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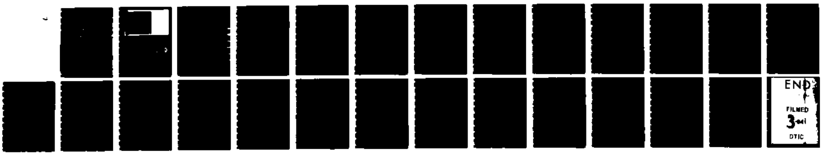

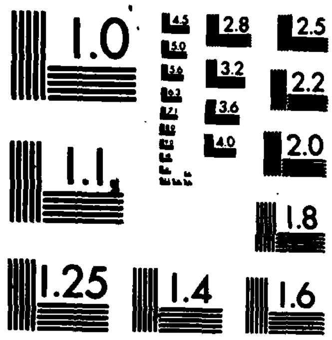

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SIMPLE COMPUTABLE BOUNDS FOR SOLUTIONS OF LINEAR COMPLEMENTARITY PROBLEMS AND LINEAR PROGRAMS

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ABSTRACT

It is shown that each feasible point of a positive semidefinite linear complementarity problem which is not a solution of the problem provides simple numerical bounds for some or all components of all solution vectors. Consequently each pair of primal-dual feasible points of a linear program which are not optimal provide simple numerical bounds for some or all components of all primal-dual solution vectors. For example each feasible point, that is $(\hat{z},\hat{w}) \geq 0$, of the linear complementarity problem $w = Mz + q \geq 0$, $z \geq 0$, $z^T w = 0$, where M is positive semidefinite, provides the following simple bound for any solution $\bar{z}$ of the linear complementarity problem:

$$\sum_{i\in I} \bar{z}_i \leq \hat{z}^T\hat{w}/\min_{i\in I} \hat{w}_i$$

where $I = \{i \mid \hat{w}_i > 0\}$. If $\hat{w} > 0$ then this inequality provides a bound on the 1-norm $\|\bar{z}\|_1$ of any solution point. Similarly each feasible point $(\hat{x},\hat{y}) \geq 0$ of the primal linear program $\min c^T x$ subject to $y = Ax - b \geq 0$, $x \geq 0$, and each feasible point $(\hat{u},\hat{v}) \geq 0$ of the dual linear program $\max b^T u$ subject to $v = -A^T u + c \geq 0$, $u \geq 0$, provide the following simple bounds for any primal optimal solution $(\bar{x},\bar{y})$ and any dual optimal solution $(\bar{u},\bar{v})$:

$$\sum_{i\in J} \bar{x}_i \leq (c^T\hat{x} - b^T\hat{u})/\min_{i\in J} \hat{v}_i, \quad \sum_{i\in I} \bar{u}_i \leq (c^T\hat{x} - b^T\hat{u})/\min_{i\in I} \hat{y}_i$$

where $J = \{i \mid \hat{v}_i > 0\}$ and $I = \{i \mid \hat{y}_i > 0\}$. If $\hat{v} > 0$ we have a bound on $\|\bar{x}\|_1$, and if $\hat{y} > 0$ we have a bound on $\|\bar{u}\|_1$. In addition we show that the existence of such numerical bounds is not only sufficient but is also necessary for the boundedness of solution vector components for both the linear complementarity problem and the dual linear programs.

AMS (MOS) Subject Classifications: 90C05, 90C20
Key Words: Linear programming, quadratic programming, bounds
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SIGNIFICANCE AND EXPLANATION

Surprisingly simple bounds are given for solutions of fundamental constrained optimization problems such as linear and convex quadratic programs. It is shown that every nonoptimal primal-dual feasible point carries within it simple numerical information which bounds some or all components of all solution vectors. The results given permit one to compute bounds without even solving the optimization problems.

The responsibility for the wording and views expressed in this descriptive summary lies with MRC, and not with the author of this report.

SIMPLE COMPUTABLE BOUNDS FOR SOLUTIONS OF LINEAR COMPLEMENTARITY PROBLEMS AND LINEAR PROGRAMS

O. L. Mangasarian

1. Introduction

The linear complementarity problem of finding a (z,w) in the 2k-dimensional real Euclidean space R^{2k} such that

$$w = Mz + q \geq 0, \; z \geq 0, \; z^T w = 0 \tag{1.1}$$

where M is a given $k \times k$ real matrix, q is a given $k \times 1$ real vector and $z^T w$ denotes the scalar product $\sum_{i=1}^{k} z_i w_i$, is a fundamental problem of mathematical programming which includes linear and quadratic programming problems, bimatrix games [1] and free boundary problems [2]. An important question of both theoretical and practical interest is the boundedness of the solution set of (1.1) which already has received attention in [8,3,6] in the form of necessary and/or sufficient conditions for this boundedness. In this work we provide simple numerical bounds for some or all components of any solution vector when M is positive semidefinite. In particular we show that each feasible point $(\hat{z},\hat{w})$, that is $(\hat{z},\hat{w}) \geq 0$, which is not a solution of (1.1), contains information on the magnitude of some or all components of all solution points. For example Theorem 2.2 provides the following simple bounds for any solution $(\bar{z},\bar{w})$ of (1.1) in terms of any feasible point $(\hat{z},\hat{w})$ when M is positive semidefinite

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$$(1.2) \qquad \left\langle \begin{aligned} \|\bar{z}_I\|_1 &\leq \hat{z}^T\hat{w} / \min \hat{w}_{i\in I} \\ \|\bar{w}_J\|_1 &\leq \hat{z}^T\hat{w} / \min \hat{z}_{i\in J} \\ \|\bar{z}_I, \bar{w}_J\|_1 &\leq \hat{z}^T\hat{w} / \min \{\hat{w}_{i\in I}, \hat{z}_{j\in J}\} \end{aligned} \right.$$

