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TECHNICAL REPORT RG-83-15

**DISTURBANCE MINIMIZATION TECHNIQUES FOR
LINEAR, TIME-INVARIANT DYNAMICAL SYSTEMS**

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20. ABSTRACT (Continue on reverse side if necessary and identify by block number) The objective of this research effort is to compile, refine and extend Disturbance Minimization Control design techniques for linear, time-invariant, regulation and servo-tracking systems. In addition, a collection of example problems are worked-out in detail to illustrate the design techniques and simulation results are used to demonstrate the performance of the resulting controllers. The results of this effort are presented in four parts. Part I, encompassing Chapters I and II, presents a review of the general			

theory of Disturbance Minimizing Control design techniques, including the ideas of disturbance modeling and control allocation. Several extant methods for minimization of disturbance effects are introduced.

Part II, encompassing Chapters III-V, presents an in-depth coverage of methods for designing control parts to effect desired disturbance minimization, for state and output set-point regulation/stabilization and state and output servo-tracking type problems.

Part III, encompassing Chapter VI, presents the conclusions and recommendations for further study. Part IV, consisting of Appendices A-G, contains a review of the concept of a matrix generalized inverse and provides detailed examples of disturbance minimizing control design which illustrate techniques presented in Parts I and II. (2007 0000)

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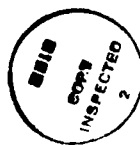


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LIST OF SYMBOLS

A	Plant coefficient matrix; dimension n by n .
$\hat{A}_{ij}$	The ij -th partition block of the matrix product $(A+FL+BK)^{-1}(A+FL)$.
$\bar{A}$	(1) Matrix defined by $(A+BK)$ (2) Matrix defined by $(A+FL+BK)$.
$\tilde{A}$	Matrix defined by $(A+FL+BK)$.
$\tilde{A}_{ij}$	The ij -th partition block of the plant A matrix after transformation from x -space to ξ -space.
$\bar{b}$	Parameter used in designing a control part which is assumed <u>a priori</u> to have the same form as the disturbance term with which it is associated.
$\bar{b}^*$	The minimum norm of $\bar{b}$.
B	Plant control distribution matrix; dimension n by r .
$\bar{B}$	Plant control matrix elements associated with critical state-variable set.
$\tilde{B}_i$	The i -th partition block of plant control distribution matrix B after transformation from x -space to ξ -space.
c	State vector of a servo-tracking dynamic process
C	Plant output coefficient matrix; dimension n by p .
$\hat{C}$	Represents the matrix product $C(A+FL)C^{-1}$.
$\mathcal{R}$	The ρ -dimensional column range space of the composite matrix $[B_1 A_1 B_1 A_1^2 B_1 \dots A_1^{(n-p-1)} B_1]$, $A_1 = PAP^\dagger$, $B_1 = PB$.
ξ	Element of an alternate state space in which the plant output y becomes a state sub-vector.

LIST OF SYMBOLS (Continued)

ξ_{sp}	A set-point in ξ -space corresponding to a given set-point in x -space.
D	Disturbance coefficient matrix; dimension ρ by ρ .
δ	Represents a general disturbance term.
e_{ss}	A variable, associated with output servo-tracking problems, defined as $(\Theta c - x)$.
E	Output control distribution matrix; dimension m by r .
ϵ	(1) Set-point error $x_{sp} - x$ or $y_{sp} - y$. (2) Servo-command error $x_c - x$ or $y_c - y$.
ϵ_{ss}	Steady-state value of ϵ .
ϵ_{ss}^*	The minimum norm value of ϵ_{ss} .
ϵ_{xss}	Steady-state error vector in x -space.
ϵ_y	Error vector in output space.
ϵ_{yss}	Steady-state error vector in output space.
ϵ_{ξ}	Error vector defined in ξ -space.
$\epsilon_{\xi ss}$	Steady-state error vector in ξ -space.
F	Disturbance input matrix; dimension n by p .
$\bar{F}$	Partition block of re-ordered F matrix corresponding to disturbance elements which cannot be completely absorbed.
$\tilde{F}_i$	The i -th partition block of external disturbance input matrix F after transformation from x -space to ξ -space.
G	Disturbance distribution matrix in plant output; dimension m by p .
γ^i	The i -th component of $\sum_{j=1}^n a_{ij} x_{spj} + \sum_{j=1}^p f_{ij} h^j$ where a_{ij} is the ij -th element of A , x_{spj} is the j -th row of the set-point vector, f_{ij} is the ij -th element of the F matrix and h^j is the j -th row of H .

LIST OF SYMBOLS (Continued)

$\gamma_{R(B)}$	The component of γ which lies within the column range space of the B matrix.
$\gamma_{R(B)}^\perp$	The component of γ which lies within the orthogonal complement of the column range space of B.
H	Output matrix in dynamic model of external disturbances; dimension p by ρ .
$\bar{H}$	Partition block of re-ordered H matrix corresponding to disturbance elements which cannot be completely absorbed.
K	(1) Gain matrix used in feedback stabilization of plant, e.g. $u_p = -Kx$. (2) Linear transformation matrix between x-space and ξ -space.
L	State coupling matrix in general output equation for external disturbance; dimension p by n.
$\bar{L}$	Partition block of re-ordered L matrix corresponding to disturbance elements which cannot be completely absorbed.
λ	Eigenvalue symbol.
M	(1) State coupling matrix in disturbance state differential equation; dimension ρ by n. (2) An n by p matrix composed of a set of basis vectors for $N(C)$.
$N(\cdot), \cup(\cdot)$	Denotes the nullspace of the matrix contained in the brackets.
μ	Sparse sequence of impulses associated with servo-command dynamical process model.
ω	Impulsive forcing function in the differential equation describing the external disturbance.
P	(1) An (n-p) by n matrix, with $\rho(P) = n-p$, whose rows form a basis for $N(C)^\perp$. (2) A weighting matrix for a quadratic norm.
$P(s)$	Numerator of the Laplace transformation of the external disturbance differential equation.
$\Phi(t, t_0)$	Plant state transition matrix.
Q	A weighting matrix for a quadratic norm.

LIST OF SYMBOLS (Continued)

$Q(s)$	Denominator of the Laplace transformation of the external disturbance differential equation.
R	Output matrix in dynamic model of servo-tracking process; dimension m by u .
R^i	An i -dimensional real vector space.
$R(\cdot), \rho(\cdot)$	The column range space of the matrix contained within the brackets.
S	Servo-command process dynamic system matrix; dimension u by u .
σ	Sparse sequence of impulses associated with external disturbance dynamical process model.
T	Linear transformation matrix between x -space and ξ -space.
$\hat{\theta}$	Represents the matrix term $(A+FL-S)$.
u	Composite control vector in plant equations.
U	Utility of external disturbance. A positive utility indicates assistance by w in minimizing disturbance effects.
u_d	Control part of u associated with minimization of disturbance effects.
u_d^*	The minimum norm value of u_d .
u_{dc}	The control part of u_d associated with minimization of servo-command disturbance effects.
u_{dc}^*	The minimum norm value of u_{dc} .
u_{dcp}	The control part of u_d associated with minimization of "coupling" disturbance $(\tilde{A}_{12}\xi_2)$ effects.
u_{ds}	The control part of u_d associated with minimization of set-point disturbance effects.
u_{ds}^*	The minimum norm value of u_{ds} .
u_{dw}	The control part of u_d associated with minimization of external disturbance effects.
u_{dw}^*	The minimum norm value of u_{dw} .

LIST OF SYMBOLS (Concluded)

u_p	Control part of u associated with stabilization of the basic plant response.
v_1	Component of plant state x lying in $N(C)^\perp$.
v_2	Component of plant state x lying in $N(C)$.
w	External disturbance input vector; dimension p by 1 .
x	Plant state vector; dimension n by 1 .
$\hat{x}$	State reconstructor estimate of plant state vector x .
x_c	State servo-command input vector.
x_{sp}	Set-point vector in state space.
x_{ss}	Steady-state value of plant state vector.
$x_{ss_{N(C)^\perp}}$	Component of x_{ss} lying in $N(C)^\perp$.
y	Plant output vector; dimension m by 1 .
y_c	Output servo-command vector.
y_{sp}	Set-point vector in plant output space.
y_{ss}	Steady-state value of plant output y .
z	External disturbance state vector; dimension ρ by 1 .
$\hat{z}$	Composite state reconstructor estimate of external disturbance state vector z .
ζ	Represents a general disturbance control vector.
$(\cdot)^{-1}$	Denotes matrix inverse of matrix in brackets.
$(\cdot)^T$	Denotes matrix transpose of matrix in brackets.
$(\cdot)^\dagger$	Denotes matrix generalized inverse of matrix in brackets.
$(\cdot)^\perp$	Denotes the orthogonal complement of matrix in brackets.
$ (\cdot) _Q^2$	The square of the weighted quadratic norm of the quantity in brackets.

PART I

GENERAL THEORY

31

CHAPTER I
AN OVERVIEW OF DISTURBANCES
IN CONTROL PROBLEMS

1.1 Summary of Chapter I

This chapter discusses the general classes of systems, disturbances and controllers considered in this study. The types of control problems involved are delineated and the notions of disturbance modeling, control allocation, state determination and state reconstruction are discussed. Some preliminary concepts relative to the mathematics of disturbance minimization are defined.

