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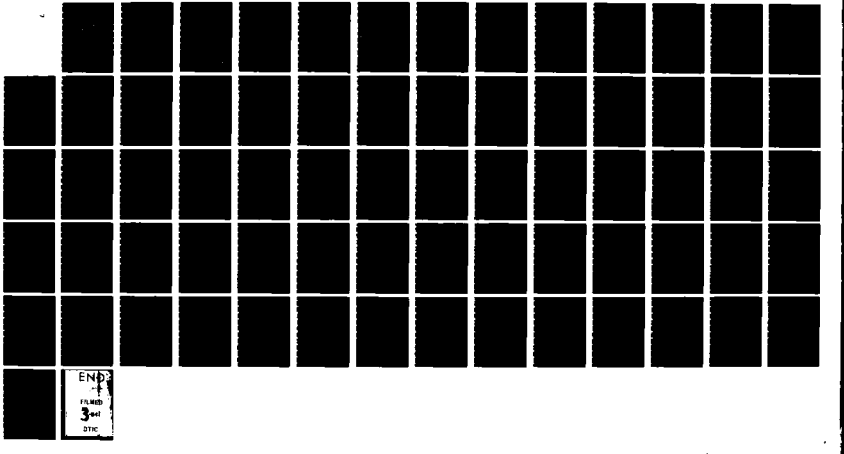

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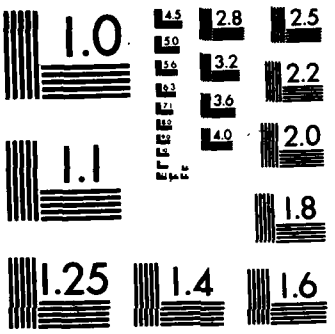

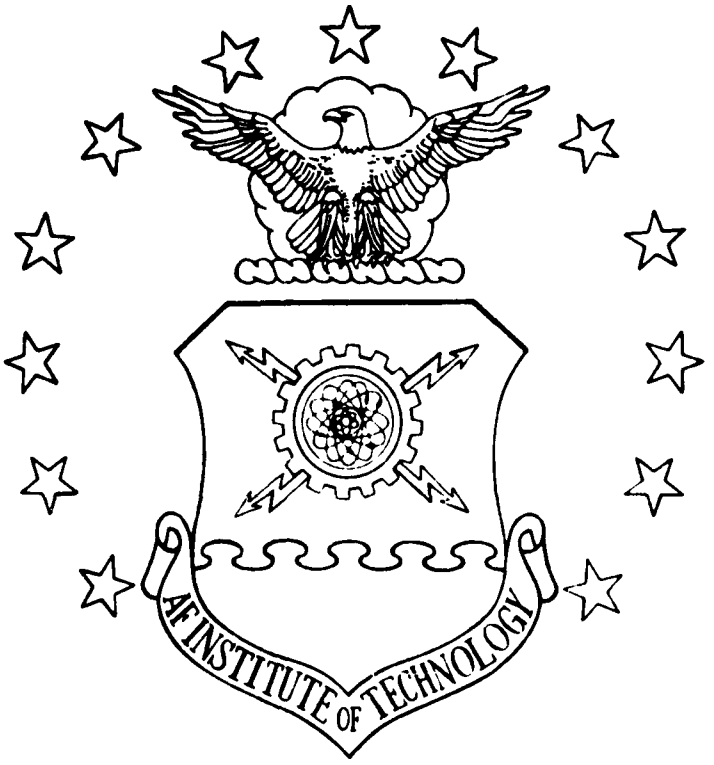

INTEGRAL GREEN'S FUNCTION SOLUTIONS TO DIFFERENTIAL EQUATIONS

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NUMERICAL GREEN'S FUNCTION SOLUTIONS TO DIFFERENTIAL EQUATIONS

THESIS

Presented to the Faculty of the School of Engineering
of the Air Force Institute of Technology
Air University
in Partial Fulfillment of the
Requirements for the Degree of
Master of Science

by
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PREFACE

This report is the result of ~~my~~ the author's investigations into the numerical solution of ordinary and partial differential equations. Green's Functions were used to convert the differential equations into integral form. The method of central finite differences is a common, well known method. The Method of Weighted Residuals was used as a third method of solution. The main purpose of this thesis was to compare the accuracy and advantages and disadvantages of the integral equation Green's Function method to the other methods. A second purpose was to add to ~~my~~ the author's own knowledge on how to solve differential equations. This effort also served to strengthen ~~my~~ his own abilities to write computer programs. I ~~have~~ has tried to give enough details on ~~my~~ his derivations and equations used in programing that others could use ~~them~~ these in solving their own problems with these methods.

I would like to thank Dr Bernard Kaplan, my adviser, for his guidance and advice throughout my effort. This thesis was sponsored by Nick Pagano, AFML/MLBM.

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ABSTRACT

Several methods of numerically solving differential equations in one and two dimensions were compared. The main method used was an integral equation solution using Green's Functions which were turned into matrix equations by using the Trapezoid Rule. The method of finite differences was used to turn differential equations into matrix equations which were solved using the Gaussian elimination method. Both a Laplacian Green's function and a Helmholtz Green's function were used to solve the one dimensional problem by turning it into integral equations. These integral equations were turned into matrix equations using the Trapezoid Rule. The finite difference method and the Laplacian Green's function method gave exactly the same results. The Helmholtz Green's Function gave slightly better results. Two types of the Method of Weighted Residuals were studied: the Galerkin Method and the Collocation Method. Both of them gave much better results than finite differences and the Green's Function methods did. The Collocation Method gave better results than the Galerkin method.

For the two dimensional problem an attempt was made to use Method of Weighted Residuals to reduce the partial differential equation to an ordinary differential equation which was then solved using the Green's Function method. Two different sries were used. In both cases the results

were very bad and did not get better with more points. The method finite differences gave reasonably good results for the partial differential equation. The method of separation of variables was examined to see if it could be used to reduce the partial differential equation to an ordinary differential equation whch would then be solved by the Green's Function Method. The analysis showed that this approach would not work very easily.

The programs were run on an Apple III personal computer and were written in UCSD PASCAL.

1. INTRODUCTION

1.1 Background

Many of the real world problems of interest to engineers and physicists can be expressed in mathematical form as differential equations or partial differential equations. These equations relate differentials of functions in time and space to each other, to other driving forces and to boundary or initial conditions. This thesis is only concerned with boundary condition type problems.

Boundary conditions can be classified into several types. Dirichlet boundary conditions specify the value of the unknown function everywhere on some boundary. Neumann boundary conditions specify the normal derivative (gradient) everywhere on the boundary. Mixed boundary conditions contain Dirichlet boundary conditions on part of the boundary and Neumann boundary conditions on part of the boundary. Cauchy boundary conditions specify both the function and its derivative everywhere on the boundary. Only certain types of boundary conditions will lead to unique solutions for different differential equations (Ref 5:706). Even when a unique solution can be shown to exist, finding it may be very difficult analytically. For these problems numerical approximation techniques are necessary.

This thesis will use the following numerical methods to

solve differential equations:

(1) The method of central finite differences(CFD)

(2) Green's Functions

(3) The Method of Weighted Residuals

Also this thesis will use the Gauss Elimination method to solve the matrix equations that arise and the Trapezoid rule to numerically evaluate any integrals.

1.2 One-Dimensional Case

A simple ordinary differential equation of the Helmholtz type was chosen. This permitted the use of two different Green's Functions (Laplacian and Helmholtz), the method of finite differences and two different weighted residual methods (Galerkin and Collocation). The relative accuracy of each of these methods will be compared. The Green's Function approach relies on defining a special function that incorpates the boundary conditions of the problem and which permits changing the differential equation into an integral equation which is sometimes easier to solve. The method of differences replaces the derivatives with differences between the values of the unknown function at nearby points and then solves the resulting set of linear equations. The Weighted Residual methods approximate the unknown function by some series of functions with unknown coefficients, substitutes it in the equation, and applies a

weighting factor, and then tries to solve the resulting equations for the coefficients.

1.3 Two-Dimensional Problem

A two dimensional heat conduction type problem was chosen with a separable driving function and functional dependence on one boundary and zeroes on the other boundaries. The central finite difference method was used again. Then attempts were made to reduce the partial differential equation to ordinary differential equations that could be solved by the Green's Function Method.

1.4 Purpose

One purpose of this thesis was to learn more about some of the various numerical techniques used to solve differential equations, do numeical integration, and solve large sets of linear equations. Another was to compare various methods to each other in hopes of learning if the Green's Function integral method was any better to use than the CFD method which is in common use. Hajdin and Krajcinovic (Ref 1 and 2) contend that since numerical integration formulas are more accurate than numerical differntiation formulas, that methods like the Green's Function method should be more accurate and more useful in

complex problems.

1.5 Plan of Development

First the theory and formulas will be developed for the various methods for the one-dimensional case. Next the algorythyms and computer programs will be written and tested. The programs will generate the data to compare the various methods. The procedures developed here will be needed for the two-dimensional problem. Then the theory and formulas for the two-dimensional problem will be developed. Finally the programs for the two-dimensional problem will be written, tested, and used to generate data.

1.6 Equipment

The programs will be written in PASCAL and run on my Apple III personel computer which has 128K bytes of memory.

2. THEORY

2.1 Green's Functions

Green's Functions are a common tool that arise in many branches of physics. Discussions on their use and derivation can be found in most math physics and electromagnetism textbooks (Ref 4-7). They are found in the solution of differential equations and integral equations.

