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In a budget allocation problem, ~~a given budget must be allocated~~ each year to the procurement of aircraft and ships. It is desired that this budget be allocated in such a way as to keep the force mix between aircraft and ships at some constant level. This paper develops a methodology for optimally achieving this goal and presents software capable of doing this optimization.

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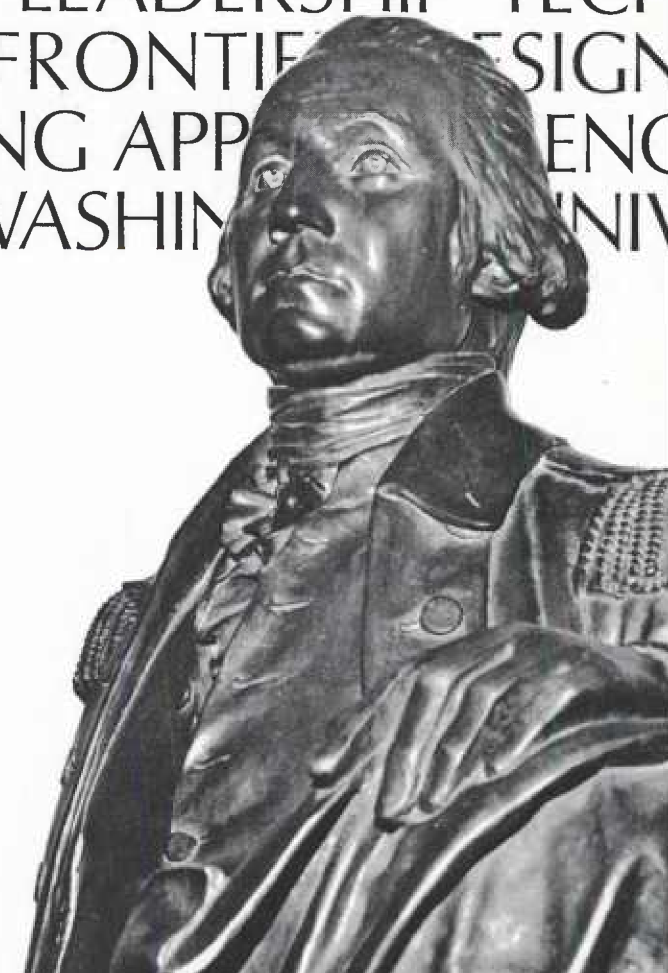

INSTITUTE FOR MANAGEMENT SCIENCE AND ENGINEERING

SCHOOL OF ENGINEERING AND APPLIED SCIENCE

OPTIMAL ALLOCATION OF A BUDGET
TOWARD PROCUREMENT OF
AIRCRAFT AND SHIPS

by

Mark R. Unkenholz

RESOURCE DYNAMICS
GWU/IMSE/Serial T-475/83
24 August 1983

THE GEORGE WASHINGTON UNIVERSITY
School of Engineering and Applied Science
Washington, DC 20052

Institute for Management Science and Engineering

Research Supported
by
Contract N00014-83-K-0215
Project NR 047 624
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In a budget allocation problem, a given budget must be allocated each year to the procurement of aircraft and ships. It is desired that this budget be allocated in such a way as to keep the force mix between aircraft and ships at some constant level. This paper develops a methodology for optimally achieving this goal and presents software capable of doing this optimization.

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OPTIMAL ALLOCATION OF A BUDGET TOWARD PROCUREMENT OF AIRCRAFT AND SHIPS

Mark R. Unkenholz

Prepared For OR-191: Problems in Operations Research

In fulfillment of a requirement of the
Bachelor's Program in Operations Research
at George Washington University

Professor: A. V. Fiacco

April 25, 1983

Sponsor: Prof. Rolf Clark
Dept. of Operations Research
George Washington University

PERSPECTIVE AND ACKNOWLEDGMENTS

This paper is the result of research conducted in conjunction with a course in Operations Research at George Washington University (OR-191 Problems in Operations Research) in fulfillment of a requirement for the Bachelor's program in Operations Research.

As part of this course, the students had to locate a "real world" Operations Research problem and, together with the help of a contact person with detailed knowledge of the specific problem to be solved, formulate a solution to the problem. In this project, the role of contact person was fulfilled by Dr. Rolf Clark, GWU. The problem described and solved in this paper arose as a result of his research into the underlying structure of U.S. Naval procurement and maintenance.

I would like to thank Dr. Clark for his help in defining and clarifying the problem and his valuable advice concerning various solutions proposed during the course of the research. I would also like to thank Dr. A.V. Fiacco, GWU, for his guidance and the perspective which he gave to this research.

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OPTIMAL ALLOCATION OF A BUDGET TOWARD PROCUREMENT OF AIRCRAFT AND SHIPS

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INTRODUCTION

Each year the U.S. Navy is faced with the problem of allocating its procurement budget to various items. For the Navy, the two primary items are ships and aircraft. This paper will restrict its attention to these two items and the procurement budget associated with them. Thus, each year a certain proportion of that budget must be allotted to the purchase of aircraft, with the remainder being spent on ships.

But what should that proportion be? The answer to that question depends on what the Navy considers to be an optimal force mix between aircraft and ships (i.e., the ratio of the number of aircraft to the number of ships). The objective of any budget allocation plan should be to achieve this optimal force mix ratio between aircraft and ships.

Constraining the Navy's attainment of this optimal force mix, however, is the procurement budget available and the respective costs of aircraft and ships over a number of years. If the available budget is small, the Navy may have to make a trade-off between aircraft and ship procurement, allowing one force (either aircraft or ships) to shrink in order to expand the other and so obtain the desired force mix ratio in that way. On the other hand, if available budgets are large, thus allowing an increase in both forces, the Navy could obtain its desired force mix by expanding both forces by some ratio and attain the goal with both forces at a higher level. In either case there arises the need to allocate the available budget in such a way as to stay as close to the desired ratio as possible over some planning period. While the Navy does not specifically plan forces based on a set ship to aircraft ratio, the historical facts indicate that a relatively constant ship to aircraft ratio results.

Thus, the objective of the optimal budgeting plan considered here will be, given information on budgets and costs over a planning period, the minimization of the deviation of the actual force mix ratio from the desired force mix ratio for all years in the planning period.

PARAMETERS

In developing an optimal budget allocation many parameters will need to be taken into account. An explanation of several of the most important ones follows.

First, there are the respective retirement rates for aircraft and ships. Ships in the fleet last, on average, 30 years. So, assuming the ages of the fleets ships to be spread somewhat uniformly between 0 and 30 years, one would expect to lose 1/30 (3.3%) of the fleet each year. Currently there are approximately 550 ships in the fleet. In order keep the fleet at its current level, 18 (550/30) new ships must be added each year. As for aircraft, the average aircraft lasts 15 years and so roughly 1/15 or 6.7% of the Navy's air fleet will retire each year. Currently the are 3500 aircraft supported by the Navy, so, to maintain current air force levels, approximately 235 new aircraft will be needed each year.

Second, there is the problem of spiraling aircraft costs. Since aircraft costs are increasing at a rate greater than either ship costs or the procurement budget growth rate, it becomes increasingly difficult, in later years, to maintain the desired force mix without allowing the number of ships in the fleet to decline.

A third parameter,bearing on an optimal budget allocation, is the leadtime for delivery of aircraft and ships. It is estimated that the leadtime for aircraft is 2 years and for ships 5 years. Hence, a budget allocation in a given year affects the ratio of aircraft to ships both two and five years from the allocation year. These differential leadtimes lead to an extensive interrelationship between the various years of budget allocation. For instance, the force mix ratio in year i is affected by the ratio in year i-1 and the budget allocations in years i-2 and i-5 which are in turn affected by actions or conditions in other years of the planning period. Thus, looking at any one year's budget and analyzing a certain allocation's effects on any one other year would not be valid. The budget allocation scheme must be viewed as a system spread over the many years of the planning period.

Finally, there is, to reiterate, the procurement budget's affect on the desired force mix. If the budgets are large, relative to costs, force levels will grow while if budgets are small levels will fall. This fluctuation of force levels with respect to the available budget will necessarily need to be taken into account when determining the optimal budget allocation (i.e., the budget allocations should be different for growing and contracting force levels).

THE MODEL

Now that some of the parameters affecting the optimal budget allocation have been explained, a description of the model to calculate the optimal proportion of the budget to be spent on aircraft each year, with remainder being spent on ships, can be made. The assumptions of the developed model are:

1) Retirement rates for aircraft and ships are constant (i.e., a fixed percentage of the total fleet of aircraft and ships will be retired each year);
2) Leadtimes for aircraft and ships are 2 and 5 years respectively;
3) There is a fixed planning period (some # of years) over which we are concerned about the force mix ratio;
4) The available budgets and costs of aircraft and ships are known for all years in the planning period.

The objective in choosing the proportion of the budget to be spent on aircraft, with the remainder being spent on ships, will be to minimize the sum of the squared deviations of the actual force mix ratio from the desired force mix ratio. In other words, we want a series of proportions, r_i, one for each year in the planning period, such that these r_i minimize the sum of the squared deviations of the actual ratio from the desired one. The reason for the squared deviation is the fact that ratios either above (yielding a positive deviation) or below (yielding a negative deviation) are considered equally costly, so a deviation of -1 or +1 should be treated the same. This is done through squaring the deviation. This method will force the actual ratio to stay as close as possible to the desired force mix ratio. This objective is then subject to the available budgets, costs of aircraft and ships, and further to the constraint that the r_i must, logically, be between 0 and 1.

Formulated in this manner, the problem of finding the proportion of the budget to be spent on aircraft each year is a nonlinear programming problem with the r_i proportions as the decision variables. From here, using the elements of multivariable calculus and optimization theory it is possible to develop a concise format for the stated problem, implement it on a computer, and solve it for any set of input data.

INPUTS

The inputs to the developed model are as follows:

1) Number of years in the planning period (up to 30)
2) Desired force mix ratio (aircraft to ships)
3) Current numbers of aircraft and ships
4) Available procurement budget (for aircraft and ships) for each year in the planning period
5) Aircraft cost in each year (average)
6) Ship cost in each year (average)
7) Retirement rates for aircraft and ships
8) Number of aircraft ordered 1 and 2 years ago
9) Number of ships ordered 1,2,3,4 and 5 years ago

It should be noted that inputs 8 and 9 represent backorders of aircraft and ships which will arrive during the planning period but which are not under the control of the optimal budget allocation scheme.

AN EXAMPLE

With these concepts in mind and with the developed software, it is now possible to construct a numerical example and show how the optimal budget allocation scheme might better achieve the desired force mix ratio than a less flexible approach.