1.2 Linear Dynamical Systems

The class of systems to be considered in this study are "linear dynamical systems" (LDS), so-called because the vector differential equation for the state $x(t)$ is a linear differential equation and the transformation between the state space and output space is linear. These systems will be represented by equations of the general form

$$\begin{aligned}\dot{x}(t) &= Ax(t) + Bu(t) + Fw(t) \\ y(t) &= Cx(t) + Eu(t) + Gw(t)\end{aligned}\tag{1.1}$$

where $x(t)$ is the plant state vector and is an n -vector, $u(t)$ is the plant control input vector and is an r -vector, $w(t)$ is the plant disturbance vector and is a p -vector, $y(t)$ is the plant output vector and is an m -vector and A , B , F , C , E , G are appropriate size, known

matrices which are possibly time-varying. If all elements of the matrices A, B, F, C, E, G are constant with respect to time, the system is called time-invariant, otherwise, it is called time-varying. The general solution of the first of Equations (1.1), in the case of a time-invariant system, can be written as [27]:

$$x(t) = e^{A(t-t_0)} x_0 + \int_{t_0}^t e^{A(t-\tau)} (Bu(\tau) + Fw(\tau)) d\tau. \quad (1.2)$$

1.3 Nature of Disturbances in Control Problems

Disturbances are defined as the uncontrollable inputs which act on a dynamical system. There are many varieties of disturbance inputs which can be associated with a controlled system and they are, for the most part, completely unpredictable in magnitude and in their arrival times.

In practice, additive disturbances, i.e., disturbances which are represented by terms added to the plant state equations, can be grouped into two major classes: (I) external disturbances due to motivating effects external to the plant, (II) internal disturbances due to motivating effects arising from the physical characteristics of plant subsystems or internal dynamics of the plant. In general, these latter disturbances occur within the physical confines of the plant, or along the plant boundary.

Each disturbance group can be further subdivided into two categories: (a) noise disturbances - characterized by random and erratic behavior with relatively high-frequency content and (b) waveform structured disturbances - characterized by a degree of waveform regularity which

can be described, piecewise in time, by differential equations forced by sparse sequences of impulses. Some examples of these varieties of disturbances, as they arise in aerospace-type control problems, are as follows:

I. External Disturbances

- a. Noise: jamming, glint
- b. Waveform: gravity, winds, target maneuvers

II. Internal Disturbances

- a. Noise: electrically generated
- b. Waveform: friction, time delays, quantization, vibrations, subsystem biases, time drifts.

The nature of these disturbances may be either completely known (through direct prior or real-time observation or test), completely unknown (random-like), or partially known.

In addition to additive disturbances, any effect which causes a perturbation in the plant parameters can be considered as a disturbance input. For example, if one or more parameters in the plant A matrix change, due perhaps to aging, one can write the perturbed A matrix A_1 as

$$A_1 = A + \delta A = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} + \begin{bmatrix} \delta a_{11} & \delta a_{12} \\ \delta a_{21} & \delta a_{22} \end{bmatrix},$$

where $\delta Ax(t)$ can be considered as a disturbance.

Disturbance Accommodating Control Theory is basically concerned with accommodating those external and internal disturbances which have

waveform structure. This study will be concerned primarily with additive type disturbances having waveform structure.

1.4 Disturbance Modeling

Johnson [8,9,10,11,12,13,17] introduced the idea of mathematically describing uncertain waveform-structured disturbances $w(t)$ by representing them as a weighted linear combination of known basis functions of the form

$$w(t) = \sum_{i=1}^n c_i f_i(t) , \quad (1.3)$$

where the weighting coefficients c_i are completely unknown constants which can change in magnitude in a random, once-in-a-while, fashion. The basis functions $f_i(t)$ are completely known because they are chosen by the designer based on the waveform patterns exhibited (or thought to be exhibited) by the disturbance and the first step in obtaining a model of the disturbance is typically to identify an appropriate set of basis functions [17].

Given that the basis functions have been identified, the next step is to attempt to find an impulsive forced linear differential equation of the form

$$\frac{d^{\rho} w}{dt^{\rho}} + \beta_{\rho} \frac{d^{\rho-1} w}{dt^{\rho-1}} + \dots + \beta_2 \frac{dw}{dt} + \beta_1 w = w(t) \quad (1.4)$$

for which Equation (1.3) is a solution. For instance, if the disturbance waveform is composed of a random-like superposition of constant value segments and exponentially varying segments, the basis functions would be selected as $f_1(t) = 1$, $f_2(t) = e^{\alpha t}$, α

known, so that

$$w(t) = c_1 1 + c_2 e^{\alpha t} . \quad (1.5)$$

Taking the Laplace transform of Equation (1.5) yields

$$w(s) = c_1/s + c_2/(s-\alpha) , \quad c_1, c_2 = \text{constant}, \quad (1.6)$$

and expressing Equation (1.6) as a ratio of polynomials over the least common denominator gives

$$w(s) = \frac{c_1(s-\alpha) + c_2 s}{s(s-\alpha)} = \frac{(c_1 + c_2)s - c_1 \alpha}{s(s-\alpha)} = \frac{P(s)}{Q(s)} . \quad (1.7)$$

If, as in [17], one imagines $w(s)$ to be the output of an imaginary linear dynamical system which has the transfer function $G(s) = 1/Q(s)$, and which is subject to initial conditions $w(0)$, $\dot{w}(0)$ giving rise to $P(s)$, then by expressing $Q(s)$ in Equation (1.7) as

$$Q(s) = s^2 - \alpha s \quad (1.8)$$

it is seen that $w(t)$ can be imagined as satisfying an impulsive forced second-order differential equation of the form

$$\frac{d^2 w}{dt^2} - \alpha \frac{dw}{dt} = \omega(t) , \quad (1.9)$$

where the parameter α is completely known because it depends only on the known basis functions. A "state model" for the disturbance is obtained by expressing Equation (1.9) as an equivalent set of first-order differential equations,

$$\begin{aligned} w &= z_1 \\ \dot{z}_1 &= z_2 + \sigma_1(t) \\ \dot{z}_2 &= \alpha z_2 + \sigma_2(t) \end{aligned} \quad (1.10)$$

where the action of $\omega(t)$ in Equation (1.9) is represented in Equations (1.10) by $\sigma_1(t)$ and $\sigma_2(t)$ which are random sequences of impulses.

More generally, $Q(s)$ has the form

$$Q(s) = s^{\rho} + \beta_{\rho} s^{\rho-1} + \text{----} + \beta_2 s + \beta_1 \quad (1.11)$$

and Equation (1.9) has the general linear form

$$\frac{d^{\rho} w}{dt^{\rho}} + \beta_{\rho} \frac{d^{\rho-1} w}{dt^{\rho-1}} + \text{----} + \beta_2 \frac{dw}{dt} + \beta_1 w = \omega(t) . \quad (1.12)$$

The general form of the disturbance state model for Equation (1.4) is

$$\begin{aligned} w &= Hz + Lx \\ \dot{z} &= Dz + Mx + \sigma(t) \end{aligned} \quad (1.13)$$

where z is the ρ -dimensional disturbance "state" vector, $\sigma(t)$ is a sparsely populated vector impulse sequence which plays the same role as the impulsive sequence $\omega(t)$ in Equation (1.4), and H, L, D, M are appropriate size, known matrices. For the example of Equation (1.5), Equations (1.13) become

$$\begin{pmatrix} \dot{z}_1 \\ \dot{z}_2 \end{pmatrix} = \begin{bmatrix} 0 & 1 \\ 0 & \alpha \end{bmatrix} \begin{pmatrix} z_1 \\ z_2 \end{pmatrix} + \sigma(t) \quad (1.14)$$

$$w = (1, 0)z . \quad (1.15)$$

A more complete and detailed exposition of the disturbance waveform modeling procedure is presented in [17].

1.5 Types of Control Problems

Given the plant model, Equations (1.1), and the disturbance model, Equations (1.13), we now examine the various types of control problems which may be encountered in practice.

1.5.1 State Control Problems. The class of problems known as state control problems involves the design of controller action to drive some or all of the plant states to a predetermined specification or target condition. One category of such problems involves either state stabilization or set-point regulation design. For the stabilization

problem the set-point is the origin, while for regulation problems the set-point can be any point in state space. In order for state set-point regulation to be successfully accomplished, the target set-point x_{sp} must be a "regulatable" set-point. By a regulatable set-point is meant a set-point for which a control $u(t)$ can be found such that the plant state $x(t)$ can be steered to x_{sp} (i.e., the system is controllable to x_{sp}) and maintained at x_{sp} thereafter. If the target set-point is not regulatable, then it will not be possible to successfully accomplish state set-point regulation.

The other category of state control problems is referred to as state servo-tracking problems and involves design of a control input function $u(t)$ such that $x(t)$ tracks an input state servo-command $x_c(t)$ with high fidelity. In order for state servo-tracking to be successfully accomplished, the input servo-command must be a "trackable" command. A trackable command is a command for which a control $u(t)$ can be found such that the plant state $x(t)$ can be steered to $x_c(t)$ and maintained thereafter "near" $x_c(t)$.

1.5.1.1 State Set-Point Regulation/Stabilization. For the state set-point regulation/stabilization problem, the control objective is to steer the plant state vector $x(t)$ to a given set-point vector x_{sp} . The control $u(t)$ should be designed to achieve this goal for all initial conditions $x(0)$ and for all disturbances, external or internal.

For our purposes, this problem will be reformulated by defining the control error $e(t)$ to be

$$e(t) = x_{sp} - x(t) \quad (1.16)$$

The dynamics of the control error can then be expressed as (using the first of Equations (1.1), with A, B, F time-invariant, for $\dot{x}(t)$)

$$\dot{\epsilon}(t) = \dot{x}_{sp} - \dot{x}(t) = A\epsilon(t) - Bu(t) - Ax_{sp} - Fw(t) . \quad (1.17)$$

Equation (1.17) is structurally similar to the plant model, Equation (1.1); however, an additional "set-point" disturbance term Ax_{sp} appears in Equation (1.17). The control objective can now be reformulated as: choose $u(t)$ to attain $\epsilon(t) \rightarrow 0$, for all $\epsilon(0)$, for all x_{sp} , and for all $w(t)$, if possible. Otherwise, choose $u(t)$ so as to minimize $\epsilon_{ss} = \lim_{t \rightarrow \infty} \epsilon(t)$ according to some specified minimization criterion.

1.5.1.2 State Servo-tracking. For the state servo-tracking problem, the control objective is to steer the system state $x(t)$ so as to follow, or track, a time-varying input servo-command vector $x_c(t)$. The control $u(t)$ should be designed to accomplish this task in the face of all plant state initial conditions $x(0)$ and all disturbances $w(t)$.