where $I = \{i \mid \hat{w}_i > 0\}$, $J = \{i \mid \hat{z}_i > 0\}$, $\bar{z}_I := \bar{z}_{i\in J}$ and $\|\cdot\|_1$ denotes the 1-norm. Note that if either I or J is empty then $(\hat{z},\hat{w})$ is a solution of the linear complementarity problem (1.1). On the other hand if $\hat{w} > 0$, then $\bar{z}_I = \bar{z}$ and (1.2) provides a bound on the 1-norm $\|\bar{z}\|_1$ of any solution $(\bar{z},\bar{w})$ of (1.1). Similarly if $\hat{z} > 0$, then $\bar{w}_J = \bar{w}$ and (1.2) provides a bound on $\|\bar{w}\|_1$. Theorem 2.2 also provides a necessary and sufficient characterization for the boundedness of $\bar{z}_I$, $\bar{w}_J$ and $(\bar{z}_I, \bar{w}_J)$ for $I, J \subset \{1,\ldots,k\}$ where $(\bar{z},\bar{w})$ is any solution of (1.1) and M is positive semidefinite. In particular it shows that $\bar{z}_I$ is bounded if and only if there exists a feasible point $(\hat{z},\hat{w}) \geq 0$ such that $\hat{w}_I > 0$; $\bar{w}_J$ is bounded if and only if there exists a feasible point $(\hat{z},\hat{w}) \geq 0$ such that $\hat{z}_J > 0$; and $(\bar{z}_I, \bar{w}_J)$ is bounded if and only if there exists a feasible point $(\hat{z},\hat{w}) \geq 0$ such that $(\hat{z}_J, \hat{w}_I) > 0$. Theorem 2.2 can be used, as in Algorithm 2.6, to determine which components if any of the solution set are bounded, without solving the linear complementarity problem (1.1). Theorem 2.2 also provides necessary conditions for the boundedness of solution components of (1.1) when M is copositive plus, that is M satisfying (1.5)-(1.6) below. In Theorem 2.8 we give bounds for the unique solution of the positive definite linear complementarity problem.

Because a linear programming problem is a special case of the linear complementarity problem [1], the bounds of Section 2 can be used to obtain bounds for solutions of the dual linear programs

$$(1.3a)\qquad \min\ c^Tx \quad \text{s.t.}\ \ y = Ax - b \geq 0,\ x \geq 0$$

$$(1.3b)\qquad \max\ b^Tu \quad \text{s.t.}\ \ v = -A^Tu + c \geq 0,\ u \geq 0$$

where A is an $m \times n$ real matrix, c and b are $n \times 1$ and $m \times 1$ real vectors respectively. In [7] Robinson and in [5] this author both gave bounds for solutions of linear programs which involved a constant which was difficult to evaluate in general. By contrast in Section 3 we provide bounds for solutions of (1.3) which involve no constants or parameters. For example Theorem 3.1 provides the following simple bounds for any solution $(\bar{x},\bar{y}) - (\bar{u},\bar{v})$ of the dual linear programs (1.3) in terms of any pair $(\hat{x},\hat{y}) - (\hat{u},\hat{v})$ of primal-dual feasible points:

$$(1.4)\qquad \left\{\begin{aligned}
\|\bar{x}_{J_1}\|_1 &\leq (c^T\hat{x} - b^T\hat{u})/\min\ \hat{v}_{i\in J_1} \\
\|\bar{y}_{I_1}\|_1 &\leq (c^T\hat{x} - b^T\hat{u})/\min\ \hat{u}_{i\in I_1} \\
\|\bar{x}_{J_1}, \bar{y}_{I_1}\|_1 &\leq (c^T\hat{x} - b^T\hat{u})/\min\ \{\hat{v}_{j\in J_1}, \hat{u}_{i\in I_1}\} \\
\|\bar{u}_{I_2}\|_1 &\leq (c^T\hat{x} - b^T\hat{u})/\min\ \hat{y}_{i\in I_2} \\
\|\bar{v}_{J_2}\|_1 &\leq (c^T\hat{x} - b^T\hat{u})/\min\ \hat{x}_{i\in J_2} \\
\|\bar{u}_{I_2}, \bar{v}_{J_2}\|_1 &\leq (c^T\hat{x} - b^T\hat{u})/\min\ \{\hat{y}_{i\in I_2}, \hat{x}_{j\in J_2}\}
\end{aligned}\right.$$

where $J_1 = \{i \mid \hat{v}_i > 0\}$, $J_2 = \{i \mid \hat{x}_i > 0\}$, $I_1 = \{i \mid \hat{u}_i > 0\}$ and $I_2 = \{i \mid \hat{y}_i > 0\}$. The bounds (1.4) show that every pair of primal-dual feasible points which are not optimal (that is at least $\hat{y} \neq 0$ or $\hat{v} \neq 0$) can provide <u>some</u>

information on the size of the solution set of the dual linear programs (1.3). Note that if either $I_1 \cup J_2$ or $J_1 \cup I_2$ is empty then $(\hat{x},\hat{y}) - (\hat{u},\hat{v})$ is optimal. If on the other hand $\hat{v} > 0$, then $\bar{x}_{J_1} = \bar{x}$ and (1.4) provides a bound on the 1-norm $\|\bar{x}\|_1$ of any solution $(\bar{x},\bar{y})$ of (1.3a). Similarly if $\hat{y} > 0$, then $\bar{u}_{I_2} = \bar{u}$ and (1.4) provides a bound on the 1-norm $\|\bar{u}\|_1$ of any solution $(\bar{u},\bar{v})$ of (1.3b). In Theorem 3.4 we consider a nonsymmetric dual linear programming pair and provide numerical bounds for its solution set.

We describe briefly now our notation. All vectors will be column vectors unless transposed to a row vector by a superscript T. For a vector x in the n-dimensional Euclidean space R^n, $\|x\|$ will denote an arbitrary but fixed norm and $\|x\|_p$ will denote the p-norm $\|x\|_p := (\sum_{i=1}^{n} |x_i|^p)^{\frac{1}{p}}$ where $1 \leq p < \infty$ and $\|x\|_\infty := \max_{1\leq i\leq n} |x_i|$. For an $m \times n$ real matrix A, A_i denotes the ith row and $A_{\cdot j}$ denotes the jth column, while $\|A\|_p$ denotes the matrix norm subordinate to the vector norm $\|\cdot\|_p$, that is $\|A\|_p = \max_{\|x\|_p=1} \|Ax\|_p$. The consistency condition $\|Ax\|_p \leq \|A\|_p \|x\|_p$ follows immediately from this definition of a matrix norm. For a subset $J \subset \{1,\ldots,n\}$, x_J or $x_{i\epsilon J}$, will denote those components x_i of the vector x in R^n such that $i \epsilon J$. Similarly for $I \subset \{1,\ldots,m\}$, A_I will denote those rows A_i of A such that $i \epsilon I$, while $A_{\cdot J}$ will denote those columns $A_{\cdot j}$ of A such that $j \epsilon J$. A vector of ones in any real finite dimensional Euclidean space will be denoted by e. A $k \times k$ real (not necessarily symmetric) matrix M is said to be copositive [1] if