Consider this:

1) A planning period of 20 years
2) A desired ratio of 7 aircraft to 1 ship
3) Current force levels of 3500 aircraft and 550 ships
4) Available budget starting at 23 billion, increasing at 3% per year
5) Aircraft cost starting at 40 million, increasing at 4% per year
6) Ship cost starting at 552 million, increasing at 3% per year
7) Retirement rates for aircraft and ships, 6.7% and 3.3% respectively
8) Backorders of aircraft and ships to just cover retirements, 235 aircraft per year, 18 ships

The results of the optimal budget allocation program are given below and on the following page. The r(i) column represents the proportion of the budget which should be spent on aircraft each year. Notice how close the optimal allocation scheme allows the actual ratio to stay to the desired ratio. Also notice the variablility in the proportion spent on aircraft. This is primarily due to the changing ratio of ship cost to aircraft cost.

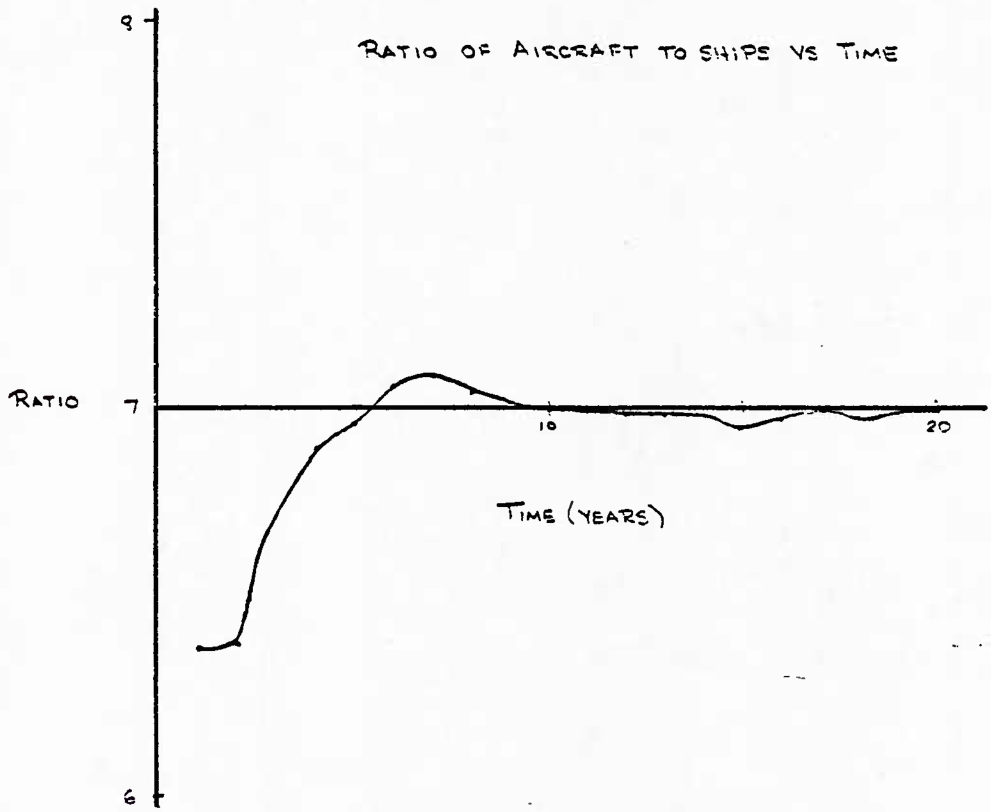
OPTIMAL BUDGET ALLOCATION PROGRAM

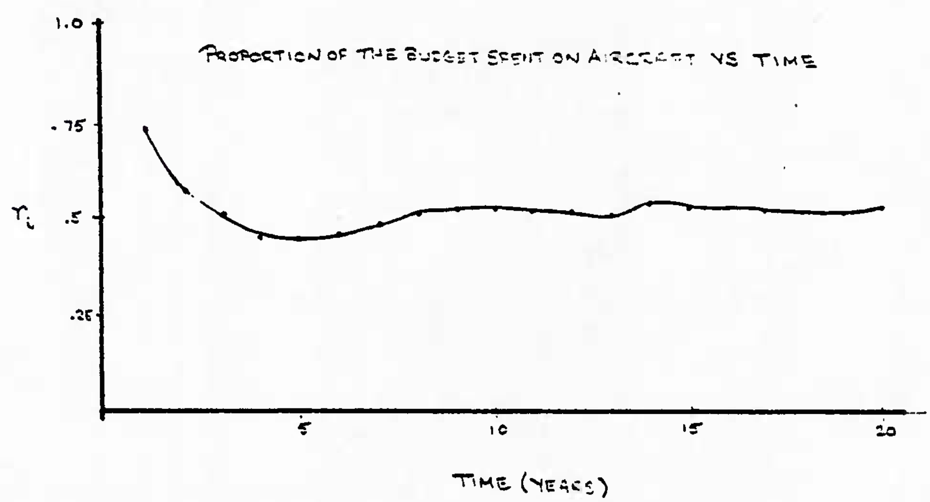
Desired Ratio = 7.00

Retirement Rate for Aircraft = 0.067
Retirement Rate for Ships = 0.033

Year	Budget (mil)	Cost Aircraft (mil)	Cost Ships (mil)	# of Aircraft Ordered	# of Ships Ordered	# of Aircraft	# of Ships	r(i)	Ratio
1	23000.00	40.00	552.00	415	11	3500	549	0.723	6.38
2	23690.00	41.60	568.56	328	17	3500	548	0.577	6.39
3	24400.70	43.26	585.62	282	20	3681	547	0.502	6.73
4	25132.72	44.99	603.19	260	22	3762	546	0.467	6.89
5	25886.70	46.79	621.28	258	22	3792	545	0.467	6.96
6	26663.30	48.67	639.92	257	22	3798	538	0.470	7.06
7	27463.20	50.61	659.12	265	21	3801	537	0.489	7.08
8	28287.09	52.64	678.89	272	20	3804	540	0.508	7.04
9	29135.70	54.74	699.26	278	19	3814	544	0.523	7.01
10	30009.77	56.93	720.23	275	19	3831	548	0.523	6.99
11	30910.07	59.21	741.84	271	19	3852	551	0.520	6.99
12	31837.37	61.58	764.10	267	20	3869	554	0.517	6.98
13	32792.49	64.04	787.02	258	20	3881	556	0.505	6.98
14	33776.26	66.60	810.63	278	18	3888	557	0.548	6.98
15	34789.55	69.27	834.95	273	18	3885	558	0.545	6.96
16	35833.23	72.04	860.00	271	18	3902	559	0.545	6.98
17	36908.23	74.92	885.80	267	19	3914	560	0.542	6.99
18	38015.47	77.92	912.37	264	19	3923	562	0.542	6.98
19	39155.94	81.03	939.74	261	19	3927	562	0.542	6.99
20	40330.61	84.27	967.93	260	18	3928	562	0.545	6.99

Optimal Allocation Results





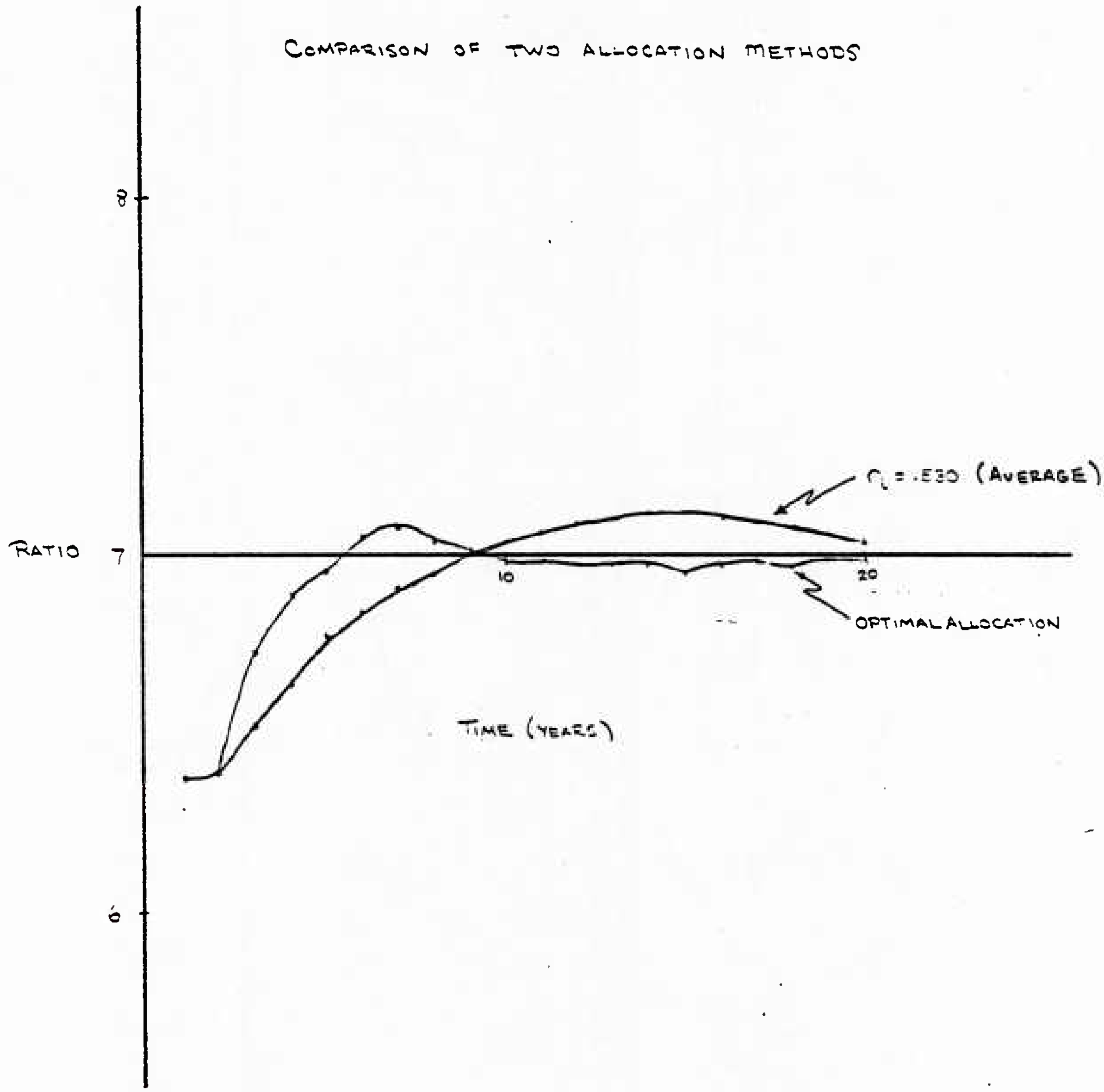
In contrast to that, consider the results of setting r(i) equal to the average proportion in the optimal plan for all years in the planning period. This results, see below, in a ratio of aircraft to ships which varies substantially from the desired ratio. Thus, in this example, the efficiencies gained through the mathematical programming formulation of the problem can be seen through its ability to remain very close to its objective while other methods cannot seem to equal this ability.