As in Equations (1.16) and (1.17), one can reformulate the servo-tracking problem in terms of an error vector,

$$\epsilon(t) = x_c(t) - x(t), \quad (1.18)$$

where $\epsilon(t)$ is governed by (using the first of Equations (1.1) for $\dot{x}(t)$)

$$\dot{\epsilon}(t) = \dot{x}_c(t) - \dot{x}(t) = A\epsilon(t) - Bu(t) - Ax_c(t) + \dot{x}_c(t) - Fw(t). \quad (1.19)$$

Equation (1.19) contains not only the external disturbance $w(t)$ but also a "servo-command disturbance", $\dot{x}_c(t) - Ax_c(t)$. The control objective is thus reformulated as: design $u(t)$ such that $\epsilon(t) \rightarrow 0$ if possible, otherwise, design $u(t)$ so as to minimize $\epsilon(t)$ according to some specified minimization criterion.

1.5.2 Output Control Problems. Output control problems involve manipulation of the control to steer the plant output $y(t)$ to some predetermined specification and are divided into output set-point regulation/stabilization and output servo-tracking type problems. For

output stabilization, the output set-point is the origin while for output set-point regulation the set-point can be any point y_{sp} in the output space. For $y(t)=Cx(t)$, C a constant matrix, in order for output set-point regulation to be successfully achieved the output set-point must lie within the column range space of the output matrix C ; otherwise, no state vector x can be found which will result in the desired output. In addition, it must be possible to find a permissible control vector $u(t)$ which will steer $x(t)$ to a plant state which will transform, under C , to the desired y_{sp} and will maintain it there.

Output servo-tracking problems involve design of a control which will steer the plant output $y(t) = Cx(t)$ such that it tracks an input servo-command $y_c(t)$ with high fidelity. In order for output servo-tracking to be successfully accomplished, the input servo-command must be a "trackable" command, i.e., it must be possible to find a permissible control $u(t)$ which will steer $y(t)$ to $y_c(t)$ and maintain it thereafter "near" $y_c(t)$.

1.5.2.1 Output Set-Point Regulation/Stabilization. For output set-point regulation/stabilization, the object of the control action is to drive the plant output vector $y(t)$ to a given set-point y_{sp} . The controller must be designed to accomplish this desired objective in the face of initial conditions $x(0)$ and disturbance inputs $w(t)$.

In many cases one finds that $y = Cx$. For such cases, one design method for the output control problem involves finding an evolution equation which governs $y(t)$. For that purpose it is convenient to elevate the output $y(t)$ to the status of a state sub-vector by finding a non-singular linear transformation K such that

$$x = K\xi = \begin{bmatrix} K_1 & K_2 \end{bmatrix} \xi, \quad K = \text{constant matrix}, \quad (1.20)$$

maps x to a new state vector ξ where

$$\xi = K^{-1}x = \begin{bmatrix} \bar{K}_1 & \bar{K}_2 \end{bmatrix}^T x. \quad (1.21)$$

In the new coordinate system, one requires that

$$\bar{K}_1 x = y = Cx. \quad (1.22)$$

For this to be possible, one condition which must be met is that $\text{rank}(C) = m$, i.e., C must represent an onto transformation from the plant state space to the plant output space.

Given that the linear transformation K can be found, the new state vector will be

$$\xi = (\xi_1 \mid \xi_2)^T = (y \mid \xi_2)^T \quad (1.23)$$

and it is then easy to show that the evolution equation for $\xi(t)$ is

(using the first of Equations (1.1) for $\dot{x}(t)$)

$$\begin{aligned} \dot{\xi}(t) &= K^{-1}\dot{x}(t) = K^{-1}(Ax(t) + Bu(t) + Fw(t)) \\ &= (K^{-1}AK)\xi(t) + K^{-1}Bu(t) + K^{-1}Fw(t). \end{aligned} \quad (1.24)$$

Expressing Equation (1.24) in the matrix form

$$\begin{pmatrix} \dot{\xi}_1 \\ \dot{\xi}_2 \end{pmatrix} = \begin{bmatrix} \tilde{A}_{11} & \tilde{A}_{12} \\ \tilde{A}_{21} & \tilde{A}_{22} \end{bmatrix} \begin{pmatrix} \xi_1 \\ \xi_2 \end{pmatrix} + \begin{pmatrix} \tilde{B}_1 \\ \tilde{B}_2 \end{pmatrix} u + \begin{pmatrix} \tilde{F}_1 \\ \tilde{F}_2 \end{pmatrix} w \quad (1.25)$$

and recalling that $\xi_1 = y$, one obtains the output evolution equation

$$\dot{y} = \tilde{A}_{11}y + \tilde{A}_{12}\xi_2 + \tilde{B}_1u + \tilde{F}_1w. \quad (1.26)$$

Now, given that the objective is to send $y(t)$ to some specified y_{sp} , one can again define an error vector

$$\epsilon(t) = y_{sp} - y(t) \quad (1.27)$$

and obtain the evolution equation

$$\dot{\epsilon}(t) = -\dot{y}(t) = \tilde{A}_{11}\epsilon(t) - \tilde{B}_1u(t) - \tilde{A}_{11}y_{sp} - \tilde{A}_{12}\xi_2(t) - \tilde{F}_1w(t). \quad (1.28)$$

The error, Equation (1.28), now contains the external disturbance, the "set-point" disturbance and a "coupling" disturbance due to ξ_2 . The output control objective is now defined as, send $\epsilon(t) \rightarrow 0$ and maintain it there for all $\epsilon(t_0)$, y_{sp} , $\xi_2(t_0)$ and $w(t)$.

1.5.2.2 Output Servo-tracking. For the case of output servo-tracking, the control objective is to force the plant output vector $y(t)$ to follow an input servo-command vector $y_c(t)$ in the face of all initial conditions $y(0)$ and disturbance inputs $w(t)$.

If one defines

$$\epsilon(t) = y_c(t) - y(t) \quad (1.29)$$

then it follows that

$$\dot{\epsilon}(t) = \dot{y}_c(t) - \dot{y}(t) \quad (1.30)$$

Utilizing the results of Section 1.5.2.1, it is readily found that

$$\dot{\epsilon}(t) = \tilde{A}_{11}\epsilon(t) - \tilde{B}_1 u(t) + \dot{y}_c(t) - \tilde{A}_{11}y_c(t) - \tilde{A}_{12}\xi_2(t) - \tilde{F}_1 w(t) \quad (1.31)$$

Thus, one has external, "coupling" and "servo-command" disturbances in Equation (1.31) and the control objective is to send $\epsilon(t) \rightarrow 0$ and maintain it there for all $\epsilon(0)$, for all $y_c(t)$, for all $\xi_2(t)$ and for all $w(t)$.

1.6 The Idea of Disturbance Accommodation

Disturbance Accommodation refers to a control engineering design technique, developed by C. D. Johnson of the University of Alabama in Huntsville [8] - [18], wherein a combination of waveform-mode disturbance modeling and state-variable control methods are utilized to design controllers which will: (1) absorb, (counteract) (2) minimize or

(3) constructively utilize the effects of uncertain disturbances on the plant.

Three main classes of controllers are considered within the overall cognomen of Disturbance Accommodating Control Theory. These are, (1) Disturbance Absorption Controllers (DAC), (2) Disturbance Minimization Controllers (DMC) and (3) Disturbance Utilizing Controllers (DUC). Each class of controllers has its own associated design goals and design methodology.

It is interesting to note that in the area of modern, multi-variable control, disturbance minimization problems have apparently received no previous attention in the control literature prior to the work done by Johnson. Since that time, the mathematical theories of DAC and DUC have been thoroughly developed in [8] - [18] and in [20] and the subject of Disturbance Absorption has been studied by numerous investigators. The subject of Disturbance Minimization; however, has apparently received little attention [9, 17]. Of course, within classical control theory there were certain transfer function techniques which had limited application in reducing disturbance effects [37]. The objective of the present work is to develop a general theory for the design of DMC-type disturbance accommodating controllers.

1.7 Control Effort Allocation in Disturbance Accommodating Control Problems

In disturbance accommodating control design, the control vector $u(t)$ is considered to be an ordered collection of the various independent control inputs which are available to accomplish the primary control objective and to "accommodate" the disturbances which act on the

system. As can be seen from Section 1.5, even if no external disturbances are acting, internal disturbance terms may still be present in the reformulated problem.

In the design of disturbance absorption and minimization controllers it is common practice to split (allocate) the total control $u(t)$ into two parts as follows

$$u(t) = u_p(t) + u_d(t) \quad (1.32)$$

where $u_p(t)$ is given the task of accomplishing the primary control objective and $u_d(t)$ is given the task of disturbance accommodation. The part $u_d(t)$ can be further subdivided into component vectors, as required.

In the case of Equation (1.17) for example, one had

$$\dot{\epsilon}(t) = A\epsilon(t) - Ax_{sp} - Fw(t) - Bu(t) \quad (1.17)$$

and thus if one substitutes in Equation (1.32) there obtains

$$\dot{\epsilon}(t) = A\epsilon(t) - Ax_{sp} - Fw(t) - Bu_p(t) - Bu_d(t) \quad (1.33)$$

If the designer now sets $u_p(t) = -K\epsilon(t)$, it is found that

$$\dot{\epsilon}(t) = (A + BK)\epsilon(t) - (Ax_{sp} + Fw(t) + Bu_d(t)) \quad (1.34)$$

The control matrix K should thus be designed such that the homogeneous system $\dot{\epsilon}(t) = (A + BK)\epsilon(t)$ will yield $\epsilon(t) \rightarrow 0$ "rapidly".

The second collection of terms on the right-hand side of Equation (1.34) includes external disturbance and "set-point" disturbance terms. One method for accommodating those terms is to split $u_d(t)$ into $u_d(t) = u_{dw}(t) + u_{ds}(t)$, where $u_{dw}(t)$ is designed to absorb or minimize those external disturbances associated with $Fw(t)$, and $u_{ds}(t)$ is designed to absorb or minimize the set-point disturbance represented by Ax_{sp} . The question as to whether $u_d(t)$ can be designed to absorb these disturbance effects, or must instead be designed to minimize them,

depends on the satisfaction or non-satisfaction of an "absorbability" condition.