$$(1.5) \qquad z \geq 0 \Rightarrow z^T M z \geq 0$$

M is said to be copositive plus [1] if it is copositive and

(1.6) $$z \geq 0,\ z^T M z = 0 \Rightarrow (M + M^T) z = 0$$

A $k \times k$ real (not necessarily symmetric) matrix M is said to be positive semidefinite (definite) if

$$z^T M z \geq 0\ (>0) \quad \text{for all} \quad z \neq 0$$

Note that a positive definite matrix is also positive semidefnite, while a positive semidefnite matrix is also a copositive plus matrix.

2. Bounds for Solutions of Positive Semidefinite Linear Complementarity Problems

We begin by a simple but useful identity.

2.1 Lemma Let M be a $k \times k$ real matrix and let q be a $k \times 1$ real vector. Then for any z and $\bar{z}$ in R^k such that $\bar{z}^T(M\bar{z}+q) = 0$ it follows that

$$(2.1) \qquad z^T(Mz+q) = \bar{z}^T(Mz+q) + z^T(M\bar{z}+q) + (z-\bar{z})^T M(z-\bar{z})$$

Proof By direct algebraic verification. □

Before establishing the principal result of this section, we need to define some sets. Let I and J be subsets of $\{1,2,\ldots,k\}$. Define

$$(2.2) \qquad \left\langle \begin{array}{l} S := \{(z,w) \mid z \geq 0,\ w = Mz+q \geq 0\} \\ \bar{S} := \{(z,w) \mid (z,w) \in S,\ z^T w = 0\} \\ S_{IJ} := \{(z,w) \mid (z,w) \in S,\ (z_I, w_J) > 0\} \\ \bar{S}_{IJ} := \{(z_I, w_J) \mid (z,w) \in \bar{S}\} \\ Z_I := \{(z,w) \mid (z,w) \in S,\ z_I > 0\} \\ \bar{Z}_I := \{z_I \mid (z,w) \in \bar{S}\},\ \bar{Z} := \{z \mid (z,w) \in \bar{S}\} \\ W_I := \{(z,w) \mid (z,w) \in S,\ w_I > 0\} \\ \bar{W}_I := \{w_I \mid (z,w) \in \bar{S}\},\ \bar{W} := \{w \mid (z,w) \in \bar{S}\} \end{array} \right.$$

With these definitions it is possible to characterize the boundedness of solutions of linear complementarity problems in terms of simple numerical bounds as follows.

2.2 <u>Theorem</u> Let M be a $k \times k$ copositive plus matrix, let $S \neq \phi$ and let I and J be subsets of $\{1,2,\ldots,k\}$. Then

(a) $W_I \neq \phi \Leftarrow \bar{\bar{Z}}_I$ bounded

(b) $Z_I \neq \phi \Leftarrow \bar{W}_I$ bounded

(c) $S_{JI} \neq \phi \Leftarrow \bar{\bar{S}}_{IJ}$ bounded

If in addition M is positive semidefinite then

(a′) (i) $W_I \neq \phi \Leftrightarrow$ (ii) $\bar{\bar{Z}}_I$ bounded $\Leftrightarrow$ (iii) $\left\langle \begin{array}{l} W_I \neq \phi \text{ and} \\ \|\bar{z}_I\|_1 \leq \hat{z}^T\hat{w}/\min \hat{w}_{i\in I} \\ \forall \bar{z}_I \in \bar{\bar{Z}}_I,\ \forall(\hat{z},\hat{w}) \in W_I \end{array} \right.$

(b′) (i) $Z_I \neq \phi \Leftrightarrow$ (ii) $\bar{W}_I$ bounded $\Leftrightarrow$ (iii) $\left\langle \begin{array}{l} Z_I \neq \phi \text{ and} \\ \|\bar{w}_I\|_1 \leq \hat{z}^T\hat{w}/\min \hat{z}_{i\in I} \\ \forall \bar{w}_I \in \bar{W}_I,\ \forall(\hat{z},\hat{w}) \in Z_I \end{array} \right.$

(c′) (i) $S_{JI} \neq \phi \Leftrightarrow$ (ii) $\bar{\bar{S}}_{IJ}$ bounded $\Leftrightarrow$ (iii) $\left\langle \begin{array}{l} S_{JI} \neq \phi \text{ and} \\ \left\| \begin{matrix} \bar{z}_I \\ \bar{w}_J \end{matrix} \right\|_1 \leq \hat{z}^T\hat{w}/\min\{\hat{z}_{i\in J}, \hat{w}_{i\in I}\} \\ \forall(\bar{z}_I, \bar{w}_J) \in \bar{\bar{S}}_{IJ},\ \forall(\hat{z},\hat{w}) \in S_{JI} \end{array} \right.$

<u>Proof</u> First by Lemke's algorithm [1], it follows that $\bar{\bar{S}} \neq \phi$ since $S \neq \phi$.