AVERAGE OPTIMAL ALLOCATION RESULTS

Desired Ratio = 7.00

Retirement Rate for Aircraft = 0.067
Retirement Rate for Ships = 0.033

Year	Budget (mil)	Cost Aircraft (mil)	Cost Ships (mil)	# of Aircraft Ordered	# of Ships Ordered	# of Aircraft	# of Ships	r(i)	Ratio
1	23000.00	40.00	552.00	304	19	3500	549	0.530	6.38
2	23690.00	41.60	568.56	301	19	3500	548	0.530	6.39
3	24400.70	43.26	585.62	298	19	3570	547	0.530	6.53
4	25132.72	44.99	603.19	296	19	3632	546	0.530	6.65
5	25886.70	46.79	621.28	293	19	3687	545	0.530	6.77
6	26663.30	48.67	639.92	290	19	3736	546	0.530	6.84
7	27463.20	50.61	659.12	287	19	3778	547	0.530	6.91
8	28287.09	52.64	678.89	284	19	3815	548	0.530	6.96
9	29135.70	54.74	699.26	282	19	3846	549	0.530	7.01
10	30009.77	56.93	720.23	279	19	3873	550	0.530	7.04
11	30910.07	59.21	741.84	276	19	3895	551	0.530	7.07
12	31837.37	61.58	764.10	274	19	3913	552	0.530	7.09
13	32792.49	64.04	787.02	271	19	3927	553	0.530	7.10
14	33776.26	66.60	810.63	268	19	3937	554	0.530	7.11
15	34789.55	69.27	834.95	266	19	3944	555	0.530	7.11
16	35833.23	72.04	860.00	263	19	3948	556	0.530	7.10
17	36908.23	74.92	885.80	261	19	3949	557	0.530	7.09
18	38015.47	77.92	912.37	258	19	3948	558	0.530	7.08
19	39155.94	81.03	939.74	256	19	3944	559	0.530	7.06
20	40330.61	84.27	967.93	253	19	3938	560	0.530	7.03


COMPARISON OF TWO ALLOCATION METHODS
8
RATIO
7
6
10
20
r_i = .530 (AVERAGE)
OPTIMAL ALLOCATION
TIME (YEARS)

Though the average proportion results presented in the foregoing example were not really "bad," the fact is that even these results could not have been obtained without the optimal budget allocation software. Hence, at least to a certain extent, the preceding example illustrates the efficacy of the method.

ANOTHER EXAMPLE

With the developed model an analyst or decision maker could, with a good deal of accuracy, assess the force mix impact of various changes to the model. For instance, the effects of very small budgets on absolute force levels could be studied given that the force mix was to be kept constant. Alternatively, one could examine the effects on absolute force levels of changes in the desired force mix ratio. As an example consider this:

1) The Navy wants to keep at least a 500 ship fleet
2) Budgets available are small (starting a $15 billion, increasing at 3% per yr)
3) The Navy would like to maintain a 7 to 1 ratio between aircraft and ships
4) Other model inputs are the same as in the last example

Are the above criteria in conflict with one another? Using the above information, the optimal allocation software was run and the results are printed below.

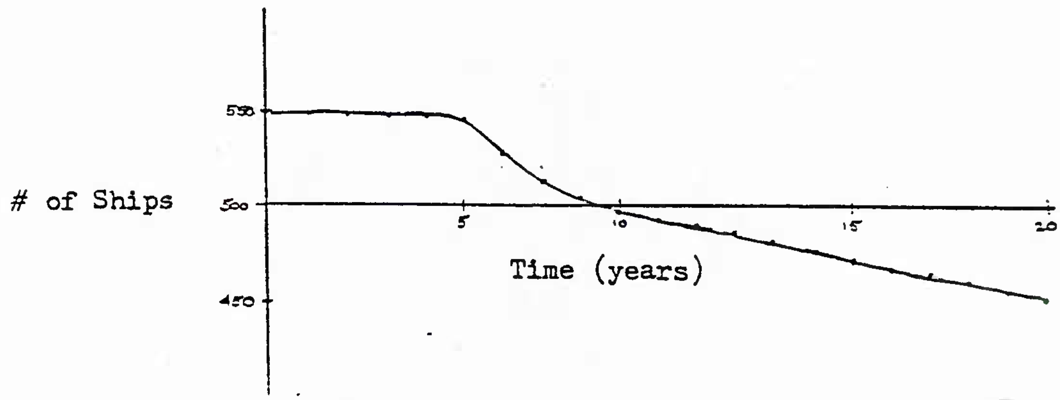
OPTIMAL BUDGET ALLOCATION PROGRAM

Desired Ratio = 7.00

Retirement Rate for Aircraft = 0.067
Retirement Rate for Ships = 0.033

Year	Budget (mil)	Cost Aircraft (mil)	Cost Ships (mil)	# of Aircraft Ordered	# of Ships Ordered	# of Aircraft	# of Ships	r(i)	Ratio
1	15000.00	40.00	552.00	375	0	3500	549	1.000	6.38
2	15450.00	41.60	568.56	307	4	3500	548	0.828	6.39
3	15913.50	43.26	585.62	260	7	3640	547	0.709	6.65
4	16390.90	44.99	603.19	221	10	3703	546	0.609	6.78
5	16882.63	46.79	621.28	177	13	3715	545	0.491	6.82
6	17389.11	48.67	639.92	182	13	3688	527	0.509	7.00
7	17910.78	50.61	659.12	191	12	3617	514	0.541	7.04
8	18448.10	52.64	678.89	197	11	3556	504	0.563	7.06
9	19001.55	54.74	699.26	197	11	3509	497	0.569	7.06
10	19571.59	56.93	720.23	199	11	3471	494	0.581	7.03
11	20158.74	59.21	741.84	197	11	3435	491	0.581	7.00
12	20763.50	61.58	764.10	179	12	3404	487	0.531	6.99
13	21386.40	64.04	787.02	183	12	3373	482	0.550	7.00
14	22028.00	66.60	810.63	189	11	3326	477	0.572	6.97
15	22688.84	69.27	834.95	187	11	3286	472	0.572	6.96
16	23369.50	72 ·4	860.00	201	10	3254	467	0.622	6.97
17	24070.58	74.92	885.80	199	10	3223	464	0.622	6.95
18	24792.70	77.92	912.37	197	10	3208	460	0.622	6.97
19	25536.48	81.03	939.74	153	13	3192	456	0.486	7.00
20	26302.58	84.27	967.93	175	11	3176	452	0.561	7.03

Number of Ships vs Time (7 to 1 Ratio)



It can be seen that given the presented criteria, a 500-ship fleet could not be maintained.

Now, however, say it was decided that a 5 to 1 ratio of aircraft to ships would be acceptable if the 500-ship fleet could be maintained. Running the model again with this new ratio the following results were obtained. It should be noticed that with the new ratio it is indeed possible to maintain a 500-ship navy, given the available budgets. Hence, the model can be used to clarify various inconsistencies in the inputs to the model allowing decision makers to adjust their thinking in such a way as to better achieve their goals.

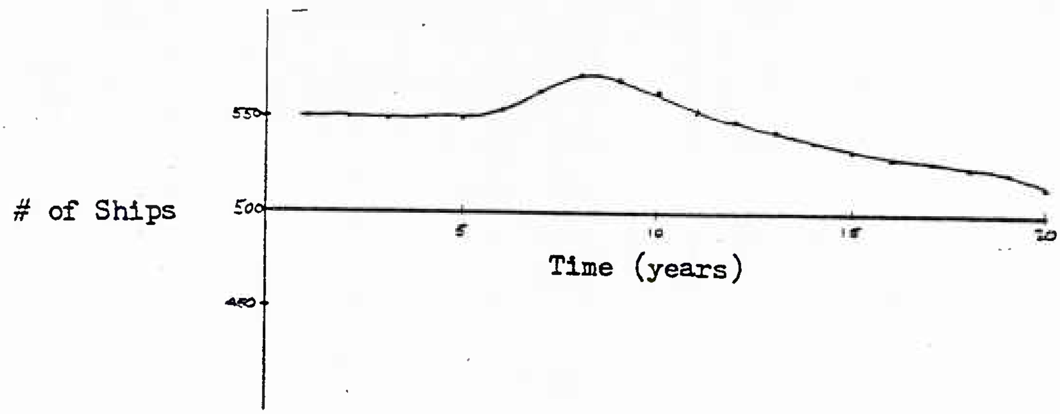
OPTIMAL BUDGET ALLOCATION PROGRAM

Desired Ratio = 5.00

Retirement Rate for Aircraft = 0.067
Retirement Rate for Ships = 0.033

Year	Budget (mil)	Cost Aircraft (mil)	Cost Ships (mil)	# of Aircraft Ordered	# of Ships Ordered	# of Aircraft	# of Ships	r(i)	Ratio
1	15000.00	40.00	552.00	0	27	3500	549	0.000	6.38
2	15450.00	41.60	568.56	0	27	3500	548	0.000	6.39
3	15913.50	43.26	585.62	0	27	3265	547	0.000	5.97
4	16390.90	44.99	603.19	147	16	3046	546	0.405	5.58
5	16882.63	46.79	621.28	188	12	2841	545	0.523	5.21
6	17389.11	48.67	639.92	203	11	2798	554	0.570	5.05
7	17910.78	50.61	659.12	185	12	2799	562	0.523	4.98
8	18448.10	52.64	678.89	167	14	2815	570	0.477	4.94
9	19001.55	54.74	699.26	186	12	2811	567	0.536	4.96
10	19571.59	56.93	720.23	159	14	2789	561	0.464	4.97
11	20158.74	59.21	741.84	157	14	2788	554	0.464	5.03
12	20763.50	61.58	764.10	156	14	2760	548	0.464	5.04
13	21386.40	64.04	787.02	142	15	2733	544	0.427	5.02
14	22028.00	66.60	810.63	156	14	2706	538	0.473	5.03
15	22688.84	69.27	834.95	164	13	2667	534	0.502	4.99
16	23369.50	72.04	860.00	170	12	2644	530	0.527	4.99
17	24070.58	74.92	885.80	181	11	2631	527	0.564	4.99
18	24792.70	77.92	912.37	133	15	2625	525	0.420	5.00
19	25536.48	81.03	939.74	145	14	2630	521	0.461	5.05
20	26302.58	84.27	967.93	151	14	2587	517	0.484	5.00

Number of Ships vs Time (5 to 1 Ratio)



SOME CONCLUSIONS

What do these examples say about the model's usefulness? To a large extent that depends on the information the user has available to input into the model. If one has all the information given a few pages ago as necessary inputs into the model and the only criteria being considered is the force mix ratio, then the results from the optimal budget allocation program should give a very good guide as to budget allocations for all years in the planning period.