One way of deriving them is by making use of the Dirac Delta function $\delta(x)$. The Dirac Delta function can be represented by the following limit:

$$\delta(x) = \lim_{g\to\infty} \frac{\sin gx}{\pi x} \tag{2.1.1}$$

Which has the following properties

$$\delta(x) = 0 \text{ for } x \neq 0 \tag{2.1.2}$$

$$\delta(x) = \infty \text{ for } x = 0 \tag{2.1.3}$$

These equations lead to the following useful result:

$$\int_{-\infty}^{\infty} \delta(x-x')f(x)\,dx = f(x') \tag{2.1.4}$$

This result will be used in the solution of the following

equation by the method of Green's Functions:

$$\frac{d^2y}{dx^2} = y'' = -F(x,y) \qquad (2.1.5)$$

Boundary Conditions: $y(0) = a \;,\; y(1) = b$ (2.1.6)

The Green's Function G(x,x') is defined by the following equations:

$$\frac{d^2G(x,x')}{dx^2} = G'' = -\delta(x-x') \qquad (2.1.7)$$

Boundary Conditions: $G(0,x') = G(1,x') = 0$ (2.1.8)

Multiply Eq 2.1.5 by G(x,x') and Eq 2.1.7 by y(x) and subtract Eq 2.1.7 from Eq 2.1.5 to get

$$Gy'' - yG'' = -F(x,y)G + y(x)\delta(x-x') \qquad (2.1.9)$$

However

$$\frac{d}{dx}[Gy' - yG'] = Gy'' - yG'' \qquad (2.1.10)$$

Therefore

$$\frac{d}{dx}[Gy' - yG'] = -F(x,y)G(x,x') + y(x)\,\delta(x-x') \qquad (2.1.11)$$

Multiply by dx, integrate from x=0 to x=1, and rearrange terms to get

$$y(x') = \left[Gy' - yG'\right]_0^1 + \int_0^1 F(x,y)\,G(x,x')\,dx \qquad (2.1.12)$$

Sustitute the boundary conditions (Eq 2.1.6 and 2.1.8) to get

$$y(x') = aG'(0,x') - bG'(1,x') + \int_0^1 F(x,y)\,G(x,x')\,dx \qquad (2.1.13)$$

Where

$$G'(x,x') = \frac{d}{dx}G(x,x') \qquad (2.1.14)$$

A more general result for three dimensions can be found by making use of Green's Theorem:

$$\int_V d^3x\,(y\nabla^2 G - G\nabla^2 y) = \oint_S da\left[y\frac{\partial G}{\partial n} - G\frac{\partial y}{\partial n}\right] \qquad (2.1.15)$$

Where n is the normal to the surface S.

$$\nabla^2 y = -F(\vec{x},y) \qquad (2.1.16)$$

$$\nabla^2 G = -\delta(\vec{x}-\vec{x}') \qquad (2.1.17)$$

for $\vec{x}$ on the surface S $\qquad G(\vec{x},\vec{x}') = 0 \qquad (2.1.18)$

Use Eq. 2.1.15 to 2.1.18 to get Eq 2.1.19 which is the three dimensional equivalent of Eq 2.1.12.

$$y(\vec{x}') = -\oint_S da \frac{\partial G(x,x')}{\partial n} y(x) + \int_V d^3x \; F(\vec{x},y)\, G(\vec{x},\vec{x}') \tag{2.1.19}$$

2.2 Finding Green's Functions

The method of finding the Green's Function G(x,x') that satisfies Eq 2.1.7 is fairly straightforward. Let

$$G(x,x') = \begin{cases} G_1(x,x') & \text{for } x<x' \\ G_2(x,x') & \text{for } x>x' \end{cases} \tag{2.2.1}$$

These functions satisfy the following equations:

$$\frac{d^2G_1}{dx^2} = 0 \quad \text{for } x<x' \tag{2.2.2}$$

$$\frac{d^2G_2}{dx^2} = 0 \quad \text{for } x>x' \tag{2.2.3}$$

Which have the general solutions of:

$$G_1 = A + Bx \quad \text{for } x<x' \tag{2.2.4}$$

$$G_2 = C + Dx \quad \text{for } x>x' \tag{2.2.5}$$

A,B,C, and D can be found by applying the following conditions:

$$G_1(0,x') = G_2(1,x') = 0 \qquad (2.2.6)$$

$$G_1(x',x') = G_2(x',x') \qquad (2.2.7)$$

$$\frac{d}{dx} G_2(x,x')\Big|_{x=x'} - \frac{d}{dx} G_1(x,x')\Big|_{x=x'} = -1 \qquad (2.2.8)$$

Eq 2.2.7 is true because we want the function G(x,x') to be continuous over the entire interval x=0 to x=1. Eq 2.2.8 comes from integrating Eq 2.1.7 once over x. Eq 2.2.8 shows that the derivative of G(x,x') is discontinuous at x=x'. Substitute the boundary conditions Eq 2.2.6 into Eq 2.2.4 and Eq 2.2.5 to get:

$$A = 0 \qquad C + D = 0 \qquad (2.2.9)$$

Use Eq 2.2.7 in Eq 2.2.4 and Eq 2.2.5 to get:

$$Bx' = C + Dx' \quad \text{or} \quad x'(B-D) = C \qquad (2.2.10)$$

Substitute Eq 2.2.4 and Eq 2.2.5 into Eq 2.2.8 to get:

$$D - B = 1 \qquad (2.2.11)$$

Eq 2.2.9,2.2.10, and 2.2.11 have the solution:

$$C = x', \quad D = -x', \quad B = 1 - x' \qquad (2.2.12)$$

Which yields the following Green's Function solution to Eq

2.1.7:

$$G(x, x') = \begin{cases} x(1-x') & \text{for } x < x' \\ x'(1-x) & \text{for } x > x' \end{cases} \qquad (2.2.13)$$

Eq 2.2.13 will be referred to in the rest of thesis as the Laplacian Green's Function since it is the solution of the one dimensional Laplacian Equation Eq 2.1.7.

Another Green's Function that will be useful is the solution to the Helmholtz Equation:

$$\frac{d^2G}{dx^2} + G(x, x') = -\delta(x, x') \qquad (2.2.14)$$

With the same homogenous boundary conditions Eq 2.2.6. This equation has the general solutions of:

$$G_1 = A\sin x + B\cos x \quad \text{for } x < x' \qquad (2.2.15)$$

$$G_2 = C\sin x + D\cos x \quad \text{for } x > x' \qquad (2.2.16)$$

Substitute Eq 2.2.6 in Eq 2.2.15 and Eq 2.2.16 to get:

$$A \cdot 0 + B \cdot 1 = 0 \Rightarrow B = 0 \qquad (2.2.17)$$

$$C\sin 1 + D\cos 1 = 0 \Rightarrow D = -C\tan 1 \qquad (2.2.18)$$

Substitute Eq 2.2.15 and Eq 2.2.16 into Eq 2.2.7 and Eq2.2.8 to get:

$$A = C\left[1 - \cot x' \tan 1\right] \tag{2.2.19}$$

$$C\left[\cos x' + \sin x' \tan 1\right] - A\cos x' = -1 \tag{2.2.20}$$

Eq 2.2.18, 2.2.19, and 2.2.20 can be solved to give:

$$G(x,x') = \begin{cases} \sin x \left[\cos x' - \dfrac{\sin x'}{\tan 1}\right] \text{ for } x < x' \\ \sin x' \left[\cos x - \dfrac{\sin x}{\tan 1}\right] \text{ for } x > x' \end{cases} \tag{2.2.21}$$

Eq 2.2.21 will be referred to in the rest of this thesis as the Helmholtz Green's Function.

A useful property and check on the solution is that most Green's Functions are symmetric, i.e. $G(x,x')=G(x',x)$, when the differential equations and boundary conditions are adjoint (Ref 5:873-874).

2.3 Method of Weighted Residuals

A second method of solving differential equations is the Method of Weighted Residuals. Consider the following differential equation where L is a differential operator:

$$L\,y(x) + F(x) = 0 \tag{2.3.1}$$

Expand y over some set of basis functions $\phi_i(x)$ to get:

$$y_N(x) = \sum_{i=1}^{N} c_i \phi_i(x) \tag{2.3.2}$$

Define the residual R(x) by:

$$R(x) \equiv L y_N + F(x) = L\left(\sum_{i=1}^{N} c_i \varphi_i(x)\right) + F(x) \qquad (2.3.3)$$

Let $W_j(x)$ be some set of weighting function. Then use the folowing set of equations to solve for c_i :

$$\int_0^1 W_j(x)\, R(x)\, dx = 0 \quad i,j = 1,2,\cdots,N \qquad (2.3.4)$$

This is the general form of the Method of Weighted Residuals. There are various names given to different choices of weighting functions $W_j(x)$:

Collocation Method: $W_j(x) = \delta(x - x_j)$ (2.3.5)

Galerkin Method $W_j(x) = \varphi_j(x)$ (2.3.6)

2.4 Method of Finite Differences

A third method to solve differential equations is the Method of Finite Differences. This method approximates derivatives with differences between values at nearby nodal points. To get these aprroximations we make use of the Taylor series to expand y(x) at the points x-h, x+h, and

x+2h:

$$y(x-h) = y(x) - h\,y'(x) + \frac{h^2}{2}y''(x) - \frac{h^3}{6}y'''(x) + \frac{h^4}{24}y^{IV}(x) - \frac{h^5}{120}y^{V}(x) + O(h^6) \quad (2.4.1)$$

$$y(x+h) = y(x) + h\,y'(x) + \frac{h^2}{2}y''(x) + \frac{h^3}{6}y'''(x) + \frac{h^4}{24}y^{IV}(x) + \frac{h^5}{120}y^{V}(x) + O(h^6) \quad (2.4.2)$$

$$y(x+2h) = y(x) + 2h\,y'(x) + 2h^2y''(x) + \frac{4}{3}h^3y'''(x) + \frac{2}{3}h^4y^{IV}(x) + \frac{4}{15}h^5y^{V}(x) + O(h^6) \quad (2.4.3)$$

Where $O(h^6)$ refers to terms containing sixth and higher order powers of h. Now subtract Eq 2.4.1 from Eq 2.4.2, subtract Eq 2.4.2 from Eq 2.4.3, and add Eq 2.4.1 to Eq 2.4.2 to get the following equations:

$$y(x+h) - y(x-h) = 2h\,y'(x) + \frac{h^3}{3}y''' + O(h^6) \quad (2.4.4)$$

$$y(x+2h) - y(x+h) = h\,y'(x) + \frac{3}{2}h^2y''(x) + O(h^6) \quad (2.4.5)$$

$$y(x+h) + y(x-h) = 2y(x) + h^2y''(x) + \frac{h^4}{12}y^{IV}(x) + O(h^6) \quad (2.4.6)$$