If one has an equation such as Equation (1.17), then complete absorption of a disturbance term δ (such as given by $\delta = Ax_{sp}$ or $\delta = Fw$) by $u_d(t)$ will ideally require that $Bu_d(t) \equiv -\delta$, $t \geq t_0$. For this to be possible, the following absorbability condition must be satisfied,

$$\text{Rank } [B|\delta] \equiv \text{Rank } [B] , \quad (1.35)$$

i.e., the column range space of δ must lie within the column range space of B . If condition (1.35) is met, then there exists a (possibly nonunique) vector ζ such that

$$B\zeta = \delta \quad (1.36)$$

and u_d can be designed as $u_d = -\zeta$.

The vector ζ is calculated using the mathematical theory of generalized inverses. The usual matrix inverse A^{-1} is defined only for square nonsingular matrices. The generalized inverse, represented by A^+ , is a generalization of the notion of A^{-1} for the class of rectangular and square-singular matrices. In the case of non-singular matrices, $A^+ = A^{-1}$. A short summary of the theory of generalized inverses is presented in Appendix A of this report. A more thorough account may be found in [3, 5, 25, 30].

For the applications in this report, the generalized inverse will be used to provide a solution ζ for consistent equations of the form of Equation (1.36) and will provide a best approximating ζ (which itself will be of minimum norm) which will give a least squares solution for inconsistent equations. Equation (1.36) is said to be inconsistent when $\text{Rank } [B|\delta]$ is greater than $\text{Rank } [B]$ and is said to be consistent when the ranks are equal. The solution for Equation (1.36)

will thus be represented symbolically in the form

$$\zeta = B^{\dagger} \delta . \quad (1.37)$$

In the case of external disturbances $w(t)$, the objective of disturbance absorption is to design a control vector $u_{dw}(t)$ to satisfy

$$Bu_{ds}(t) + Fw(t) = 0 ; \quad w(t) = Hz(t) + Lx(t) \quad (1.38)$$

or $Bu_{dw}(t) = -Fw(t) , \text{ for all } x \in R^n, z \in R^0, \quad (1.39)$

in order to completely cancel the disturbances. If the column range space of $F[H|L]$ is completely contained within the column range space of B , then there exists a matrix Γ such that $F[H|L] = B\Gamma$ and thus, if one defines

$$\zeta = -B^{\dagger} F[H|L] (z|x)^T , \quad (1.40)$$

the control

$$u_{dw} = \zeta \quad (1.41)$$

will cancel the external disturbance.

For the case of set-point disturbances Ax_{sp} , complete cancellation is possible if, and only if, there exists a u_{ds} such that

$$Bu_{ds} + Ax_{sp} = 0 . \quad (1.42)$$

Thus, if the column range space of Ax_{sp} is completely contained within the column range space of B then there exists a

$$\zeta = -B^{\dagger} Ax_{sp} \quad (1.43)$$

such that

$$u_{ds} = \zeta \quad (1.44)$$

will provide cancellation of the set-point disturbance.

If condition (1.35) is not satisfied, complete cancellation of δ is impossible and it will then be necessary to design $u_d(t)$ to minimize some specified feature of the disturbance effects. This is the main topic to be addressed in this dissertation.

The principle of splitting $u(t)$ into the sum of $u_p(t)$ and $u_d(t)$

parts is also followed in the design of state servo-tracking controllers and in output set-point and servo-tracking controllers. The final control $u(t)$ can take various forms in those problems [9,10,13,17]. In the case of disturbance utilizing controllers (DUC), the control is not split. The design process for DUC controllers produces a control which will accomplish the primary control objective and at the same time will allow the disturbances to aid in the accomplishment of this objective, if possible. For a further discussion of this latter topic see [8, 9, 17, 18].

1.8 On-Line, Real-Time State Reconstructors for Disturbance

Accommodation

One typically arrives at an idealized DAC control law expression involving terms like $u_p(t) = -Kx(t)$ or $u_{dw}(t) = -\Gamma z(t)$. In most practical situations, however, $x(t)$ and $z(t)$ are not directly measurable. Therefore, to physically implement the control law it becomes necessary to estimate (reconstruct, observe) the instantaneous state values $x(t)$, $z(t)$ from the available measurements. The device which is used to accomplish this state estimation, in those cases when it is possible, is a state reconstructor or observer.

To investigate the possibility of applying a state reconstructor we have the following [19]:

Theorem: The state of a constant coefficient linear system

$$\dot{x}(t) = Ax(t) + Bu(t)$$

$$y(t) = Cx(t) ,$$

where $x(t)$ is of dimension n , is completely constructable if, and only if,

$$\text{rank} [C^T, A^T C^T, (A^T)^2 C^T, \dots, (A^T)^{n-1} C^T] = n. \quad (1.45)$$

In other words, if a time-invariant system is completely observable, the state $x(t)$ is constructable.

In the case of time-invariant systems it has been shown [8,14, 17] that if the pair

$$\left(\left[\begin{array}{c|c} A + FL & FH \\ \hline M & D \end{array} \right], [C \mid 0] \right) \quad (1.46)$$

is completely observable, it will be possible to design a composite state reconstructor which will utilize on-line measurements of the system output $y(t)$, together with the control input $u(t)$, to generate on-line, real-time estimates $\hat{x}(t)$, $\hat{z}(t)$ of the plant and disturbance states $x(t)$, $z(t)$, such that $\hat{x}(t) \rightarrow x(t)$ and $\hat{z}(t) \rightarrow z(t)$ "rapidly".

One of the inherent features of the DAC design process is that a composite state reconstructor is designed as an integral part of the controller. Johnson [8,9,14] has developed design "recipes" for constructing both full-dimensional and reduced-order state reconstructors which provide the on-line, real-time estimates $\hat{x}(t)$ and $\hat{z}(t)$ for processing by the controller to provide the necessary components for physically realizing the control $u(t)$. Recall from Equations (1.1) and (1.13) that the most general plant state equations are

$$\dot{x}(t) = Ax(t) + Bu(t) + Fw(t)$$

$$y(t) = Cx(t) + Eu(t) + Gw(t)$$

and the disturbances are modeled by the general state equations

$$w(t) = Hz(t) + Lx(t)$$

$$\dot{z}(t) = Dz(t) + Mx(t) + \sigma(t).$$

If one combines Equations (1.1) and (1.13), the resulting equations are found to be

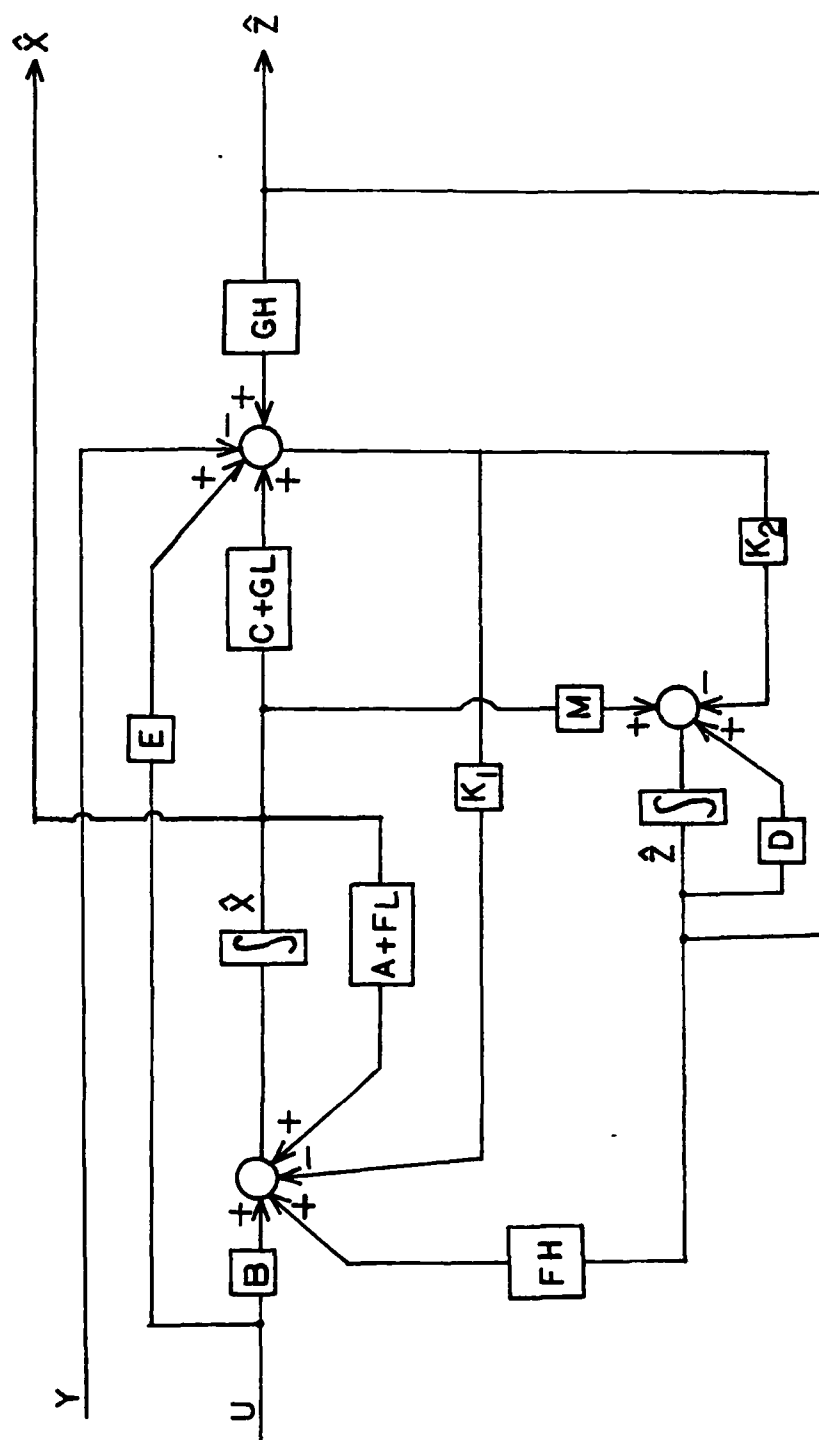


Figure 1-1. Block diagram of composite observer.

$$\begin{aligned}\dot{x}(t) &= (A + FL)x(t) + Bu(t) + FH_z(t) \\ y(t) &= (C + GL)x(t) + Eu(t) + GH_z(t) .\end{aligned}\tag{1.47}$$

Using Equations (1.47), a diagram for a state reconstructor (patterned after [21]) can be constructed as shown in Figure 1-1.