But what if information about the future is unknown (i,e., budgets and cost are unknown in the future)? Need the methodology presented here be abandoned? The answer is no, if one is willing to forecast into the future based on what has happened in the past. This was the problem faced when incorporating the optimal allocation model into a larger model of the Navy. The methodology developed to overcome the problem of the lack of future information is presented in the next section.

INCORPORATION OF THE ALLOCATION MODEL IN A HIGHER LEVEL MODEL

The initial purpose of this project was to develop a method whereby a more general model of naval budgeting, maintenance etc., could use some method of optimal allocation for its budget available to procure aircraft and ships. Prior to implementation of this method the allocation of the budget was simply a 50-50 split with small adjustments for past orders of aircraft and ships. The method could now be modified to entail the following:

1) Based on past data project budget and cost growth rates
2) Using this projection compute estimated budgets and costs over a 10-year period
3) Using these estimates and other (known) inputs, use the optimal allocation scheme to find the optimal proportion to be spent on aircraft each year in the planning period.
4) Use the computed first year proportion to allocate the present year's budget
5) Repeat the above process for each year in the model

This methodology allows a higher level model to use the optimal allocation scheme without the need for future data (i.e., future budgets and costs). Thus a model which models only one year at a time with no lookahead capability can utilize a form of optimal allocation.

The results from using this procedure should be quite good because previous years' allocations are incorporated into the present year's allocations through the backorders of ships which are inputs to the allocation model. Hence, if many aircraft are purchased in one year, this would be reflected in the back-orders of aircraft in the next year, thus decreasing the optimal proportion for this year. Also, since the budget and cost growth rates can be adjusted each year, this method of allocation should be able to provide reasonable results if the growth rate changes are not too sharp.

DERIVATION OF THE MODEL

At this point it would be prudent to describe in some detail the derivation of the model employed to develop the preceding results. This section will first describe the derivation of formulas giving the number of aircraft and ships in any year given the proportion of the budget to be spent on aircraft each year from the current year until the year in question and past data. Next, a derivation of an objective function for the model is presented, along with its gradient (which is used in the optimization). And finally a presentation of the actual optimization method is given.

DERIVATION OF THE NUMBER OF AIRCRAFT AND SHIPS IN YEAR i

To begin with consider the following definitions.

Let:

u = retirement rate for aircraft (in decimal form i.e., .03 for 3% per year)
v = retirement rate for ships
b_i = budget available for procurement of aircraft and ships in year i
ca_i = cost of aircraft in year i (average)
cs_i = cost of ships in year i
r_i = proportion of the budget spent on aircraft in year i
na_i = number of aircraft ordered in year i (to be delivered in year i+2)
ns_i = number of ships ordered in year i (to be delivered in year i+5)
A_i = number of aircraft in the fleet at year i
S_i = number of ships in the fleet at year i

Since the retirement rate for aircraft and ships was assumed to be constant and the leadtimes for aircraft and ships are 2 and 5 years respectively; then:

$$A_i = (1-u)A_{i-1} + na_{i-2}$$

$$S_i = (1-v)S_{i-1} + ns_{i-5}$$

These equations say, for instance, that the number of aircraft in year i equals the number of aircraft on hand in year i-1 still active in year i (100(1-u)%) plus the number of aircraft ordered in year i-2. In a like manner, A_{i-1} and S_{i-1} can be expressed in terms of A_{i-2} and S_{i-2} and so on to A_0 and S_0.

From this recursive expression the terms A_i and S_i can be expressed as functions of A_0 and S_0 plus the number of ships and aircraft ordered prior to year i. After some algebraic manipulation the following can be obtained:

$$A_i = (1-u)^i A_0 + \sum_{j=0}^{i-1} (1-u)^j (na_{i-j-2})$$

$$S_i = (1-v)^i S_0 + \sum_{j=0}^{i-1} (1-v)^j (ns_{i-j-5})$$

Notice that the indexes for na_{i-j-2} and ns_{i-j-5} can become negative. These negative indicies represent backorders of ships and aircraft, ordered before the planning period began.

Now if these negative indicies are pulled out of the summations the following equations are obtained:

$$A_i=(1-u)^i A_0+(1-u)^{i-1}na_{-1} +(1-u)^{i-2}na_0 + \sum_{j=0}^{i-3} (1-u)^j na_{i-j-2}$$

$$S_i=(1-v)^i S_0+(1-v)^{i-1}ns_{-4}+(1-v)^{i-2}ns_{-3}+(1-v)^{i-3}ns_{-2}+(1-v)^{i-4}ns_{-1} + (1-v)^{i-5}ns_0 + \sum_{j=0}^{i-6} (1-v)^j ns_{i-j-5}$$

The terms preceding the summation in each of the above equations are based on actions which have already occured and are in no way affected by budget allocations in the planning period (whereas the other na_i and ns_i are). The terms can be grouped for simplicity into functions X_i and Y_i for the A_i and S_i terms respectively. Hence, with this simplification:

$$A_i = X_i + \sum_{j=0}^{i-3} (1-u)^j na_{i-j-2}$$

$$S_i = Y_i + \sum_{j=0}^{i-6} (1-v)^j ns_{i-j-5}$$

Now, examining na_i and ns_i it can be seen that:

$$na_i = r_i b_i/ca_i \quad \text{and} \quad ns_i = (1-r_i)b_i/cs_i$$

Thus, the number of aircraft ordered in year i equals the proportion of the budget spent on aircraft (r_i) times the budget (B_i) (yielding the amount spent on aircraft) divided by the average unit cost of aircraft (ca_i). Likewise for ships, but here the proportion spent on ships is 1-(prop. spent on aircraft). Now, if these substitutions are made and the indexes shifted the following are true:

$$A_i = X_i + \sum_{k=1}^{i-2} (1-u)^{i-k-2} r_k b_k/ca_k$$

$$S_i = Y_i + \sum_{k=1}^{i-5} (1-v)^{i-k-5} (1-r_k)b_k/cs_k$$

Hence, if the b_k, ca_k, cs_k, X_i and Y_i are known the above equations represent the force levels for aircraft and ships in terms of the proportion of the budget to be spent on aircraft in each year of the planning period (i.e., the r_k).

DERIVATION OF THE OBJECTIVE FUNCTION

As discussed in the first section of this paper the objective of an optimal budgeting allocation scheme would be to choose proportions of the budget to be spent each year on aircraft so as to keep the actual ratio of aircraft to ships as close as possible to some desired ratio. In other words, we want to find a set of proportions (r_i) that minimize some cost function relating to the deviation of the actual ratio from the desired ratio in all the years of the planning period (plus two years since the last year's policy will have no effect until then). Since deviations above or below the desired ratio can be considered equally costly the objective might be to minimize the sum of the squared deviations of the actual ratio from the desired ratio for all years in the planning period.

Now Let:

A_i = Number of aircraft in the fleet at year i
S_i = Number of ships in the fleet at year i
k = Desired ratio
tmax = Number of years in the planning period plus 2
r = Vector of proportion of the budget to be spent on aircraft (1 through tmax-2)

The objective then becomes:

$$\text{minimize} \sum_{i=3}^{tmax} \left(\frac{A_i}{S_i} - k\right)^2$$

Notice that this summation runs from i=3 to tmax rather than i=1 to tmax. This is because prior to the third year of the planning period no budget allocation chosen during the planning period will have any effect on the ratio of aircraft to ships and hence need not be included in the objective function. Now let:

$$f_i(r) = \left(\frac{A_i}{S_i} - k\right)$$

Note also that the dependency of f_i on the vector r is through the A_i and S_i terms.

So, with this definition the objective function becomes:

$$\underset{r}{\text{minimize}} \sum_{i=3}^{tmax} (f_i(r))^2 \quad \text{and let} \quad \sum_{i=3}^{tmax} (f_i(r))^2 = D(r)$$

To minimize a nonlinear function such as this one procedure would be to choose an initial starting point (vector) for r and then, since the opposite direction of the gradient of a function is the direction of steepest descent of the function, move along the opposite direction of the gradient until the desired minimum is reached. To do this the gradient of D(r) with respect to the r_i is needed.

DERIVATION OF THE GRADIENT

From the above definition of D(r) it can be seen that:

$$\frac{\partial D}{\partial r_j} = \sum_{i=3}^{tmax} 2(f_i(r))\frac{\partial f_i(r)}{\partial r_j}$$

So,

$$\frac{1}{2}\frac{\partial D}{\partial r_j} = \sum_{i=3}^{tmax} f_i(r)\frac{\partial f_i(r)}{\partial r_j}$$

If we represent the $f_i(r)$ as a vector f and the $\frac{\partial f_i}{\partial r_j}$ likewise then:

$$\frac{1}{2}\frac{\partial D}{\partial r_j} = f \cdot \left(\frac{\partial f_3}{\partial r_j}, \frac{\partial f_4}{\partial r_j}, \ldots\ldots\ldots, \frac{\partial f_{tmax}}{\partial r_j}\right)$$

So,

$$\frac{1}{2}\nabla D = \frac{1}{2}\cdot\begin{bmatrix} \frac{\partial D}{\partial r_1} \\ \vdots \\ \frac{\partial D}{\partial r_{tmax-2}} \end{bmatrix} = \begin{bmatrix} \frac{\partial f_3}{\partial r_1} & \frac{\partial f_4}{\partial r_1} & \cdots\cdots & \frac{\partial f_{tmax}}{\partial r_1} \\ \vdots & \vdots & & \\ \frac{\partial f_3}{\partial r_{tmax-2}} & \frac{\partial f_4}{\partial r_{tmax-2}} & \cdots\cdots & \frac{\partial f_{tmax}}{\partial r_{tmax-2}} \end{bmatrix} \begin{bmatrix} f_3 \\ f_4 \\ \vdots \\ f_{tmax} \end{bmatrix}$$

Again, the f's run from 3 to tmax because no allocation policy has any effect, because of leadtimes, prior to year 3. Also the r_i run from 1 to tmax-2 because proportions chosen in years tmax-1 and tmax will have no effect in the time frame minimized over.