(a) We shall prove the contrapositive implication.

$$W_I = \phi \Longleftrightarrow Mz + q \geq 0,\ z \geq 0,\ M_I z + q_I > 0 \text{ has no solution}$$

$$\Longleftrightarrow M^T u \leq 0,\ u \geq 0,\ 0 \neq \begin{pmatrix} u_I \\ -q^T u \end{pmatrix} \geq 0 \text{ has solution}$$

(By Motzkin's theorem of the alternative [4])

$$\Longleftrightarrow M^T u \leq 0,\ u \geq 0,\ q^T u < 0 \text{ has solution, or}$$

$$M^T u \leq 0,\ u \geq 0,\ q^T u = 0,\ 0 \neq u_I \geq 0 \text{ has solution}$$

$$\Longleftrightarrow M^T u \leq 0,\ u \geq 0,\ q^T u = 0,\ 0 \neq u_I \geq 0 \text{ has solution}$$

($q^T u < 0$ alternative excluded by $S \neq \phi$)

$$\Longleftrightarrow u^T M u = 0,\ M^T u \leq 0,\ u \geq 0,\ q^T u = 0,\ 0 \neq u_I \geq 0 \text{ has solution}$$

(By copositivity of M)

$$\Longleftrightarrow Mu = -M^T u \geq 0,\ u \geq 0,\ q^T u = 0,\ 0 \neq u_I \geq 0 \text{ has solution}$$

(By copositivity plus of M)

$$\Rightarrow \bar{z} + \lambda u \in \bar{Z} \text{ for any } (\bar{z},\bar{w}) \in \bar{S}, \text{ any } \lambda > 0 \text{ and } u \geq 0,$$

$$Mu = -M^T u \geq 0,\ q^T u = 0,\ 0 \neq u_I \geq 0$$

$$\Rightarrow \bar{Z}_I \text{ unbounded.}$$

(b) We again prove the contrapositive implication.

$$Z_I = \phi \Longleftrightarrow Mz + q \geq 0,\ z \geq 0,\ z_I > 0 \text{ has no solution}$$

$$\Longleftrightarrow M^T u \leq 0,\ u \geq 0,\ 0 \neq \begin{pmatrix} (M^T u)_I \\ q^T u \end{pmatrix} \leq 0 \text{ has solution}$$

(By Motzkin's theorem)

$$\Longleftrightarrow M^T u \leq 0,\ u \geq 0,\ q^T u = 0,\ 0 \neq (M^T u)_I \leq 0 \text{ has solution}$$

(Alternative $q^T u < 0$ is excluded by $S \neq \phi$)

$$\Longleftrightarrow Mu = -M^T u \geq 0,\ u \geq 0,\ q^T u = 0,\ 0 \neq (Mu)_I = -(M^T u)_I \geq 0 \text{ has solution}$$

(By copositivity plus of M)

$$\Rightarrow \bar{z} + \lambda u \in \bar{Z} \text{ for any } (\bar{z},\bar{w}) \in \bar{S}, \text{ any } \lambda > 0 \text{ and } u \geq 0,$$

$$Mu = -M^T u \geq 0,\ q^T u = 0,\ 0 \neq (Mu)_I \geq 0$$

$$\Rightarrow \bar{W}_I \text{ unbounded.}$$

(c) $\bar{S}_{IJ}$ bounded implies $\bar{Z}_I$ bounded and $\bar{W}_J$ bounded. By (a) above it follows that $W_I \neq \phi$, and by (b) above it follows that $Z_J \neq \phi$. Let $(\hat{z},\hat{w}) \in W_I$ and let $(\tilde{z},\tilde{w}) \in Z_J$. Then

$$\left(\frac{\hat{z}+\tilde{z}}{2}, \frac{\hat{w}+\tilde{w}}{2}\right) \in Z_J \cap W_I = S_{JI}$$

(a′) The implication (i)⇐(ii) follows from (a) above. The implication (ii)⇐(iii) is evident. We now establish the implication (i)⇒(iii) by means of Lemma 2.1. Let $(\hat{z},\hat{w}) \in W_I$ and $\bar{z}_I \in \bar{Z}_I$. Then by Lemma 2.1 and the positive semidefiniteness of M we have

$$\hat{z}^T\hat{w} = \hat{z}^T(M\hat{z}+q) \geq \bar{z}^T(M\hat{z}+q) + \hat{z}^T(M\bar{z}+q) \geq \bar{z}_I^T(M\hat{z}+q)_I$$

$$\geq \|\bar{z}_I\|_1 \min \hat{w}_{i\in I}$$

Hence

$$\|\bar{z}_I\|_1 \leq \hat{z}^T\hat{w}/\min \hat{w}_{i\in I}$$

(b′) The implication (i)⇐(ii) follows from (b) above. The implication (ii)⇐(iii) is evident. We now establish (i)⇒(iii). Let $(\hat{z},\hat{w}) \in Z_I$ and let $\bar{w}_I \in \bar{W}_I$. By Lemma 2.1 and the positive semidefiniteness of M we have

$$\hat{z}^T\hat{w} \geq \bar{z}^T\hat{w} + \hat{z}^T\bar{w} \geq \hat{z}_I^T\bar{w}_I \geq \|\bar{w}_I\|_1 \min \hat{z}_{i\in I}$$

Hence

$$\|\bar{w}_I\|_1 \leq \hat{z}^T\hat{w}/\min \hat{z}_{i\in I}$$

(c′) Again the implication (i)⇐(ii) follows from (c) above. The implication (ii)⇐(iii) is evident. To establish (i)⇒(iii), let $(\hat{z},\hat{w}) \in S_{JI}$

and let $(\bar{z}_I, \bar{w}_J) \in \bar{S}_{IJ}$. Then by Lemma 2.1 and the positive semidefiniteness of M we have

$$\hat{z}^T\hat{w} \geq \bar{z}^T\hat{w} + \hat{z}^T\bar{w} \geq \bar{z}_I^T\hat{w}_I + \hat{z}_J^T\bar{w}_J \geq \left\|\begin{matrix}\bar{z}_I \\ \bar{w}_J\end{matrix}\right\|_1 \min \{\hat{z}_{i\in J}, \hat{w}_{i\in I}\}$$

Hence

$$\left\|\begin{matrix}\bar{z}_I \\ \bar{w}_J\end{matrix}\right\|_1 \leq \hat{z}^T\hat{w}/\min \{\hat{z}_{i\in J}, \hat{w}_{i\in I}\} \qquad \square$$

2.3 <u>Remark</u> The sets I and J of Theorem 2.2 above may be taken as singletons in which case the bounds in (a′), (b′) and (c′) simplify respectively to

$$\bar{z}_i \leq \hat{z}^T\hat{w}/\hat{w}_i \quad \text{for} \quad \bar{z}_i \in \bar{Z}_i, \ (\hat{z}_i, \hat{w}_i) \in W_i$$

$$\bar{w}_i \leq \hat{z}^T\hat{w}/\hat{z}_i \quad \text{for} \quad \bar{w}_i \in \bar{W}_i, \ (\hat{z}_i, \hat{w}_i) \in Z_i$$

$$\bar{z}_i + \bar{w}_j \leq \hat{z}^T\hat{w}/\min \{\hat{z}_j, \hat{w}_i\} \quad \text{for} \quad (\bar{z}_i, \bar{w}_j) \in \bar{S}_{ij}, \ (\hat{z}, \hat{w}) \in S_{ji}$$

2.4 <u>Remark</u> The positive semidefiniteness assumption plays an indispensible role in obtaining the numerical bounds of parts (a′), (b′) and (c′) of Theorem 2.2. It is unlikely that such numerical bounds can be obtained for the copositive plus case. Whether the forward implications of parts (a), (b) and (c) of Theorem 2.2 also hold under a copositive plus assumption is an open question. However when $I = \{1,2,\ldots,k\}$, the forward assumption of (a) does hold for a copositive plus M. See Theorem 2, (ii)$\Leftrightarrow$(ix) [6].

