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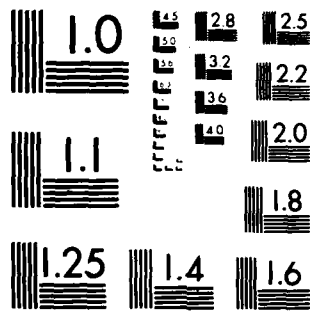
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A Comparison of Several Alternatives
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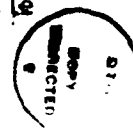
A Comparison of Several Alternatives to Maximum Likelihood
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by

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1. Introduction

The Weibull distribution is very versatile and has found many uses in reliability, and in the climatological modeling of weather elements. In this study, we compare several alternatives to maximum likelihood (ML) estimation. Somerville and Bean (1982) compared ML and least squares (LS) and found that under ideal conditions LS and ML gave substantially the same results. However, when contamination or censoring occurs, or when the wrong model is used, LS can give substantially better results. In addition, ML estimation of the parameters of the Weibull distribution requires iterative techniques. The alternatives in this study are considered for two reasons. They are more robust, and most of them are much easier to compute.

It is common to evaluate estimators on the basis of their variances and biases. Although these are important considerations, a user is frequently more interested in how well the model is going to predict probabilities. That is, we are more interested in the fit of the cumulative distribution than the values of the parameters. We will evaluate the alternatives on the basis of the fit to the cumulative distribution. We use the following form of the cumulative distribution function

$$F(x) = 1 - \exp(-\alpha x^\beta), \quad x, \alpha, \beta > 0. \quad (1)$$

2. Alternatives to Maximum Likelihood

Let $x_1, x_2, \dots, x_n$ be the ordered observations of a random sample from the Weibull distribution. We use the following form for the empirical cumulative distribution function (CDF)

$$F_n(x_i) = (i-.5)/n \quad (2)$$

and

$$F_n(x) = F_n(x_i) \quad x_i \leq x < x_{i+1}. \quad (3)$$

The estimators which follow (except for the method of moments) select (α, β) so as to minimize the "distance" between $F(x; \alpha, \beta)$ and $F_n(x)$.

2.1 Non-Linear Regression (NLR)

In non-linear regression, (α, β) are selected so that the expression

$$\sum_{i=1}^n (F(x_i; \alpha, \beta) - F_n(x_i))^2 \quad (4)$$

is minimized. This ensures that the model distribution fits the empirical CDF in the least squares sense. However, costly iterative techniques are required.

2.2 The Log-Linearization Method (LLM)

The non-linear model in 2.1 may be made linear by using logarithms.

Let

$$q_i = 1 - F_n(x_i) \quad (5)$$

$$\hat{q}_i = \exp(-\hat{\alpha} \hat{\beta} x_i) \quad (6)$$

where $\hat{\alpha}$ and $\hat{\beta}$ are estimates of α and β . Then

$$\ln(-\ln \hat{q}_i) = \ln \hat{\alpha} + \hat{\beta} \ln x_i. \quad (7)$$

We may regard this as a regression model with $\ln(-\ln q_i)$ as the dependent variable and $\ln x$ as the independent variable. Reversing the roles of dependent and independent variables, we may also write

$$\ln \hat{x}_i = \ln(-\ln q_i)/\hat{\beta} - (\ln \hat{\alpha})/\hat{\beta}. \quad (8)$$

Ordinary least squares may be used to obtain model coefficients from which estimates of (α, β) may be obtained. Using (7), the model attempts to fit the CDF while in (8) the model attempts to fit the percentiles. However, using equations (5) and (6), the non-linear least squares equation (4) can be written as

$$\sum_{i=1}^n (q_i - \hat{q}_i)^2 \quad (9)$$

and it is this sum (9) that non-linear least squares seeks to minimize. Using the log-linearization method coupled with ordinary least squares one is seeking to minimize

$$\sum_{i=1}^n (\ln(-\ln q_i) - \ln(\ln \hat{q}_i))^2 \quad (10)$$

Since the sum of squares being minimized is different, the resulting estimates of $\hat{\alpha}$ and $\hat{\beta}$ derived from log-linearization can be very different from the estimates derived using non-linear regression.