Now the preceding equation can be simplified by noticing that the matrix is the transpose of the Jacobian Matrix of the f_i and the r_i (only those applicable to the problem). If we denote this matrix by J then:

$$\tfrac{1}{2}\nabla D = J^T f$$

But what of the elements of the Jacobian Matrix? How are they computed?

<u>DERIVATION OF THE JACOBIAN MATRIX</u>

A typical element of the Jacobian matrix is $\dfrac{\partial f_i}{\partial r_j}$.

Now,

$$\frac{\partial f_i}{\partial r_j} = \frac{\partial}{\partial r_j}\left(\frac{A_i}{S_i} - k\right) = \frac{S_i\left(\frac{\partial A_i}{\partial r_j}\right) - A_i\left(\frac{\partial S_i}{\partial r_j}\right)}{(S_i)^2}$$

Further, from before we have that:

$$A_i = X_i + \sum_{j=1}^{i-2} (1-u)^{i-j-2} b_j r_j / ca_j$$

$$S_i = Y_i + \sum_{j=1}^{i-5} (1-v)^{i-j-5} b_j (1-r_j) / cs_j$$

So,

$$\frac{\partial A_i}{\partial r_j} = \begin{cases} 0 & \text{for } j > i-2 \\ b_j(1-u)^{i-j-2}/ca_j & \text{for } j \leq i-2 \end{cases}$$

$$\frac{\partial S_i}{\partial r_j} = \begin{cases} 0 & \text{for } j > i-5 \\ -(1-v)^{i-j-5} b_j/cs_j & \text{for } j \leq i-5 \end{cases}$$

Combining this information and substituting into the previous equation for the partial derivatives of the f_i we obtain the results on the following page.

ELEMENTS OF THE JACOBIAN MATRIX

For i=3

$$\frac{\partial f_3}{\partial r_j} = \begin{cases} 0 & \text{for } j>1 \\ (b_1/ca_1)\ (1/S_3) & \text{for } j=1 \end{cases}$$

For i=4

$$\frac{\partial f_4}{\partial r_j} = \begin{cases} 0 & \text{for } j>2 \\ (1-u)^{4-j-2} b_j/(ca_j\ S_4) & \text{for } j\leq 2 \end{cases}$$

For i=5

$$\frac{\partial f_5}{\partial r_j} = \begin{cases} 0 & \text{for } j>3 \\ (1-u)^{5-j-2} b_j/(ca_j\ S_5) & \text{for } j\leq 3 \end{cases}$$

For i > 5

$$\frac{\partial f_i}{\partial r_j} = \begin{cases} 0 & \text{for } j\geq i-1 \\ \dfrac{S_i(1-u)^{i-j-2}(b_j/ca_j) \quad + \quad A_i(1-v)^{i-j-5}(b_j/cs_j)}{(S_i)^2} & \text{for } 1\leq j\leq i-5 \\ (1-u)^{i-j-2} b_j/(ca_j\ S_i) & \text{for } j=i-4, i-3, i-2 \end{cases}$$

From the preceding derivations the gradient of $D(r)$ (∇D) can be computed at any point. This computation is done by first computing the Jacobian Matrix as described then multiplying this matrix by the vector of deviations (the f vector described). This multiplication yields $J^T f = \frac{1}{2}\nabla D$ and multiplying $J^T f$ by 2 yields ∇D. From this point an optimization algorithm can be described to take an initial starting point (r vector) and iteratively improve it by adjusting it by some increment in the opposite direction of the gradient until the minimum of $D(r)$ is found, to within some tolerance.

METHOD OF OPTIMIZATION

The chosen method for finding the minimum of the $D(r)$ function utilizes a very simplified gradient seach method. The algorithm used can be described as follows.

1) Set an initial proportion vector r (Chosen to be all .5)
2) Set an initial increment to adjust the r_i elements (Chosen to be .4)
3) Compute the Jacobian Matrix as described in the previous section (J)
4) Compute the vector of deviations (f) given the current r vector
5) Compute $J^T f$ (this is $\frac{1}{2}$ the gradient of the objective function)
6) Since the objective is to minimize the objective function we would like to move in the opposite direction of the gradient. So the algorithm implemented does the following.

   If

   $(J^T f)_i < 0$ then let $r_i = r_i$ + increment (if >1 $r_i=1$)

   $(J^T f)_i > 0$ then let $r_i = r_i$ - increment (if <0 $r_i=0$)

   $(J^T f)_i = 0$ then let $r_i = r_i$
7) Now, check if all the r_i are equal to either the preceding r_i or the r_i preceding that. If they all are, then the solution is oscillating between two points and so the optimum should be somewhere in between the two vectors that are being oscillated between. So, if this condition is true the increment is cut in half and the algorithm proceeds.
8) The process described above continues until the increment is smaller than some set tolerance level. (Chosen to be .001.)*

*The .001 tolerance level was chosen because it was felt that changes to the r vector smaller than that were not really significant enough to be taken into account. This increment can be easily changed if it were desireable to do so.

SOFTWARE FOR OPTIMAL ALLOCATION

This section presents the software developed for the optimal budget allocation problem discussed in this paper. The software was written in both standard FORTRAN and FORTRAN-77 for implementation on an IBM and a VAX 11/780 system respectively. The software for both systems has been validated but only the software for the IBM implementation is presented here.

THE DEVELOPED SOFTWARE

The software developed to do the optimal budget allocation is written in standard FORTRAN and consists of 4 primary routines and a routine written as a linkage between the optimal allocation routines and another model. A description of the routines follows.

ALLOCA - This routine performs the optimization described in the previous section of this paper. It should be noted that this routine may be replaced by other (possibly more sophisticated) nonlinear optimization routines if the convergence of the outlined methodology is considered too slow. I don't consider its convergence is bad, but bad is a somewhat relative term.

JACOBI - This routine computes the Jacobian Matrix (J) of the problem using the equations developed in the preceding section of this paper.

JTRANF - This routine multiplies the vector of deviations of the actual ratio from the desired ratio by the transpose of the Jacobian Matrix computed by JACOBI. This yields the vector $J^T f$, $\frac{1}{2}$ the gradient of the objective function.

COMSTR - (stands for compute strength) This routine computes the number of aircraft and ships in any year of the planning period plus two years, given past data and a vector of proportions of the budget to be spent on aircraft in each year of the planning period.

BUDALL - This is a linkage routine between a higher level model and the above optimal allocation routines. Its primary function is to simplify the inputs to the model--requiring the calling program to specify only budget and cost growth rates, and to place the calling programs inputs into a block of common storage accessed by all the optimization routines. This routine should be easily modified to suit the users precise specifications. For instance, if growth rates are not constant, the calling program could pass vectors containing the budgets and costs for all years and then have BUDALL put these inputs into the appropriate location in COMMON storage

CALLING INSTRUCTIONS

As presently formulated the calling sequence for the optimization routines is via the routine BUDALL. To access the routines the following FORTRAN statement is needed:

```
      CALL BUDALL (CURAIR,CURSHI,BUDNOW,CANOW,CSNOW,
1                  BUDG1,CAG1,CSG1,BACKA,BACKS,TIME,X,Y,
1                  RATCON,R)
```

Where the 1's represent FORTRAN continuation cards and the parameters are defined as follows.

CURAIR - The number of aircraft currently in the fleet (in year zero),INTEGER

CURSHI - The number of ships currently in the fleet (in year zero),INTEGER

BUDNOW - The budget available for procuring aircraft and ships in year 1

CANOW - The average unit cost of aircraft in year 1 of the planning period

CSNOW - The average unit cost of ships in year 1 of the planning period

BUDG1 - The growth of the available budget each year (i.e.,3% growth,budg1=.03)

CAG1 - The growth rate for aircraft cost (similar formulation as for budg1)

CSG1 - The growth rate for ship cost (same as for budg1)

BACKA - Vector of length 2 giving the backorders of aircraft,INTEGER
with those ordered 1 year ago in BACKA(1)
with those ordered 2 years ago in BACKA(2)

BACKS - Vector of length 5 giving the backorders of ships,INTEGER
with those ordered 1 year ago in BACKS(1)

with those ordered 5 years ago in BACKS(5)

TIME - The number of years in the planning period,INTEGER

X - The retirement rate for aircraft (3% retirement rate, x=.03)

Y - The retirement rate for ships (same as above)

RATCON - The desired ratio of aircraft to ships

R - A vector of length 30 in which the vector of optimal proportions will be returned, where R(i)= the proportion of the budget to be spent on aircraft in year i

If it is desired to modify BUDALL the necessary modification to the above call statement will need to be made.

COMMON BLOCK

Regardless of the modification made to the routine BUDALL to suit the users specifications, BUDALL must place values of the input parameters into the COMMON block ALLOC within that subroutine (this is a labelled COMMON block in FORTRAN). The following parameters must be put in common.*

ITIME	- The number of years in the planning period, INTEGER
B	- A vector of length 30 containing the available budget for each year in the planning period
CA	- A vector of length 30 containing the unit aircraft cost for each year in the planning period
CS	- A vector of length 30 containing the unit ship cost for each year in the planning period
P -	(1-Retirement rate for aircraft)
Q -	(1-Retirement rate for ships)
BACKA -	Same definition as previously given, INTEGER
BACKS -	" " " INTEGER
CURAIR-	" " " INTEGER
CURSHI-	" " " INTEGER
RATCON-	" " "
TOL -	The tolerance acceptable for minimization (either .001 or .002 should be acceptable)