1.9 Objectives of this Research Study

The basic objectives of this research study are twofold: (a) to compile and unify the disturbance minimization design techniques which have been previously presented in various publications, and (b) to further refine and extend existing disturbance minimization techniques.

As a part of the second objective, a number of examples will be presented to illustrate the design steps associated with different minimization techniques. Hopefully, this will permit applications-oriented readers to obtain a better understanding of the performance which disturbance minimizing controllers can provide.

CHAPTER II
DISTURBANCE MINIMIZING CONTROLLERS FOR LINEAR
DYNAMICAL SYSTEMS: GENERAL THEORY

2.1 Summary of Chapter II

This chapter introduces several techniques which can be used to accomplish disturbance minimization. These techniques include: (a) maximum partial absorption, (b) critical state variable, (c) indirect disturbance absorption and (d) norm minimization. A list of minimization criteria is presented and each of the minimization techniques is discussed in general detail.

The error evolution equations for state and output servotracking and stabilization/regulation type problems are developed. General expressions for the error and the steady-state error are given in each case, assuming regulatable set-points and trackable servocommands. The notion of "disturbable subspace" is presented and minimization of disturbance effects in the plant output is discussed.

2.2 Choice of Minimization Criterion/Technique

2.2.1 Introduction. The state equations for the class of plants being considered are given by Equations (1.1) as

$$\begin{aligned}\dot{x}(t) &= Ax(t) + Bu(t) + Fw(t) \\ y(t) &= Cx(t) + Eu(t) + Gw(t)\end{aligned}\tag{2.1}$$

and the class of assumed disturbances are modeled by Equations (1.13) as

$$\begin{aligned}w(t) &= Hz(t) + Lx(t) \\ \dot{z}(t) &= Dz(t) + Mx(t) + \sigma(t)\end{aligned}\tag{2.2}$$

where the various matrices in (2.1)-(2.2) are all assumed time-invariant.

It was shown in Section 1.7 that, with regard to state control problems, if certain conditions relating to the column range spaces of the control and disturbance distribution matrices B , F , H are satisfied, it is possible to design a control u_d to completely cancel the disturbance effects on the plant state $x(t)$. With respect to output control problems, the necessary and sufficient conditions to be met are somewhat more complicated [10,13,17].

If the conditions for disturbance absorption cannot be satisfied, then it is not possible to exactly cancel out the disturbances. Therefore, if the original design specifications called for suppression of all disturbance effects, but disturbance absorption is not possible, then the design philosophy must revert to some form of minimization mode. Within this category lie several routes by which designers may achieve their goals and many criteria which they may use to effect the minimization of disturbances. A brief outline of some of the approaches to disturbance minimization will be listed here and will be more fully described in the following sections.

Some of the disturbance minimization design techniques which have been proposed; see [9, 11; especially 17], are:

(1) Maximum partial absorption: In this method, u_d is designed so as to absorb as many individual disturbance elements w_i as possible.

(2) Critical state variable: If a particular state variable(s) is of overriding concern, u_d is chosen to counteract the total effect of disturbances on that state(s), if possible. This might not be easy since disturbance effects can act directly on the chosen variable(s)

and can also act indirectly through effects on other variables which are transmitted to the critical state(s).

(3) Indirect disturbance absorption: In those instances where the control vector cannot act directly on the chosen critical state, u_d can be chosen such that those states which are controllable are maneuvered in such a fashion as to counteract the disturbances and steer the critical state as required.

(4) Norm-minimization: With this method, u_d is chosen to minimize the norm of some combination of the control/disturbance vectors in order to reduce the disturbance effects on the plant dynamics.

The one underlying theme in all of the aforementioned techniques is, if total cancellation of disturbances cannot be achieved, a "minimization" process must occur. In other words, a control must be designed which will approximate total absorption in some specified sense. If total absorption is not possible, then the disturbance vector lies outside of the subspace defined by the column range space of the control matrix B . It is thus not possible to express the disturbance vector as some linear combination of the columns of B . However, in that case the disturbance vector can always be expressed as a direct sum of two component vectors, one lying in the column range space of B , the other lying "out" of that column range space. It thus behooves the designer to try to minimize the magnitude (norm) of the disturbance component which lies "outside of" the column range space of B . In order to achieve this, the concepts of orthogonal projectors and generalized inverses of matrices play a central role. A detailed review of these latter topics is presented in Appendix A.

2.2.2 Minimization Criteria. Given that the disturbances cannot be totally absorbed, but must be somehow "minimized", a decision must be made as to just what "minimization" means. Minimization will connote different things for different problems, but in most cases the candidate criteria fall into one of the following four categories:

(1) Counteract all those disturbance components which satisfy the condition for total cancellation and ignore the remaining components.

(2) As in (1), but counteract disturbance effects only for the set of critical states which are deemed to be of vital importance for the proper functioning of the plant.

(3) Assuming that the effect of the sum of all the disturbance components on the plant performance is to be made as small as possible, one can utilize the notion of a norm on a vector space. In this dissertation the metric defined by $\|x\|_Q^2 = x^T Q x$, $Q > 0$, will be used as a distance measure. If Q is chosen as the identity matrix I , then the metric will be the classical Euclidean norm, $\|x\| = (x^T x)^{1/2}$. Utilizing the concepts of generalized inverses (Appendix A), one can then seek to find a control u which will result in a disturbance residual $\|Bu + Fw\|_Q$ having minimum norm.

(4) If restrictions exist on the amount of control energy available to accomplish a given task, optimal control techniques may be utilized to design a controller which will accomplish disturbance minimization by minimizing some quadratic performance index involving the error and control vectors.

2.2.3 Maximum Partial Absorption Technique [17]. The purpose of the maximum partial absorption technique is to design a disturbance controller which will absorb all those individual disturbance elements $w_i(t)$ which satisfy the absorption criteria. The first step in applying this technique is to column partition the F matrix as

$$F = [f_1 \mid f_2 \mid \dots \mid f_p] \quad (2.3)$$

and check each column for satisfaction of the condition that f_i be contained in the column range space of B. If this condition is met for a particular f_i , it implies that there exists a ζ_i such that

$$f_i = B\zeta_i, \quad (2.4)$$

and thus, to cancel the one component w_i one can design the control to be

$$u_{dw}(t) = -\zeta_i w_i(t). \quad (2.5)$$

From the first equation of (2.2), it is seen that

$$w_i(t) = h_i z(t) + l_i x(t), \quad (2.6)$$

therefore,

$$u_{dw}(t) = -\zeta_i [h_i z(t) + l_i x(t)]. \quad (2.7)$$

In Equations (2.6) and (2.7), h_i and l_i represent the i -th rows of H and L, respectively and $w_i(t)$ represents the i -th component of $w(t)$.

After all columns of F have been checked in this manner, one re-orders the F matrix so that all those columns which satisfy the absorption criterion are grouped as

$$\{f_1, f_2, \dots, f_q\}, \quad (2.8)$$

q less than or equal to p , and those columns which do not are grouped as

$$\{f_{q+1}, f_{q+2}, \dots, f_p\}. \quad (2.9)$$

Then, $u_{dw}(t)$ can be expressed as

$$u_{dw}(t) = - \left(\sum_{j=1}^q \zeta_j h_j \right) z(t) - \left(\sum_{j=1}^q \zeta_j l_j \right) x(t). \quad (2.10)$$

The designer is then left with the set given by Equation (2.9), which represents the unabsorbable disturbance residual. The designer may wish to minimize the effects due to this residual. One proposed solution 20 is to design a control component, which we will denote by $u_{dw}(t)$, such that the norm given by

$$\| B u_{dw}(t) + \bar{F}[\bar{H}|\bar{L}] \|_I \quad (2.11)$$

is minimized, where $\bar{F}$ is equal to the set given by Equation (2.9), and $\bar{H}$, $\bar{L}$ comprise the corresponding rows of the re-ordered matrices H and L . Norm minimization will be discussed in a later section (see also Appendix A).

2.2.4 Critical State Variable Technique. This section will discuss disturbance minimization via the techniques of direct and indirect absorption of disturbance effects on critical state variables.

2.2.4.1 Direct Absorption. In general, every state variable in the plant state equations, (2.1), can potentially be affected by the disturbance $w(t)$ either directly by overt action of $w_i(t)$ on $\dot{x}_i(t)$ or indirectly through coupling with other states which are directly affected by disturbance terms. The technique to be described in this section gives a means of designing a control vector $u_{dw}(t)$ which will absorb the direct action of $w(t)$ on a given set of "critical" state variables.

To begin, one expands the first of Equations (2.1), using the first of Equations (2.2), as follows,

$$\begin{pmatrix} \dot{x}_1(t) \\ \dot{x}_2(t) \\ \vdots \\ \dot{x}_n(t) \end{pmatrix} = \begin{bmatrix} a_1 \\ a_2 \\ \vdots \\ a_n \end{bmatrix} x(t) + \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_n \end{bmatrix} u(t) + \begin{bmatrix} fh_1 \\ fh_2 \\ \vdots \\ fh_n \end{bmatrix} z(t) + \begin{bmatrix} fl_1 \\ fl_2 \\ \vdots \\ fl_n \end{bmatrix} x(t) \quad (2.12)$$

where a_i , b_i , fh_i , fl_i denote the i -th rows of A , B , FH and FL ,

respectively. The direct disturbance effect on $\dot{x}_i(t)$ is thus given by

$$\delta_i(t) = fh_i z(t) + fl_i x(t) . \quad (2.13)$$

To completely cancel $\delta_i(t)$, the control $u_{dw}(t)$ must satisfy

$$b_i u_{dw}(t) = -\delta_i(t) . \quad (2.14)$$

Provided that b_i is not a zero row of B, Equation (2.14) can always be satisfied for any one index i .

