been introduced for finite element methods, see e.g. Fortin and Glowinski (1982), and by Strikwerda (1983) for finite difference schemes.

Recently an interesting method for solving the finite difference equations arising from the Navier-Stokes equations has been advanced by Amit, Hall, and Porsching (1982),(see also Hageman (1982)). An essential requirement is that one be able to determine the null space of the discrete divergence operator. If this is possible then their method appears to work well. At present, this method seems limited to certain discretizations of the rectangle.

6. Determining the Accuracy of a Method.

With these various methods available an obvious question arises as to how one decides which methods are more accurate or suitable than others. This is not an easy question to answer.

Most of the numerical methods which have been presented have been tested on the driven cavity problem. However, until recently there has not been a numerical solution of this problem which is of sufficiently high accuracy to be a reliable comparison. The work of Schreiber and Keller (1983a) should provide such a standard by which accuracy can be determined for other methods. The experimental data on this problem does not appear adequate to provide good checks of accuracy. It should also be mentioned that some methods which were designed to do well on flows like the driven cavity give totally incorrect solutions due to excessive artificial viscosity (Strikwerda (1982)), see also Schreiber and Keller (1983b).

A test problem with an analytic solution to the time-dependent Navier-Stokes equations was used by Chorin (1967). This problem could be used more widely as a test case.

The advantage of using analytic solutions to test numerical schemes is that this is the only way one can determine the true accuracy of the scheme. As shown in section 2 schemes which use the pressure equation approach can be zeroth-order accurate, similarly for certain upwind differencing schemes (see Strikwerda (1982)). If these schemes had been adequately tested either they would not have been published or we would know how much confidence to put in them.

It is very easy to obtain analytic solutions to the system (1.1). For example, for any smooth function $\psi(x,y,t)$ one can generate solutions in two dimensions by prescribing

(6.1) $$u = \psi_y, \; v = -\psi_x$$

and setting the pressure arbitrarily. The force $\vec{f}(x,y,t)$ is determined so as to make (6.1) a solution. For example, with (6.1) one can set

$$p = \tfrac{1}{2}(\psi_x{}^2 + \psi_y{}^2)$$
$$f_1 = -\frac{1}{R}(\nabla^2\psi)_y - \psi_x\nabla^2\psi$$
$$f_2 = +\frac{1}{R}(\nabla^2\psi)_x - \psi_y \; \psi_y\nabla^2\psi$$

or

$$p = \tfrac{1}{2}(\psi_x{}^2 + \psi_y{}^2) + \psi\nabla^2\psi$$
$$f_1 = -\frac{1}{R}(\nabla^2\psi)_y - \psi(\nabla^2\psi)_x$$
$$f_2 = +\frac{1}{R}(\nabla^2\psi)_x - \psi(\nabla^2\psi)_y.$$

A paper which is notable for its careful analysis of the accuracy is that by Aubert and Deville (1983). They present a scheme which is fourth-order accurate in the velocities and third-order in the pressure. The accuracy is well demonstrated with a series of test problems.

7. Conclusions

In this review we have examined methods of solving the incompressible Navier-Stokes equations. It is hoped that the comments and observations given here will stimulate others to develop more accurate and efficient numerical methods for these important equations.

REFERENCES

R. Amit, C.A. Hall, T.A. Porsching, (1980). An application of network theory to the solution of implicit Navier-Stokes difference equations, J. Comp. Phys., 40, pp. 183-201.

X. Aubert, M. Deville (1983). Steady viscous flows by compact differences in boundary-fitted coordinates, J. Comp. Phys., 49, pp. 490-522.

A. Brandt, N. Dinar (1979). Multi-grid solutions to elliptic flow problems, Proc. of Conference on Numerical Solutions of Part. Diff. Equations, Math. Res. Center, Madison, WI, October 1978.

K. Bube, J. Strikwerda (1983). Interior regularity estimates for elliptic systems of difference equations, SIAM J. Num. Anal., 20, pp. 653-670.

A. J. Chorin (1967). A numerical method for solving incompressible viscous flow problems, J. Comp. Phys., 2, pp. 12-26.

A. J. Chorin (1968). Numerical solution of the Navier-Stokes equations, Math. Comput., 22, pp. 745-762.

A. J. Chorin (1969). On the convergence of discrete approximations to the Navier-Stokes equations, Math. Comp., 23, pp. 341-353.

M. Fortin, R. Glowinski (1982). Methodes de Lagrangian Augmente, Methodes Mathematiques de I'Informatique, vol. 9, Dunod.

L. Frank (1968) Difference operators in convolutions, Soviet Math. Dokl., 9, pp. 831-834.

K. N. Ghia, W. L. Hankey, J. K. Hodge (1977). Study of incompressible Navier-Stokes equations in primitive variables using implicit numerical technique, AIAA paper 77-648.

L. A. Hageman (1982). A family of iterative procedures for solving the semi-implicit Navier-Stokes difference equations DOE Research and Development Report,. WAPD-TM-1525.

F. H. Harlow, J. E. Welch (1964). Numerical calculation of time-dependent viscous incompressible flow of fluid with free surface, Physics of Fluids, i, pp. 2181-2189.