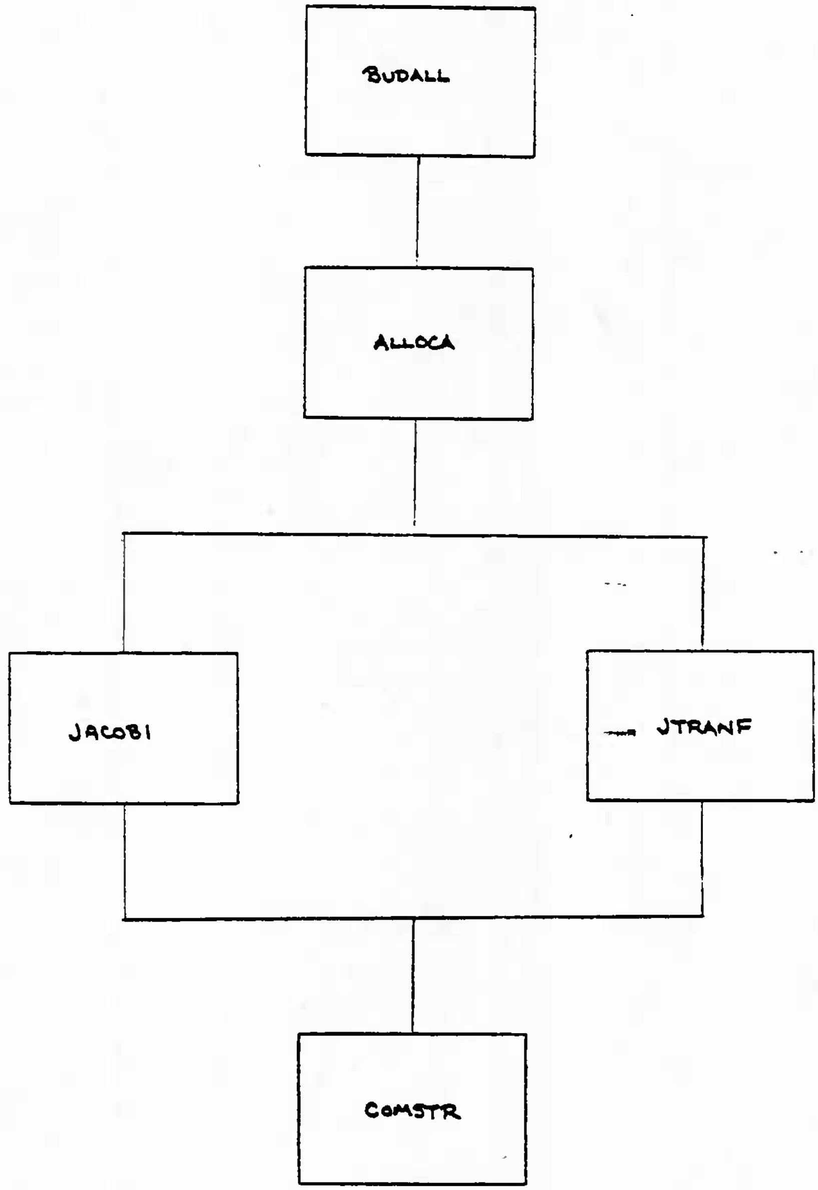
The common statement should take the following form.

```
      COMMON/ALLOC/ITIME,B(30),CA(30),CS(30),P,Q,BACKA(2),BACKS(5)
1              ,CURAIR,CURSHI,RATCON,TOL
```

The above variables also represent most of the important variables for each of the routines. For further information on the software, one should either examine the FORTRAN code and the comments contained therein, or refer to the flow diagrams on the following pages.

*It should be noted that in the code for BUDALL there is a slight modification to some of the variable names in the common block. This is because it was desired to use the same variable names in the parameter list passed as were used throughout the program. FORTRAN doesn't allow a variable to be in both a parameter list and a common statement, so to circumvent this problem the variable in this routine alone had their names changed from what is above.

HIERARCHY DIAGRAM FOR OPTIMAL BUDGET ALLOCATION PROGRAM



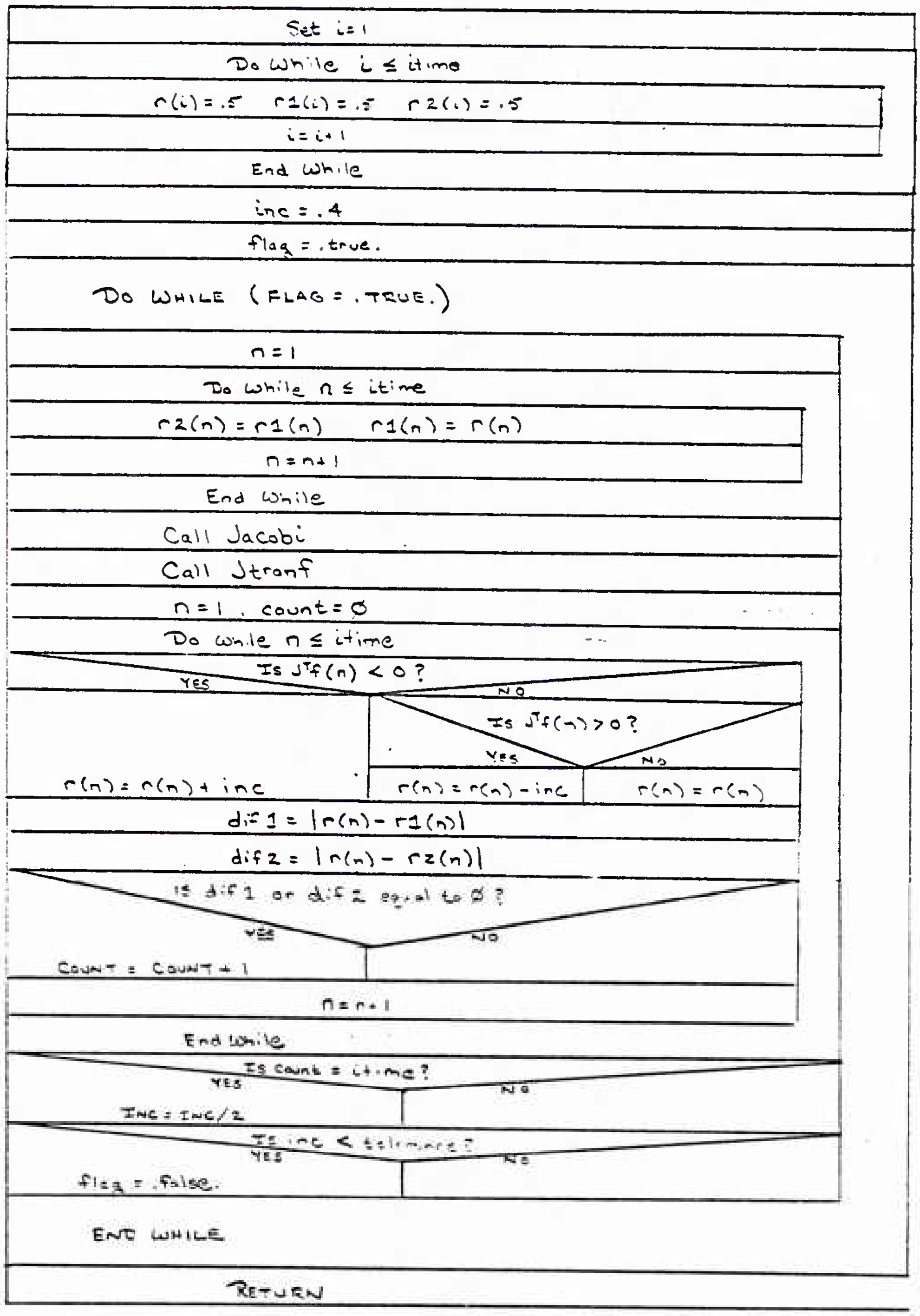
BUDALL

```
P = 1-Retirement Rate for Aircraft   (P = 1-X)
Q = 1-Retirement Rate for Ships   (Q = 1-Y)
Minimization Tolerance (tol) = .001   (or .002)
BUDGRW = BUDG1 + 1
CAGRW = CAG1 + 1
CSGRW = CSG1 + 1
CURRAI = CURAIR
CURRSH = CURSHI
RAT = RATCON
B(1) = BUDNOW
CA(1) = CANOW
CS(1) = CSNOW
TMAX = TIME + 2
i = 2
Do While i ≤ tmax
      B(i) = BUDGRW x B(i-1)
      CA(i) = CAGRW x CA(i-1)
      CS(i) = CSGRW x CS(i-1)
      i = i + 1
End While
BA(1) = BACKA(1)
BA(2) = BACKA(2)
i = 1
Do While i ≤ 5
      BS(i) = BACKS(i)
      i = i - 1
End While
ITIME = TIME
CALL ALLOCA(R, TMAX)
Return
```

Significant Variables: (Not Explained Elsewhere)

TMAX - # of Years in the Planning Period plus 2 Years, INTEGER

ALLOCA



Significant Variables: (Not Explained Elsewhere)

r(i) - Vector of length 30 to contain the proportion of the budget to be spent each year on aircraft

JACOBI

```
Zero out the J matrix
Call Comstr ( air , ships , r )
J(3,1) = b(1)/(ca(1) x ships(3)) , J(4,1) = (p x b(1))/(ca(1) x ships(4)), J(5,1) = (p^2 b(1))/(ca(1) x ships(5))
J(3,2) = Ø.Ø  J(4,2) = b(2)/(ca(2) x ships(4)) , J(5,2) = p x b(2)/(ca(2) x ships(5))
J(3,3) = Ø.Ø  J(4,3) = Ø.Ø  J(5,3) = b(3)/(ca(3) x ships(5))
i = 4
Do while i ≤ itime + 2
    J(3,i) = Ø.Ø  J(4,i) = Ø.Ø  J(5,i) = Ø.Ø
    i = i + 1
End while
i = 6
Do while i ≤ itime + 2
    k = 1
    Do while k ≤ i - 5
        a = float(air(i)) , s = float(ships(i)) , x = p^(i-k-2) , y = q^(i-k-5)
        J(i,k) = (((s·x·b(k)/ca(k)) + (a·y·b(k)/cs(k))) / (s·s))
        k = k + 1
    End while
    k = i - 4
    Do while k ≤ i - 2
        J(i,k) = (p^(i-k-2) b(k)) / (ca(k) · ships(i))
        k = k + 1
    End while
    i = i + 1
End while
i = 1
Do while i ≤ 30
    k = 1
    Do while k ≤ 30
        J(i,k) = J(i+2,k)
        k = k + 1
    End while
    i = i + 1
End while
Return
```

Significant Variables: (Not Explained Elsewhere)

- air(i) - vector of length 30 to contain the # of aircraft in the fleet at year i in air(i)
- ships(i) - vector of length 30 to contain the # of ships in the fleet at year i in ships(i)

JTRANF

```
Call Comstr (a, s, r)
i = 1
Do while i ≤ itime
    k = 1 , JTF(i) = 0.0
    Do while k ≤ itime
        a1 = float (a(k+2))
        s1 = float (s(k+2))
        JTF(i) = JTF(i) + (J(k,i)*((a1/s1) - ratcon))
        k = k + 1
    End while
    i = i + 1
End while
Return
```

Significant Variables: (Not Explained Elsewhere)

a(i) - a vector of length 30 to contain the # of aircraft in each year

s(i) - " " " ships " "

JTF(i) - vector of length 30 to contain the elements of the vector J^TF

COMSTR

```
Zero out a(i) and S(i) vectors
a(1) = ifix ((p*curair) + backa(2))
S(1) = ifix ((q x cursh) + backs(5))
a(2) = ifix ((p x a(1)) + backa(1))
s(2) = ifix ((q x s(1)) + backs(4))
s(3) = ifix ((q x s(2)) + backs(3))
s(4) = ifix ((q x s(3)) + backs(2))
s(5) = ifix ((q x s(4)) + backs(1))
i = 3
Do while i ≤ itime + 2 = (tmax)
        a(i) = ifix ((p * a(i-1)) + (r(i-2) * b(i-2) / ca(i-2))
        i = i + 1
End while
i = 6
Do while i ≤ itime + 2 = (tmax)
        s(i) = ifix ((q x s(i-1)) + ((1 - r(i-5)) b(i-5) / ca(i-5)))
        i = i + 1
End while
Return
```

Significant Variables: (Not Explained Elsewhere)

- a(i) - vector of length 30 to contain the # of aircraft in the fleet at year i in a(i)
- s(i) - vector of length 30 to contain the # of ships in the fleet at year i in s(i)

CONCLUSIONS

Given the preceding results, what conclusions can be drawn? The model of the budget allocation process as developed and solved allows for the optimal allocation of a budget, over some time span, toward procurement of aircraft and ships so as to keep the ratio of aircraft to ships (force mix) as close as possible to some desired level, regardless of what that level may be. The developed software does this by minimizing the sum of the squared deviation of the actual ratio from the desired ratio for all years in the planning period.