These equations can be solved for y'(x) and y''(x) to get:

$$y'(x) = \frac{y(x+h) - y(x-h)}{2h} - \frac{h^2}{6}y'''(x) + O(h^4) \quad (2.4.7)$$

$$y'(x) = \frac{y(x+2h) - y(x+h)}{h} - \frac{3}{2}h\,y''(x) + O(h^3) \quad (2.4.8)$$

$$y''(x) = \frac{y(x+h) + y(x-h) - 2y(x)}{h^2} - \frac{h^2}{12}y^{IV}(x) + O(h^4) \quad (2.4.9)$$

The method of finite differences takes h to be some small number and neglects the second terms in these equations to get the following equations:

$$y'(x) \cong \frac{y(x+h)-y(x-h)}{2h} \qquad (2.4.10)$$

$$y'(x) \cong \frac{y(x+2h)-y(x+h)}{h} \qquad (2.4.11)$$

$$y''(x) \cong \frac{y(x+h)+y(x-h)-2y(x)}{h^2} \qquad (2.4.12)$$

Eq 2.4.10 is known as the central finite difference (CFD) approximation for y'(x). It has an associated error of order h^2. Eq 2.4.11 is known as the forward finite difference (FFD) approximation for y'(x). It has an associated error of order h. Therefore, using the CFD method with symmetric points around x should give more accurate results than the FFD method. Eq 2.4.12 is the CFD approximation to y''(x). It has an associated error of order h^2.

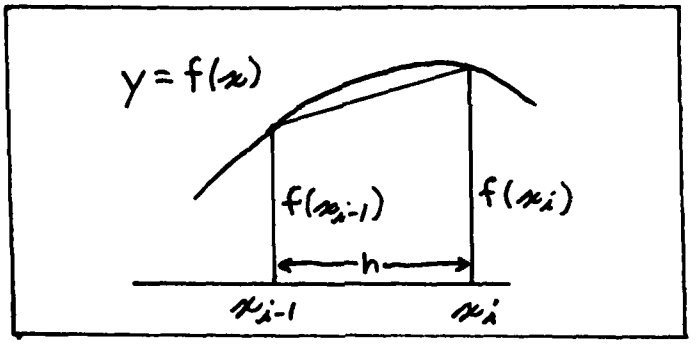
2.5 Integration Methods

When the Green's Function Method and the Method of Weighted Residuals are used to solved differential equations, integrals will need to be evaluated numerically. There are many methods of numerically calculating integrals. Generally they work by taking the values of the integrand f(x) at n+1 points, finding a polynomial of nth order that fits these n+1 points and then calculating the area under this polynomial. This procedure is repeated for small intervals until the range of integration is covered.

Trapezoid Rule

The Trapezoid Rule is the simplest method of doing numerical integration and is the method which will be used in this thesis. It uses first order polynomials (i.e. straight lines) to connect points on the curve y=f(x). These lines form trapezoids whose areas are easily calculated. The area under a typical trapezoid with a base length of h and sides of $f(x_{i-1})$ and $f(x_i)$ is:

$$\text{Area} = \frac{h}{2}\left[f(x_{i-1}) + f(x_i)\right] \qquad (2.5.1)$$



Figue 1. Trapezoid Area

So the combined total area of all trapezoids, over the interval $x_1 = a$ to $x_N = b$ is:

$$\text{Total Area} = \frac{h}{2}\left[f(x_1) + f(x_2)\right] + \frac{h}{2}\left[f(x_2) + f(x_3)\right] + \cdots + \frac{h}{2}\left[f(x_{N-1}) + f(x_N)\right] \qquad (2.5.2)$$

Therefore, the total area under the Trapezoid Rule is

$$\int_a^b f(x)\,dx = \frac{h}{2}\left[f(x_1) + 2f(x_2) + \cdots + 2f(x_{N-1}) + f(x_N)\right] \qquad (2.5.3)$$

This approximation has an associated error of order h^3.

Simpson's Rule

The next simplest method of numerical integration is Simpson's Rule which uses polynomials of order 2 (i.e. parabolas) to fit f(x). It can omly be used the number of intervals used to divide x=a to x=b is even. The resulting equation is (Ref 6:350-351):

$$\int_a^b f(x)\,dx = \frac{h}{3}\left[f(x_1) + 4f(x_2) + 2f(x_3) + 4f(x_4) + \cdots + 2f(x_{N-2}) + 4f(x_{N-1}) + f(x_N)\right] \qquad (2.5.4)$$

This approximation has an associated error of order h^5. An earlier thesis (Ref 3:26-30) found that the trapezoid rule gave more accurate results than Simpson's Rule for problems involving linear Green's functions. This is because at the point x=x' a parabola includes more incorrect area than trapezoids do for a linear Green's Function. Therefore, because it is simpler, can be used for any number of nodal points (Simpson's Rule requires even number of points), and because it gives better results for the Laplacian Green's Function this thesis will use the Trapezoid Rule when doing numerical integration.

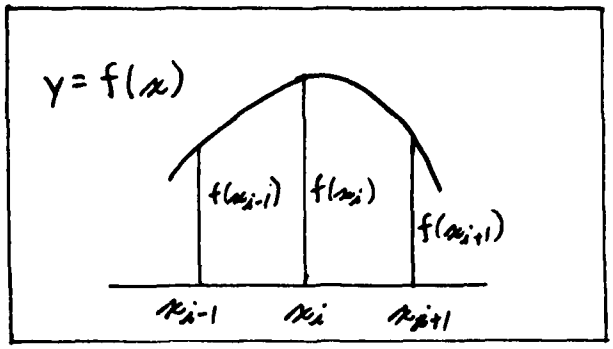


Figure 2. Parabloid Area

2.6 Converting Integral Equations to Matrix Equations

In sections 2.1,2.4, and 2.5 we have discussed various methods of solving differential equations. It will simplify notation and make it easier to see the methods of solution by adopting matrix notation and methods. Consider the following equation:

$$y(x) = F(x) + \int_0^1 G(x,x')\,f(x')\,dx' \qquad (2.6.1)$$

Use the Trapezoid Rule Eq 2.5.3 to replace the definite integral:

$$y(x_i) = F(x_i) + \sum_{j=1}^{N} D_j\, G(x_i,x_j)\, f(x_j) \qquad (2.6.2)$$

$$D_1 = D_N = \frac{h}{2} \qquad D_2 = D_3 = \cdots = D_{N-1} = h \qquad (2.6.3)$$

$$\text{let} \quad y(x_i) \equiv y_i \;;\; F(x_i) \equiv F_i \;;\; G(x_i,x_j) \equiv G_{ij} \qquad (2.6.4)$$
$$f(x_i) \equiv f_i$$

Use these results and definitions to write the following

matrix equation:

$$\begin{pmatrix} y_1 \\ \vdots \\ y_N \end{pmatrix} = \begin{pmatrix} F_1 \\ \vdots \\ F_N \end{pmatrix} + \begin{pmatrix} G_{11}f_1 & G_{12}f_2 & \cdots & G_{1N}f_N \\ \vdots & & & \vdots \\ G_{N1}f_N & \cdots & & G_{NN}f_N \end{pmatrix} \qquad (2.6.5)$$

In more compact form Eq 2.6.6 can be written as:

$$\bar{y} = \bar{F} + \overline{\overline{(Gf)}}\,\bar{D} \qquad (2.6.6)$$

Another equation that will be needed is the following integral equation:

$$y(x) = F(x) + \int_0^1 G(x,x')\,y(x')\,dx' \qquad (2.6.7)$$

Using the Trapezoid Rule Eq 2.6.7 can be written as:

$$y_i = F_i + \sum_{j=1}^{N} G_{ij}\,D_j\,y_j \qquad (2.6.8)$$

Or in matrix form as:

$$\bar{y} = \bar{F} + \bar{\bar{G}}\,\bar{\bar{D}}\,\bar{y} \qquad (2.6.9)$$

Where: $$\bar{\bar{G}} = \begin{pmatrix} G_{11} & \cdots & G_{1N} \\ \vdots & & \vdots \\ G_{N1} & \cdots & G_{NN} \end{pmatrix} \qquad (2.6.10)$$

$$\bar{\bar{D}} = \begin{pmatrix} D_1 & & 0 \\ & D_2 & \\ & & \ddots \\ 0 & & D_N \end{pmatrix} = [D_i \delta_{ij}] \qquad (2.6.11)$$

Eq 2.6.9 can be rewritten as:

$$[\bar{\bar{I}} - \bar{\bar{G}}\,\bar{\bar{D}}]\,\bar{y} = \bar{F} \qquad (2.6.12)$$

2.7 Converting CFD Into Matrix Equations

In section 2.4 the formulas for central finite differences were developed. In this section they will be used to turn the following differential equation into a matrix equation:

$$\frac{d^2y(x)}{dx^2} + g(x)\,y(x) = f(x) \qquad (2.7.1)$$

Sustitute Eq 2.4.12 into Eq 2.7.1 to get:

$$\frac{y(x+h) + y(x-h) - 2y(x)}{h^2} + g(x)y(x) = f(x) \qquad (2.7.2)$$

Make the following notation changes:

$$y(x_i) \equiv y_i \;;\; g(x_i) \equiv g_i \;;\; f(x_i) \equiv f_i \tag{2.7.3}$$

The resulting general equation is:

$$y_{i+1} + y_{i-1} + y_i(g_i - 2) = h^2 f_i \qquad \text{for } i = 2, 3, \cdots, N-1 \tag{2.7.4}$$

Let the boundary conditions on y(x) be:

$$y_1 = A \quad \text{and} \quad y_N = B \tag{2.7.5}$$

Eq 2.7.4 and Eq 2.7.5 generate the following set of linear equations:

$$\begin{aligned}
y_1 &= A \\
y_1 + (g_2 - 2)y_2 + y_3 &= h^2 f_2 \\
y_2 + (g_3 - 2)y_3 + y_4 &= h^2 f_3 \\
&\vdots \\
y_{N-2} + (g_{N-1} - 2)y_{N-1} + y_N &= h^2 f_{N-1} \\
y_N &= B
\end{aligned} \tag{2.7.6}$$

