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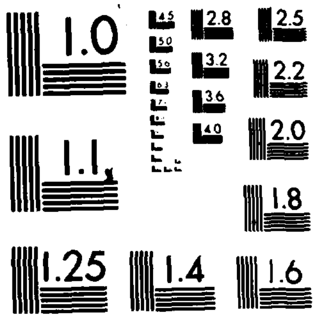

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A BOUNDARY MODULUS OF CONTINUITY FOR A CLASS OF SINGULAR PARABOLIC EQUATIONS

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E. DiBenedetto*

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Abstract

Parabolic equations describing diffusion phenomena with change of phase are considered. It is demonstrated that weak solutions are continuous up to the parabolic boundary of the domain of definition. The continuity is quantitatively described by a modulus determined a priori only in terms of the data.

AMS(MOS) Subject Classification: 35K63, 35R35, 35R05, 35K55.

Key Words: Singular equations, free boundary porous medium, modulus of continuity.

Work Unit Number 1 - Applied Analysis

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SIGNIFICANCE AND EXPLANATION

The equations considered in the paper represent a mathematical model for diffusion processes involving change of phase. Typical are the Stefon problem and the porous flow. We demonstrate that the weak solutions are continuous up to the closure of the domain of definition, and supply a quantitative estimate of the modulus of continuity. Such a quantitative estimate provides with a compactness tool to construct solutions in spaces of continuous functions.

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A BOUNDARY MODULUS OF CONTINUITY FOR A CLASS OF SINGULAR PARABOLIC EQUATIONS

E. DiBenedetto*

1. Introduction

We will be concerned with the derivation of a boundary modulus of continuity for weak solutions of singular parabolic equations with principal part in divergence form, of the type

$$\text{(1.1)} \qquad \frac{\partial}{\partial t}\beta(u) - \operatorname{div}\vec{a}(x,t,u,\nabla_x u) + b(x,t,u,\nabla_x u) \ni 0 ,$$

when non-homogeneous Dirichlet data are assigned on the parabolic boundary of a cylindrical domain $\Omega \times (0,T]$. Here β is a maximal monotone graph in $\mathbb{R} \times \mathbb{R}$ with a singularity at the origin.

Questions of regularity for weak solutions u of (1.1) have been considered in [3,4]. In these papers, beside statements of interior regularity, we derived a boundary modulus of continity at $t = 0$, and on $\partial\Omega \times (0,T]$, when (1.1) is associated with (non-linear) variational boundary data, or with homogeneous Dirichlet data. The case of non-homogeneous data, presented unsuspected difficulties and was left open. It is our purpose in this paper to fill the gap and complete the theory.

The problem has been tackled by W. Ziemer [9] who demonstrated the continuity of u up to the closure of $\Omega \times (\varepsilon,T]$, $\forall\varepsilon > 0$; the methods employed however do not seem to give a modulus.

On the other hand, quantitative statements about the regularity of u are important for the following reasons. First, the known regularity techniques, require that $u_t \in L_2(\Omega_T)$. The

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estimates are all independent of $\|u_t\|_{2,\Omega_T}$, but such information is necessary to justify some of the intermediate calculations.

Now this is not natural for weak solutions of (1.1) (even if $\beta(u) = u$, see [6]), whereas it can be verified for a sequence of approximating problems. Second, the a priori knowledge of a uniform modulus of continuity is useful in compactness arguments. We refer to [1,7,8] for applications to existence theory as well as asymptotic behaviour.

We will consider graphs β of two kinds, both arising from physics.

I. Graphs of Stefan type: β is given by

$$\beta(s) = \begin{cases} \beta_1(s) \ , & s > 0 \\ [-\nu, 0] \ , & s = 0 \\ \beta_2(s) - \nu \ , & s < 0 \ , \end{cases}$$

where $\nu > 0$ is a given constant and β_i , $i = 1,2$ are monotone increasing functions in their respective domain of definition, a.e. differentiable and

$$0 < \alpha_0 \le \beta_i'(s) \le \alpha_1 \ ; \ i = 1,2$$

for two positive constants α_0 , α_1 . Moreover $\beta_i(0) = 0$.

II. Graphs of porous media type: β is continuous, monotone increasing in $\mathbb{R}$ and $\beta(0) = 0$. With $\beta'(s)$ we denote the Dini numbers

$$\beta'(s) = \begin{cases} \limsup\limits_{h \downarrow 0} \dfrac{\beta(s) - \beta(s-h)}{h} , & s > 0 \\ \limsup\limits_{h \downarrow 0} \dfrac{\beta(s+h) - \beta(s)}{h} , & s < 0 , \end{cases}$$

and on $s \to \beta'(s)$ assume the following.

(i) $0 < \alpha_0 \le \beta'(s) , \forall s \in \mathbb{R}\backslash\{0\}$

(ii) $\liminf\limits_{|s| \to 0} \beta'(s) = \infty$

(iii) There exists an interval $[-\delta_0, \delta_0]$ around the origin such that $\beta'(s) \le \beta'(r)$ for $s \in \mathbb{R}\backslash[-\delta_0, \delta_0]$ and $r \in [-\delta_0, \delta_0]$, and $\beta'(\cdot)$ is decreasing over $(0, \delta_0]$ and increasing over $[-\delta_0, 0)$.

(iv) there exists a constant γ such that

$$\frac{\beta(s)}{s} \le \gamma\beta'(s) \quad \forall s \neq 0 .$$

The model example for such a β is

$$\beta(s) = |s|^{\frac{1}{m}} \text{ sign } s , \; m > 1 , \tag{1.2}$$

which occurs in filtration of gases in porous media, where the flow obeys a polytropic law.

Note that no symmetry requirement has been made on β around the origin.

Next we formulate our hypotheses and state our main result. The notation of [3,6] is adopted.

Let Ω be an open set of $\mathbb{R}^N$; $0 < T < \infty$ and set $\Omega_T \equiv \Omega \times (0,T]$, $S = \partial\Omega \times (0,T]$. The domain Ω is assumed to satisfy

[A_1] $\exists \alpha_* \in (0,1)$, $R_0 > 0$ such that $\forall x_0 \in \partial\Omega$ and every ball $B(x_0,R)$, centered at x_0 with radius $R < R_0$,

$$\text{meas}\,[\Omega \cap B(x_0,R)] \le (1-\alpha_*)\text{meas}\, B(x_0,R) \ .$$

On the space part of the operator in (1.1) we assume

[A_2] $\vec{a} : \Omega_T \times \mathbb{R}^{N+1} \to \mathbb{R}^N$; $b : \Omega_T \times \mathbb{R}^{N+1} \to \mathbb{R}$ are measurable and

$$\vec{a}(x,t,u,\vec{p}) \cdot \vec{p} \ge c_0(|u|)|\vec{p}|^2 - \phi_0(x,t) \tag{1.3}$$

$$|\vec{a}(x,t,u,\vec{p})| \le \mu_0(|u|)|\vec{p}| + \phi_1(x,t) \tag{1.4}$$

$$|b(x,t,u,\vec{p})| \le \mu_1(|u|)|\vec{p}|^2 + \phi_2(x,t) \ , \tag{1.5}$$

where $c_0(\cdot) : \mathbb{R}^+ \to \mathbb{R}^+$ is continuous, decreasing and strictly positive; $\mu_i(\cdot) : \mathbb{R}^+ \to \mathbb{R}^+$ are continuous and increasing, $i = 0 , 1$; ϕ_i , $i = 0,1,2$ are non-negative and satisfy

$$\|\phi_0 + \phi_1^2 + \phi_2\|_{\hat{q},\hat{r},\Omega_T} \le \mu_2 \ . \tag{1.6}$$

Here μ_2 is a given constant and $\hat{q}$, $\hat{r}$ are positive numbers, linked by

$$(1.7)\quad \begin{cases} \frac{1}{\hat{r}} + \frac{N}{2\hat{q}} = 1 - \kappa_1 \\ \hat{q} \in \left[\frac{N}{2(1-\kappa_1)}, \infty\right] \ , \ \hat{r} \in \left[\frac{1}{1-\kappa_1}, \infty\right] \ , \ \kappa_1 \in (0,1) \\ \hat{q} \in (1,\infty) \ , \ \hat{r} \in \left[\frac{1}{1-\kappa_1}, \frac{1}{1-2\kappa_1}\right], \ \kappa_1 \in (0,\tfrac{1}{2}) \ . \end{cases}$$

By a solution of (1.1) we mean a function $u \in W_2^{1,1}(\Omega_T)$, defined by $u \in \beta^{-1}(w)$, where $w \in L^{\infty}(0,T,L_2(\Omega))$ is such that $w \subset \beta(u)$, the inclusion being intended in the sense of graphs, and w and u satisfy

$$(1.8)\quad \int_{\Omega} w(x,\tau)\phi(x,\tau)dx\Big|_{t_1}^{t} + \int_{t_1}^{t}\int_{\Omega}\{-w\phi_t + \vec{a}(x,\tau,u,\nabla_x u)\cdot \nabla_x\phi$$
$$+ b(x,\tau,u,\nabla_x u)\phi\}dx\,d\tau = 0$$

for all $\phi \in \overset{\circ}{W}{}_2^{1,1}(\Omega_T)$ and all intervals $[t_1,t] \subset (0,T]$.

We assume u takes on boundary values on S

$$(1.9)\quad u(x,t) = f(x,t) \ ; \ (x,t) \in S \ ,$$

in the sense of the traces, and suppose that

$[A_3]$ f is continuous on S, with modulus of continuity $\omega_f(\cdot)$.

Let u be a weak solution of (1.1) satisfying (1.9) and assume that

$[A_4]$ $\quad u \in L_{\infty}(\Omega_T)$, and $\|u\|_{\infty,\Omega_T} \le M$.

We can now state our main result.