Now, consider the case when more than one state is considered to be critical. Suppose that $\{x_i(t), x_j(t), \dots, x_l(t)\}$, $1 \leq i < j < \dots < l \leq n$, are chosen as critical states and the matrix equations are re-ordered so that the critical states are grouped as $\{x_1(t), x_2(t), \dots, x_q(t)\}$, $q \leq n$. The direct disturbance effects on the critical states can be represented as

$$\delta_k(t) = fh_k z(t) + fl_k x(t), \quad k = 1, 2, \dots, q. \quad (2.15)$$

In order to cancel out this set of disturbance equations, it must be possible to choose $u_{dw}(t)$ to simultaneously satisfy the set of equalities

$$b_k u_{dw}(t) = -\delta_k(t), \quad k = 1, 2, \dots, q. \quad (2.16)$$

This will be possible if the set (2.15) satisfies the following condition for complete cancellation,

$$\text{Rank}[\bar{B} \mid \bar{\delta}(t)] = \text{Rank}[\bar{B}] , \quad (2.17)$$

where $\bar{B}$ represents the collection of control matrix elements associated with the critical state variables in the re-ordered equations and $\bar{\delta}$ corresponds to the direct disturbance components on those critical state variables, i.e., $\bar{B} = (b_1, \dots, b_q)^T$, $\bar{\delta} = (\delta_1(t), \dots, \delta_q(t))^T$.

2.2.4.2 Indirect Absorption. If the direct absorption condition, Equation (2.14), fails, i.e., if $b_i = 0$, then $\delta_i(t)$ cannot be directly absorbed by the control vector. However, by use of the

indirect absorption technique [17], it might be possible to come in through the back door, so to speak, and coerce some other plant state(s) $x_i(t)$ into performing the role of a surrogate disturbance absorber.

This method is entirely dependent on the problem set-up and constraints and its success depends on the ability to maneuver certain controllable, unrestricted, uncritical states in such a manner as to accomplish the desired end result.

2.2.5 Norm Minimization. It is in connection with the methods to be described in this section that generalized matrix inverses (Appendix A) make their invaluable appearance. As mentioned in Section 1.7, generalized inverses are useful in obtaining solutions, or best approximate solutions, for equations involving rectangular or square-singular matrices where the usual matrix inverse A^{-1} does not exist.

For those uncritical or critical state variables which cannot be totally absolved from disturbance contamination, or for those cases where no particular variable $x_i(t)$ is more important (but instead the group result is of primary concern) the objective of disturbance minimization is to design a control which will result in a minimal deviation of the plant from its primary goal. Some design proposals along this line have been as follows [9,11,17]:

- (1) One can choose $u_{dw}(t)$ to minimize

$$\|Bu_{dw}(t) + Fw(t)\|_I, \quad (2.18)$$

as was mentioned in Section 2.2.3. This would give a minimal magnitude of the residual disturbance vector acting on the plant state rate, $\dot{x}(t)$. The general solution for a $u_{dw}(t)$ which minimizes (2.18) is

$$u_{dw}(t) = -B^+Fw(t) + (I - B^+B)q. \quad (2.19)$$

where q is an arbitrary vector and $(\cdot)^\dagger$ denotes the Moore-Penrose generalized inverse of $(\cdot)$. Setting $q = 0$ in Equation (2.19) yields the particular minimizing control which itself has minimum norm, namely

$$u_{dw}^*(t) = -B^\dagger Fw(t). \quad (2.20)$$

(2) If the disturbance effects on both $x(t)$ and $y(t)$ are to be considered, where $y(t)$ is as described by the second equation of Equations (2.1), one can, for instance, choose $u_{dw}(t)$ to minimize

$$\left\| \begin{bmatrix} B \\ - \\ E \end{bmatrix} u_{dw}(t) + \begin{bmatrix} F \\ - \\ G \end{bmatrix} w(t) \right\|_I. \quad (2.21)$$

In this instance, that control which minimizes Equation (2.21) and which itself has minimum norm is given by

$$u_{dw}^*(t) = - \begin{bmatrix} B \\ - \\ E \end{bmatrix}^\dagger \begin{bmatrix} F \\ - \\ G \end{bmatrix} w(t). \quad (2.22)$$

(3) For minimization of disturbance effects in the system output one can, for instance, choose $u_{dw}(\tau)$ to minimize

$$\| C\Phi(t, \tau) [Bu_{dw}(\tau) + Fw(\tau)] \|_I. \quad (2.23)$$

(4) Finally, one can choose to minimize some quadratic form of the limiting terminal state, x_∞ as $t \rightarrow \infty$, i.e., $x_\infty^T W x_\infty$, where W is a positive or positive semidefinite matrix. This approach assumes that $w(t)$ and $x(t)$ approach constants as $t \rightarrow \infty$.

The above design approaches are all concerned with minimizing the effects of external disturbances. However, the same type of reasoning can be applied to the "set-point" disturbances and "servo-command" disturbances which appear in Equations (1.17) and (1.19). For example, in those cases one can choose $u_{ds}(t)$ to minimize

$$\| Bu_{ds}(t) + Ax_{sp} \|_Q \quad (2.24)$$

or choose $u_{dc}(t)$ to minimize

$$\|Bu_{dc}(t) + \dot{x}_c(t) - Ax_c(t)\|_Q. \quad (2.25)$$

Notice that for Equations (2.18), (2.21), (2.24) and (2.25), the norm minimization process actually is minimizing the disturbance component which acts on the plant state rate $\dot{x}(t)$.

On the other hand, suppose that the problem at hand is to steer $x(t)$ such that, in the steady-state, one has x_{ss} as close as possible to x_{sp} ; or to steer $x(t)$ such that $x(t) \rightarrow x_c(t)$ and thereafter remains as close as possible to $x_c(t)$. In these cases, the quantity of concern is actually the difference between where the plant state is and where one would like to have it go. That difference can be expressed as

$$\epsilon_{sp}(t) = x_{sp} - x(t) \quad (2.26)$$

or

$$\epsilon_c(t) = x_c(t) - x(t). \quad (2.27)$$

Therefore, the disturbance effect to be minimized should be the disturbance contribution to the steady-state value of ϵ_{sp} and the disturbance contribution to the instantaneous value of ϵ_c , respectively.

Thus, for a stabilization/set-point regulation problem, if one were to choose u_d to minimize $\|\epsilon_{sp_{ss}}\|_I$ the result would be a minimization of the "miss distance" from the required set-point which is achieved in the face of the disturbances acting upon the plant. Given a servo-command problem, minimizing $\|\epsilon_c(t)\|_I$ would minimize the instantaneous offset between the commanded and achieved states in the face of the disturbances.

The preceding development assumes that the specified set-points or servo-commands represent states which are achievable by the plant and represent errors which are the result of incomplete disturbance absorption. However, the possibility also arises of having set-points or

servo-command inputs which are themselves unreachable, exclusive of the disturbance effects. An example of the latter would be the case $y = Cx$ where the specified output set-point y_{sp} does not lie within the column range space of the C matrix. In such cases, the minimum Euclidean "miss distance" or offset will be attained by steering the state or output vectors to the orthogonal projection of the given set-point or servo-command vector on the range space of the control matrix, i.e., minimize the error norm between the given and achievable (see Appendix A and [3, 4, 5, 23, 25, 34]).

2.3 Error Evolution Equations

As suggested in the previous section, it perhaps would be effective to re-formulate the disturbance control problem in terms of the error vector and associated error dynamics so as to work directly with the quantity whose minimization is the ultimate goal of the design process. In this section, we will formulate general evolution equations for the error dynamics and present the general solution for the error vector for state and output set-point regulation/stabilization and servo-tracking problems.

2.3.1 State Set-Point Regulation/Stabilization. For the state set-point regulation/stabilization problem the error dynamics were given by Equation (1.17) as

$$\dot{\epsilon}(t) = A\epsilon(t) - Bu(t) - Ax_{sp} - Fw(t) . \quad (1.17)$$

Utilizing the control allocation approach discussed in Section 1.7, one obtains the result

$$\dot{\epsilon}(t) = (A+BK)\epsilon(t) - Ax_{sp} - Fw(t) - Bu_d(t) . \quad (1.34)$$

If the pair (A, B) is completely controllable [14], there exists a K

such that the solution of

$$\dot{\epsilon}(t) = (A + BK)\epsilon(t) \quad (2.28)$$

will achieve $\epsilon(t) \rightarrow 0$ promptly.

As mentioned previously in Section 1.7, one proposed method of controlling the disturbance terms in Equation (1.34) is to split $u_d(t)$ into $u_d(t) = u_{dw}(t) + u_{ds}(t)$. This results in the error dynamics being represented as

$$\dot{\epsilon}(t) = (A+BK)\epsilon(t) - (Ax_{sp} + Bu_{ds}(t)) - (Fw(t) + Bu_{dw}(t)) \quad (2.29)$$

where $u_{ds}(t)$ and $u_{dw}(t)$ can now be designed using one of the methods presented in Section 2.3. As an alternative method, one can leave $u_d(t)$ intact and design it to accommodate the combination of the disturbances in Equation (2.29). In this case, the resulting equation will be the same as Equation (1.34). These two methods will be considered in a later chapter, (see Sections 3.4, 3.5 of Chapter 3).