Eq 2.7.6 can be written in either of the following matrix forms:

$$\begin{pmatrix} 1 & 0 & 0 & 0 & 0 & \cdots & \\ 1 & (g_2-2) & 1 & 0 & 0 & \cdots & \\ 0 & 1 & (g_3-2) & 1 & 0 & \cdots & \\ \vdots & & & \ddots & & & \\ & & & & 1 & (g_{N-1}-2) & 1 \\ & & & & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} y_1 \\ y_2 \\ y_3 \\ \vdots \\ y_{N-1} \\ y_N \end{pmatrix} = \begin{pmatrix} A \\ h^2 f_2 \\ h^2 f_3 \\ \\ h^2 f_{N-1} \\ B \end{pmatrix} \qquad (2.7.7)$$

$$\begin{pmatrix} (g_2-2) & 1 & 0 & 0 & 0 & \cdots & \\ 1 & (g_3-2) & 1 & 0 & 0 & \cdots & \\ 0 & 1 & (g_4-2) & 1 & 0 & \cdots & \\ \vdots & \vdots & & \ddots & & & \\ & & & & 1 & (g_{N-1}-2) & \end{pmatrix} \begin{pmatrix} y_2 \\ y_3 \\ y_4 \\ \vdots \\ y_{N-1} \end{pmatrix} = \begin{pmatrix} h^2 f_2 - A \\ h^2 f_3 \\ h^2 f_4 \\ \\ h^2 f_{N-2} \\ h^2 f_{N-1} - B \end{pmatrix} \qquad (2.7.8)$$

$$\bar{\bar{A}}\,\bar{y} = \bar{B} \qquad (2.7.9)$$

2.8 Matrix Inversion

In sections 2.6 and 2.7 the following form of matrix equation has arisen (Eq 2.6.12 and Eq 2.7.9):

$$\bar{\bar{A}}\,\bar{x} = \bar{B} \qquad (2.8.1)$$

Where $\bar{\bar{A}}$ is the coefficient matrix, $\bar{x}$ is the unknown column matrix, and $\bar{B}$ is constant column matrix. The problem is to determine $\bar{x}$.

The most commonly employed method is the Gauss elimination method (Ref 4:1-4). This method makes use of the

fact that the following algeraic manipulations performed on a set of linear equations leaves their solution unchanged:

(1) Multiplication or division of any equation (row) by a constant

(2) Adding or subtracting one equation(row) to another equation(row).

Consider as an example the following set of four equations:

$$\begin{pmatrix} A_{11} & A_{12} & A_{13} & A_{14} \\ A_{21} & A_{22} & A_{23} & A_{24} \\ A_{31} & A_{32} & A_{33} & A_{34} \\ A_{41} & A_{42} & A_{43} & A_{44} \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} = \begin{pmatrix} B_1 \\ B_2 \\ B_3 \\ B_4 \end{pmatrix} \qquad (2.8.2)$$

The Gauss elimination method is simply a sequential application of row operations (1) and (2) above. The goal is to reduce all elements below the diagonal elements to zero. First subtract A_{21}/A_{11} times the first row from the second row, subtract A_{31}/A_{11} times the first row from the third row, and subtract A_{41}/A_{11} times the first row from the fourth row. This produces Eq 2.8.3 where all the elements in the first column are zero except for the first element:

$$\begin{pmatrix} A_{11} & A_{12} & A_{13} & A_{14} \\ 0 & A'_{22} & A'_{23} & A'_{24} \\ 0 & A'_{32} & A'_{33} & A'_{34} \\ 0 & A'_{42} & A'_{43} & A'_{44} \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} = \begin{pmatrix} B_1 \\ B'_2 \\ B'_3 \\ B'_4 \end{pmatrix} \qquad (2.8.3)$$

Next start with the second row and follow the same steps to reduce the third and fourth elements of the second column to zero. Finally, use the third row to reduce the fourth element of the third column to zero:

$$\begin{pmatrix} A_{11} & A_{12} & A_{13} & A_{14} \\ 0 & A_{22}' & A_{23}' & A_{24}' \\ 0 & 0 & A_{33}'' & A_{34}'' \\ 0 & 0 & 0 & A_{44}''' \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} = \begin{pmatrix} B_1 \\ B_2' \\ B_3'' \\ B_4''' \end{pmatrix} \qquad (2.8.4)$$

Eq 2.8.4 can now be used to find the solution matrix by simple substitution:

$$A_{44}''' x_4 = B_4''' \Rightarrow x_4 = B_4'''/A_{44}''' \qquad (2.8.5)$$

$$A_{33}'' x_3 + A_{34}'' x_4 = B_3'' \Rightarrow x_3 = \frac{B_3'' - A_{34}'' x_4}{A_{33}''} \qquad (2.8.6)$$

$$x_2 = (B_2 - A_{24}' x_4 - A_{23}' x_3)/A_{22}' \qquad (2.8.7)$$

$$x_1 = (B_1 - A_{14} x_4 - A_{13} x_3 - A_{12} x_2)/A_{11} \qquad (2.8.8)$$

During these operations the key elements are the diagonal elements which are known as the pivot elements. The method does not work if a pivot element is/becomes zero. Also if the pivot element is small compared to the other elements below it it will led to round-off errors. The solution is to switch rows around to put the largest element in the column at or below the diagonal in the diagonal position. This

comparison is done when the method is done with the the previous column.

Another thing to note is the types of matrices to be solved are different. The Green's Function method and the Method of Weighted Residuals generate 'dense' matrices where most of the elements are nonzero. The central finite difference method on the other hand generates a tridiagonal matrix where only the diagonal elements and the elements to each side are nonzero. The solution of the tridiagonal matrix is much simpler because of the fewer number of operations needed (of order N versus order N^3) and computer memory storage requirements can be reduced drastically from N*N to 3N. Also because of the fewer number of operations needed to solve a tridiagonal matrix, the accumlated round-off errors will be much smaller.

3. ONE DIMENSIONAL PROBLEM

3.1 Computer Programs

The computer programs used to solve a one dimensional problem for the various methods used were all very similiar. The flow diagrams appears in Figure 3.

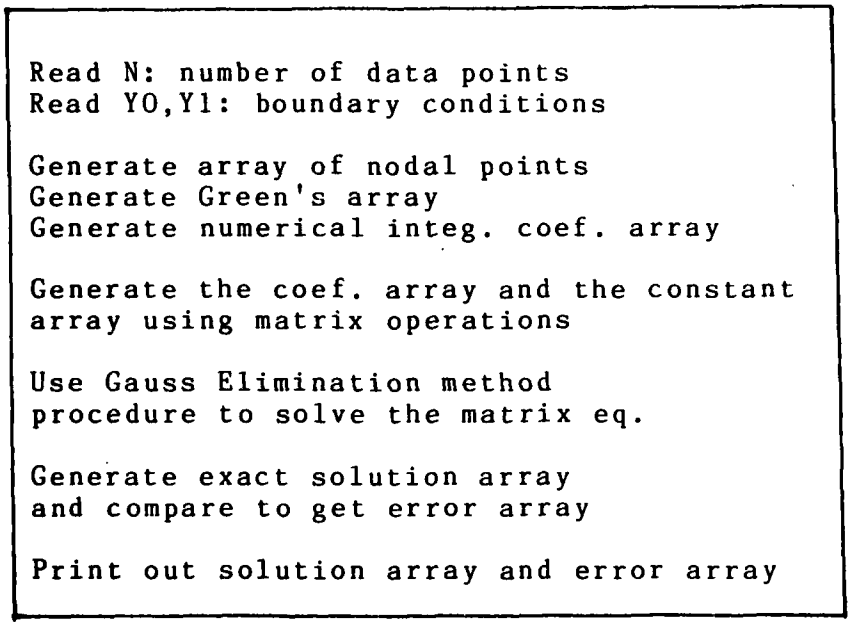


Fig. 3 Flow Diagram for Computer Programs

All that needed to be changed for each method used were the procedures that generated the matrices needed and the appropiate products of matrices.

3.2 Problem

The following one dimensional differential equation

will be solved by the Green's Function method using both Laplacian and Helmholtz Green's Functions, the method of central finite differences (CFD), and by the Method of Weighted Residuals (MWR) using collocation and Galerkin:

$$\frac{d^2y(x)}{dx^2} = y''(x) = -y(x) + x \qquad (3.2.1)$$

Boundary Conditions: $y(0) = y(1) = 0$ (3.2.2)

This equation was chosen beause it is the simplest second order differential equation whose Laplacian Green's Function solution gives a integral equation and also has a simple Helmholtz Green's Function solution. It will also show the essential features of CFD and MWR methnods.