<u>Theorem</u>: Let β be of type I or II, and let u be a weak solution of (1.1) satisfying (1.9), and let $[A_1]$ - $[A_4]$ hold. Then

(a) u is continuous in $\bar{\Omega} \times (0,T]$, and $\forall \varepsilon > 0$ there exists a continuous non-decreasing function $\omega : \mathbb{R}^+ \to \mathbb{R}^+$, depending upon ε, ω_f and the data in $[A_1]$ - $[A_4]$, such that

$$|u(x_1,t_1) - u(x_2,t_2)| \le \omega(|x_1-x_2| + |t_1-t_2|^{\frac{1}{2}}) ,$$

$$\forall (x_i,t_i) \in \bar{\Omega} \times [\varepsilon,T] , \; i = 1,2 .$$

(b) If $\omega_f(r) = cr^\lambda$ for some $c > 0$ and $\lambda \in (0,1)$, then the function $\omega(\cdot)$ is given by

$$\omega(r) = (\ln|\ln c_1 r|)^{c_2} \qquad (r \text{ small}) ,$$

where c_1, c_2 are constants depending only upon the data and ε; $c_1 > 0$, $c_2 < 0$.

(c) Suppose in addition that $u(x,0) = u_0(x)$ in the sense of the traces over $\Omega \times \{0\}$, and u_0 is continuous in $\bar{\Omega}$ with modulus of continuity ω_0. Then u is continuous in $\bar{\Omega}_T$, and there exists a continuous non-decreasing function $\omega : \mathbb{R}^+ \to \mathbb{R}^+$ depending upon ω_0, ω_f, and the data in $[A_1]$ - $[A_4]$, such that

$$|u(x_1,t_1) - u(x_2,t_2)| \le \omega(|x_1-x_2| + |t_1-t_2|^{1/2}) ,$$

$$\forall (x_i,t_i) \in \bar{\Omega}_T , \ i = 1,2 .$$

(d) If $\omega_f(r)$, $\omega_0(r) = cr^{\lambda}$, then

$$\omega(r) = (\ln|\ln c_1 r|)^{c_2}$$

Remark 1.1. If the boundary datum f is only known to be continuous on a subdomain $S' \subset S$, then the interior continuity of u can be extended up to S' , and for every $\varepsilon > 0$, a modulus of continuity ω_ε for u can be deduced, in the set ($\tilde{S}'$ is the complement of S'),

$$\Omega(S',\varepsilon) \equiv \{(x,t) \in \Omega_T : t > \varepsilon , \ \mathrm{dist}((x,t),\tilde{S}') \ge \varepsilon\} .$$

Remark 1.2 Since the modulus of continuity can be determined a priori in terms of $\omega_f(\cdot)$ and the various constants in $[A_1]$-$[A_4]$, the theorem could be stated, (as a variant) as equicontinuity of approximating sequences, along the lines of similar situations in [3,4]. It could also be used as a device to construct solutions in appropriate spaces of continuous functions (see [1]). Since the arguments are technically involved, we wish to describe euristically their meaning in order to single out the idea of the proof which is new with respect to previous argument. We introduce some symbolism first.

Let $(x_0,t_0) \in S$ and for $R \le R_0$ denote with $B(R)$ the ball $B(R) \equiv \{|x-x_0| < R\}$ and with Q_R^η , $\eta > 0$, the cylinder

$$Q_R^\eta \equiv B(R) \equiv \{t_0-\eta R^2 , t_0\} .$$

If $\sigma_1, \sigma_2 \in (0,1)$ denote by $Q_R^\eta(\sigma_1,\sigma_2)$ the cylinder

$$Q_R^\eta(\sigma_1,\sigma_2) \equiv B(R-\sigma_1 R) \times \{t_0 - \eta(1-\sigma_2)R^2 , t_0\} .$$

Let $(x,t) \to \zeta(x,t)$ be a cutoff function in Q_R^η such that

(i) $\zeta(x,t) = 1 , (x,t) \in Q_R^\eta(\sigma_1,\sigma_2)$

(ii) $\zeta \geq 0$ and $\zeta = 0$ on $\partial B(R)$ and for $t = t_0 - \eta R^2$

(iii) $|\nabla_x \zeta| \leq (\sigma_1 R)^{-1} , 0 \leq \zeta_t \leq (\sigma_2 \eta R^2)^{-1}$.

In what follows ζ will denote always such a cutoff funct[illegible]

We will be interested in that portion of Q_R contained in Ω_T . Therefore we set

$$\Omega_R = \Omega \cap B(R)$$

$$C_R^\eta = Q_R^\eta \cap \Omega_T$$

$$C_R^\eta(\sigma_1,\sigma_2) = Q_R^\eta(\sigma_1,\sigma_2) \cap \Omega_T$$

Notice that ζ vanishes on the parabolic boundary of Q_R^η but not on the parabolic boundary of C_R^η . If $(x_0,t_0) \in S$ is fixed, we will take R to be so small that $t_0 - \eta R^2 > 0$.

Consider the simple case

$$(1.10) \qquad \begin{cases} \beta(u)_t - \Delta u \ni 0 & \text{in } \Omega_T \\ u = f & \text{on } S \end{cases}$$

If $k > \sup_{Q_R^\eta \cap S} f$, then $(u-k)^+ \zeta^2 \in \overset{\circ}{W}{}_2^{1,1}(\Omega_{t_0})$; analogously if $k < \inf_{Q_R^\eta \cap S} f$, $-(u-k)^- \zeta^2 \in \overset{\circ}{W}{}_2^{1,1}(\Omega_{t_0})$.

We multiply (1.10) formally by $\pm(u-k)^\pm \zeta^2$ and integrate over $\Omega_R \times \{t_0 - \eta R^2, t\}$. Carrying on formal calculations, the first term gives

$$\int_{\Omega_R}\int_{t_0-\eta R^2}^{t} \pm \frac{\partial}{\partial t}\beta(u)(u-k)^\pm \zeta^2\,dx\,d\tau$$

$$= \int_{\Omega_R}\int_{t_0-\eta R^2}^{t} \frac{\partial}{\partial t}\left(\int_0^{(u-k)^\pm} \beta'(s\pm k)s\,ds\right)\zeta^2\,dx\,d\tau$$

$$\geq \int_{\Omega_R\times\{t\}}\left(\int_0^{(u-k)^\pm} \beta'(s\pm k)s\,ds\right)\zeta^2(x,t)\,dx$$

$$- C\iint_{C_R} (u-k)^\pm \zeta\zeta_t\,dx\,d\tau\ .$$

For the second term we have

$$\int_{t_0-\eta R^2}^{t}\int_{\Omega_R} \mp\Delta u(u-k)^\pm\ \zeta^2\,dx\,d\tau \geq \int_{t_0-\eta R^2}^{t}\int_{\Omega_R} |\nabla(u-k)^\pm|^2\zeta^2\,dx\,d\tau$$

$$- C\iint_{C_R^\eta} (u-k)^{\pm 2}|\nabla\zeta|^2\,dx\,d\tau\ .$$

Combining these calculations and recalling the properties of $(x,t) \to \zeta(x,t)$ we have

$$\sup_{t_0-\eta(1-\sigma_2)R^2\leq t\leq t_0} \int_{\Omega_{R-\sigma_1 R}\times\{t\}}\left(\int_0^{(u-k)^\pm} \beta'(s\pm k)s\,ds\right)dx \tag{1.11}$$

$$+ \iint_{C_R^\eta(\sigma_1,\sigma_2)} |\nabla(u-k)^\pm|^2\,dx\,d\tau \leq \frac{C(\eta,\sigma_1,\sigma_2)}{R^2}\iint_{C_R^\eta} (u-k)^\pm\,dx\,d\tau\ .$$

Thus the presence of the singularity of β in (1.10) has the effect of producing, on the right hand side of (1.11), the L_1 norm of $(u-k)^{\pm}$ instead of the L^2-norm.

If u were known to satisfy the inequality

$$(1.12) \quad \sup_{t_0-\eta(1-\sigma_2)R^2\le t\le t_0} \int_{\Omega_{R-\sigma_1 R}} (u-k)^{\pm^2}(t)\,dx + \iint_{C_R(\sigma_1,\sigma_2)} |\nabla(u-k)^{\pm}|^2\,dx\,d\tau$$

$$\le \frac{C(\eta,\sigma_1,\sigma_2)}{R^2} \iint_{C_R^{\eta}} (u-k)^{\pm^2}\,dx\,d\tau \ ,$$

for all $R > 0$ and $\sigma_i \in (0,1)$, $i = 1,2$, then by classical theory (see for example [6]), u would be Hölder continuous at (x_0,t_0) . Let us compare (1.11) and (1.12).

The first term on the left hand side of (1.11) can be estimated below by

$$(1.13) \quad [\inf_{\mathbb{R}} \beta'(s)] \sup_{t_0-\eta(1-\sigma_2)R^2\le t\le t_0} \int_{\Omega_{R-\sigma_1 R}} (u-k)^{\pm^2}\,dx \ ,$$

and therefore the left hand side of (1.11) has the same aspect as the companion in (1.13). In fact an estimate below like (1.13) was the starting point in the proof of interior continuity in [2,3,4,8,9].

As for the right hand side of (1.11), if k is near zero, on the set $[u \text{ near zero}]$, the term $(u-k)^{\pm}$ is much larger than $(u-k)^{\pm^2}$; therefore near the set of singularity for β , the right hand side of (1.11) gives an estimate poorer than (1.12).

In the proof of interior regularity this difficulty was dealt with by choosing appropriate levels k for u. Near the boundary our choice of levels k is limited to those k for which $(u-k)^{\pm} \in \overset{\circ}{W}{}_2^{1,1}(\Omega_T)$ consequently the difficulty is more severe.