The general solution to Equation (2.29) is given by

$$\epsilon(t) = \Phi_{\epsilon}(t, t_0)\epsilon(t_0) - \int_{t_0}^t \Phi_{\epsilon}(t, \tau)(Ax_{sp} + Bu_{ds}(\tau) + Fw(\tau) + Bu_{dw}(\tau))d\tau \quad (2.30)$$

where $\Phi_{\epsilon}(t, t_0)$ is the state transition matrix for $(A+BK)$ and is the unique solution of Equation (2.28) with $\Phi_{\epsilon}(t_0, t_0) = I$. Since $(A+BK)$ is a time-invariant matrix, the state transition matrix $\Phi_{\epsilon}(t, t_0)$ can be expressed as

$$\Phi_{\epsilon}(t, t_0) = e^{(A+BK)(t-t_0)} \quad (2.31)$$

If $u_p(t)$ has been properly designed, then

$$\lim_{t \rightarrow \infty} \Phi_{\epsilon}(t, t_0)\epsilon(t_0) \rightarrow 0, \text{ for all } \epsilon(t_0), \quad (2.32)$$

in Equation (2.30). To obtain the contribution of the remaining terms on the right-hand side of Equation (2.30) to the steady-state

error, one can re-express them and take the limit as follows

$$\lim_{t \rightarrow \infty} \left\{ - \int_{t_0}^t \phi_{\epsilon}(t, \tau) (Ax_{sp} + Bu_{ds}(\tau)) d\tau - \int_{t_0}^t \phi_{\epsilon}(t, \tau) (Fw(\tau) + Bu_{dw}(\tau)) d\tau \right\} . \quad (2.33)$$

If x_{sp} is a regulatable set-point, then the first integral will be bounded, i.e., there will exist an $M = M(t)$ such that $\int_{t_0}^t g(\tau) d\tau \leq M$ for each $t > t_0$, where $g(\tau)$ represents the integrand of the first integral in Equation (2.33). The second integral, however, might be unbounded for some disturbances [8, 9] since $Fw(t)$ cannot necessarily be cancelled by suitable choice of $u_{dw}(t)$.

If x_{sp} is a regulatable set-point, and the external disturbances are such that $\lim_{t \rightarrow \infty} w(t) = w_{\infty} = \text{constant}$ exists, the second integral in Equation (2.33) is then bounded and one may find ϵ_{ss} , the steady-state of $\epsilon(t)$, as follows. Since a steady-state for $\epsilon(t)$ implies that $\dot{\epsilon}(t) = 0$, one may set the right-hand side of Equation (2.29) to zero and solve the resulting expression for ϵ_{ss} . This will result in the following expression for ϵ_{ss} :

$$\epsilon_{ss} = (A + BK)^{-1} (Ax_{sp} + Fw_{\infty} + Bu_{dw} + Bu_{ds}) . \quad (2.34)$$

Since $u_p = -K\epsilon$ is chosen such that $(A + BK)$ has all eigenvalues in the left-half plane, then $(A + BK)$ is a nonsingular matrix and hence is invertible.

One can now choose some function of ϵ_{ss} to minimize and Equation (2.34) can be used to solve for the necessary control vector. For example, if norm minimization is the method chosen, u_{ds} and u_{dw} can be chosen to minimize

$$\begin{aligned} & ||(A+BK)^{-1}(Ax_{sp} + Bu_{ds})||_Q \\ \text{and} & ||(A+BK)^{-1}(Fw_{\infty} + Bu_{dw})||_Q, \end{aligned} \quad (2.35)$$

respectively. Generalized inverse techniques can be utilized in each instance to solve for the necessary u_{ds} and u_{dw} .

This method of designing the disturbance control components will also enable the designer to see which, if any, disturbance components can be completely absorbed if the critical state variable method or the maximum partial absorption approach is of interest.

2.3.2 State Servo-Tracking. For the state servo-tracking problem, the error dynamics were given in Equation (1.19) as:

$$\dot{\epsilon}(t) = A\epsilon(t) - Bu(t) - Ax_c(t) + \dot{x}_c(t) - Fw(t) \quad (1.19)$$

where $\epsilon(t) = x_c(t) - x(t)$. Splitting $u(t)$ as before, and again letting $u_p(t) = -K\epsilon(t)$, one obtains

$$\dot{\epsilon}(t) = (A+BK)\epsilon(t) - (Ax_c(t) - \dot{x}_c(t) + Bu_{dc}(t)) - (Fw(t) + Bu_{dw}(t)). \quad (2.36)$$

In DAC theory, the state servo-command is assumed to be modeled by a linear dynamic process [9,10] similar to the disturbance model, namely

$$\begin{aligned} x_c(t) &= Rc(t) \\ \dot{c}(t) &= Sc(t) + \mu(t) \end{aligned} \quad (2.37)$$

where $c(t)$ is the servo-command "state" vector and $\mu(t)$ plays the same role as does $\sigma(t)$ in the disturbance model. An accurate estimate $\hat{c}(t)$ of $c(t)$ is assumed available from a state reconstructor operating on $x_c(t)$.

Differentiating Equation (2.37) one obtains

$$\dot{x}_c(t) = R\dot{c}(t) = RSc(t) + R\mu(t) \quad (2.38)$$

and substitution of Equation (2.38) into Equation (2.36) yields

$$\begin{aligned} \dot{\epsilon}(t) = & (A+BK)\epsilon(t) - ((AR-RS)c(t) - R\mu(t) + Bu_{dc}(t)) \\ & - (Fw(t) + Bu_{dw}(t)) . \end{aligned} \quad (2.39)$$

The term $\mu(t)$ denotes a sequence of completely unknown, randomly occurring delta functions in the servo-command model. Each occurrence of a $\mu_i(t)$ will impart a jump discontinuity to $\epsilon(t)$, i.e., will impose a sequence of unknown random-like "initial conditions" on ϵ during the control interval. Therefore, the task of designing $u_d(t)$ for Equation (2.39) is equivalent to the task of designing it for Equation (2.39) with the term $R\mu(t)$ removed, (see [10]).

The general solution to Equation (2.39) will thus be:

$$\begin{aligned} \epsilon(t) = & \phi_{\epsilon}(t, t_0)\epsilon(t_0) - \int_{t_0}^t \phi_{\epsilon}(t, \tau)(Fw(\tau) + Bu_{dw}(\tau))d\tau \\ & - \int_{t_0}^t \phi_{\epsilon}(t, \tau)((AR - RS)c(\tau) + Bu_{ds}(\tau))d\tau . \end{aligned} \quad (2.40)$$

In the problem at hand, $u_p(t)$ should be so designed that the transients due to initial plant errors and to errors engendered by discontinuities in the servo-command process settle out "rapidly" with respect to plant time constants and to the arrival times of the $\mu(t)$ impulses.

With respect to the integral terms on the right-hand side of Equation (2.40), the design process is similar to that discussed at the end of Section 2.3.1. In this case, however, since the servo-command can change, the upper limit of integration should be "infinite" only to the extent that it is far enough removed from the arrival time of an impulse in the servo-command, say t_0 , for the error transients mentioned above to settle out to a "steady-state" before the next impulse arrival, i.e., the disturbance control should settle out the

disturbance effects within the same time period as that taken for the plant initial conditions to settle out. This will also be a function of the reconstructor settling times for the state, disturbance and servo-command processes.

If one assumes that $x_c(t)$ is a trackable state servo-command input and applies the same line of reasoning as in Section 2.3.1, with the same cautions regarding unabsorbable disturbances in the limit, with respect to Equation (2.39) the following expression for ϵ_{ss} can be found,

$$\epsilon_{ss} = (A+BK)^{-1} \{ ((AR-RS)c_\infty + Bu_{dc}) + (Fw_\infty + Bu_{dw}) \} . \quad (2.41)$$

One can now choose to use norm minimization, critical state variable or maximum partial absorption techniques to design u_d to minimize some function of ϵ_{ss} .

2.3.3 Output Set-Point Regulation/Stabilization. The plant model considered for output set-point regulation/stabilization is given as

$$\begin{aligned} \dot{x}(t) &= Ax(t) + Bu(t) + Fw(t) \\ y(t) &= Cx(t) \end{aligned} \quad (2.42)$$

and the control objective is to stabilize $y(t)$ to the origin or to regulate $y(t)$ to a given non-zero set-point y_{sp} .

If C were an invertible matrix, an evolution equation for $\dot{y}(t)$ could be obtained as

$$\dot{y}(t) = C\dot{x}(t) = CAC^{-1}y(t) + CBu(t) + CFw(t) \quad (2.43)$$

where $x(t) = C^{-1}y(t)$. Then, the error dynamics would be given by

$$\dot{\epsilon}(t) = -\dot{y}(t) = CAC^{-1}\epsilon(t) - CAC^{-1}y_{sp} - CBu(t) - CFw(t) . \quad (2.44)$$

In view of Equation (2.44), one can design $u_p(t)$ as $u_p(t) = -K\epsilon(t)$

and $u_d(t)$ can be designed to minimize the set-point and external disturbance effects. If C is indeed an invertible matrix, this is the procedure which is recommended.

However, in most practical applications C is a rectangular matrix, hence its inverse does not exist. Thus, the error evolution equation could not be written as in Equation (2.44) but could only be obtained in the form

$$\dot{\epsilon}(t) = -CAx(t) - CBu(t) - CFw(t) . \quad (2.45)$$

The procedure outlined in Section 1.5.2.1 is one method which has been proposed for dealing with this problem when C has maximal rank.