The solution to Eq 3.2.1 is relatively easy. It is to see that the particular solution $y_p(x) = x$ satisfies the inhomogeneous Eq 3.2.1 This leaves the solution of the following homogenous equation to be found:

$$y'' + y = 0 \qquad (3.2.3)$$

Eq 3.2.3 can be solved by assuming that y(x) is a linear combination of exponentials. Substitute the following equation (Eq 3.2.4) in Eq 3.2.3 to get Eq 3.2.5:

$$y(x) = e^{Dx} \qquad (3.2.4)$$

$$(D^2+1)e^{Dx}=0 \tag{3.2.5}$$

Divide by the exponential to get:

$$D^2+1=0 \Rightarrow D=\pm i \tag{3.2.6}$$

Therefore the homogenous solution is:

$$y_h(x)=A'e^{ix}+B'e^{-ix}=A\cos x+B\sin x \tag{3.2.7}$$

Therefore the general solution is:

$$y(x)=y_p+y_h=x+A\cos x+B\sin x \tag{3.2.8}$$

Apply the boundary conditions Eq 3.2.2 to get:

$$y(0)=0=0+A\cdot 1+B\cdot 0 \tag{3.2.9}$$

$$y(1)=0=1+A\cos 1+B\sin 1 \tag{3.2.10}$$

Which have the solution:

$$A=0 \quad \text{and} \quad B=-\frac{1}{\sin 1} \tag{3.2.11}$$

Which gives the general solution as:

$$y(x) = x - \frac{\sin x}{\sin 1} \tag{3.2.12}$$

3.3 Green's Function Integral Equations

We know from section 2.1 and Eq 2.1.5 and Eq 2.1.12 that the Eq 3.2.1 can be put in the following form:

$$y(x') = \left[Gy' - yG'\right]_0^1 + \int_0^1 [y - x]\, G(x, x')\, dx \tag{3.3.1}$$

Use the boundary conditions:

$$y(0) = y(1) = G(0, x') = G(1, x') = 0 \tag{3.3.2}$$

to simplify Eq 3.3.1 to the following form:

$$y(x') = \int_0^1 (y - x)\, G(x, x')\, dx \tag{3.3.3}$$

The Laplacian Green's Function is (from Eq 2.2.13):

$$G_L(x, x') = \begin{cases} x(1 - x') & \text{for } x < x' \\ x'(1 - x) & \text{for } x > x' \end{cases} \tag{3.3.4}$$

Eq 3.3.3 can be turned into the following matrix equation (see section 2.6):

$$\bar{y} = \bar{\bar{G}}_L \bar{\bar{D}}\, \bar{y} - \bar{\bar{D}}\, \bar{x} \qquad (3.3.5)$$

Let $$\bar{B} \equiv -\bar{\bar{G}}_L \bar{x} \qquad (3.3.6)$$

Eq 3.3.5 can be put in the following form using matrix methods:

$$(\bar{\bar{I}} - \bar{\bar{G}}_L \bar{\bar{D}})\bar{y} = \bar{B} \qquad (3.3.7)$$

Eq 3.3.7 has the form of:

$$\bar{\bar{A}}\, \bar{y} = \bar{B} \qquad (3.3.8)$$

Where $$\bar{\bar{A}} = \bar{\bar{I}} - \bar{\bar{G}}_L \bar{\bar{D}} \qquad (3.3.9)$$

Eq 3.3.8 can be solved using Gaussian elimination (see section 2.8) to get the Laplacian Green's Function solution to y(x).

Another method of solving Eq 3.2.1 is to use the Helmholtz Green's Function. The Helmholtz Equation is:

$$y'' + y = x \qquad (3.3.10)$$

The Helmholtz Green's Function is (Eq 2.2.21):

$$G_H(x,x') = \begin{cases} \sin x \left[\cos x' - \frac{\sin x'}{\tan 1}\right] & \text{for } x < x' \\ \sin x' \left[\cos x - \frac{\sin x}{\tan 1}\right] & \text{for } x > x' \end{cases} \qquad (3.3.11)$$

The equivalent of Eq 3.3.3 is the following equation:

$$y(x) = -\int_0^1 x' G_H(x,x') dx' \qquad (3.3.12)$$

Which has the matrix form of (using the Trapezoid Rule):

$$\bar{y} = -\bar{\bar{G}}\,\bar{\bar{D}}\,\bar{x} \qquad (3.3.13)$$

Eq 3.3.13 can be solved easily by matrix multiplications without having to use Gaussian elimination.

3.4 Central Finite Difference Solution

The next method used to solve Eq 3.2.1 is the method of central finite differences (CFD) (see sections 2.4 and 2.7). Use Eq 2.4.12 to get:

$$y'' = \frac{y_{i+1} + y_{i-1} - 2y_i}{h^2} = -y_i + x_i \qquad (3.4.1)$$

This equation simplifies to the following set of linear equations:

$$y_{i-1} + (h^2 - 2)y_i + y_{i+1} = +h^2 x_i \qquad (3.4.2)$$

With the boundary condition equations:

$$y_1 = y_N = 0 \tag{3.4.3}$$

Eq 3.4.2 and Eq 3.4.3 can be turned into the following matrix equation and definitions:

$$\bar{\bar{A}}\,\bar{y} = \bar{B} \tag{3.4.4}$$

$$A = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & \cdots \\ 1 & (h^2-2) & 1 & 0 & 0 & \cdots \\ 0 & 1 & (h^2-2) & 1 & 0 & \cdots \\ \vdots & & & \ddots & & \\ & & & 1 & (h^2-2) & 1 \\ & & & & 0 \quad 0 & 1 \end{pmatrix} \tag{3.4.5}$$

$$\bar{y} = [y_1 \cdots y_N] \tag{3.4.6}$$

$$\bar{B} = [x_1 \cdots x_N] \tag{3.4.7}$$

Eq 3.4.4 can be solved by the Gauss elimination method to get the solution for y(x) at the nodal points.

3.5 Method of Weighted Residuals Solution

Eq 3.2.1 can also be solved using the Method of

Weighted Residuals (see section 2.3). Chose a set of basis functions which satisfy the boundary conditions $y(0)=y(1)=0$:

$$\phi_i(x) = x^i(1-x) \tag{3.5.1}$$

Substitute Eq 3.5.1 into Eq 2.3.2 to get an approximation to y(x):

$$y_m(x) = \sum_{j=1}^{m} a_j x^j (1-x) \tag{3.5.2}$$

The residual is:

$$R_m(x) \equiv y_m'' + y_m - x \tag{3.5.3}$$

$$R_m(x) = \sum_{j=1}^{m} a_j [\, j(j+1)x^{j-2} - j(j+1)x^{j-1}] \tag{3.5.4}$$
$$+ \sum_{j=1}^{m} a_j x^j (1-x) - x$$

$$R_m(x) = -x + \sum_{j=1}^{m} a_j [\, x^j(1-x) \tag{3.5.5}$$
$$+ j x^{j-2}\{j-1-x(j+1)\}]$$

Galerkin Method

For the Galerkin Method we use the functions in Eq 3.5.1 as the weighting functions and integrate from x=0 to x=1 and set the Residual equal to zero to get:

$$\int_0^1 R_m(x)\, q_i(x) = 0 \tag{3.5.6}$$

Substitute Eq 3.5.5 to get:

$$0 = \int_0^1 dx\; x^i(1-x)\left\{-x + \sum_{j=1}^{m} a_j\left[x^j(1-x) + j x^{j-2}\left\{j-1-x(j+1)\right\}\right]\right. \tag{3.5.7}$$

Simplify Eq 3.5.7 to get:

$$0 = \int_0^1 dx \left\{-x^{i+1}(1-x) + \sum_{j=1}^{m} a_j x^i(1-x)\left[x^j(1-x) + j x^{j-2}\left\{j-1-x(j+1)\right\}\right]\right. \tag{3.5.8}$$

Use the following result (Eq 3.5.9) to simplify Eq 3.5.8 and get Eq 3.5.10:

$$\int_0^1 x^m dx = \left[\frac{x^{m+1}}{m+1}\right]_0^1 = \frac{1}{m+1} \tag{3.5.9}$$

$$0 = \frac{1}{i+3} - \frac{1}{i+2} + \sum_{j=1}^{m} a_j\left[\frac{j(j-1)}{i+j-1} - \frac{2j^2}{i+j} + \frac{j(j+1)+1}{i+j+1} - \frac{2}{i+j+2} + \frac{1}{i+j+3}\right] \tag{3.5.10}$$

Eq 3.5.10 can be turned into the following matrix equation and definitions:

$$\bar{\bar{A}}\,\bar{a} = \bar{B} \tag{3.5.11}$$

$$A_{ij} = \frac{j(j-1)}{i+j-1} - \frac{2j^2}{i+j} + \frac{j(j+1)+1}{i+j+1} - \frac{2}{i+j+2} + \frac{1}{i+j+3} \tag{3.5.12}$$

$$B_i = \frac{1}{i+2} - \frac{1}{i+3} \tag{3.5.13}$$

Where $\bar{a}$ is the column matrix of a_i . Eq 3.5.11 can be solved using the Gauss elimination method.

Collocation Method

The collocation method uses a set of displaced Dirac Delta functions as the weighting functions in Eq 3.5.6:

$$\int_0^1 R_m(x)\,\delta(x-x_i)\,dx = 0 \qquad i = 1, 2, \cdots, N \tag{3.5.14}$$

Substitute Eq 3.5.5 into Eq 3.5.14 to get:

$$0 = -x_i + \sum_{j=1}^{m} a_j \left[x_i^j (1-x_i) + j\, x_i^{j-2} \left\{ j-1-x_i(j+1) \right\} \right] \tag{3.5.15}$$

Eq 3.5.15 can be put into the form of Eq 3.5.11 with the following definitions:

$$A_{ij} = x_i^j (1-x_i) + j\, x_i^{j-2} \left[j-1-x_i(j+1) \right] \tag{3.5.16}$$

$$B_i = +x_i \tag{3.5.17}$$

Eq 3.5.11 can again be solved by the Gauss Elimination method to get coef a_j .

To get y(x) for both Galerkin and Collocation methods substitute the solutions into Eq 3.5.2 and the values of the nodal points in x:

$$Ym(xi) = \sum_{j=1}^{m} aj\, xi^{j}(1-xi) \qquad (3.5.18)$$

3.6 Numerical Results and Comparisons

This section will present the numerical results and errors of the Laplacian Green's Function, Helmholtz Green's Function, central finite difference (CFD), Galerkin, and Collocation methods.

First, Table 1 shows results from the CFD method, the Laplacian and Helmholtz Green's Function methods and the exact value of y(x). Results are shown for one, three and five interior nodal points. Second, Table 2 shows the relative percentage errors of the results in Table 1.