On the other hand if k and u are both near zero, $\beta'(s\pm k)$ in (1.11) is large. Therefore the new tool will consist in deriving an estimate below for the left hand side of (1.11) which takes in account the largeness of β' near the singularity, and then balance off the largeness of β' on the left hand side of (1.11) with the relative largeness of $(u-k)^{\pm}$ on the right hand side. These remarks are only formal since β is not differentiable. The difficulty consists in finding a technical way of translating the above observations (which hold only on the portion of Q_R where u is near zero), in terms of integrals over Q_R^{η}. This will be accomplished by introducing a sequence of time dilations (this will be the role of the parameter η) and by using a more refined version of the Sobolev embedding of $V^{1,0}(Q_R^{\eta})$ into $L_{q,r}(Q_R^{\eta})$ (see section 2 for the precise statement).

If β is a graph of porous medium type, then $\beta'(s)$ is large for s in a neighborhood of zero, whereas if β is of Stefan type then "$\beta'(s)$" is finite except "at zero". Therefore it is not surprising that the latter presents more difficulties.

We will concentrate on graphs of type I and will indicate later how to modify the arguments to include β of type II.

Section 2 contains some preliminary lemmas.

The proof of the theorem will be given in section 3 and will be extended to β of porous medium type in section 4.

With γ we will denote a generic non-negative constant depending upon the data in $[A_1]$ - $[A_4]$ only.

2. Basic estimates

If v is a function defined in $B(R)$ and $k \in \mathbb{R}$ we set

$$A^+_{k,R} \equiv \{x \in B(R) \mid v(x) > k\} \; ; \; A^-_{k,R} \equiv \{x \in B(R) \mid v(x) < k\} \, .$$

The proof of the following lemma can be found in [6].

<u>Lemma 2.1</u> Let $v \in W^1_1(B(R))$ and let k , ℓ be real numbers such that $\ell > k$. Then

$$(\ell-k)\,\mathrm{meas}\, A^-_{k,R} \le \frac{\gamma R^{N+1}}{\mathrm{meas}\{B(R)\setminus A^-_{\ell,R}\}} \int_{A^-_{\ell,R}\setminus A^-_{k,R}} |\nabla v|\,dx \, ,$$

where γ is a constant depending only upon the dimension N .

The following embedding lemma is essentially proved in [6]. We reproduce the proof in a particular case, in order to derive a special constant dependence, needed in what follows.

<u>Lemma 2.2</u> Let $v \in \overset{\circ}{V}{}^{1,2}(Q^\eta_R)$. Then there exists a constant γ , independent of v , R and η , such that for every $\lambda > 0$

$$\lambda^{\frac{2}{N+2}} \|v\|^2_{2,Q^\eta_R} \le \gamma[\mathrm{meas}\,[v=0] \cap Q^\eta_R]^{\frac{2}{N+2}} \, .$$

$$\left\{ \lambda \sup_{t_0-\eta R^2 \le t \le t_0} \|v\|^2_{2,B(R)} + \|\nabla_x v\|^2_{2,Q^\eta_R} \right\} .$$

<u>Proof</u> By the Gagliardo-Nirenberg inequality,

(2.1) $$\|v\|_{q,B(R)} \le \gamma \|\nabla_x v\|^{\alpha}_{2,B(R)} \, \|v\|^{1-\alpha}_{2,B(R)} ,$$

for a.e. $t \in [t_0 - \eta R^2, t_0]$, where

$$\alpha = N(\tfrac{1}{2} - \tfrac{1}{q}) \ ; \ q > 2 \ , \ \alpha \in (0,1) \ .$$

Choosing

$$q = \frac{2(N+2)}{N} \quad ; \ \alpha = \frac{N}{N+2}$$

we have

$$\|v\|^{q}_{q,B(R)} \le \gamma \|\nabla_x v\|^{2}_{2,B(R)} \, \|v\|^{4/N}_{2,B(R)} \ .$$

Integrating over $[t_0 - \eta R^2, t_0]$ and taking the q-th rooth we obtain

(2.2) $$\|v\|_{q,Q^\eta_R} \le (\sup_t \|v\|_{2,B(R)})^{\frac{2}{N+2}} \, \|\nabla_x v\|^{\frac{N}{N+2}}_{2,Q^\eta_R} \ .$$

Let now λ be any positive number. Then

(2.3) $$\lambda^{\frac{2}{N+2}} \iint_{Q^\eta_R} v^2 \le \lambda^{\frac{2}{N+2}} \|v\|^2_{q,Q^\eta_R} \, [\text{meas } [v \neq 0] \cap Q^\eta_R]^{\frac{2}{N+2}} ,$$

and by (2.2)

$$\lambda^{\frac{2}{N+2}} \|v\|^2_{q,Q^\eta_R} \le \gamma\left\{(\lambda^{\frac{1}{2}} \sup_t \|v\|_{2,B(R)})^{\frac{2}{N+2}} \|\nabla_x v\|^{\frac{N}{N+2}}_{2,Q^\eta_R}\right\}^2$$

$$\le \gamma\{\lambda \sup_t \|v\|^2_{2,B(R)} + \|\nabla_x v\|^2_{2,Q^\eta_R}\} .$$

Substituting this last estimate in (2.3), the lemma follows.

Remark: In a similar way one can show the inequality

$$\lambda^{1-\frac{2}{r}} \|v\|^2_{q,r,Q_R} \le \gamma\{\lambda \sup_t \|v\|^2_{2,B(R)} + \|\nabla_x v\|^2_{Q^\eta_R}\}$$

valid for all $\lambda > 0$ and q , $r > 1$ satisfying

$$\frac{1}{r} + \frac{N}{2q} = \frac{N}{4} .$$

If $(x_0,t_0) \in S$ is fixed, we choose R to be so small that $R < R_0$ and

$$t_0 - R^{2(1-N\kappa)} > 0 ,$$

where

(2.4) $$\kappa = 2\kappa_1/N ,$$

and κ_1 is defined in $[A_2]$.

Let such R be fixed and consider the cylinder

$$Q_{R^{1-N\kappa}} \equiv \{|x-x_0| < R^{1-N\kappa}\} \times \{t_0 - R^{2(1-N\kappa)} , t_0\} .$$

Set

$$(2.5) \qquad \mu^+ = \sup_{Q_R 1-N\kappa} u \quad ; \quad \mu^- = \inf_{Q_R 1-N\kappa} u \quad ; \quad \omega(R^{1-N\kappa}) \geq \operatorname*{osc}_{Q_R 1-N\kappa} u \ .$$

Also set

$$f^+ = \sup_{Q_R^{1-N\kappa} \cap S} f \quad ; \quad f^- = \inf_{Q_R^{1-N\kappa} \cap S} f$$

(2.6)

$$\omega_f(R^{1-N\kappa}) = f^+ - f^- = \operatorname*{osc}_{Q_R^{1-N\kappa} \cap S} f \ .$$

We will consider cylinders Q_ρ^η coaxial with $Q_R 1-N\kappa$ and with same "vertex" (x_0, t_0) ; i.e. we will impose

$$(2.7) \qquad \rho < \eta^{-\frac{1}{2}} R^{(1-N\kappa)} \ .$$

If k is a number satisfying $k \geq f^+$, and ζ is a cutoff function in Q_ρ^η and $Q_\rho^\eta(\sigma_1, \sigma_2)$, then

$$(u-k)^+ \ \zeta^2 \in \overset{\circ}{W}{}_2^{1,1}(\Omega \times (t_0 - \eta\rho^2, t_0))$$

and therefore it can be used as a test function in (1.8) for $[t_1, t] \subset [t_0 - \eta\rho^2, t_0]$.

Analogously, if $k \leq f^-$, then $-(u-k)^- \zeta^2 \in \overset{\circ}{W}{}_2^{1,1}(\Omega \times (t_0 - \eta\rho^2, t_0))$.

The functions $\pm(u-k)^{\pm}\zeta^2$ introduced above are defined only in C_ρ^η. We will consider them to be defined in all Q_ρ^η, by extending them to be equal to zero on $Q_\rho^\eta \setminus C_\rho^\eta$.

Next by a suitable change of variable, it is no loss of generality to assume β is given by

$$(2.7) \qquad \beta(u) = \begin{cases} u & , u > 0 \\ [-\nu, 0] & , u = 0 \\ u - \nu & , u < 0 \end{cases} .$$

This is demonstrated in [3] to which we refer for details.

By taking $\pm(u-k)^{\pm}\zeta^2$ in (1.8) we proved in [3], that the following inequality holds.

$$(2.8) \quad \sup_{t_0-\eta(1-\sigma_2)\rho^2\le t\le t_0} \left\{ \int_{B(\rho-\sigma_1\rho)\times\{t\}} (u-k)^{\pm^2} dx + \Phi^{\pm}(k, t_0 - \eta\rho^2, t, \zeta) \right\}$$

$$+ \|\nabla_x (u-k)^{\pm}\|^2_{2,Q_\rho^\eta(\sigma_1,\sigma_2)} \le \gamma[(\sigma_1\rho)^{-2} + (\sigma_2\eta\rho^2)^{-1}] \|(u-k)^{\pm}\|^2_{2,Q_\rho^\eta}$$

$$+ \gamma \left\{ \int_{t_0-\eta\rho^2}^{t_0} [\text{meas } A^{\pm}_{k,\rho}(\tau)]^{\frac{r}{q}} d\tau \right\}^{\frac{2}{r}(1+\kappa)} ,$$

where

$$(2.9) \quad \Phi^{\pm}(k, t_0 - \eta\rho^2, t, \zeta) = -\int_{B(\rho)} \nu(x,\tau)\chi[u\le 0][\pm(u-k)^{\pm}]\zeta^2(x,\tau)dx \Big|_{t_0-\eta\rho^2}^{t}$$

$$+ \int_{t_0-\eta\rho^2}^{t} \int_{B(\rho)} \nu(x,\tau)\,\chi[u\le 0]\frac{\partial}{\partial t}[\pm(u-k)^{\pm}\zeta^2]dx\, d\tau ;$$

(2.10) $\quad \nu(x,t)$ is a selection out of $[-\nu,0]$, for $u(x,\tau) = 0$ and $0 \le \nu(x,\tau) \le \nu$;

(2.11) $\quad \frac{1}{r} + \frac{N}{2q} = \frac{N}{4}$, and the admissible range of r , q is

$$q \in (2, \frac{2N}{N-2}], \quad r \in [2,\infty) \quad \text{for} \quad N \ge 3$$

$$q \in (2,\infty) \quad , r \in (2,\infty) \quad \text{for} \quad N = 2$$

$$q \in (2,\infty) \quad , r \in [4,\infty) \quad \text{for} \quad N = 1 .$$

Remark 2.1

Suppose (2.8) are written for $k > f^+ \ge 0$ and $(u-k)^+$, then a simple computation shows that

$$\Phi^+(k,t_0 - n\rho^2, t, \zeta) = 0 . \tag{2.12}$$

Analogously if $k \le f^- \le 0$ and (2.8) are written for $(u-k)^-$, we see that (2.12) holds with Φ^+ replaced by Φ^- . Therefore from (2.8) we deduce the following lemma.