2.3 Weighted Least Squares (WLS)

If we can weight the regression using the log-linearization method in such a way that the weighted distance metric is the same as for the non-linear regression method, then we can achieve the results of the non-linear regression without using the costly iterative technique. That is, by putting $u = \ln(-\ln q_i)$, $v = \ln x_i$, $a = \ln \alpha$ and $b = \beta$, (7) becomes $u = a + bv + e$ and we solve for values of a and b which minimize $\sum w_i^2 e_i^2$.

Also, a weighted log-linear result using (7) and using weights w_i may be obtained by minimizing

$$\sum_{i=1}^n w_i^2 (\ln(-\ln q_i) - \ln(\ln \hat{q}_i))^2 \quad (11)$$

Equating expressions (9) and (11), and solving for w_i we have

$$1/w_i = (\ln(-\ln q_i) - \ln(-\ln \hat{q}_i)) / (q_i - \hat{q}_i) \quad (12)$$

As $\hat{q}_i \rightarrow q_i$, $1/w_i \rightarrow (\ln(-\ln q_i))'$ (13)

where the prime indicates the derivative with respect to q_i , and we have $w_i = -q_i \ln q_i$.

Thus, using (11) with $w_i = -q_i \ln q_i$, we have a weighted least squares method (using ordinary least squares) which is approximately equivalent to using non-linear least squares. (Similar results are obtained when the roles of the independent and dependent variables are reversed and (8) is used. In that case it is easy to show that $w_i = x_i$.)

2.4 Generalized Least Squares (GLS)

Engeman and Keefe (1982) describe a method which takes into account the fact that the observations must be ordered to calculate the empirical CDF or percentiles. Putting

$$\hat{y}_i = \ln \hat{x}_i, \hat{z}_i = \ln(-\ln q_i), \hat{a} = -(\ln \hat{\alpha})/\hat{\beta}, \hat{b} = 1/\hat{\beta} \quad (14)$$

(8) may be written as

$$\hat{y}_i = \hat{a} + \hat{b} z_i. \quad (15)$$

However, the variance-covariance matrix of $Y' = (y_1, y_2, \dots, y_n)$ is not $\sigma^2 I$ but $\sigma^2 V$. An approximation to V is given by

$$v_{ij} = (1-q_i)/(q_i \ln q_i \ln q_j) \quad \text{for } i \leq j. \quad (16)$$

The resulting estimates from Engeman and Keefe are obtained from the equation

$$(\hat{a}, \hat{b})' = (Z'V^{-1}Z)^{-1}(Z'V^{-1}Y), \quad (17)$$

where Z consists of a column of ones and a column of the values $\ln(-\ln q_i)$.
Knowing estimates of a and b , we can find estimates of α and β since

$$\begin{aligned} \hat{\alpha} &= \exp(-\hat{a}/\hat{b}) \\ \hat{\beta} &= 1/\hat{b}. \end{aligned} \quad (18)$$

2.5 Methods of Moments (MM)

Menon (1963) used the method of moments to obtain

$$\begin{aligned} \hat{b} &= (6s^2/\pi^2)^{1/2} \\ \hat{a} &= \bar{x}' + /5772 b \end{aligned} \quad (19)$$

where s^2 and $\bar{x}'$ are the sample variance and mean, respectively, of $\ln x_i$.
We then calculate

$$\begin{aligned} \hat{\alpha} &= \exp(-a/b) \\ \hat{\beta} &= 1/b. \end{aligned} \quad (20)$$

3. Simulation Results

The five estimation methods described in the previous section, the log-linear (LL), weighted least squares (WLS), non-linear regression (NLR), generalized least squares (GLS), and Menon's methods of moments (MM) were used on a number of generated data sets. All of the methods except MM could be approached from two directions. The $\ln x$ could be considered the dependent variable or $\ln(-\ln q)$ could be considered as the dependent

variable. The LL, WLS, and NLR generally gave better results when $\ln(-\ln q)$ was used as the dependent variable. GLS gave much better results when $\ln x$ was used as the dependent variable. Therefore, we present results for $\ln(-\ln q)$ as the dependent variable when LL, WLS, or NLR was used, and $\ln x$ when GLS was used.

Samples of size $n = 25$ were generated from each of the distributions in the study. For each distribution $N = 30$ replications were used. Several Weibull distributions were used with $\alpha = .001, .01, .1, 1, 10$ and $\beta = .5, 1, 2, 4$. Since a number of the combinations yielded nearly the same results, we report only the following (α, β) combinations: $(1,1)$, $(10,1)$, $(1,2)$, and $(1,4)$.