A. Krzivickii, O. A. Ladyzhenskaya (1966). The method of nets for the non-stationary Navier-Stokes equations, Proc. of the Steklov Inst., 92, pp. 105-112.

O. A. Ladyzhenskaya (1963). The mathematical Theory of Viscous Incompressible Flows, Translated by R. A. Silverman, Gordan and Breach, New York.

N. S. Liu, E. Krause (1979). Calculation of incompressible viscous flows in vessels with moving boundaries, Acta Mech., 33, pp. 21-32.

S. V. Patankar, D. B. Spalding (1972). A calculation procedure for heat, mass, and momentum transfer in three dimensional parabolic flows, Internat. J. Heat and Mass Trans., 15, pp. 1787-1806.

J. Patterson, J. Imberger (1980). Unsteady natural convection in a rectangular cavity, J. Fluid Mech., 100, pp. 65-86.

C. S. Peskin (1977). Numerical analysis of blood flow in the heart, J. Comp. Phys., 25, pp. 220-252.

G. D. Raithby, G. E. Schneider (1979). Numerical solution of problems in incompressible fluid flow: Treatment of the velocity-pressure coupling, Num. Heat Trans., 2, 417-440.

P. Roache (1972). Computational Fluid Dynamics, Hermosa Publ., Albuquerque, NM.

S. Rubin, J. Harris, ed. (1975). Numerical studies of incompressible viscous flow in a drive cavity, NASA SP-378.

R. Schreiber, H. B. Keller (1983a). Driven cavity flows by efficient numerical techniques, J. Comp. Phys. 49, pp. 310-333.

R. Schreiber, H. B. Keller (1983b). Spurious solutions in driven cavity calculations, J. Comp. Phys., 49, pp. 165-172.

J. C. Strikwerda (1982). An iterative method for solving the Stokes equations, to appear.

T. D. Taylor, E. Ndefo (1970). Computation of viscous flow in a channel by the method of splitting, Proc. Second Int. Conf. Numer. Methods Fluid Dynamics, p. 356-364.

R. Temam (1979). Navier-Stokes Equations, North Holland Publ. Co., Amsterdam.

J. F. Thompson (1980). Numerical solution of flow problems using body-fitted coordinate systems, Computational Fluid Dynamics, Hemisphere Publ. Corp. Washington, D.C., ed. W. Kollman.

J. A. Viecelli, (1971). A computing method for incompressible flows bounded by moving walls, J. Comp. Phys., 8, pp. 119-143.

N. N. Yanenko (1971). The Method of Fractional Steps; the Solution of Problems of Mathematical Physics in Several Variables. Translated by M. Holt, Springer-Verlag, Berlin

JCS/jgb

SECURITY CLASSIFICATION OF THIS PAGE (When Data Entered)

REPORT DOCUMENTATION PAGE

READ INSTRUCTIONS BEFORE COMPLETING FORM

1. REPORT NUMBER 2584	2. GOVT ACCESSION NO. AD-A136353	3. RECIPIENT'S CATALOG NUMBER
4. TITLE (and Subtitle) Finite Difference Methods for the Incompressible Navier-Stokes Equations - A Survey		5. TYPE OF REPORT & PERIOD COVERED Summary Report - no specific reporting period
		6. PERFORMING ORG. REPORT NUMBER
7. AUTHOR(s) John C. Strikwerda		8. CONTRACT OR GRANT NUMBER(s) DAAG29-80-C-0041
9. PERFORMING ORGANIZATION NAME AND ADDRESS Mathematics Research Center, University of Wisconsin, 610 Walnut Street, Madison, Wisconsin 53706		10. PROGRAM ELEMENT, PROJECT, TASK AREA & WORK UNIT NUMBERS Work Unit Number 3 - Numerical Analysis and Scientific Computing
11. CONTROLLING OFFICE NAME AND ADDRESS U. S. Army Research Office, P.O. Box 12211, Research Triangle Park, North Carolina 27709		12. REPORT DATE October 1983
		13. NUMBER OF PAGES 18
14. MONITORING AGENCY NAME & ADDRESS(if different from Controlling Office)		15. SECURITY CLASS. (of this report) UNCLASSIFIED
		15a. DECLASSIFICATION/DOWNGRADING SCHEDULE

16. DISTRIBUTION STATEMENT (of this Report)

Approved for public release; distribution unlimited.

17. DISTRIBUTION STATEMENT (of the abstract entered in Block 20, if different from Report)

18. SUPPLEMENTARY NOTES

19. KEY WORDS (Continue on reverse side if necessary and identify by block number)

Navier-Stokes Equations, Finite Difference Methods

20. ABSTRACT (Continue on reverse side if necessary and identify by block number)

This paper is a survey of finite difference techniques for solving the incompressible Navier-Stokes equations. We consider what features are important for schemes in order to be useful in scientific and engineering applications. In so far as possible we try to point out the strengths and weaknesses of various methods.

DD FORM 1 JAN 73 1473 EDITION OF 1 NOV 65 IS OBSOLETE

UNCLASSIFIED

SECURITY CLASSIFICATION OF THIS PAGE (When Data Entered)

LMED

8