Lemma 2.3 The functions $(u-k)^\pm$ satisfy the inequality:

$$\|(u-k)^\pm\|^2_{V^{1,0}(Q^n_\rho(\sigma_1,\sigma_2))} \le \gamma[(\sigma_1\rho)^{-2} + (\sigma_2 n\rho^2)^{-1}]\|(u-k)^\pm\|^2_{2,Q^n_\rho} . \tag{2.13}$$

$$+ \gamma\left\{\int_{t_0-n\rho^2}^{t_0} [\text{meas } A^\pm_{k,\rho}(\tau)]^{\frac{r}{q}} d\tau\right\}^{\frac{2}{r}(1+\kappa)} ,$$

for all $\rho < \eta^{-\frac{1}{2}} R^{1-N\kappa}$ and σ_1 , $\sigma_2 \in (0,1)$, provided that

(i) If (2.13) is written for $(u-k)^+$, k satisfies $k \geq f^+ \geq 0$

(ii) If (2.13) is written for $(u-k)^-$, k satisfies $k \leq f^- \leq 0$.

Let us assume now that (2.8) are written for $(u-k)^-$ and k satisfies

$$0 < k \leq f^- .$$

In such a case $\Phi^- \neq 0$ and we will estimate it as follows.

$$\Phi^-(k,t_0-\eta\rho^2,t,\zeta^2) = \int_{B(\rho)\cap[u\leq 0]} \nu(x,t)(u-k)^- \zeta^2(x,t)dx$$

$$- \int_{t_0-\eta\rho^2}^{t} \int_{B(\rho)} \nu \frac{\partial}{\partial t} u^- \zeta^2 \, dx \, d\tau$$

$$- 2\int_{t_0-\eta\rho^2}^{t} \int_{B(\rho)\cap[u\leq 0]} \nu(x,\tau)(u-k)^- \zeta \zeta_t \, dx \, d\tau$$

$$\geq \nu k \operatorname{meas}\{B(\rho-\sigma_1\rho)\cap[u(\cdot,t)\leq 0]\} - \frac{2k}{\sigma_2\eta\rho^2} \iint_{Q_\rho^\eta} \chi[u\leq 0]dx\, d\tau .$$

Substituting this estimate below in (2.8) proves the following lemma.

<u>Lemma 2.4</u> Let $0 < k \leq f^-$. Then $(u-k)^-$ satisfies the inequality

$$(2.14)\quad \sup_{t_0-\eta(1-\sigma_2)\rho^2\le t\le t_0}\Big\{\|(u-k)^-\|^2_{2,B(\rho-\sigma_1\rho)}(t)$$

$$+ \nu k \text{ meas } B(\rho-\sigma_1\rho)\cap[u(\cdot,t)\le 0]\Big\}$$

$$+ \|\nabla_x(u-k)^-\|^2_{2,Q^\eta_\rho(\sigma_1,\sigma_2)}$$

$$\le \gamma[(\sigma_1\rho)^{-2} + (\sigma_2\eta\rho^2)^{-1}]\,\|(u-k)^-\|^2_{2,Q^\eta_\rho}$$

$$+ 2k(\sigma_2\eta\rho^2)^{-1}\text{ meas}(Q^\eta_\rho\cap[u\le 0])$$

$$+\gamma\Big\{\int_{t_0-\eta\rho^2}^{t_0}[\text{meas } A^-_{k,\rho}(\tau)]^{r/q}d\tau\Big\}^{\frac{2}{r}(1+\kappa)}\ ,$$

for all $\rho < \eta^{-\frac{1}{2}} R^{1-N\kappa}$ and all σ_1 , $\sigma_2 \in (0,1)$.

We demonstrated in [3] that the previous inequalities hold if the levels k are subject to the further restriction

$$(2.15)\qquad \|(u-k)^{\pm}\|_{\infty,Q^\eta_\rho} \le \delta\ ,$$

where δ is a given number depending only upon the constants in assumptions $[A_1]$ - $[A_4]$.

If in the definition of Q^η_ρ the parameter $\eta = 1$ we simply write Q_ρ .

We have $Q_{2R} \subset Q_{R^{1-N\kappa}}$ if $R < 2R^{1-N\kappa}$ which holds if R is sufficiently small.

We let s_0 be the smallest positive integer for which

$$(2.16)\qquad \frac{\omega}{2^{s_0}} \le \frac{2M}{2^{s_0}} < \delta\ ,$$

where δ is introduced in (2.15).

Lemma 2.5 Assume that f^- is such that $\mu^- + \frac{\omega}{2^{s_0}} \geq f^-$. There exists a number $s^* > s_0$ such that if

$$\text{(2.17)} \qquad \text{meas}\{[u \leq 0] \cap Q_{2R}\} \leq \frac{\omega}{2^{s^*}} \, \text{meas}\{[u < \mu^- + \frac{\omega}{2^{s^*+1}}] \cap Q_{R/2}\}$$

and

$$\text{(2.18)} \qquad |\mu^-| < \frac{\omega}{2^{s^*}} ,$$

then either

$$\text{(i)} \qquad \frac{\omega}{2^{s^*}} < R^{N\kappa/2} , \text{ or}$$

$$\text{(ii)} \qquad \forall (x,t) \in Q_{R/2} , \; u(x,t) \geq \mu^- + \frac{\omega}{2^{s^*+2}} .$$

The number s^* can be determined a priori only in dependence of the data and it is independent of ω and R.

Proof Consider (2.14) written for $\frac{R}{2} \leq \rho \leq 2R$, $\eta = 1$ and $k = \mu^- + \frac{\omega}{2^s}$, for $s^* \geq s > s_0$.

Using (2.17) - (2.18) we have for every $\frac{R}{2} \leq \rho \leq 2R$,

$$\frac{k}{\sigma_2 \rho^2} \text{ meas}\{[u \leq 0] \cap Q_\rho\} \leq \frac{1}{\sigma_2 \rho^2} \left(\frac{\omega}{2^s}\right)^2 \text{meas}\{[u < \mu^- + \frac{\omega}{2^{s+1}}] \cap Q_\rho\} .$$

Now

$$\iint_{Q_\rho} [(u-(\mu^- + \frac{\omega}{2^s}))^-]^2 dx\, d\tau \geq (\frac{\omega}{2^{s+1}})^2 \text{ meas}\{[u < \mu^- + \frac{\omega}{2^{s+1}}] \cap Q_\rho\}$$

and therefore

$$\frac{k}{\sigma_2 \rho^2} \text{ meas}\{[u \leq 0] \cap Q_\rho\} \leq \frac{\gamma}{\sigma_2 \rho^2} \iint_{Q_\rho} [(u - (\mu^- + \frac{\omega}{2^s}))^-]^2 dx\, d\tau \ .$$

Using this estimate, from (2.14) we deduce that for all $s^* \geq s > s_0$, all $\frac{R}{2} \leq \rho \leq 2R$ and all σ_1, $\sigma_2 \in (0,1)$ we have

(2.19) $$\|(u-(\mu^- + \frac{\omega}{2^s}))^-\|^2_{V^{1,0}(Q_\rho(\sigma_1,\sigma_2))}$$

$$\leq \gamma[(\sigma_1\rho)^{-2} + (\sigma_2\rho^2)^{-1}] \ \|(u - (\mu^- + \frac{\omega}{2^s}))^-\|^2_{2,Q_\rho}$$

$$+ \gamma\left\{\int_{t_0-\rho^2}^{t_0} [\text{meas } A^-_{\mu^- + \frac{\omega}{2^s},\rho}(\tau)]^{\frac{r}{q}} d\tau\right\}^{\frac{2}{r}(1+\kappa)} \ .$$

From (2.19) it follows by classical theory (see [6] page 110-128) that there exist s^* which can be determined only in dependence of the various constants in (2.19) and the number α in $[A_1]$, such that either (i) holds or

$$u(x,t) \geq \mu^- + \frac{\omega}{2^{s^*+2}} \quad , \quad \forall (x,t) \in Q_{R/2} \ .$$

<u>Lemma 2.6</u> There exists a number c_0 depending only upon the data such that if

$$\text{meas}\{(x,t) \in Q_{R/2} \mid u(x,t) < \bar{\mu} + \frac{\omega}{2^{s^*+2}}\} \leq c_0\left(\frac{\omega}{2^{s^*}}\right)^{\frac{N+2\kappa_1}{2\kappa_1}} \times \text{meas } [Q_{R/2}] \ ,$$

then either

(i) $\frac{\omega}{2^{s^*}} < R^{N\kappa/2}$, or

(ii) $u(x,t) \geq \mu^- + \frac{\omega}{2^{s^*+4}}$, $\forall (x,t) \in Q_{R/4}$.