Table 1. CFD, Laplacian, and Helmholtz Results for Different N and at Several Points

N	X	Y CFD	Y LAP	Y HELM	Y EXACT
1	0.5000	-0.07412857	-0.07142857	-0.06828782	-0.06974697
3	0.2500	-0.04427401	-0.04427401	-0.04378418	-0.04401368
	0.5000	-0.07015590	-0.07015590	-0.06938332	-0.06974697
	0.7500	-0.06040304	-0.06040305	-0.05974305	-0.06005621
5	0.1667	-0.03056336	-0.03056336	-0.03041289	-0.03048348
	0.3333	-0.05564811	-0.05564810	-0.05537471	-0.05550328
	0.5000	-0.06992782	-0.06992781	-0.06958544	-0.06974697
	0.6667	-0.06837619	-0.06837618	-0.06804304	-0.06820101
	0.8333	-0.04640671	-0.04640670	-0.04618201	-0.04628921

Table 2. Relative Percentage Errors of CFD, Laplacian, and Helmholtz Methods for Different N and at Several Points

N	X	Y CFD	Y LAP	Y HELM
1	0.5000	2.411	2.411	-2.092
3	0.2500	0.591	0.591	-0.521
	0.5000	0.586	0.586	-0.521
	0.7500	0.577	0.577	-0.521
5	0.1667	0.262	0.262	-0.232
	0.3333	0.261	0.261	-0.231
	0.5000	0.259	0.259	-0.232
	0.6667	0.257	0.257	-0.232
	0.8333	0.254	0.254	-0.232

Table 3. Relative Percentage Error Trends at X=.5 for Different N for CFD, Laplacian, and Helmholtz Methods

N	H	Y CFD	Y LAP	Y HELM
1	0.50000	2.41101	2.41101	-2.09206
3	0.25000	0.58629	0.58630	-0.52138
5	0.16666	0.25929	0.25928	-0.23159
9	0.10000	0.09313	0.93086	-0.08335
19	0.05000	0.02343	0.02313	-0.02094
29	0.03333	0.01052	0.01039	-0.00930

Table 3 shows the relative errors at x=0.5 as the number of nodal points (N) increases and the step size (H) decreases. It can be seen that the size of the errors decreases for smaller steps as would be expected. The most unexpected result in Tables 1,2, and 3 is that the CFD method and the Laplacian Green's Function method give exactly the same results (differences in the last decimal place are due to different accumulated round-off error). A

similiar result was gotten in a previous thesis (Ref 3:62-65). The Helmholtz Green's Function method gave slightly better results in all cases than the CFD and Laplacian Green's Function.

Table 4. Laplacian, Galerkin, and Collocation Results for Different N and at Several Points

N	X	Y LAP	Y GAL	Y COLLOC	Y EXACT
3	0.2500	-0.04427401	-0.04403237	-0.04396500	-0.04401368
	0.5000	-0.07015590	-0.06974640	-0.06974170	-0.06974697
	0.7500	-0.06040304	-0.06003848	-0.06009403	-0.06005621
5	0.1667	-0.03056336	-0.03048322	-0.03048368	-0.03048348
	0.3333	-0.05564811	-0.05550319	-0.05550334	-0.05550328
	0.5000	-0.06992782	-0.06974734	-0.06974698	-0.06974697
	0.6667	-0.06820101	-0.06820088	-0.06820092	-0.06820101
	0.8333	-0.04640671	-0.04628902	-0.04628904	-0.04628921

Table 5. Relative Percentage Errors of Lapacian, Galerkin, and Collocation Methods for Different N and at Several Points

N	X	Y LAP	Y GAL	Y COL
3	0.2500	0.59147	0.04247	-0.11058
	0.5000	0.58630	-0.00081	-0.00756
	0.7500	0.57751	-0.02952	0.62973
5	0.1667	0.26204	-0.00087	0.00025
	0.3333	0.26095	-0.00015	0.00011
	0.5000	0.25929	0.00053	0.00001
	0.6667	0.25687	-0.00019	-0.00013
	0.8333	0.25385	-0.00041	-0.00036

Table 6. Relative Percentage Error Trends at x=0.5 for Different N for Laplacian, Galerkin, and Collocation

N	H	Y LAP	Y GAL	Y COLLOC
3	0.25000	0.58629	-0.00081	-0.00756
5	0.16666	0.25929	0.00053	0.00001
9	0.10000	0.09313	0.00037	-0.00001
19	0.05000	0.02342	-0.00064	-0.00003
29	0.03333	0.01052	0.00004	0.00002

From Table 5 it can be seen that the Galerkin and Collocation methods are much more accurate then the Laplacian Green's Function Method (and CFD method). However, the size of the error varies by a large amount at various nodal points. From Table 6 it can be seen that there is nothing to be gained by going to large numbers of nodal points since the size of the error stopped decreasing at N=9 for the Galerkin method and at N=5 for the Collocation method. One potential problem with using the Galerkin method in more complicated probims is that doing the integral in Eq 3.5.6 will be more difficult. In fact, numerical integration maybe necessary to evaluate Eq 3.5.6 and this will decrease the accuracy of the final results. On the other hand, the Collocation method does not suffer from this problem of evaluating Eq 3.5.6 since it uses Eq 3.5.14 instead which is always easy to integrate because of the Dirac Delta function.

4. TWO DIMENSIONAL DIFFERENTIAL EQUATION

4.1 Problem

The following form of equation often arises in heat conduction problems and in electromagnetism:

$$\nabla^2 T + k^2 T = \frac{\partial^2 T}{\partial x^2} + \frac{\partial^2 T}{\partial y^2} + k^2 T = 0 \qquad (4.1.1)$$

If k*k takes the following form the above equation can be separated:

$$k^2 = a^2 + f(x) + g(y) \qquad (4.1.2)$$

Let $$T(x,y) = X(x)\cdot Y(y) \qquad (4.1.3)$$

Susitute Eq 4.1.2 and Eq 4.1.3 into Eq 4.1.1 to get:

$$\frac{d^2 X}{dx^2} + [b^2 + f(x)]X(x) = 0 \qquad (4.1.4)$$

$$\frac{d^2 Y}{dy^2} + [a^2 - b^2 + g(y)]Y(y) = 0 \qquad (4.1.5)$$

To get a problem that would result in an integral equation let us start with a T(x,y) and find out what k*k is:

$$T(x,y) = x\,y\cos\frac{\pi x}{2}\sin\frac{\pi y}{2} \qquad (4.1.6)$$

$$\frac{\partial^2 T}{\partial x^2} = x\cos\frac{\pi x}{2}\left[-\left(\frac{\pi}{2}\right)^2 y\sin\frac{\pi y}{2} + \pi\sin\frac{\pi y}{2}\right] \qquad (4.1.7)$$

$$\frac{\partial^2 T}{\partial y^2} = \left[-\left(\frac{\pi}{2}\right)^2 x \cos\frac{\pi x}{2} - \pi \sin\frac{\pi x}{2}\right] y \sin\frac{\pi y}{2} \qquad (4.1.8)$$

Therefore:

$$\frac{\partial^2 T}{\partial x^2} + \frac{\partial^2 T}{\partial y^2} = -\frac{\pi^2}{2} T - \pi y \sin\frac{\pi x}{2} \sin\frac{\pi y}{2} + \pi y \cos\frac{\pi x}{2} \cos\frac{\pi x}{2} \qquad (4.1.9)$$

Use Eq 4.1.10 and Eq 4.1.11 to simplify Eq 4.1.9 into Eq 4.1.12:

$$\pi y \sin\frac{\pi x}{2} \sin\frac{\pi y}{2} = \frac{\pi T}{x} \tan\frac{\pi x}{2} \qquad (4.1.10)$$

$$\pi x \cos\frac{\pi x}{2} \cos\frac{\pi y}{2} = \frac{\pi T}{y} \cot\frac{\pi y}{2} \qquad (4.1.11)$$

$$\therefore \frac{\partial^2 T}{\partial x^2} + \frac{\partial^2 T}{\partial y^2} + \left[\frac{\pi^2}{2} + \frac{\pi}{x}\tan\frac{\pi x}{2} - \frac{\pi}{y}\cot\frac{\pi y}{2}\right] T = 0 \qquad (4.1.12)$$

Eq 4.1.12 is the two dimensioal differential equation which will be used to try out the methods of solving it. The boundary conditions are:

$$T(0,y) = T(1,y) = T(x,0) = 0 \qquad (4.1.13)$$

$$T(x,1) = x \cos\frac{\pi x}{2} \qquad (4.1.14)$$

The exact solution to Eq 4.1.12 is Eq 4.1.6. Eq 4.1.12 can be put inthe following more compact form:

$$\frac{\partial^2 T}{\partial x^2} + \frac{\partial^2 T}{\partial y^2} + F(x,y)\,T(x,y) = 0 \qquad (4.1.15)$$

$$F(x,y) \equiv \frac{\pi^2}{2} + \frac{\pi}{x}\tan\frac{\pi x}{2} - \frac{\pi}{y}\cot\frac{\pi y}{2} \qquad (4.1.16)$$

4.2 Central Finite Difference Solution

Figure 3 illustrates the grid of points spaced h apart in x and y which will be used in the method of Central Finite Differences to solve Eq 4.1.15.