The following corollary which is a direct consequence of part (a′) of Theorem 2.2 provides a practical method for determining which components of the solution set are bounded and which are not without solving the linear complementarity problem (1.1).

<u>2.5 Corollary</u> Let M be a $k \times k$ positive semidefinite matrix and let $S \neq \phi$. There exists a partition $I \cup L$ of $\{1,2,\ldots,k\}$ such that

(2.3) $\bar{Z}_I$ is bounded, $\bar{Z}_L$ is unbounded

or equivalently such that

(2.4) $W_I \neq \phi,\ W_L = \phi$

One way to determine the partition $I \cup L$ of the above corollary for a given linear complementarity problem is to solve at most $N(I)$ linear programs, where $N(I)$ is the number of elements in I, as in the following Algorithm 2.6. This algorithm determines the partition $I \cup L$ of $\{1,2,\ldots,k\}$ for a positive semidefinite linear complementarity problem (1.1) such that $\bar{Z}_I$ is bounded and $\bar{Z}_L$ is unbounded, by determining W_I such that $W_I \neq \phi$ and W_L such that $W_L = \phi$. The algorithm which does <u>not</u> solve the linear complementarity problem, solves at most $N(I)$ (but potentially considerably fewer) linear programs.

2.6 <u>Algorithm</u> (Determination of $I \cup L = \{1,2,\ldots,k\}$ such that $\bar{Z}_I$ is bounded, $\bar{Z}_L$ is unbounded, for a positive semidefinite M)

Step 0: Set $j = 0$, $I_0 = \phi$, $L_0 = \{1,2,\ldots,k\}$

Step 1: Solve the LP: $\max \sum_{j \in L_j} (Mz+q)_j$ s.t. $Mz + q \geq 0$, $z \geq 0$

If LP is infeasible, LCP (1.1) is infeasible. Stop.

If LP max = 0, set $I = I_j$, $L = \{1,2,\ldots,k\}\backslash I_j$. Stop.

If $0 < \text{LP max} < \infty$, set $z(\lambda) = \bar{z}$ where $\bar{z}$ is an LP solution.

If LP max $\to \infty$, set $z(\lambda) = \bar{z} + \lambda\bar{d}$ where $\bar{z} + \lambda\bar{d}$ is feasible for all $\lambda > 0$ and $\sum_{j \in L_j} M_j\bar{d} > 0$.

Set $I_{j+1} = I_j \cup \{i \mid M_i z(\lambda) + q_i > 0,\ \lambda \to \infty\}$

$L_{j+1} = \{1,2,\ldots,k\}\backslash I_{j+1}$

Step 2: $j + 1 \rightarrow j$

Step 3: Go to Step 1.

2.7 Remark The LP solutions of Algorithm 2.6 can be used in conjunction with Theorem 2.2 (a'iii) to give numerical bounds for $\|\bar{z}_I\|_1$, $\bar{z}_I \in \bar{Z}_I$.

When M is positive definite, additional simple bounds can be obtained as follows.

2.8 Theorem Let M be a $k \times k$ positive definite matrix with $\alpha > 0$ being the smallest eigenvalue of $\frac{M+M^T}{2}$ and $\beta > 0$ the smallest eigenvalue of $\frac{M^{-1}+(M^{-1})^T}{2}$. Then the unique solution $(\bar{z},\bar{w})$ of the linear complementarity (1.1) is bounded by

$$\text{(2.5a)} \qquad \max\{0, \|\hat{z}\|_2 - (\hat{z}^T\hat{w}/\alpha)^{\frac{1}{2}}\} \leq \|\bar{z}\|_2 \leq \|\hat{z}\|_2 + (\hat{z}^T\hat{w}/\alpha)^{\frac{1}{2}}$$

$$\text{(2.5b)} \qquad \max\{0, \|\hat{w}\|_2 - (\hat{w}^T\hat{z}/\beta)^{\frac{1}{2}}\} \leq \|\bar{w}\|_2 \leq \|\hat{w}\|_2 + (\hat{w}^T\hat{z}/\beta)^{\frac{1}{2}}$$

for any feasible $\hat{z} \geq 0$, $\hat{w} = M\hat{z} + q \geq 0$.

Proof By Lemma 2.1 we have that

$$\hat{z}^T\hat{w} \geq (\hat{z}-\bar{z})^T M(\hat{z}-\bar{z}) \geq \alpha\|\hat{z}-\bar{z}\|_2^2$$

Hence

$$\|\bar{z}\|_2 \leq \|\hat{z}\|_2 + \|\hat{z}-\bar{z}\|_2 \leq \|\hat{z}\|_2 + (\hat{z}^T\hat{w}/\alpha)^{\frac{1}{2}}$$

which gives the second inequality of (2.5a). The first inequality of (2.5a) follows from

$$\|\hat{z}\|_2 \leq \|\bar{z}\|_2 + \|\hat{z}-\bar{z}\|_2 \leq \|\bar{z}\|_2 + (\hat{z}^T\hat{w}/\alpha)^{\frac{1}{2}}$$

To obtain (2.5b) note first that M is nonsingular, for if it were singular then $Mz = 0$ for some $z \neq 0$ and consequently $z^TMz = 0$ which contradicts the positive definiteness of M. Furthermore for $0 \neq y \in R^k$, we have $z = M^{-1}y \neq 0$ and

$$y^TM^{-1}y = z^TM^TM^{-1}Mz = z^TM^Tz = zMz > 0$$

Hence M^{-1} is positive definite. For a nonsingular M the linear complementarity problem (1.1) is equivalent to

$$z = M^{-1}w - M^{-1}q,\ w \geq 0,\ w^Tz = 0 \tag{2.6}$$

Hence (2.5a) of this theorem applied to (2.6) yields (2.5b). $\square$

3. Bounds for Solutions of Linear Programs

We begin this section with some results which are direct consequences of Section 2. These results follow by considering the pair of dual linear programs (1.3) as a linear complementarity problem with a skew-symmetric, and hence, positive semidefinite matrix. Later on in this section we shall obtain bounds for solutions of linear programs with explicit equality constraints.