Proof. This is exactly lemma 3.1 of [3] page 144.

3. Proof of the theorem: β of type I

The proof is based on the following proposition.

Proposition 3.1: There exists positive numbers $\bar{s} \in \mathbb{N}$, δ^* , $\tilde{\gamma}_i$, $i = 1,2$, a , $A > 1$ depending only upon the data such that

$$\operatorname*{osc}_{Q_{R/4}} u \leq \max\left\{\left(1 - \frac{1}{2^{\bar{s}+A/\omega^a(R^{1-N\kappa})}}\right)\omega(R^{1-N\kappa}) \; ; \right.$$

$$\left. \left[\frac{\tilde{\gamma}_1}{|\ell n\ \tilde{\gamma}_2 R|}\right]^{\delta^*} ; \; 2^{\bar{s}}\,\omega_f(R^{1-N\kappa})\right\} .$$

Proof Without loss of generality we may assume that

$$|\mu^-| \leq \mu^+ . \tag{3.1}$$

If the reverse inequality occurs, the arguments are similar. To simplify the symbolism we write ω for $\omega(R^{1-N\kappa})$. Since obviously

$$\omega = \mu^+ - \mu^- > \frac{\omega}{2^{s_0-1}}$$

where s_0 is introduced in (2.16), from (3.1) we deduce

$$\mu^+ - \frac{\omega}{2^{s_0}} > \left|\frac{\omega}{2^{s_0}} + \mu^-\right| \geq 0 . \tag{3.2}$$

Also we may assume that

$$f^+ - f^- = \omega_f(R^{1-N\kappa}) < \omega/2^{s_0+2} , \tag{3.3}$$

otherwise the proposition becomes trivial.

We claim that at least one of the following two inequalities is satisfied,

$$\begin{cases} \mu^+ - \dfrac{\omega}{2^{s_0+2}} \geq f^+ \\ \mu^- + \dfrac{\omega}{2^{s_0+2}} \leq f^- \end{cases} . \tag{3.4}$$

In fact if both are violated

$$\mu^+ - \mu^- - \frac{\omega}{2^{s_0+1}} < \omega_f(R^{1-N\kappa}) , \tag{3.5}$$

contradicting (3.3).

If the first of (3.4) is verified, then the function $(x,t) \to (u(x,t) - (\mu^+ - \frac{\omega}{2^s}))^+$, in view of Remark 2.1, satisfies (2.13) for all $s \geq s_0 + 2$, all $0 < \rho < R^{1-N\kappa}$ and all $\sigma_1 , \sigma_2 \in (0,1)$. Now (2.13) is precisely inequality (8.1) of [6] page 123, and hence the proposition is a consequence of the results of [6].

If the second of (3.4) holds and

(3.6) $$\mu^- + \frac{\omega}{2^{s_0+2}} < 0 ,$$

then $(x,t) \to (u(x,t) - (\mu^- + \frac{\omega}{2^s}))^-$ verifies (2.13) for all $s \ge s_0 + 2$, and again the proposition follows by classical theory.

The case to consider is therefore when the following occurs,

(3.7) $$\mu^+ - \frac{\omega}{2^{s_0+2}} < f^+$$

(3.8) $$\mu^- + \frac{\omega}{2^{s_0+2}} \le f^-$$

(3.9) $$|\mu^-| \le \frac{\omega}{2^{s_0+2}} .$$

In such a case by (3.3) and (3.2)

(3.10) $$f^- > f^+ - \frac{\omega}{2^{s_0+2}} \ge \mu^+ - \frac{\omega}{2^{s_0+1}} > \frac{\omega}{2^{s_0+1}} .$$

Let s^* be the number claimed by lemma 2.5. We may assume that

(3.11) $$\frac{\omega}{2^{s^*}} > R^{N\kappa/2}$$

otherwise the proposition is trivial for the choice $\bar{s} = s^*$.

Also we may assume that $\exists(\bar{x},\bar{t}) \in Q_{R/4}$ such that

(3.12) $$u(\bar{x},\bar{t}) < \mu^- + \omega/2^{s^*+4} .$$

In fact if $y(x,t) \in Q_{R/4}$, $u(x,t) \geq \mu^- + \omega/2^{s^*+4}$, we have

$$-\inf_{Q_{R/4}} u \leq -\mu^- - \omega/2^{s^*+4} ,$$

and adding $\sup_{Q_{R/4}} u$ on the left hand side and μ^+ on the right hand side we obtain

$$\operatorname*{osc}_{Q_{R/4}} u \leq \omega(R^{1-N\kappa})\left(1 - \frac{1}{2^{s^*+4}}\right) ,$$

and again the proposition becomes trivial, for the choice $\bar{s} = s^* + 4$ and any $A > 1$.

Consequently in view of (3.12) the assumptions of lemma 2.6 must be violated and therefore we must have

$$(3.13) \qquad \operatorname{meas}\{(x,t) \in Q_{R/2} \mid u(x,t) < \mu^- + \frac{\omega}{2^{s^*+2}}\}$$

$$> c_0\left(\frac{\omega}{2^{s^*}}\right)^{\frac{N+2\kappa_1}{2\kappa_1}} \operatorname{meas}[Q_{R/2}] .$$

Analogously also the hypotheses of lemma 2.5 must be violated, i.e. either

$$(3.14) \quad \operatorname{meas}\{[u \leq 0] \cap Q_{2R}\} > \left(\frac{\omega}{2^{s^*}}\right) \operatorname{meas}\{[u < \mu^- + \frac{\omega}{2^{s^*+1}}] \cap Q_{R/2}\}$$

or

$$(3.15) \qquad |\mu^-| > \frac{\omega}{2^{s^*}} ,$$

or both.

Suppose that (3.15) holds. Then for all $s > s^*$ we have that

$$k = \mu^- + \frac{\omega}{2^s} < 0$$

and therefore $(x,t) \to (u(x,t) - (\mu^- + \frac{\omega}{2^s}))^-$, satisfies (2.13) for all $s > s^*$, $0 < \rho < R^{1-N\kappa}$, $\eta = 1$, and all σ_1 , $\sigma_2 \in (0,1)$. By the results of [6], there exists $s_1 \in \mathbb{N}$, $s_1 > s^*$ such that

$$u(x,t) \geq \mu^- + \frac{\omega}{2^{s_1}} , \ \forall (x,t) \in Q_{R/2} ,$$

and this implies the proposition, for the choice $\bar{s} = s_1$, and every $A > 1$.

Consequently we may assume that (3.13) and (3.14) both hold.

Combining these two inequalities, in what follows, we may assume that for all

$$2R \leq \rho \leq 8R$$

$$\text{meas}\{[u\leq 0] \cap Q_\rho\} > \gamma_0 \omega^b \, \text{meas}[Q_\rho] , \tag{3.16}$$

where

$$b = \frac{N+4\kappa_1}{2\kappa_1} = \frac{1+2\kappa}{\kappa} , \tag{3.17}$$

$$\gamma_0 = \frac{c_0}{2^{s^* b}} \inf_{2R\leq\rho\leq 8R} \frac{\text{meas}[Q_{R/2}]}{\text{meas}[Q_\rho]} .$$

Since s^* is determined only in terms of the data, we have that in (3.16) γ_0 and b are defined only upon the data and are independent of ω and R.

Notice that if R is small enough $8R < R^{1-N\kappa}$ and hence $Q_\rho \subset Q_{R^{1-N\kappa}}$, $2R \le \rho \le 8R$.

Lemma 3.2 Let $\eta \ge 1$, $2R \le \rho \le 8R$ and let (3.16) hold. Then $\forall s > s^*$

$$(3.18)\quad \sup_{t_0-\eta\rho^2\le t\le t_0} \text{meas}\{[u(\cdot,t)\le 0]\cap B_\rho\}$$
$$\ge \left(\frac{2^s}{\omega}\right)^2 \gamma_0 \omega^{b+1} \sup_{t_0-\eta\rho^2\le t\le t_0} \int_{B_\rho\times\{t\}} \left[u-\left(\mu^- + \frac{\omega}{2^s}\right)\right]^{-2} dx .$$

Proof From (3.16), $\exists t^* \in [t_0-\rho^2, t_0]$ such that

$$\text{meas}\{[u(\cdot,t^*)\le 0]\cap B_\rho\} \ge \gamma_0\omega^b \,\text{meas}[B_\rho] .$$

Now for all $t \in [t_0-\eta\rho^2, t_0]$, $(\eta\rho^2 < R^{2(1-N\kappa)})$

$$\text{meas}\,[B_\rho] \ge \text{meas}\{[u(\cdot,t) < \mu^- + \frac{\omega}{2^{s-1}}]\cap B_\rho\}$$
$$\ge \left(\frac{2^s}{\omega}\right)^2 \int_{B_\rho\times\{t\}} \left[\left(u-\left(\mu^- + \frac{\omega}{2^s}\right)\right)^-\right]^2 dx ,$$

and the lemma follows.

All the subsequent arguments will be carried over cylinders Q_ρ^η, $\eta > 1$ and will make use of inequality (2.14) written for

$$2R \le \rho \le 8R ,$$

$\sigma_1, \sigma_2 \in (0,1)$ such that $2R \le \rho - \sigma_1\rho \le 8R$, and for the levels

$$k = \mu^- + \frac{\omega}{2^s}, \quad s > s^*,$$

provided that $|\mu^-| < \omega/2^{s+1}$.