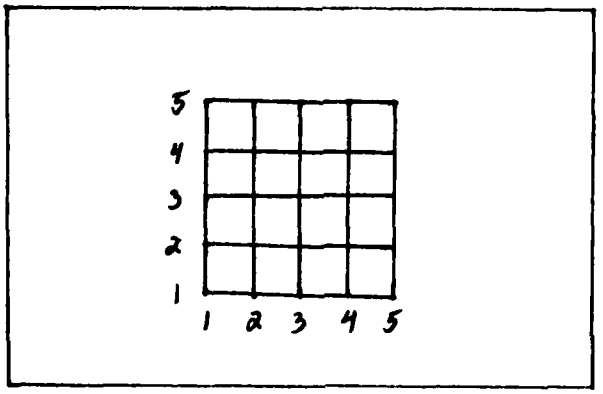


Figure 3. Two Dimensional Grid of Points

Recalling Eq 2.4.12 we can write the following equations:

$$\frac{\partial^2 T}{\partial x^2} = \frac{T_{I-1,J} + T_{I+1,J} - 2T_{I,J}}{h^2} \qquad (4.2.1)$$

$$\frac{\partial^2 T}{\partial y^2} = \frac{T_{I,J-1} + T_{I,J-1} - 2T_{I,J}}{h^2} \qquad (4.2.2)$$

Where $T_{IJ} \equiv T(x_I, y_J)$ (4.2.3)

Let $F_{IJ} = F(x_I, y_J)$ (4.2.4)

Subsitute to get:

$$\frac{T_{I-1,J} + T_{I+1,J} + T_{I,J-1} + T_{I,J+1} - 4T_{I,J}}{h^2} + F_{IJ}T_{IJ} = 0 \quad (4.2.5)$$

Multiply by h*h to get:

$$T_{I-1,J} + T_{I+1,J} + T_{I,J-1} + T_{I,J+1} + (h^2F_{IJ} - 4)T_{I,J} = 0 \quad (4.2.6)$$

Eq 4.2.6 upon careful examination can be converted into the following matrix equation:

$$\begin{pmatrix} (h^2F_{22}-4) & 1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 1 & (h^2F_{23}-4) & 1 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & (h^2F_{24}-4) & 0 & 0 & 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & (h^2F_{32}-4) & 1 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 1 & (h^2F_{33}-4) & 1 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 1 & (h^2F_{34}-4) & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 & 0 & (h^2F_{42}-4) & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 1 & (h^2F_{43}-4) & 1 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 1 & (h^2F_{44}-4) \end{pmatrix} \begin{pmatrix} T_{22} \\ T_{23} \\ T_{24} \\ T_{32} \\ T_{33} \\ T_{34} \\ T_{42} \\ T_{43} \\ T_{44} \end{pmatrix} = \begin{pmatrix} -T_{12}-T_{21} \\ -T_{13} \\ -T_{14}-T_{25} \\ -T_{31} \\ 0 \\ -T_{35} \\ -T_{51}-T_{15} \\ -T_{53} \\ -T_{45}-T_{54} \end{pmatrix} \quad (4.2.7)$$

The ones are located next to the diagonal and at N elements away from the diagonal element. Any of these elements will be zero instead if they correspond with elements on the boundary of the problem. The following conditions incorpate the boundary conditions (see Eq4.1.13 and Eq 4.1.14):

$$T_{1J} = T_{5J} = T_{I1} = 0 \quad (4.2.8)$$

$$T_{I5} = X_I \cos\left(\frac{\pi x_I}{a}\right) \quad (4.2.9)$$

Eq 4.2.7 has the form of:

$$\bar{\bar{A}}\,\bar{T} = \bar{B} \tag{4.2.10}$$

Eq 4.2.10 can be solved by the Gauss elimination to get the solution for T at the nodal points.

4.3 Method of Weighted Residuals Solution

Let T(x,y) be approximated by the following expansion:

$$T_N(x,y) = \sum_{i=1}^{N} x^i (1-x)\, c_i(y) \tag{4.3.1}$$

Then use Eq 4.3.1 to reduce Eq 4.1.15 to an ordinary differential equation which will be solved by the method Green's Functions:

$$\frac{\partial^2 T_N}{\partial y^2} = \sum_{i=1}^{N} x^i (1-x)\, c_i''(y) \tag{4.3.2}$$

$$\frac{\partial^2 T_N}{\partial x^2} = \sum_{i=1}^{N} c_i(y)\, i\, x^{i-2}\left[i-1-x(i+1)\right] \tag{4.3.3}$$

The residual R(x) is:

$$R(x,y) = \frac{\partial^2 T_N}{\partial x^2} + \frac{\partial^2 T_N}{\partial y^2} + F(x,y)\, T_N(x,y) \tag{4.3.4}$$

Substitute Eq 4.3.2 and Eq. 4.3.3 into Eq.4.3.4 and simplify

to get:

$$R(x,y) = \sum_{i=1}^{N} \left\{ c_i''(y)\, x^i (1-x) + c_i(y) \left[i x^{i-2} \{ i-1 - x(i+1) \} + F(x,y)\, x^i (1-x) \right] \right\} \tag{4.3.5}$$

Multiply by the shifted Dirac Delta weighting functions and integrate to get (see Eq 3.5.1):

$$0 = \sum_{i=1}^{N} \left\{ c_i''(y)\, x_j^{\,i} (1-x_j) + c_i(y) \left[i x_j^{\,i-2} \{ i-1 - x_j(i+1) \} + F(x_j,y)\, x_j^{\,i} (1-x) \right] \right\} \tag{4.3.6}$$

The only way that this equation can be solved is if each term for a different i value is equal to 0:

$$0 = c_i''(y)\, x_j^{\,i} (1-x_j) + c_i(y) \left[i x_j^{\,i-2} \{ i-1 - x_j(i+1) \} + F(x_j,y)\, x_j^{\,i} (1-x) \right] \tag{4.3.7}$$

This equation can be simplified into the following differential equation:

$$c_i''(y) + c_i(y)\, f_{ij}(y) = 0 \tag{4.3.8}$$

Where $$f_{ij}(y) = \frac{i[i-1-x_j(i+1)]}{x_j^2(1-x_j)} + F(x_j,y) \tag{4.3.9}$$

Next solve Eq 4.3.8 using a Laplacian Green's Function to get (see Eq 2.1.12):

$$C_i(y) = \left[G\, c_i'(y') - C_i(y') \frac{d}{dy'} G \right]_{y'=0}^{y'=1} + \int_0^1 C_i(y')\, f_{ij}(y')\, G(y,y')\, dy' \qquad (4.3.10)$$

Boundary Conditions are:

$$G(y,0) = G(y,1) = 0 \;;\; C_i(0) = 0 \qquad (4.3.11)$$

$$C_i(1) \neq 0 \qquad (4.3.12)$$

The Laplacian Green's Function is (see Eq 2.2.13):

$$G_L(y,y') = \begin{cases} y(1-y') & \text{for } y < y' \\ y'(1-y) & \text{for } y > y' \end{cases} \qquad (4.3.13)$$

The derivative of the Green's Function at y'=1 is:

$$\frac{d}{dy} G_L(y,y')\Big|_{y'=1} = -y \qquad (4.3.14)$$

Therefore Eq 4.3.10 becomes:

$$C_i(y) = y\, C_i(1) + \int_0^1 C_i(y')\, f_{ij}(y')\, G(y,y')\, dy' \qquad (4.3.15)$$

This can be put into the matrix form:

$$\bar{C}_i = \bar{y}\, C_i(1) + \overline{\overline{(Gf)}}\; \bar{\bar{D}}\, \bar{C}_i \qquad (4.3.16)$$

Therefore:

$$\left(\bar{\bar{I}} - \overline{\overline{(Gf)}}\, \bar{\bar{D}} \right) \bar{C}_i = \bar{y}\, C_i(1) \qquad (4.3.17)$$

To find the boundary conditions at y=1 use the following expansion:

$$x_j \cos\frac{\pi x_j}{2} = \sum_{i=1}^{N} x_j^i (1-x_j) C_i(1) \qquad (4.3.18)$$

This equation can be turned into the following matrix equation:

$$\overline{\overline{A}}\ \overline{C_i(1)} = \overline{B} \qquad (4.3.19)$$

$$A_{ij} = x_j^i (1-x_j) \qquad B_j = x_j \cos\frac{\pi x_j}{2} \qquad (4.3.20)$$

Eq 4.3.19 can be solved by Gauss elimination to get the to substitute into Eq 4.3.17 which can then be solved for . The main probem with this method is that turns out to be a function of the x used in the displaced Dirac Delta Functions. As a result we have to solve N ordinary differential equations for for each x for a total of N*N differentials equations solved (where N is the number of nodes along each axis).

4.4 Alternate MWR Solution

Another solution to Eq 4.1.15 uses the following expansion for T(x,y):

$$T_N(x,y) = \sum_{i=1}^{N} \sin(i\pi x)\, c_i(y) \tag{4.4.1}$$

This expansion gives the following expansion for the residual:

$$R(x,y) = \sum_{i=1}^{N} \left[c_i''(y) - i^2\pi^2 c_i(y) + F(x,y)\, c_i(y)\right] \cdot \sin(i\pi x) \tag{4.4.2}$$

Use the displaced Dirac Delta fuctions as weighting functions and integrate to get:

$$0 = \sum_{i=1}^{N} \sin(i\pi x_j)\left[c_i''(y) - i^2\pi^2 c_i(y) + c_i(y) F(x_j,y)\right] \tag{4.4.3}$$

Again the only way this equation can be satisfied is if each term for constant i is equal to zero:

$$c_i''(y) + c_i(y)\left[F(x_j,y) - (i\pi)^2\right] \tag{4.4.4}$$

Where F(x,y) is Eq 4.1.16. This equation can be changed into (see Eq 4.3.13):

$$c_i(y) = y\, c_i(1) + \int_0^1 dy'\, c_i(y')\left[F(x_j,y) - (i\pi)^2\right] \cdot G(y,y') \tag{4.4.5}$$

Eq 4.4.4 can be turned into the following matrix equation:

$$\bar{c}_i = \bar{y}\, c_i(1) + \bar{\bar{H}}\,\bar{\bar{D}}\,\bar{c}_i \tag{4.4.6}$$

$$(H_i)_{kl} = F(x_j,y_l)\, G(y_k,y_l) \tag{4.4.7}$$

Eq 4.4.6 can be solved to get:

$$(\bar{\bar{I}} - \bar{H}\,\bar{\bar{D}})\,\bar{C}_i = \bar{y}\,C_i(1) \qquad (4.4.8)$$

This equation can be solved by the Gauss elimination method.

4.5 Results

This section presents the results of solving of Eq 4.1.15 by the Method of Central Finite Diffferences and the Method of Weighted Residuals using a polynomial series (Eq 4.3.1) and sine series (Eq 4.4.1) to reduce the partial differential equation to an ordinary differential equation which is then solved by the Green's Function method. Table 7 shows the results of the CFD method and shows the exact result on the line below for 2x2 and 3x3 interior nodal points. Table 8 shows how the relative percentage error near the center decreases as the number of nodal points increases. 6x6 nodal points was as large as the program run on this Apple III could handle. The CFD method generates a tridiagonal matrix of size N*N by N*N which needs a number of operation of order N*N to solve on a computer.