Contamination and censoring are two factors which can significantly affect the estimates. One form of contamination was simulated by randomly taking 8% or 2 out of 25 of the observations and transforming them by $a + \sqrt{b}x$. Values of (a,b) were $(1,1)$, $(.5,2)$ and $(0,4)$. Only $a = .5$, $b = 2$ is reported in Table 3.2 since the other values yielded similar results and this set of values gives the most contrast between the methods. Censoring was simulated by keeping a percent of the smallest observations. We used 40%, 60%, and 80% as percentages of observations kept.

The robustness of the estimation methods was also tested by using the Weibull model on data generated from another distribution. We used the normal distribution with $\mu = 3.5$, $\sigma^2 = 1$ and $\mu = 6$, $\sigma^2 = 1$, the log-normal with $\mu = 0$, $\sigma^2 = 1$, and the "log-Cauchy" distribution. The log-Cauchy distribution was obtained by generating a Cauchy observation x and transforming it by e^x . The location and scale parameters were 0 and .1, respectively.

Error evaluations were recorded for all of the different cases. The root mean square (RMS) was calculated with respect to the true underlying distribution, and the RMS of the parameters was also calculated. The formulas used are given by

$$RMS = \left(\sum_{j=1}^N \sum_{i=1}^n (F(x_{ij}) - F(x_{ij}; \alpha_j, \beta_j))^2 / Nn \right)^{1/2} \quad (21)$$

$$RMS_{\alpha} = \left(\sum_{j=1}^N (\alpha - \alpha_j)^2 / N \right)^{1/2} \quad (22)$$

$$RMS_{\beta} = \left(\sum_{j=1}^N (\beta - \beta_j)^2 / N \right)^{1/2}. \quad (23)$$

$F(x_{ij})$ is the value of the true underlying distribution for the i^{th} observation in the j^{th} generated sample. The sample estimates for α_j and β_j are α and β using the j^{th} sample. Tables 3.1 through 3.3 give the results for the simulated problems.

TABLE 3.1

RMS, RMS_{α} , RMS_{β} For the Fit to the Known Weibull Distribution

(All combinations of (α, β) gave essentially the same results)

	<u>LL</u>	<u>WLS</u>	<u>GLS</u>	<u>NLR</u>	<u>MM</u>
RMS	.056	.061	.055	.064	.059
RMS_{α}	.230	.228	.221	.263	.252
RMS_{β}	.216	.212	.194	.251	.252

TABLE 3.2

RMS, RMS_{α} , RMS_{β} For the Fit to the Known Weibull Distribution When 8% of the Generated Values Have Been Replaced by a Contaminated Value

	<u>LL</u>	<u>WLS</u>	<u>GLS</u>	<u>NLR</u>	<u>MM</u>
$(\alpha = 1, \beta = 1)$					
RMS	.060	.059	.057	.060	.060
RMS_{α}	.202	.183	.199	.194	.210
RMS_{β}	.206	.200	.194	.277	.233
$(\alpha = 10, \beta = 1)$					
RMS	.074	.065	.072	.063	.072
RMS_{α}	3.81	4.01	4.39	5.75	3.68
RMS_{β}	.150	.160	.179	.197	.147
$(\alpha = 1, \beta = 2)$					
RMS	.072	.062	.069	.061	.070
RMS_{α}	.228	.180	.203	.187	.228
RMS_{β}	.348	.369	.369	.402	.359
$(\alpha = 1, \beta = 4)$					
RMS	.122	.080	.116	.063	.112
RMS_{α}	.328	.186	.269	.192	.324
RMS_{β}	1.16	1.15	1.50	.78	.87

Note: A contaminated observation is $.5 + 2x$, where x is the true observation: all other contaminations gave similar results.