Taking in account lemma 2.4, from (2.14) we have

$$(3.18)\quad (\mu^- + \frac{\omega}{2^s})\left(\frac{2^s}{\omega}\right)^2 \omega^{b+1} \gamma_0 \sup_{t_0-\eta(1-\sigma_2)\rho^2 \le t \le t_0} \int_{B(\rho-\sigma_1\rho)\times\{t\}} [(u-(\mu^- + \omega/2^s))^-]^2\,dx$$

$$+ \|\nabla_x (u-(\mu^- + \frac{\omega}{2^s}))^-\|^2_{2,Q^\eta_\rho(\sigma_1,\sigma_2)}$$

$$\le \gamma[(\sigma_1\rho)^{-2} + (\sigma_2\rho^2)^{-1}]\, \|(u-(\mu^- + \frac{\omega}{2^s}))^-\|^2_{2,Q^\eta_\rho}$$

$$+ \frac{\gamma(\mu^- + \frac{\omega}{2^s})}{\sigma_2 \eta\rho^2}\, \text{meas}\{[u \le 0] \cap Q^\eta_\rho\}$$

$$+ \gamma\left\{\int_{t_0-\eta\rho^2}^{t_0} [\text{meas}\, A^-_{\mu^- + \frac{\omega}{2^s},\rho}(\tau)]^{\frac{r}{q}}\, d\tau\right\}^{\frac{2}{r}(1+\kappa)},$$

for all $s > s^*$ and $\eta > 1$ such that

$$(3.19)\qquad \eta\rho^2 < R^{2(1-N\kappa)}, \text{ and}$$

$$(3.20)\qquad |\mu^-| < \frac{\omega}{2^{s+1}}.$$

In the inequality (3.18), s need not be integer, as long as $s > s^*$.

<u>Lemma 3.3</u> Let $\theta_0 \in (0,1)$ be arbitrary but fixed. There exist two constants γ_1, γ_2 depending upon the data but independent of ω, R, θ_0, an integer $s_* > s^*$, $s_* = s_*(\omega,\theta_0)$ and $\eta = \eta(\omega)$ such that if

$$|\mu^-| < \omega/2^{s_*+1} ,$$

then either

(i) $$\omega^{1+b}\,\theta_0 \le \frac{\gamma_1}{\ln(\gamma_2/R)} , \text{ or}$$

(ii) $$\text{meas}\{(x,t) \in Q_{4R}^{\eta} \mid u(x,t) < \mu^- + \omega/2^{s^*}\} < \theta_0 \ \text{meas}[Q_{4R}^{\eta}] .$$

<u>Proof</u> Let s_* and η to be chosen and for simplicity set

$$\rho_0 = 4R .$$

We apply Lemma 2.1 to the function $u(\cdot,t)$ for $t \in [t_0-\eta\rho_0^2 , t_0]$, for the levels

$$\ell = \mu^- + \omega/2^s , \; k = \mu^- + \omega/2^{s+1} , \; s_* \ge s \ge s^* .$$

We observe that in view of assumption $[A_1]$

$$\text{meas}\{B(\rho_0)\backslash A^-_{\mu^- + \frac{\omega}{2^s},\rho_0}(t)\} \ge \alpha_* \text{meas}[B(\rho_0)]$$

for all $s \geq s^*$ and all $t \in [t_0-\eta\rho_0^2, t_0]$.

For notational simplicity also set

$$A_s(t) = A^-_{\mu^- + \frac{\omega}{2^s}, \rho_0}(t) \text{ , and}$$

$$A_s = \int_{t_0-\eta\rho^2}^{t_0} [\text{meas } A_s(\tau)] d\tau .$$

Then from Lemma 2.1 we have

$$\text{(3.21)} \qquad \frac{\omega}{2^{s+1}} A_{s+1}(t) \leq \gamma\rho_0 \int_{A_s(t)\setminus A_{s+1}(t)} |\nabla u| \, dx ,$$

for a new constant γ depending upon the data, and for all $t \in [t_0-\eta\rho_0^2, t_0]$. We integrate over $[t_0-\eta\rho_0^2, t_0]$, square both sides and use Hölder inequality on the right hand side to obtain

$$\text{(3.22)} \quad \left(\frac{\omega}{2^{s+1}}\right)^2 (\text{meas } A_{s+1})^2 \leq \gamma\rho_0^2 \iint_{Q^\eta_{\rho_0}} |\nabla(u-(\mu^- + \frac{\omega}{2^s}))^-|^2 \, dx \, d\tau$$

$$\times \text{ meas}[A_s \setminus A_{s+1}] .$$

The integral on the right hand side is majorized by using (3.18) written over $Q^\eta_{\rho_0}$, $Q^\eta_{2\rho_0}$. In this case $\sigma_1 = \sigma_2 = \frac{1}{2}$ and if $\eta > 1$ is chosen to satisfy

$$\text{(3.23)} \qquad 4\rho_0^2\eta < R^{2(1-N\kappa)} ,$$

then $Q^\eta_{2\rho_0} \subset Q_{R^{1-N\kappa}}$. Therefore taking in account the assumptions of the lemma and (2.11) we have

$$\iint_{Q^{\eta}_{\rho_0}} |\nabla(u-(\mu^- + \frac{\omega}{2^s}))^-|^2 \,dx\,d\tau \le \gamma \frac{1}{\rho_0^2} \left(\frac{\omega}{2^s}\right)^2 \text{meas}[Q^{\eta}_{\rho}]$$

$$+ \gamma \frac{1}{\rho_0^2} \left(\frac{\omega}{2^s}\right) \eta^{-1} \text{ meas } [Q^{\eta}_{\rho_0}] + \gamma \eta^{\frac{2}{r}(1+\kappa)} \rho_0^{N+N\kappa} .$$

We substitute this estimate in (3.22) and divide by $(\omega/2^{s+1})^2$ to obtain

$$(3.24)_s \qquad [\text{meas } A_{s+1}]^2 \le \gamma\left(1 + \frac{2^s}{\omega} \eta^{-1} + \eta^{\frac{2}{r}(1+\kappa)-1} \left(\frac{2^s}{\omega}\right)^2 R^{N\kappa}\right)$$

$$\times \text{ meas}[Q^{\eta}_{\rho_0}] \text{ meas } [A_s \setminus A_{s+1}] .$$

Inequality $(3.24)_s$ holds $\forall \eta > 1$, $\forall s > s^*$ as long as $|\mu^-| < \omega/2^{s+1}$. Let $s_* > s^*$ to be selected and assume that

$$|\mu^-| < \omega/2^{s_*+1} .$$

Then adding $(3.24)_s$ for $s = s^*$, $s^* + 1, \ldots, s_*-1$, we obtain

$$(3.25) \qquad [\text{meas } A_{s_*}]^2 \le \frac{\gamma}{(s_* - s^* - 1)} \left(1 + \frac{2^{s^*}}{\omega} \eta^{-1} + \eta^{\kappa} \left(\frac{2^{s^*}}{\omega}\right)^2 R^{N\kappa}\right) .$$

$$\times \left(\text{meas } [Q^{\eta}_{\rho_0}]\right)^2 ,$$

where we have used the fact that since $\eta > 1$ and $r > 2$ $\eta^{\frac{2}{r}(1+\kappa)-1} \le \eta^{\kappa}$.

Next we choose

(3.26) $$\eta = 2^{s_*} \omega^b .$$

This choice is admissible if $Q^{\eta}_{\rho_0} \subset Q_{R^{1-N\kappa}}$, i.e.

(3.27) $$2^{s_*}\omega^b < 2^{-6} R^{-2N\kappa} .$$

Moreover recalling the definition (3.17) of b,

$$\eta^{\kappa}\left(\frac{2^{s_*}}{\omega}\right)^2 R^{N\kappa} < \frac{\gamma}{\omega^{b+1}} 2^{s_*(2+\kappa)} R^{N\kappa}$$

$$\leq \gamma/\omega^{b+1} ,$$

if in addition to (3.27) we require that

(3.28) $$2^{s_*(2+\kappa)} R^{N\kappa} \leq 1 .$$

Suppose for the moment that (3.27) - (3.28) are satisfied; then substituting these estimates in (3.25) we deduce for a new constant γ

(3.29) $$[\text{meas } A_{s_*}]^2 \leq \frac{\gamma}{\omega^{b+1}(s_*-s^*)} (\text{meas } Q^{\eta}_{\rho_0})^2 .$$

Therefore to prove the lemma we have only to choose s_* so large that

(3.30) $$s_* \geq s^* + \frac{\gamma}{\omega^{b+1}\theta_0^2} .$$

Having now chosen s_* as in (3.30), if (3.28) is violated we have

$$2^{s^*} 2^{(\gamma/\omega^{b+1}\theta_0^2)} > R^{-N\kappa/(2+\kappa)} ,$$

and therefore by taking logarithms we see that there exist two constants γ_1, γ_2 which are independent of ω, θ_0, R such that

$$\omega^{1+b}\theta_0^2 \le \frac{\gamma_1}{\ln \gamma_2/R} .$$

A similar argument holds if (3.27) is violated, and therefore the lemma is proved.

Remark Lemma 3.3 says that if ω is not comparable to $1/|\ln R|$, then the set where u is close to its minimum, can be made arbitarily small relatively to a cylinder whose dimensions are comparable to $R^{1-\kappa N}$.

Lemma 3.4 There exists $\theta_0 = \theta_0(\omega)$, $s_* = s_*(\omega)$, $\eta = \eta(\omega)$, $\tilde{\gamma}_1$, γ_2, $\delta^* > 0$ such that if $|\mu^-| < \omega/2^{s_*+1}$, then either

(i) $$\omega \le \left[\frac{\tilde{\gamma}_1}{\ln(\gamma_2/R)}\right]^{\delta^*} , \text{ or}$$

(ii) $$u(x,t) \ge \mu^- + \frac{\omega}{2^{s_*+2}} , \quad \forall (x,t) \in Q_{2R}^{\eta(\omega)} .$$

<u>Proof</u> As before we set $\rho_0 = 4R$. Define

$$\rho_n = \frac{\rho_0}{2} + \frac{\rho_0}{2^{n+2}} \; ; \; \bar{\rho}_n = \frac{\rho_0}{2} + \frac{3\rho_0}{2^{n+4}} \; , \; n = 0,1,2,\ldots ,$$

and consider the cylinders $Q^{\eta}_{\rho_n}$ and

$$\bar{Q}^{\eta}_n = \{|x-x_0| < \bar{\rho}_n\} \times \{t_0 - \eta\rho^2_{n+1} , t_0\} ,$$

which satisfy the inclusion

$$Q^{\eta}_{\rho_{n+1}} \subset \bar{Q}^{\eta}_n \subset Q^{\eta}_{\rho_n} .$$

Construct smooth cutoff functions $x \to \zeta_n(x)$ as follows.