The results presented in this study are generally quite problem specific, dealing with the particular problem of allocating a budget between purchases of aircraft and purchases of ships so as to hold the force mix constant. If one is seeking answers to such questions or if one is interested in studying the interactions of the model's inputs or their effect on force levels, the model and solution method presented in this paper should serve as a useful tool. If, on the other hand, one is looking for a methodology for solving other, similar, allocation problems, the approach presented here may be useful as a guide, because of its simplicity.

Throughout this study, it was felt that one prime objective in the development of the foregoing model was that the model should remain as uncomplicated as possible while still yielding a correct solution. This, to a large extent, was accomplished.

Likewise, the prime objective of the study: The optimal allocation of a budget toward procurement of aircraft and ships in such a way as to keep the force mix constant, was achieved.

Appendix I - Some further results

On the following two pages are the results of two runs of the optimal budget allocation software. The inputs to these runs are the same as for the first example in section one except that the desired ratio for these runs is 8 for the first and 5 for the second. These results are presented to further demonstration of the ability of the optimal allocation software to achieve its objective of keeping the force mix between aircraft and ships at the desired level.

OPTIMAL BUDGET ALLOCATION PROGRAM

Desired Ratio = 5.00

Retirement Rate for Aircraft = 0.067
Retirement Rate for Ships = 0.033

Year	Budget (mil)	Cost Aircraft (mil)	Cost Ships (mil)	# of Aircraft Ordered	# of Ships Ordered	# of Aircraft	# of Ships	r(i)	Ratio
1	23000.00	40.00	552.00	0	41	3500	549	0.000	6.38
2	23690.00	41.60	568.56	0	41	3500	548	0.000	6.39
3	24400.70	43.26	585.62	24	39	3265	547	0.044	5.97
4	25132.72	44.99	603.19	181	28	3046	546	0.325	5.58
5	25886.70	46.79	621.28	274	20	2866	545	0.497	5.26
6	26663.30	48.67	639.92	267	21	2855	568	0.488	5.03
7	27463.20	50.61	659.12	250	22	2938	590	0.463	4.98
8	28287.09	52.64	678.89	235	23	3008	610	0.438	4.93
9	29135.70	54.74	699.26	226	23	3057	617	0.425	4.95
10	30009.77	56.93	720.23	197	26	3087	617	0.375	5.00
11	30910.07	59.21	741.84	189	26	3106	617	0.363	5.03
12	31837.37	61.58	764.10	264	20	3095	619	0.512	5.00
13	32792.49	64.04	787.02	254	20	3076	622	0.497	4.95
14	33776.26	66.60	810.63	231	22	3134	625	0.456	5.01
15	34789.55	69.27	834.95	210	24	3178	630	0.419	5.04
16	35833.23	72.04	860.00	211	23	3196	635	0.425	5.03
17	36908.23	74.92	885.80	209	23	3192	634	0.425	5.03
18	38015.47	77.92	912.37	219	22	3189	634	0.450	5.03
19	39155.94	81.03	939.74	223	22	3184	635	0.462	5.01
20	40330.61	84.27	967.93	233	21	3190	638	0.487	5.00

OPTIMAL BUDGET ALLOCATION PROGRAM

Desired Ratio = 8.00

Retirement Rate for Aircraft = 0.067
Retirement Rate for Ships = 0.033

Year	Budget (mil)	Cost Aircraft (mil)	Cost Ships (mil)	# of Aircraft Ordered	# of Ships Ordered	# of Aircraft	# of Ships	r(i)	Ratio
1	23000.00	40.00	552.00	575	0	3500	549	1.000	6.38
2	23690.00	41.60	560.56	536	2	3500	548	0.942	6.39
3	24400.70	43.26	585.62	369	14	3840	547	0.655	7.02
4	25132.72	44.99	603.19	259	22	4119	546	0.464	7.54
5	25886.70	46.79	621.28	248	22	4212	545	0.448	7.73
6	26663.30	48.67	639.92	288	19	4189	527	0.527	7.95
7	27463.20	50.61	659.12	261	21	4156	512	0.483	8.12
8	28287.09	52.64	678.89	274	20	4166	509	0.511	8.18
9	29135.70	54.74	699.26	288	19	4148	514	0.542	8.07
10	30009.77	56.93	720.23	303	17	4144	520	0.577	7.97
11	30910.07	59.21	741.84	297	17	4154	522	0.570	7.96
12	31837.37	61.58	764.10	299	17	4179	526	0.580	7.94
13	32792.49	64.04	787.02	287	18	4196	529	0.561	7.93
14	33776.26	66.60	810.63	303	16	4214	530	0.598	7.95
15	34789.55	69.27	834.95	283	18	4218	530	0.564	7.96
16	35833.23	72.04	860.00	275	18	4238	530	0.555	8.00
17	36908.23	74.92	885.80	273	18	4237	530	0.555	7.99
18	38015.47	77.92	912.37	278	17	4229	530	0.570	7.98
19	39155.94	81.03	939.74	286	16	4218	529	0.592	7.97
20	40330.61	84.27	967.93	293	16	4213	529	0.612	7.96

Appendix II - FORTRAN code for optimal allocation

Routines listed:

BUDALL - Linkage Routine

ALLOCA - Optimal Allocation Optimization Routine

JACOBI - Computes the Jacobian Matrix of the Problem

JTRANF - Computes $J^T f$ (the gradient of the objective function)