By considering the dual linear programs (1.3) as a linear complementarity problem [1] defined by (1.1) and

$$(3.1)\qquad M:=\begin{pmatrix} 0 & -A^T \\ A & 0 \end{pmatrix},\quad q:=\begin{pmatrix} c \\ -b \end{pmatrix},\quad z:=\begin{pmatrix} x \\ u \end{pmatrix},\quad w:=\begin{pmatrix} v \\ y \end{pmatrix}$$

the following theorem is a direct consequence of Theorem 2.2.

3.1 Theorem Assume that the dual linear programs (1.3) are both feasible and hence both solvable. Let caret variables $(\hat{x},\hat{y})$, $(\hat{u},\hat{v})$ denote primal and dual feasible vectors respectively, that is

$$(3.2)\qquad \hat{y} = A\hat{x} - b \geq 0,\ \hat{x} \geq 0,\ \hat{v} = -A^T\hat{u} + c \geq 0,\ \hat{u} \geq 0$$

and let bar variables $(\bar{x},\bar{y})$, $(\bar{u},\bar{v})$ denote primal and dual optimal vectors respectively, that is

$$(3.3)\qquad \bar{y} = A\bar{x} - b \geq 0,\ \bar{x} \geq 0,\ \bar{v} = -A^T\bar{u} + c \geq 0,\ \bar{u} \geq 0,\ c^T\bar{x} - b^T\bar{u} = 0$$

Let $J \subset \{1,2,\ldots,n\}$ and let $I \subset \{1,2,\ldots,m\}$. Then the following equivalences hold:

$$\text{(a1)}\quad \exists \hat{v}_J > 0 \Longleftrightarrow \forall \bar{x}_J \text{ bounded} \Longleftrightarrow \left\langle \begin{array}{l} \exists \hat{v}_J > 0;\ \forall(\bar{x},\hat{x}),\ \forall \hat{u} \ \text{ s.t. } \hat{v}_J > 0: \\ \qquad\qquad \|\bar{x}_J\|_1 \le (c^T\hat{x} - b^T\hat{u})/\min \hat{v}_{j\in J} \end{array}\right.$$

$$\text{(a2)}\quad \exists \hat{y}_I > 0 \Longleftrightarrow \forall \bar{u}_I \text{ bounded} \Longleftrightarrow \left\langle \begin{array}{l} \exists \hat{y}_I > 0;\ \forall(\bar{u},\hat{u}),\ \forall \hat{x} \ \text{ s.t. } \hat{y}_I > 0: \\ \qquad\qquad \|\bar{u}_I\|_1 \le (c^T\hat{x} - b^T\hat{u})/\min \hat{y}_{i\in I} \end{array}\right.$$

$$\text{(b1)}\quad \exists \hat{x}_J > 0 \Longleftrightarrow \forall \bar{v}_J \text{ bounded} \Longleftrightarrow \left\langle \begin{array}{l} \exists \hat{x}_J > 0;\ \forall(\bar{v},\hat{u}),\ \forall \hat{x} \ \text{ s.t. } \hat{x}_J > 0: \\ \qquad\qquad \|\bar{v}_J\|_1 \le (c^T\hat{x} - b^T\hat{u})/\min \hat{x}_{j\in J} \end{array}\right.$$

$$\text{(b2)}\quad \exists \hat{u}_I > 0 \Longleftrightarrow \forall \bar{y}_I \text{ bounded} \Longleftrightarrow \left\langle \begin{array}{l} \exists \hat{u}_I > 0;\ \forall(\bar{y},\hat{x}),\ \forall \hat{u} \ \text{ s.t. } \hat{u}_I > 0: \\ \qquad\qquad \|\bar{y}_I\|_1 \le (c^T\hat{x} - b^T\hat{u})/\min \hat{u}_{i\in I} \end{array}\right.$$

$$\text{(c1)}\quad \exists \hat{v}_J > 0, \hat{u}_I > 0 \Longleftrightarrow \forall(\bar{x}_J,\bar{y}_I) \text{ bounded} \Longleftrightarrow \left\langle \begin{array}{l} \exists \hat{v}_J>0,\ \hat{u}_I>0;\ \forall(\bar{x},\hat{x}),\ \forall \hat{u} \text{ s.t. } \hat{v}_J>0,\ \hat{u}_I>0: \\ \qquad \left\| \begin{matrix} \bar{x}_J \\ \bar{y}_I \end{matrix} \right\|_1 \le (c^T\hat{x} - b^T\hat{u})/\min \{\hat{v}_{j\in J}, \hat{u}_{i\in I}\} \end{array}\right.$$

$$\text{(c2)}\quad \exists \hat{y}_I > 0, \hat{x}_J > 0 \Longleftrightarrow \forall(\bar{u}_I,\bar{v}_J) \text{ bounded} \Longleftrightarrow \left\langle \begin{array}{l} \exists \hat{y}_I>0,\ \hat{x}_J>0;\ \forall(\bar{u},\hat{u}),\ \forall \hat{x} \text{ s.t. } \hat{y}_I>0,\ \hat{x}_J>0: \\ \qquad \left\| \begin{matrix} \bar{u}_I \\ \bar{v}_J \end{matrix} \right\|_1 \le (c^T\hat{x} - b^T\hat{u})/\min \{\hat{y}_{i\in I}, \hat{x}_{j\in J}\} \end{array}\right.$$

3.2 <u>Corollary</u> The quantity $c^T\hat{x}$ in parts (a1), (b2) and (c1) of Theorem 3.1 can be replaced by any upper bound α to $\min\ c^Tx$ s.t. $Ax \ge b$, $x \ge 0$, while the quantity $b^T\hat{u}$ in parts (a2), (b1) and (c2) of Theorem 3.1 can be replaced by any lower bound β to $\max\ b^Tu$ s.t. $A^Tu \le c$, $u \ge 0$.