(i) $$\zeta_n(x) = 1 \; , \; |x-x_0| < \rho_{n+1}$$

(ii) $$\zeta_n(x) = 0 \; , \; |x-x_0| > \tfrac{1}{2}[\rho_n + \rho_{n+1}] = \bar{\rho}_n \; ,$$

(iii) $$|\nabla\zeta_n(x)| < 2^{n+4}/\rho_0 .$$

Define

(3.31) $$k_n = \mu^- + \frac{\omega}{2^{s_*+1}} + \frac{1}{2^n}\frac{\omega}{2^{s_*+1}} \; , \; n = 0,1,2,\ldots$$

and let $s_n \in [s_*, s_*+1]$ be defined by

$$k_n = \mu^- + \frac{\omega}{2^{s_n}} .$$

Let also

(3.32) $$y_n = \iint_{Q^{\eta}_{\rho_n}} [(u-k_n)^-]^2 dx\, d\tau$$

$$(3.33)\qquad z_n = \left\{\int_{t_0-\eta\rho_n^2}^{t_0} [\text{meas } A^-_{k_{n+1},\rho_n}(\tau)]^{\frac{r}{q}} d\tau\right\}^{\frac{2}{r}} .$$

The lemma will be proved if we can choose a priori $\theta_0(\omega)$, $s_* = s_*(\omega)$, $\eta = \eta(\omega)$ for which

$$Y_n \to Y_\infty = \iint_{Q_{2R}^{\eta(\omega)}} [(u-(\mu^- + \omega/2^{s_*(\omega)+1}))^-]^2 dx\, d\tau = 0$$

Define

$$Y_n = \left(\frac{2^{s_*}}{\omega}\right)^2 \frac{1}{\text{meas}[Q^\eta_{\rho_0}]} \iint_{Q^\eta_{\rho_n}} [(u-k_n)^-]^2 dx\, d\tau$$

$$Z_n = \eta^{-\frac{2}{r}} \frac{z_n}{\text{meas}[B(\rho_0)]} .$$

The quantities Y_n , Z_n satisfy the following recursion inequalities

$$(3.34)\qquad Y_{n+1} \le \frac{\gamma}{\omega^{b+1}} 2^{4n} \{Y_n^{1+\frac{2}{N+2}} + Y_n^{\frac{2}{N+2}} Z_n^{1+\kappa}\}$$

$$(3.35)\qquad Z_{n+1} \le \frac{\gamma}{\omega^{b+1}} 2^{4n}\{Y_n + Z_n^{1+\kappa}\} .$$

To prove (3.34) we apply the embedding lemma 2.1 to the function $(u-k_{n+1})^-\zeta_n$ over the cylinder $\bar{Q}^\eta_n$, for the choice of λ

$$(3.36)\qquad \lambda = 2^{s_*(\omega)} \omega^b .$$

We have

$$(3.37) \qquad \lambda^{\frac{2}{N+2}} Y_{n+1} \le \lambda^{\frac{2}{N+2}} \iint_{\bar{Q}_n^\eta} [(u-k_{n+1})^-]^2 \zeta_n^2 \, dx\, d\tau$$

$$\le \gamma(\text{meas}\,[u<k_{n+1}] \cap Q_{\rho_n}^\eta)^{\frac{2}{N+2}}$$

$$\times \Big\{ \lambda \sup_{t_0-\eta\rho_{n+1}^2 \le t \le t_0} \int_{B(\bar{\rho}_n)\times\{t\}} [(u-k_{n+1})^-]^2 dx$$

$$+ \| \nabla_x (u-k_{n+1})^- \|^2_{2,\bar{Q}_n^\eta}$$

$$+ \iint_{\bar{Q}_n^\eta} [(u-k_{n+1})^-]^2 \, |\nabla \zeta_n|^2 \, dx\, d\tau \Big\} .$$

We observe that

$$(3.38) \quad Y_n = \iint_{Q_{\rho_n}^\eta} [(u-k_n)^-]^2 dx\, d\tau \ge (k_{n+1}-k_n)^2 \ \text{meas}([u<k_{n+1}] \cap Q_{\rho_n}^\eta)$$

$$\ge 2^{-2n-4} \left(\frac{\omega}{2^{s_*}}\right)^2 \text{meas}([u<k_{n+1}] \cap Q_{\rho_n}^\eta) .$$

We majorize the first two terms in brackets on the right hand side of (3.37) by making use of (3.18) written for $s = s_{n+1}$, $|\mu^-| < \omega/2^{s_*(\omega)+2}$, over the pair of cylinders $\bar{Q}_n^\eta$ and $Q_{\rho_n}^\eta$, for which

$$(\sigma_1 \rho_n)^{-2} = \rho_0^{-2}\, 2^{2(n+4)} \ ; \ (\sigma_2 \rho_n)^{-2} \le \rho_0^{-2}\, 2^{n+3} .$$

Using these remarks in (3.37) we deduce, since $s_n \in [s_*, s_*+1]$,

$$(3.39)\qquad \lambda^{\frac{2}{N+2}} Y_{n+1} \le \gamma\left(\frac{2^{s*}}{\omega}\right)^{\frac{4}{N+2}} 2^{\frac{4}{N+2}n} Y_n^{\frac{2}{N+2}} \left\{ \frac{2^{2n}}{\rho_0^2} Y_{n+1} + 2^{2n}\left(\frac{\omega}{2^{s*}}\right)\eta^{-1} \operatorname{meas}([u<k_{n+1}] \cap Q^{\eta}_{\rho_n}) + z_n^{1+\kappa} \right\} .$$

We divide by $(\frac{\omega}{2^{s*}})^2$ meas $[Q^{\eta}_{\rho_0}]$ and use (3.38) and the definition of Y_n , Z_n to obtain

$$(3.40)\qquad \lambda^{\frac{2}{N+2}} Y_{n+1} \le 2^{4n} \eta^{\frac{2}{N+2}} \left\{ \left(1 + \frac{2^{s*}}{\omega}\eta^{-1}\right) Y_n^{1+\frac{2}{N+2}} + \eta^{\frac{2}{r}(1+\kappa)-1} \left(\frac{2^{s*}}{\omega}\right)^2 R^{N\kappa} Y_n^{\frac{2}{N+2}} Z_n^{1+\kappa} \right\} .$$

Next we choose

$$(3.41)\qquad \eta = 2^{s*(\omega)}\omega^{b} = \lambda .$$

Such a choice is possible if $\eta\rho_0^2 < R^{2(1-N\kappa)}$, i.e. if

$$(3.42)\qquad 2^{s*(\omega)}\omega^{b} < 2^{-6} R^{-2N\kappa} .$$

Moreover

$$\left(\frac{2^{s*}}{\omega}\right)^2 \eta^{\frac{2}{r}(1+\kappa)-1} R^{N\kappa} \le \gamma/\omega^{b+1} ,$$

if in addition to (3.42) we require that

$$(3.43)\qquad 2^{s*(\omega)[2+\kappa]} R^{N\kappa} \le 1 .$$

Assuming (3.42), (3.43) for the moment, inequality (3.34) follows.

Next to prove (3.35) we observe that $\forall\lambda > 0$, for the embedding lemma 2.1 and the remark following it

$$\lambda^{1-\frac{2}{r}} z_{n+1}(k_{n+1} - k_{n+2})^2 \le \lambda^{1-\frac{2}{r}} \|(u-k_{n+1})^-\|^2_{q,r,\bar{Q}_n^\eta}$$

$$\le \gamma\Big\{\lambda \sup_{t_0-\eta\rho^2_{n+1}\le t\le t_0} \int_{B(\bar{\rho}_n)\times\{t\}} [(u-k_{n+1})^-]^2 dx$$

$$+ \|\nabla(u-k_{n+1})^-\|^2_{2,\bar{Q}_n^\eta} + \frac{2^{n+4}}{\rho_0^2} \|(u-k_{n+1})^-\|^2_{2,\bar{Q}_n^\eta}\Big\}.$$

Estimating the right hand side as above and choosing λ as in (3.41), inequality (3.35) follows at once.

From inequalities (3.34) - (3.35) it follows, by virtue of Lemma 5.7 of [6] page 96, that $Y_n, Z_n \to 0$ as $n \to \infty$ if

$$Y_0 \le d \; ; \; Z_0^{1+\kappa} \le d, \tag{3.44}$$

where

$$d = \gamma^* \omega^{(1+b)\left(\frac{N+2}{2}\right)}, \tag{3.45}$$

for a (small) constant γ^* independent of R and ω.