Table 7. 2-D CFD/Exact Solutions at Each Point for N=2 and N=3

N		x=.3333	x=.6666	
2	y=.6666	0.169342	0.196272	
		0.166667	0.192450	
	y=.3333	0.483753	0.561300	
		0.481125	0.555556	
		x=.2500	x=.5000	x=.7500
3	y=.7500	0.161330	0.247200	0.201088
		0.160041	0.244981	0.198874
	y=.5000	0.082402	0.126317	0.102830
		0.081660	0.125000	0.101474
	y=.2500	0.022097	0.033879	0.027586
		0.022097	0.033825	0.027459

Table 8. Relative Percentage Errors Near the Center for 2-D CFD for Various N

N	Error(%)
2	1.98
3	1.05
4	0.78
5	0.50
6	0.40

Table 9 shows the results using the polynomial series, the sine series, and the exact result (in that order) at each nodal point for 2x2 and 3x3 interior nodal points. Table 10 shows the examples of the best and worst relative percentage errors for both methods for 2x2, 3x3, 4x4, and 6x6 interior nodal points.

As can seen the MWR method combined with the Green's

Function method did not give very good results. In fact, the best results came in the 2x2 case and got worse for for larger results. Besides the bad accuracy, another problem is that the MWR plus Green's Function method uses considerably more operations. The differential equations for $c_i(y)$ (Eq 4.3.6 and Eq 4.4.3) depend on the value of the x_j chosen in the Dirac Delta weighting function used in the Collocation Method , so there are N equations for N x_j values. Each of these equations generates a dense matrix of order N which require N*N*N operations each to solve. As a result the total number of operations required is of order N to the fifth for MWR/Collocation plus Green's Function method versus N*N for the CFD method.

Table 9. MWR Results:x (1-x), sin(iπ x), and Exact at Each Point for N=2 and N=3

N		x=.3333	x=.6667	
2	y=.6667	0.057980	0.200274	
		0.055061	0.192389	
		0.055556	0.192450	
	y=.3333	0.057221	0.018779	
		0.041218	0.152943	
		0.048113	0.166667	
		x=.2500	x=.5000	x=.7500
3	y=.7500	0.028060	0.104077	0.202004
		0.032901	0.118357	0.216602
		0.027459	0.101474	0.198874
	y=.5000	0.028303	0.110059	0.230413
		0.030039	0.114432	0.233472
		0.033825	0.1.2500	0.244981
	y=.2500	-0.027734	-0.077424	0.027208
		0.018811	0.072418	0.150224
		0.022097	0.081660	0.160041

Table 10. Relative Percentage Errors for 2-D MWR for Different N and Two Weighting Functions

N	x (1-x)		sin(iπ x)	
	Best	Worst	Best	Worst
2	4.07	18.93	-0.03	-14.33
3	1.57	-225.53	-8. 5	19.82
4	0.66	-164.09	-3.61	51.89
6	-0.70	101.90	-4.89	189.95

4.6 Separation of Variables

The method of separation of variables can be used to get the analytic solutions to partial differential equations. Another possible method of solving a 2-D partial

diffential equation would be to use the method of separation of variables to turn the partial differential equation into two ordinary differential equations which are then solved using the Green's Functon method. Consider a general heat conduction problem given in figure 5.

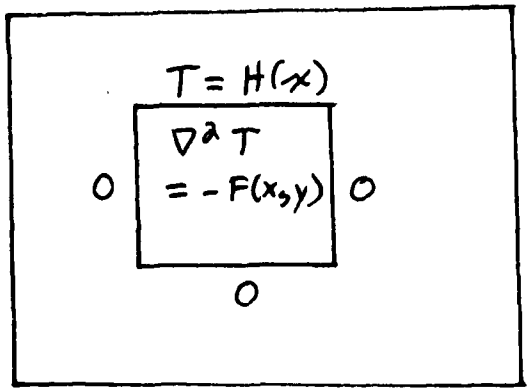


Figure 5. General Heat Conduction Problem

$$\nabla^2 T(x,y) = -F(x,y) \tag{4.6.1}$$

Boundary Conditions: $T(0,y) = T(1,y) = T(x,0) = 0$ (4.6.2)

$$T(x,1) = H(x)$$

A partial differential equation can be only separated if it takes the following form (Ref 4:498-499):

$$\nabla^2 T(x,y) + [a^2 + f(x) + g(y)]\, T(x,y) = 0 \tag{4.6.3}$$

Comparing Eq 4.6.1 and Eq 4.6.3 we see that only if

F(x,y) takes the following form will Eq 4.6.1 be separable:

$$F(x,y) = [a^2 + f(x) + g(y)]\, T(x,y) \qquad (4.6.4)$$

Letting T be separable we get the following results:

$$T(x,y) = X(x)\, Y(y) \qquad (4.6.5)$$

$$X''(x) + [b^2 + f(x)]\, X(x) = 0 \qquad (4.6.6)$$

Boundary Conditions: $X(0) = X(1) = 0$ (4.6.7)

$$Y''(y) + [a^2 - b^2 + g(y)]\, Y(y) = 0 \qquad (4.6.8)$$

Boundary Conditions: $Y(0) = Y(1) = 0$ (4.6.9)

Eq 4.6.6 and Eq 4.6.7 could now be solved using the Green's Function method. The problem that arises is that there is a pair of differentila equations (Eq 4.6.6 and Eq 4.6.8) to be solved for each choice of the separation constant a*a. As an example, the 2-D problem that we have been considering in the previous sections (Eq 4.1.12) becomes:

$$X''(x) + \left(a^2 + \frac{\pi}{x} \tan \frac{\pi x}{2}\right) X(x) = 0 \qquad (4.6.10)$$

Boundary Conditions: $X(0) = X(1) = 0$ (4.6.11)

$$Y''(y) + \left(b^2 - a^2 + \frac{\pi}{y} \cot \frac{\pi y}{2}\right) Y(y) = 0 \qquad (4.6.12)$$

Boundary Conditions: $Y(0) = Y(1) = 0$ (4.6.13)

For this problem there is only one valid choice for the separation constant:

$$a^2 = \frac{\pi^2}{4} \tag{4.6.10}$$

The difficulty that arises in the general problem, is finding the proper choice(s) for the separation constant and the proper weighting of each valid solution such that the weighted sum of products of individual solutions matches the boundary conditions. Combining this difficulty with the fact that potentially a large number of ordinary differential equations will have to be solved we see that this method is not a very viable or attractive method of solving partial differential equations.

5. CONCLUSIONS AND RECOMMENDATIONS

The purpose ot this thesis was to investigate a number of techniques that can be used to solve ordinary and partial differential equations. The main interest was on the use of Green's Functions in solving differential equations. The Method of Weighted Residuals was also used. The method of central finite differences was used since it is a commonly used method and it was desired to see how the other methods compared to it.

5.1 One Dimensional Problem

A one dimensional equation was used that could be solved using both the Laplacian Green's Function and the Helmholtz Green's Function. The Laplacian solution is an integral equation. The one dimensional equation also had nice solutions using the Galerkin and Collocation methods which are subtypes of the Method of Weighted Residuals. The Galerkin and Collocation methods both gave very accurate results for only a small number of points used. The Collocation method gave better results than the Galerkin method. The central finite difference method was easy to accomplish for this problem. The results were that CFD method and the Laplacian Green's Function method gave exactly the same results (to within accumulated round-off

error). This result was surprising considering the completely different matrices in each method. However, a previous thesis had also gotten a similiar result. The Helmholtz Green's Function method gave slightly better results than the CFD and Laplacian methods. The Galerkin and Collocation methods both gave much better results than the CFD or either Green's Function method.

5.2 Two Dimensional Problem

In the two dimensional case a partial differential equation was chosen that was separable and had the form of the Helmholtz Eq. The CFD method gave reasonably good results. Using the Method of Weighted Residuals to reduce the partial differential equation to an ordinary differential equation gave very bad results for both approximations that were tried. The method of separaion of variables was considered as a way of reducing a partial differential equation to ordinary differential equations which would then be solved using Green' Function method. This method was not actually solved because of difficulties in choosing the separation constants and in weighting the various solutions that each sepration constant generates. A two dimensional Green's Function solution was looked at during thesis preparation but the results were not presented in this thesis. The main difficulty was in the summation of

the infinite semi-periodic series expansion for the Green's Function.

5.3 Recommendations

When solving a one dimensional ordinary differential equation, the Weighted Residuals Methods should be considered. They appear to give very good results. They have the advantage that an approximate functional form is generated which can be used to generate an answer at any point and not just at the nodal points used. One disadvantage of Weighted Residual Methods is that the sizes of the relative errors vary considerably over the range of integration. The Method of central finite differences involves fewer operations than Weighted Residual Methods and the Green's Function methods. For simple problems, it is preferable to the Green's Function methods. For really complicated problems the Green's Functions methods may give better results, but this was not actualy shown to be true.

Using a Weighted Residual method to reduce a two dimensional partial differential equation to a one dimensional differential equation which is then solved by Green's Functions does not appear to work. The central finite difference method is much preferable as a method of solution to the two dimensional problem.

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18. SUBJECT TERMS (Continue on reverse if necessary and identify by block number): Partial Differential Equations, Green's Functions, Finite Element Analysis, Weighting Functions Residuals

19. ABSTRACT (Continue on reverse if necessary and identify by block number)

Several methods of numerically solving differential equations in one and two dimensions were compared. The main method used was a Green's Function integral solution. This method was compared with a central finite difference (CFD) solution and with two Weighted Residual methods: Galerkin and Collocation. In a one dimensional problem the Laplacian Green's Function method gave results that were the same as the CFD method. The Helmholtz Green's Function method gave slightly better results than the Laplacian Green's Function. The Galerkin and Collocation methods gave much better results than the Green's Function For a two dimensional problem the CFD method gave good results. The Collocation method was used to reduce the partial differential equation to an ordinary differential equation which was then solved by the Green's Function method. This method turned out to not work. Separation of Variables to reduce the partial differential equation to an ordinary differential equation which is then solved by the Green's Function method was also considered. However, this approach was shown not to be useful for most two dimensional problems.

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