<u>Proof</u> To prove the first part, set $\hat{x}$ in (a1), (b2) and (c1) equal to a solution $\bar{x}$ of (1.3a) and note that $c^T\hat{x} = c^T\bar{x} \le \alpha$. To prove the second part, set $\hat{u}$ in (a2), (b1) and (c2) equal to a solution $\bar{u}$ of (1.3b) and note that $-b^T\hat{u} = -b^T\bar{u} \le -\beta$. $\square$

3.3 Remark When the index sets I, J are taken as singletons, the first equivalence in each of the statements (a1) to (c2) of Theorem 3.1 reduce to Theorem 3b of Williams [11]. In [10] Williams characterizes boundedness of components of feasible, but not optimal, points of linear constraint sets. In [9] Williams characterizes the boundedness of the totality of all the components (in contrast with individual components) of optimal points of linear programs. None of Williams' characterizations contain quantitative bounds like ours.

We turn our attention now to the nonsymmetric pair of dual linear programs

(3.4a) $$\min\ c^T x \quad \text{s.t.}\ Ax = b,\ x \geq 0$$

(3.4b) $$\max\ b^T u \quad \text{s.t.}\ v = -A^T u + c \geq 0$$

and establish the following bounds for their solutions.

3.4 Theorem Assume that the dual linear programs (3.4) are both feasible and hence both solvable. Let caret variables denote primal and dual feasible vectors, that is

(3.5) $$A\hat{x} = b,\ \hat{x} \geq 0,\ \hat{v} = -A^T\hat{u} + c \geq 0$$

and let bar variables denote primal and dual optimal vectors, that is

(3.6) $$A\bar{x} = b,\ \bar{x} \geq 0,\ \bar{v} = -A^T\bar{u} + c \geq 0,\ c^T\bar{x} - b^T\bar{u} = 0$$

Let $J \subset \{1,2,\ldots,n\}$ and $I \subset \{1,2,\ldots,m\}$. Then the following equivalences hold:

(a1) (i) $\exists \hat{v}_J > 0 \Longleftrightarrow$ (ii) $\forall \bar{x}_J$ bounded $\Longleftrightarrow$ (iii) $\left\langle \begin{array}{l} \exists \hat{v}_J > 0;\ \forall(\bar{x},\hat{x}),\ \forall \hat{u} \text{ s.t. } \hat{v}_J > 0: \\ \|\bar{x}_J\|_1 \leq (c^T\hat{x} - b^T\hat{u})/\min \hat{v}_{j\in J} \end{array} \right.$

(a2) (i) $\exists \hat{x} > 0$ and rows of A lin. indep. $\Longleftrightarrow$ (ii) $\forall \bar{u}$ bounded

$\Longleftrightarrow$ (iii) $\left\langle \begin{array}{l} \exists \hat{x} > 0 \text{ and rows of } A \text{ lin. indep; } \forall(\bar{u},\hat{u}),\ \forall \hat{x} > 0: \\ \|\bar{u}\|_1 \leq \|(AA^T)^{-1}A\|_1 (\|c\|_1 + (c^T\hat{x} - b^T\hat{u})/\min\limits_{1\leq i\leq n} \hat{x}_i) \end{array} \right.$

(b1) (i) $\exists \hat{x}_J > 0 \Longleftrightarrow$ (ii) $\forall \bar{v}_J$ bounded $\Longleftrightarrow$ (iii) $\left\langle \begin{array}{l} \exists \hat{x}_J > 0;\ \forall(\bar{v},\hat{u}),\ \forall \hat{x} \text{ s.t. } \hat{x}_J > 0: \\ \|\bar{v}_J\|_1 \leq (c^T\hat{x} - b^T\hat{u})/\min \hat{x}_{j\in J} \end{array} \right.$

<u>Proof</u>. (a1): (ii) $\Leftarrow$ (iii): Evident.

(i) $\Leftarrow$ (ii): We establish the contrapositive implication.

$$\nexists \hat{v}_J > 0 \Longleftrightarrow -A^Tu + c\zeta \geq 0,\ (-A^Tu + c\zeta)_J > 0,\ \zeta > 0 \text{ has no solution}$$

$$\Longleftrightarrow -Ax - A_{\cdot J}z_J = 0,\ c^Tx + c_J^Tz_J + \eta = 0,\ x \geq 0,\ 0 \neq (z_J,\eta) \geq 0, \text{ has solution}$$

(By Motzkin's theorem)

$$\Longleftrightarrow -Ax = 0,\ c^Tx + \eta = 0,\ x \geq 0,\ 0 \neq (x_J,\eta) \geq 0 \text{ has solution}$$

$\Rightarrow$ $(\eta = 0)$ For each solution $\bar{x}$ of (3.4a), $\bar{x} + \lambda x$ is also a solution for any $\lambda > 0$, where $Ax = 0$, $c^Tx = 0$, $x \geq 0$, $0 \neq x_J \geq 0$

($\eta > 0$ excluded, because it implies (3.4a) is unbounded below, which is ruled out by primal-dual feasibility assumption)

$\Rightarrow \exists$ unbounded $\bar{x}_J$

(i) $\Rightarrow$ (iii): $c^T\hat{x} \geq c^T\bar{x} = b^T\bar{u} = b^T\bar{u} + \bar{x}^T\bar{v} \geq b^T\hat{u} + \bar{x}^T\hat{v} \geq b^T\hat{u} + \bar{x}_J^T\hat{v}_J$

$$\geq b^T\hat{u} + \|\bar{x}_J\|_1 \min \hat{v}_{j\in J}$$

Hence

$$\|\bar{x}_J\|_1 \leq (c^T\hat{x} - b^T\hat{u})/\min \hat{v}_{j\in J}$$

(a2): (ii) $\Leftarrow$ (iii): Evident.

(i)$\Leftarrow$(ii): We shall prove the contrapositive implication.