The first of (3.44) holds if

$$(3.46)\quad Y_0 \le \left(\frac{2^{s_*(\omega)}}{\omega}\right)^2 \frac{1}{\text{meas}[Q^\eta_{\rho_0}]} \iint_{Q^\eta_{\rho_0}} [(u-(\mu^- + \omega/2^{s_*(\omega)}))^-]^2\, dx\, d\tau$$

$$\le \frac{\text{meas}[u<\mu^- + \frac{\omega}{2^{s_*}}] \cap Q^\eta_{\rho_0}}{\text{meas } Q^\eta_{\rho_0}} \le d = \gamma^*\omega^{(1+b)(\frac{N+2}{2})} .$$

To satisfy the second of (3.44), we observe that if $r \le q$

$$Z_0 \le \rho_0^{-N} \left\{\frac{1}{\eta} \int_{t_0-\eta\rho_0^2}^{t_0} [\text{meas } A^-_{\mu^- + \frac{\omega}{2^{s_*}},\rho_0}(\tau)]^{\frac{r}{q}} d\tau\right\}^{\frac{2}{r}}$$

$$\le \left(\frac{\text{meas }([u<\mu^- + \frac{\omega}{2^{s_*}}] \cap Q^\eta_{\rho_0})}{\text{meas }[Q^\eta_{\rho_0}]}\right)^{\frac{2}{q}}$$

and if $r > q$

$$Z_0 \le \left(\frac{\text{meas}[u<\mu^- + \frac{\omega}{2^{s_*}}] \cap Q^\eta_{\rho_0}}{\text{meas}[Q^\eta_{\rho_0}]}\right)^{\frac{2}{r}} .$$

Therefore both inequalities in (3.44) are satisfied if $s_*(\omega)$ is chosen in such a way that

$$(3.47)\quad \frac{\text{meas}[u<\mu^- + \frac{\omega}{2^{s_*}}] \cap Q^\eta_{\rho_0}}{\text{meas }[Q^\eta_{\rho_0}]} \le \gamma_*\omega^{\frac{(1+b)(N+2)}{4}\max\{r;q\}} .$$

Next select

$$(3.48) \qquad \theta_0 = \theta_0(\omega) = \gamma_* \omega^{\frac{(1+b)(N+2)}{4}\max\{r,q\}} ,$$

and then $\theta_0(\omega)$ being fixed choose $s_*(\omega)$ according to (3.30) and $\eta = \eta(\omega)$ according to (3.26).

By virtue of Lemma 3.3 if such choices are admissible then (3.47) is verified and part (ii) of the lemma is proved. If such choices are not admissible then

$$\omega^{1+b}\,\theta_0 \le \frac{\gamma_1}{\ln {}^{\gamma}2/R}$$

and in view of (3.48)

$$\omega \le \left[\frac{\tilde{\gamma}_1}{\ln {}^{\gamma}2/R}\right]^{\delta^*} , \quad \tilde{\gamma}_1 = \gamma_1/\gamma_* ,$$

where

$$\delta^* = \left[(1+b) + \frac{(1+b)(N+2)}{2}\max\{r,q\}\right]^{-1} ,$$

The lemma is proved.

Since $Q_{R/4} \subset Q_{\rho_0}^{\eta(\omega)}$, lemma 3.4 implies the following Corollary.

<u>Corollary 3.5</u> If the assumptions of lemma 3.4 are satisfied, then either

$$\operatorname*{osc}_{Q_{R/4}} u \le \omega(R^{1-N\kappa})\left(1 - \frac{1}{2^{s^* + {}^{\gamma}/\omega^{\delta^*-1}}}\right)$$

or

$$\operatorname*{osc}_{Q_{R/4}} u < \left[\frac{\tilde{\gamma}_1}{\ln {}^{\gamma}2/R}\right]^{\delta^*} .$$

Let now $s_*(\omega)$ be the number determined in Lemmas 3.3, 4. This number can be determined *a priori* and is independent of R ; it only depends upon ω .

Lemma 3.6 Assume that $|\mu^-| \geq \frac{\omega}{2^{s_*+1}}$. Then there exists $m \in \mathbb{N}$, independent of ω and R such that either

$$u(x,t) \geq \mu^- + \omega/2^{s_*+m} \qquad \forall (x,t) \in Q_{R/4}$$

or

$$\omega \leq \left[\frac{\tilde{\gamma}}{\dfrac{1}{\ln \tilde{\gamma} 2/R}}\right]^{\delta^*} .$$

Proof: The levels $k = \mu^- + \frac{\omega}{2^{s_*+n}}$ are negative $\forall n > 1$ and therefore the functions $(x,t) \to (u(x,t) - (\mu^- + \omega/2^{s_*+n}))^-$ satisfy (2.13) for all $n \geq 1$, $0 < \rho < R^{1-N\kappa}$ and all σ_1 , $\sigma_2 \in (0,1)$. The lemma follows now from the classical results of [6].

The proposition is proved as soon as we choose

$$\bar{s} = s^*+m \; ; \; a = \delta^{*-1} , \; A = \gamma \; .$$

Proof of the theorem. Since $Q_{R^{1+N\kappa}} \subset Q_{R/4} \subset Q_{R^{1-N\kappa}}$ if we set

$$R_* = R^{1-N\kappa} ,$$

Proposition 3.1 can be stated as

$$\underset{Q_{R_*^{1+\sigma}}}{\text{osc}}\, u \le \max\left\{\omega(R_*)\left(1 - \frac{1}{2^{\bar{s}+A/\omega^a(R_*)}}\right) ; \left[\frac{\tilde{\gamma}_1}{\ln \tilde{\gamma}_2/R_*}\right]^{\delta^*} ; \omega_f(R_*)\right\} ,$$

where

$$\sigma = 2N\kappa/(1-N\kappa) .$$

Let $R_* \le R_0$ be fixed and define two sequences of numbers as follows

$$R_1 = R_* \; ; \; R_{n+1} = R_n^{1+\sigma} \; , \; n = 1,2,\ldots \quad \omega_1 = 2M \quad \text{and}$$

$$\omega_{n+1} = \max\left\{\omega_n\left(1 - \frac{1}{2^{\bar{s}+A/\omega_n^a}}\right) ; \left[\frac{\tilde{\gamma}_1}{\ln \tilde{\gamma}_2/R_n}\right]^{\delta^*} ; \omega_f(R_n)\right\} .$$

Since R_n, $\omega_n \to 0$ as $n \to \infty$ the first part of the theorem is proved. The quantitative nature of the constants $\bar{s}$, A, σ, $\tilde{\gamma}_i$, $i = 1,2$, proves the second part by making use of an argument of [5] page 31-38.

4. Proof of the theorem: β of type II

We indicate here how the previous arguments can be modified to include the case of β of porous medium type. The notations in (2.5) - (2.6) are mantained. We assume for simplicity that β is differentiable; the general case can be recovered by the approximation procedure described in [4].

If $k \ge f^+$ or if $k \le f^-$, the function $(u-k)^{\pm}$ satisfies the following inequalities, which are analogous to (2.8) (see [4] for details).

$$(4.1)\qquad \sup_{t_0-\eta\rho^2\le t\le t_0}\int_{B(\rho-\sigma_1\rho)\times\{t\}}\left(\int_0^{(u-k)^{\pm}}\xi\beta'(k\pm\xi)\,d\xi\right)dx$$

$$+\ \|\nabla_x(u-k)^{\pm}\|^2_{2,Q_\rho^\eta}\le\gamma[(\sigma_1\rho)^{-2}+(\sigma_2\rho^2)^{-1}]\,\|(u-k)^{\pm}\|^2_{2,Q_\rho^\eta}$$

$$+\ \frac{\gamma}{\sigma_2\eta\rho^2}\iint_{Q_\rho^\eta}\left(\int_0^{(u-k)^{\pm}}\xi\beta'(k\pm\xi)\,d\xi\right)dx\,d\tau$$

$$+\ \gamma\left\{\int_{t_0-\eta\rho^2}^{t_0}[\text{meas }A^{\pm}_{k,\rho}(\tau)]^{\frac{r}{q}}\,d\tau\right\}^{\frac{2}{r}(1+\kappa)}.$$

We recall that $Q_\rho^\eta\subset Q_{R^{1-N\kappa}}$ and that $(x_0,t_0)\in S$.

We wish to show that in the cylinder $\mathcal{Q}_{3/4}$ the oscillation of u has decreased in a way quantitatively described by proposition 3.1.

The first part of the analysis in the proof of proposition 3.1 can be carried over to the present situation with minor changes.

The case to consider is when $|\mu^-|$ is small when compared to ω i.e. say $|\mu^-|<\omega/2^{s+1}$ for large s.

In such a case we write (4.1) for

$$k=\mu^-+\frac{\omega}{2^s}>0\ ,$$

and $(u-k)^-$, provided that $\mu^-+\frac{\omega}{2^s}\le f^-$.

We estimate the first term on the left hand side of (4.1) as follows

$$\int_{B(\rho-\sigma_1\rho)} \left(\int_0^{(u-k)^-} \xi\beta'(k-\xi)d\xi\right)dx$$

$$\geq \tfrac{1}{2}\beta'\left(\mu^- + \frac{\omega}{2^s}\right) \int_{B(\rho-\sigma_1\rho)} [(u-k)^-]^2 dx$$

$$\geq \tfrac{1}{2}\beta'\left(\frac{\omega}{2^s}\right) \int_{B(\rho-\sigma_1\rho)} [(u-k)^-]^2 dx \ .$$

The analogous term on the right hand side is estimated *above* by using (iv) in the definition II of β . We have

$$\frac{\gamma}{\sigma^2\eta\rho^2}\,\beta\left(\frac{\omega}{2^s}\right) \iint_{Q_\rho^\eta} (u-k)^- \,dx\,d\tau$$

$$\leq \frac{\tilde{\gamma}}{\sigma_2\rho^2}\left(\beta'\left(\frac{\omega}{2^s}\right)\eta^{-1}\right)\left(\frac{\omega}{2^s}\right)^2 \text{meas}([u<k] \cap Q_\rho^\eta) \ .$$

We remark that *in order* to control the growth of $\beta'(\frac{\omega}{2^s})$ with s , η has to be chosen to be of the same order. We may now complete the argument by proving, for the present case, lemmas analogous to lemmas 3.2 - 3.5, which lead to proposition 2.1. We leave to the reader the few technical modifications needed.

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Parabolic equations describing diffusion phenomena with change of phase are considered. It is demonstrated that weak solutions are continuous up to the parabolic boundary of the domain of definition. The continuity is quantitatively described by a modulus determined a priori only in terms of the data.

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