TABLE 3.3

RMS Values For the Fit to the Known Weibull Distribution Where
a Percentage of the Largest Values Have Been Removed or Censored

<u>Distrib.</u>	<u>% Censored</u>	<u>LL</u>	<u>WLS</u>	<u>GLS</u>	<u>NLR</u>	<u>MM</u>
Weibull $\alpha=1, \beta=4$	60	.123	.107	.100	.105	-
	40	.089	.080	.079	.081	-
	20	.070	.067	.063	.069	-
	0	.056	.061	.055	.064	.059
Normal $N(6,1)$	60	.104	.092	.097	.091	-
	40	.080	.074	.076	.075	-
	20	.064	.066	.064	.065	-
	0	.061	.062	.059	.063	.063
Log Normal $\mu=0, \sigma^2=1$	60	.120	.099	.111	.113	-
	40	.091	.078	.082	.080	-
	20	.072	.071	.070	.069	-
	0	.071	.067	.067	.067	.073
Log Cauchy*	0	.165	.118	.132	.075	.153

*The Log-Cauchy was formed using the Cauchy probability density function for x given by $s/(x^2 + (x-m)^2)$ where $x = .1$, $m = 0$. If x has a Cauchy pdf, then the Log-Cauchy variable is $\exp(x)$.

Table 3.1 gives a comparison of the various methods under ideal conditions. That is, the model distribution is correct, and there is no contamination or censoring of the data. The values are essentially the same for all (α, β) combinations with the exception of $\alpha = 10$, $\beta = 1$, in

which case the RMS_{α} is approximately 20 times larger than other RMS , RMS_{α} , and RMS_{β} values but the pattern is the same. The GLS method out-performs the other methods with respect to the fit of the CDF and the estimation of (α, β) . It is interesting to note that the LL method is close to GLS. This is important for cases in which computer storage space is limited and the sample size is large since GLS does require an additional $n \times n$ matrix.

Table 3.2 illustrates the effects of contamination. WLS and NLR appear to be very stable with respect to the fit of the CDF. The LL, GLS, and MM tend to degrade for larger α or β values.

Table 3.3 shows that GLS does slightly better than WLS and NLR when the underlying distribution is Weibull and censoring is present. Also worth noting is that the LL method appears to be more adversely affected by censoring than the other methods.

Table 3.3 also shows the robustness properties of the estimators. It is interesting that the Weibull can approximate a normal distribution quite well when no censoring is present for any estimation methods. Another interesting case is that of the log-Cauchy distribution, Only NLR was able to give satisfactory results, and actually the fit is surprisingly good considering the underlying distribution.

4. SUMMARY AND CONCLUSIONS

We have examined the small sample properties of five methods of parameter estimation for the two parameter Weibull distribution using simulation. It is clear that the GLS is the best estimator and LL is a close second best with respect to the fit of the CDF and the RMS of the parameters under ideal conditions. When contamination is present, WLS and NLR are very stable whereas the other methods give much poorer results

especially when α or β are larger than 2. NLR also gives much better results than the other methods when the underlying distribution is the log-Cauchy distribution. WLS, GLS, and NLR perform well when the underlying distribution is the Log-Normal distribution.

Computationally LL and MM are certainly the least expensive. They are easy to use even on a hand calculator, and the results they give are not far from the GLS under ideal conditions. However, LL and MM do not offer the robustness of the other three methods. The robustness of the WLS, GLS, and NLR costs in terms of complexity and storage. NLR requires a fairly complex program or a software package to implement. GLS requires an additional $n \times n$ matrix which can be prohibitive for large sample sizes. This problem can probably be overcome by grouping data if storage is a problem. WLS requires only an additional n weights to the LL method.

The WLS method appears to be the most cost effective method of the five examined in this study. The NLR method appears to be the most robust. If we consider the log-Cauchy to be an extreme case of a wrong model using the Weibull, then we may conclude that NLR is an extremely robust procedure.

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The method which we denote by WLS (weighted least squares) was first suggested to us by Major A. Boehm of the U.S. Air Force.

All the programs used to make the simulations reported on in this manuscript were written by David Van Brackle. The programs were written in FORTRAN 77 and run on a DEC-VAX-11/780.

6. REFERENCES

- Engeman, R. M. and T. J. Keefe 1982: Generalized Least Squares Estimation of the Weibull Distribution, Communications in Statistics, Theory and Methods, 11(19), 2181-2193.
- Menon, M. H. 1963: Estimation of the Shape and Scale Parameters of the Weibull Distribution, Technometrics 5, 175-182.
- Somerville, P. N. and S. J. Bean 1982: A Comparison of Maximum Likelihood and Least Squares For the Estimation of a Cumulative Distribution, J. Statist. Comput. Simul. Vol. 14, 229-239.