Rows of A lin. dep. or $\not\exists \hat{x} > 0$ such that $A\hat{x} = b$

$\Longleftrightarrow$ Rows of A lin. dep. or $0 \neq \begin{pmatrix} -A^T u \\ b^T u \end{pmatrix} \geq 0$ has solution

(By Motzkin's theorem)

$\Longleftrightarrow$ Rows of A lin. dep. or $0 \neq -A^T u \geq 0$, $b^T u = 0$ has solution (Case of $-A^T u \geq 0$, $b^T u > 0$, ruled out because it implies (3.4b) is unbounded above which is impossible by primal-dual feasibility assumption)

$\Rightarrow$ For each solution $\bar{u}$ of (3.4b), $\bar{u} + \lambda u$ is also a solution for any $\lambda > 0$ where either $b^T u = 0$, $A^T u = 0$, $u \neq 0$ or $b^T u = 0$, $0 \neq -A^T u \geq 0$.

$\Rightarrow \exists$ unbounded $\bar{u}$

(i)$\Rightarrow$(iii): Since $A^T\bar{u} = c - \bar{v}$ and rows of A are linearly independent it follows that $\bar{u} = (AA^T)^{-1}A(c - \bar{v})$ and hence

$$\|\bar{u}\|_1 \leq \|(AA^T)^{-1}A\|_1(\|c\|_1 + \|\bar{v}\|_1)$$

But

$$b^T\hat{u} \leq c^T\bar{x} = c^T\bar{x} - \bar{v}^T\bar{x} \leq c^T\hat{x} - \bar{v}^T\hat{x} \leq c^T\hat{x} - \|\bar{v}\|_1 \min_{1\leq i\leq n} \hat{x}_i$$

Hence

$$\|\bar{v}\|_1 \leq (c^T\hat{x} - b^T\hat{u})\Big/\min_{1\leq i\leq n} \hat{x}_i$$

and consequently

$$\|\bar{u}\|_1 \leq \|(AA^T)^{-1}A\|_1(\|c\|_1 + (c^T\hat{x} - b^T\hat{u})\Big/\min_{1\leq i\leq n} \hat{x}_i)$$

(a3): (ii) $\Leftarrow$ (iii): Evident.

(i) $\Leftarrow$ (ii): We shall prove the contrapositive implication.

$\nexists \hat{x}$ such that $\hat{x}_J > 0$, $\hat{x} \geq 0$ and $A\hat{x} = b$

$\Longleftrightarrow A^Tu + v = 0$, $b^Tu \geq 0$, $v \geq 0$, $\begin{pmatrix} v_J \\ b^Tu \end{pmatrix} \neq 0$, has solution

(By Motzkin's theorem)

$\Longleftrightarrow A^Tu + v = 0$, $b^Tu = 0$, $v \geq 0$, $v_J \neq 0$ has solution

(Case of $b^Tu > 0$, $A^Tu + v = 0$, $v \geq 0$, ruled out because it implies (3.4b) is unbounded above which is impossible by primal-dual feasibility assumption)

$\Rightarrow$ For each solution $(\bar{u},\bar{v})$ of (3.4b), $(\bar{u} + \lambda u, \bar{v} + \lambda v)$ is also a solution for any $\lambda > 0$ where $A^Tu + v = 0$, $b^Tu = 0$, $v \geq 0$, $v_J \neq 0$.

$\Rightarrow \exists$ *unbounded* $\bar{v}_J$

(i) $\Rightarrow$ (iii): $b^T\hat{u} \leq c^T\bar{x} = c^T\bar{x} - \bar{v}^T\bar{x} \leq c^T\hat{x} - \bar{v}^T\hat{x} \leq c^T\hat{x} - \bar{v}_J\hat{x}_J$

$$\leq c^T\hat{x} - \|\bar{v}_J\|_1 \min \hat{x}_{j\in J}$$

Hence

$$\|\bar{v}_J\|_1 \leq (c^T\hat{x} - b^T\hat{u})/\min \hat{x}_{j\in J} \qquad \square$$

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It is shown that each feasible point of a positive semidefinite linear complementarity problem which is not a solution of the problem provides simple numerical bounds for some or all components of all solution vectors. Consequently each pair of primal-dual feasible points of a linear program which are not optimal provide simple numerical bounds for some or all components of all primal-dual solution vectors. For example each feasible point, that is $(\hat{z},\hat{w}) \geqq 0$, of the linear complementarity problem $w = Mz + q \geqq 0,\ z \geqq 0,\ z^T w = 0$, where M is positive semidefinite, provides the following simple bound for any

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ABSTRACT (continued)

solution $\bar{z}$ of the linear complementarity problem:

$$\sum_{i\in I} \bar{z}_i \leqq \hat{z}^T\hat{w}/\min \hat{w}_{i\in I}$$

where $I = \{i|\hat{w}_i > 0\}$. If $\hat{w} > 0$ then this inequality provides a bound on the 1-norm $\|\bar{z}\|_1$ of any solution point. Similarly each feasible point $(\hat{x},\hat{y}) \geqq 0$ of the primal linear program $\min c^Tx$ subject to $y = Ax - b \geqq 0$, $x \geqq 0$, and each feasible point $(\hat{u},\hat{v}) \geqq 0$ of the dual linear program $\max b^Tu$ subject to $v = -A^Tu + c \geqq 0$, $u \geqq 0$, provide the following simple bounds for any primal optimal solution $(\bar{x},\bar{y})$ and any dual optimal solution $(\bar{u},\bar{v})$:

$$\sum_{i\in J} \bar{x}_i \leqq (c^T\hat{x} - b^T\hat{u})/\min \hat{v}_{i\in J}, \quad \sum_{i\in I} \bar{u}_i \leqq (c^T\hat{x} - b^T\hat{u})/\min \hat{y}_{i\in I}$$

where $J = \{i|\hat{v}_i > 0\}$ and $I = \{i|\hat{y}_i > 0\}$. If $\hat{v} > 0$ we have a bound on $\|\bar{x}\|_1$, and if $\hat{y} > 0$ we have a bound on $\|\bar{u}\|_1$. In addition we show that the existence of such numerical bounds is not only sufficient but is also necessary for the boundedness of solution vector components for both the linear complementarity problem and the dual linear programs.

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