COMSTR - Computes the number of aircraft and ships in the fleet at any year given a particular allocation policy over some planning period

```
C ****************************************************************
C *                                                              *
C *   ROUTINE BUDALL - THIS IS A LINKAGE ROUTINE BETWEEN        *
C *       THE OPTIMAL BUDGET ALLOCATION ROUTINES AND THE        *
C *       USER'S SOFTWARE.                                       *
C *                                                              *
C ****************************************************************
C
      SUBROUTINE BUDALL (CURAIR,CURSHI,BUDNOW,CANOW,CSNOW,
     1                   BUDG1,CAG1,CSG1,BACKA,BACKS,TIME,X,Y,
     1                   RATCON,R)
      INTEGER BACKA(2),BACKS(5),CURAIR,CURSHI,TMAX,TIME
      INTEGER CURRAI,CURRSH,BA,BS
      REAL R(30)
      COMMON/ALLOC/ITIME,B(30),CA(30),CS(30),P,Q,BA(2),BS(5),
     1              CURRAI,CURRSH,RAT,TOL
C
C  P AND Q EQUAL THE PROPORTION OF THE FLEET ACTIVE IN YEAR I
C  STILL ACTIVE IN YEAR I+1 (1-RETIREMENT RATE)
C
      P=1-X
      Q=1-Y
C
C  THIS IS THE TOLERANCE LEVEL FOR THE OPTIMIZATION ALGORITHM
C
      TOL=.0020
      BUDGRW=1+BUDG1
      CAGRW=1+CAG1
      CSGRW=1+CSG1
C
C  INFORMATION FROM CALLING PROGRAM PUT INTO COMMON STORAGE
C
      CURRAI=CURAIR
      CURRSH=CURSHI
      RAT=RATCON
      B(1)=BUDNOW
      CA(1)=CANOW
      CS(1)=CSNOW
C
C  SET TMAX = # OF YEARS OVER WHICH THE OBJECTIVE IS MINIMIZED
C
      TMAX=TIME+2
C
C  FILL IN BUDGET AND COST VECTORS FOR ALL YEARS IN THE PLANNING PERIOD
C
      DO 100 I=2,TMAX
         B(I)=BUDGRW*B(I-1)
         CA(I)=CAGRW*CA(I-1)
         CS(I)=CSGRW*CS(I-1)
  100 CONTINUE
C
C  MORE PARAMETERS PUT INTO COMMON
C
      BA(1)=BACKA(1)
      BA(2)=BACKA(2)
      DO 200 I=1,5
         BS(I)=BACKS(I)
  200 CONTINUE
      ITIME=TIME
```

```
C
C  CALL OPTIMIZATION ROUTINE ALLOCA TO COMPUTE OPTIMAL PROPORTION OF
C  THE BUDGET TO BE SPENT ON AIRCRAFT FOR EACH YEAR IN THE PLANNING
C  PERIOD.  THE RESULT IS RETURNED IN THE VECTOR R WHERE THE OPTIMAL
C  PROPORTION FOR YEAR I IS RETURNED IN R(I).
C
         CALL ALLOCA (R,TMAX)
         RETURN
         END
```

```
C ******************************************************************
C *                                                                *
C *    ROUTINE ALLOCA - THIS ROUTINE CARRIES OUT THE OPTIMIZ-      *
C *         ATION OF THE PROPORTION OF THE BUDGET SPENT ON         *
C *         AIRCRAFT EACH YEAR.  FOR DETAILS ON THE THEORY         *
C *         BEHIND THE ALGORITHM SEE THE ACCOMPANYING PAPER        *
C *                                                                *
C *    THE ARRAYS J(30,30) AND JTF(30) ARE USED TO CONTAIN THE     *
C *    THE JACOBIAN MATRIX AND GRADIENT (J'F) RESPECTIVELY         *
C *                                                                *
C *    IT SHOULD BE NOTED THAT THIS ROUTINE MAY BE REPLACED BY     *
C *    OTHER NON-LINEAR OPTIMIZATION ROUTINES WHILE STILL UTIL-    *
C *    IZING THE ROUTINES JACOBI AND JTRANF TO COMPUTE THE         *
C *    GRADIENT OF THE NON-LINEAR OBJECTIVE FUNCTION.              *
C *                                                                *
C ******************************************************************
C
      SUBROUTINE ALLOCA (R,TMAX)
      REAL J(30,30),JTF(30),P,Q,INC,R(30),R1(30),R2(30)
      INTEGER BACKA,BACKS,TMAX,CURAIR,CURSHI
      COMMON/ALLOC/ITIME,B(30),CA(30),CS(30),P,Q,BACKA(2),BACKS(5)
     1        ,CURAIR,CURSHI,RATCON,TOL
      LOGICAL FLAG
      INTEGER COUNT
C
C  INITIALIZE PROPORTION TO .5
C
      DO 100 I=1,ITIME
         R(I)=0.5
         R1(I)=0.5
         R2(I)=0.5
  100 CONTINUE
      INC=.4
      FLAG=.TRUE.
C
C  LOOP UNTIL THE INCREMENT THAT THE R(I) ARE ADJUSTED BY IS LESS
C  THAN THE TOLERANCE (THEN FLAG SET FALSE)
C
  200 IF (.NOT.FLAG) GO TO 1000
         DO 250 N=1,ITIME
            R2(N)=R1(N)
            R1(N)=R(N)
  250    CONTINUE
C
C  CALL ROUTINES JACOBI AND JTRANF TO COMPUTE THE GRADIENT
C
         CALL JACOBI (J,R,TMAX)
         CALL JTRANF (J,JTF,R,TMAX)
         COUNT=0
C
C  COMPUTE A NEW R VECTOR AND TEST IF SOLUTION OSCILLATING.  THIS IS
C  DONE BY TESTING WHETHER THE ELEMENTS OF THE CURRENT SOLUTION EQUAL
C  EITHER THE PREVIOUS SOLUTION OR THE ONE PRECEDING THAT.
C
         DO 300 N=1,ITIME
            IF (JTF(N).LT.0.0) R(N)=R(N)+INC
            IF (JTF(N).GT.0.0) R(N)=R(N)-INC
            IF (R(N).LT.0.0) R(N)=0.0
            IF (R(N).GT.1.0) R(N)=1.0
```

```
                  DIF1=ABS(R(N)-R1(N))
                  DIF2=ABS(R(N)-R2(N))
                  IF ((DIF1.LT.0.000001).OR.(DIF2.LT.0.000001))
     1              COUNT=COUNT+1
  300       CONTINUE
            IF (COUNT.EQ.ITIME) INC=INC/2
            IF (INC.LT.TOL) FLAG=.FALSE.
            GO TO 200
 1000 CONTINUE
      RETURN
      END
```

```
C ******************************************************************
C *                                                                *
C *    ROUTINE JACOBI - THIS ROUTINE COMPUTES THE JACOBIAN         *
C *        MATRIX OF THE PROBLEM AS DESCRIBED IN THE ACCOMP-       *
C *        ANYING PAPER.                                           *
C *                                                                *
C *    THE ROUTINE IS PASSED A VECTOR R CONTAINING THE CURRENT*
C *    SOLUTION (VECTOR OF OPTIMAL PROPORTIONS                     *
C *                                                                *
C ******************************************************************
C
      SUBROUTINE JACOBI(J,R,TMAX)
      INTEGER BACKA,BACKS,CURAIR,CURSHI
      INTEGER AIR(30),SHIPS(30),TMAX
      REAL J(30,30),R(30)
      COMMON/ALLOC/ITIME,B(30),CA(30),CS(30),P,Q,BACKA(2),BACKS(5)
     1              ,CURAIR,CURSHI,RATCON,TOL
C
C  CALL ROUTINE COMSTR TO DETERMINE THE NUMBER OF AIRCRAFT AND SHIPS
C  IN THE FLEET AT ANY YEAR IN THE PLANNING PERIOD GIVEN THE CURRENT
C  SOLUTION
C
      CALL COMSTR (AIR,SHIPS,R,TMAX)
C
C  ZERO OUT THE J MATRIX
C
      DO 20 I=1,30
         DO 10 N=1,30,5
            J(I,N)=0.0
            J(I,N+1)=0.0
            J(I,N+2)=0.0
            J(I,N+3)=0.0
            J(I,N+4)=0.0
  10     CONTINUE
  20  CONTINUE
C
C  COMPUTATION OF THE JACOBIAN MATRIX ELEMENTS
C
      J(3,1)=B(1)/(CA(1)*SHIPS(3))
      J(4,1)=(P*B(1))/(CA(1)*SHIPS(4))
      J(5,1)=(P*P*B(1))/(CA(1)*SHIPS(5))
      J(3,2)=0.0
      J(4,2)=B(2)/(CA(2)*SHIPS(4))
      J(5,2)=(P*B(2))/(CA(2)*SHIPS(5))
      J(3,3)=0.0
      J(4,3)=0.0
      J(5,3)=B(3)/(CA(3)*SHIPS(5))
      DO 800 I=4,ITIME
         J(3,I)=0.0
         J(4,I)=0.0
         J(5,I)=0.0
 800  CONTINUE
      DO 1000 I=6,TMAX
         NUM=I-5
         DO 900 K=1,NUM
            A=FLOAT(AIR(I))
            S=FLOAT(SHIPS(I))
            X=P**(I-K-2)
            Y=Q**(I-K-5)
```

```
                J(I,K)=(((S*X*B(K)/CA(K))+(A*Y*B(K)/CS(K)))/(S*S))
   900       CONTINUE
             NUM1=I-4
             NUM2=I-2
             DO 950 K=NUM1,NUM2
                J(I,K)=((P**(I-K-2))*B(K))/(CA(K)*SHIPS(I))
   950       CONTINUE
  1000    CONTINUE
C
C  HERE THE MATRIX IS SHIFTED SO THAT THE PARTIAL DERIVATIVE OF
C  THE 3 YEAR'S DEVIATION WITH RESPECT TO THE FIRST YEAR'S PRO-
C  PORTION IS FOUND IN J(1,1).  THIS AIDS IN THE COMPUTATION OF
C  JTF(I) THE GRADIENT OF THE OBJECTIVE FUNCTION
C
          DO 1500 I=1,28
             DO 1250 K=1,30
                J(I,K)=J(I+2,K)
  1250       CONTINUE
  1500    CONTINUE
  2000    CONTINUE
          RETURN
          END
```

```
C ******************************************************************
C *                                                                *
C *   ROUTINE JTRANF - THIS ROUTINE TAKES THE TRANSPOSE OF         *
C *       THE J MATRIX COMPUTED BY THE ROUTINE JACOBI AND          *
C *       POST-MULTIPLIES IT BY THE VECTOR OF DEVIATION OF         *
C *       THE ACTUAL RATIO FROM THE DESIRED RATIO GIVEN THE        *
C *       CURRENT SOLUTION.  THIS YIELDS THE GRADIENT OF THE       *
C *       OF THE OBJECTIVE FUNCTION.                               *
C *                                                                *
C ******************************************************************
C
      SUBROUTINE JTRANF (J,JTF,R,TMAX)
      INTEGER BACKA,BACKS,CURAIR,CURSHI,TMAX
      INTEGER A(30),S(30)
      REAL J(30,30),JTF(30),R(30)
      COMMON/ALLOC/ITIME,B(30),CA(30),CS(30),P,Q,BACKA(2),BACKS(5)
     1              ,CURAIR,CURSHI,RATCON,TOL
      CALL COMSTR (A,S,P,TMAX)
      DO 500 I=1,ITIME
         JTF(I)=0.0
         DO 250 K=1,ITIME
            A1=FLOAT(A(K+2))
            S1=FLOAT(S(K+2))
            JTF(I)=JTF(I)+(J(K,I)*((A1/S1)-RATCON))
  250    CONTINUE
  500 CONTINUE
 1000 CONTINUE
      RETURN
      END
```

```
C ********************************************************************
C *                                                                  *
C *   ROUTINE COMSTR - THIS ROUTINE COMPUTES THE NUMBER OF           *
C *       AIRCRAFT AND SHIPS IN THE FLEET FOR ANY YEAR IN            *
C *       THE PLANNING PERIOD PLUS TWO YEARS GIVEN THE FOL-          *
C *       LOWING INFORMATION:                                        *
C *            CURRENT NUMBER OF AIRCRAFT AND SHIPS                  *
C *            BACKORDERS OF AIRCRAFT 1 AND 2 YEARS AGO              *
C *            BACKORDERS OF SHIPS 1,2,3,4 AND 5 YEARS AGO           *
C *            BUDGETS FOR PROCUREMENT FOR ALL YEARS IN THE          *
C *               PLANNING PERIOD                                    *
C *            AVERAGE UNIT COSTS OF AIRCRAFT AND SHIPS FOR          *
C *               ALL YEARS IN THE PLANNING PERIOD                   *
C *            A VECTOR R CONTAIN THE PROPORTION OF THE BUD-         *
C *               GET TO BE SPENT ON AIRCRAFT IN EACH YEAR           *
C *                                                                  *
C ********************************************************************
C
      SUBROUTINE COMSTR (A,S,R,TMAX)
      INTEGER BACKA,BACKS,CURAIR,CURSHI,A(30),S(30),TMAX
      REAL R(30)
      COMMON/ALLOC/ITIME,B(30),CA(30),CS(30),P,Q,BACKA(2),BACKS(5)
     1             ,CURAIR,CURSHI,RATCON,TOL
C
C  FOR FURTHER INFORMATION ON THESE COMPUTATIONS SEE ACCOMPANYING PAPER.
C
      A(1)=IFIX((P*CURAIR)+BACKA(2))
      S(1)=IFIX((Q*CURSHI)+BACKS(5))
      A(2)=IFIX((P*A(1))+BACKA(1))
      S(2)=IFIX((Q*S(1))+BACKS(4))
      S(3)=IFIX((Q*S(2))+BACKS(3))
      S(4)=IFIX((Q*S(3))+BACKS(2))
      S(5)=IFIX((Q*S(4))+BACKS(1))
      DO 500 I=3,TMAX
         A(I)=IFIX((P*A(I-1))+(R(I-2)*B(I-2)/CA(I-2)))
  500 CONTINUE
      DO 800 I=6,TMAX
         S(I)=IFIX((Q*S(I-1))+((1-R(I-5))*B(I-5)/CS(I-5)))
  800 CONTINUE
 1000 CONTINUE
      RETURN
      END
```

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