It is recalled that the linear transformation process outlined in Section 1.5.2.1 results in an expression for $\dot{y}(t)$ in the new coordinate system and the associated error dynamics were given by Equation (1.28) as

$$\dot{\epsilon}(t) = \tilde{A}_{11}\epsilon(t) - \tilde{B}_1u(t) - \tilde{A}_{11}y_{sp} - \tilde{A}_{12}\xi_2(t) - \tilde{F}_1w(t) , \quad (1.28)$$

where, in the stabilization problem, $\tilde{A}_{11}y_{sp} = 0$ since $y_{sp} = 0$ in that case. Splitting the control $u(t)$ into parts $u_p(t)$ and $u_d(t)$, the general solution for $\epsilon(t)$ is given by

$$\begin{aligned} \epsilon(t) = & \phi_{\epsilon}(t, t_0)\epsilon(t_0) - \int_{t_0}^t \phi_{\epsilon}(t, \tau)(\tilde{A}_{11}y_{sp} + \tilde{A}_{12}\xi_2(\tau) + \tilde{B}_1u_{ds}(\tau))d\tau \\ & - \int_{t_0}^t \phi_{\epsilon}(t, \tau)(\tilde{F}_1w(\tau) + \tilde{B}_1u_{dw}(\tau))d\tau . \end{aligned} \quad (2.46)$$

The control part $u_p(t)$ should be designed as in Section 2.3.1 such that

$$\lim_{t \rightarrow \infty} \phi_{\epsilon}(t, t_0)\epsilon(t_0) \rightarrow 0. \quad (2.47)$$

If one assumes that y_{sp} is a regulatable set-point and applies the same line of reasoning as in Section 2.3.1, with the same cautions regarding unabsorbable disturbances, with respect to Equation (2.44), then the following expression for ϵ_{ss} is obtained when C is an invertible matrix,

$$\epsilon_{ss} = (AC^{-1} + BK)^{-1}(AC^{-1}y_{sp} - Fw_{\infty} - Bu_d) \quad (2.48)$$

When C is not invertible, an expression for ϵ_{ss} is found from Equation (1.28) as

$$\epsilon_{ss} = (\tilde{A}_{11} + \tilde{B}_1 K)^{-1}(\tilde{B}_1 u_d + \tilde{A}_{11} y_{sp} + \tilde{A}_{12} \xi_{2\infty} + \tilde{F}_1 w_{\infty}) \quad (2.49)$$

where the vector $\xi_2(t)$ is a coupling disturbance which obeys the differential equation given by Equation (1.25). The control u_d can be designed, using Equation (2.48) or (2.49), to minimize some function of ϵ_{ss} .

Another approach which can be used if C is a rectangular matrix of rank less than m or a square singular matrix, is to make use of generalized inverses in the following manner. Assuming that the plant output is represented by $y(t) = Cx(t)$, the control objective for the output set-point problem is to find an x_{sp} corresponding to a given y_{sp} such that $y_{sp} = Cx_{sp}$ and then solve the problem as a state set-point problem. If C is of rank m , then the column range space of C will encompass the entire output space and an appropriate x_{sp} can be found for any given y_{sp} in the output space. However, if rank C is less than m , the column range space of C will be a subspace of the output space and not the entire space. Hence, the possibility exists that a given y_{sp} may lie outside the column range space of C and thus not be regulatable, i.e., there will be no x_{sp} such that $Cx_{sp} = y_{sp}$. In this case, the generalized inverse could be used to find an x_{sp} , or a family of x_{sp} 's, which would minimize the norm of the error between the given y_{sp} and the

attained Cx_{sp} , i.e., $\|\delta\|_Q = \|y_{sp} - Cx_{sp}\|_Q$. Ideally, one would like to have this error equal to zero, but assuming this is not possible, a set of solutions which will give the best approximation in a least squares sense, for $Q = I$, is

$$x_{sp} = C^+ y_{sp} + (I - C^+ C)q \quad (2.50)$$

where q is an arbitrary vector and the second term on the right-hand side of Equation (2.50) lies in the nullspace of C . The particular solution which itself is of minimum norm is given by Equation (2.50) with $q=0$, i.e.,

$$x_{sp}^* = C^+ y_{sp} \quad (2.51)$$

Thus, if $\text{rank } C < m$ and the external disturbances were zero, steering $x(t)$ to the result given by Equation (2.51) would produce an error of minimum norm, given that y_{sp} is unregulatable. The presence of external disturbance terms would provide a variation about x_{sp} and the total error would be due to the external disturbances and δ . By using the result of Equation (2.51) with the method presented in Section 2.3.1, a solution could then be found for this problem.

2.3.4 Output Servo-Tracking. In this section, as in Section 2.3.3, it is assumed that C is not necessarily invertible. In addition, it is assumed that the servo-command process is modeled as

$$\begin{aligned} y_c(t) &= Rc(t) \\ \dot{c}(t) &= Sc(t) + \mu(t) \end{aligned} \quad (2.52)$$

In light of Equations (2.52), the definition given in Section 1.5.2 for a "trackable" servo-command can be made more precise. In order for theoretically exact servo-tracking to be possible, i.e., for $y_c(t)$ to be a "trackable" servo-command, the following necessary "exact trackability" condition must be satisfied [10],

$$R(R) \subseteq R(C) . \quad (2.53)$$

In other words, it must be possible to express R as

$$R \equiv C\Theta \quad (2.54)$$

for some, possibly nonunique, matrix Θ . Satisfaction of condition (2.53) or (2.54) implies that, for some $x(t)$, it will be possible to have

$$y_c(t) = Rc(t) = Cx(t) = y(t),$$

$$\text{i.e.,} \quad C\Theta c(t) - Cx(t) = C(\Theta c(t) - x(t)) = 0 . \quad (2.55)$$

If one employs the coordinate transformation approach so that the plant output $y(t)$ is made a "sub-state" vector in the new coordinate system, the output servo-tracking error dynamics are given by Equation (1.31) as

$$\dot{\epsilon}(t) = \tilde{A}_{11}\epsilon(t) - \tilde{B}_1 u(t) + \dot{y}_c(t) - \tilde{A}_{11}y_c(t) - \tilde{A}_{12}\xi_2(t) - \tilde{F}_1 w(t) . \quad (1.31)$$

Therefore, substituting Equation (2.52) into (1.31), splitting the control vector and disregarding $\mu(t)$ (see Section 2.3.2) results in

$$\begin{aligned} \dot{\epsilon}(t) = & (\tilde{A}_{11} + \tilde{B}_1 \tilde{K})\epsilon(t) - \{\tilde{B}_1 u_{dc}(t) + \tilde{A}_{12}\xi_2(t) - (RS - \tilde{A}_{11}R)c(t)\} \\ & - (\tilde{B}_1 u_{dw}(t) + \tilde{F}_1 w(t)) . \end{aligned} \quad (2.56)$$

The general solution to Equation (2.56) can be expressed as

$$\begin{aligned} \epsilon(t) = & \Phi_\epsilon(t, t_0)\epsilon(t_0) - \int_{t_0}^t \Phi_\epsilon(t, \tau) \{\tilde{B}_1 u_{dc}(\tau) + \tilde{A}_{12}\xi_2(\tau) \\ & - (RS - \tilde{A}_{11}R)c(\tau)\} d\tau - \int_{t_0}^t \Phi_\epsilon(t, \tau) (\tilde{B}_1 u_{dw}(\tau) + \tilde{F}_1 w(\tau)) d\tau . \end{aligned} \quad (2.57)$$

As in previous sections, it is assumed that $u_p(t) = -\tilde{K}\epsilon(t)$ has been designed such that $\Phi_\epsilon(t, t_0)\epsilon(t_0) \rightarrow 0$ rapidly in relation to system time constants.

Suppose that a given $y_c(t)$ does not satisfy the exact trackability condition (2.54). In that case, the input servo-command does not lie entirely within the column range space of C and it will not be possible to steer $y(t)$ to $y_c(t)$ even if there are no external disturbances acting on the plant. One can decompose $y_c(t)$ into component vectors in the output space and if this is carried out by letting one component be the orthogonal projection of $y_c(t)$ onto the column range space of C then the remaining component, lying in the orthogonal complement of the column range space of C , will be of minimum norm compared to that which would be obtained from any other decomposition of $y_c(t)$. If the system is completely state controllable, then it will be possible to find some $x_c(t)$, or set of states $x_c(t)$, such that $y(t) = Cx_c(t)$ will equal the component of $y_c(t)$ lying in the column range space of C . Given that some $x_c(t)$ exists, if a permissible control $u(t)$ can be found which will steer $x(t)$ to $x_c(t)$, then the norm of the output servo-tracking error $\|y_c(t) - Cx_c(t)\|_Q$ will be minimized.

To be more succinct, if one wishes to minimize

$$\|\delta(t)\| = \|y_c(t) - Cx_c(t)\|_Q, \quad (2.58)$$

then one must solve for an appropriate $x_c(t)$. The set of states $x_c(t)$ which minimize the norm of $\delta(t)$, for instance for $Q = I$, is given by

$$x_c(t) = C^+ y_c(t) + (I - C^+ C)q \quad (2.59)$$

where q is an arbitrary vector and the second term on the right-hand side of Equation (2.59) lies in the nullspace of C . The solution of minimum norm is given by Equation (2.59) with $q=0$, i.e.,

$$x_c^*(t) = C^+ y_c(t). \quad (2.60)$$

One can then solve the original output servo-command problem as a state servo-command problem by using $x_c^*(t)$ as the input state servo-command

vector with the method of solution presented in Section 2.3.2.

2.4 Minimization of Disturbance Effects in the Plant Output

Prior to considering a minimization technique for this type problem, a brief digression is in order for the purpose of defining a certain subspace.

In the design of a conventional disturbance absorbing controller [17] to completely cancel the disturbance effects in the plant output corresponding to Equations (2.1), the general solution for $y(t)$ is given as

$$\begin{aligned} y(t; x_0, t_0, w) = & \bar{C}\bar{\Phi}(t, t_0)x_0 + \bar{C} \int_{t_0}^t \bar{\Phi}(t, \tau)Bu_p(\tau)d\tau + E\tilde{u}_p(t) + Eu_d(t) \\ & + Gw(t) + \bar{C} \int_{t_0}^t \bar{\Phi}(t, \tau)(Bu_d(\tau) + Fw(\tau))d\tau \end{aligned} \quad (2.61)$$

where $\bar{C} = C + EK$ and $\bar{\Phi}(t, t_0)$ represents the state transition matrix for $\bar{A} = A + BK$. In order to completely cancel the disturbance effects in the output, $u_d(t)$ must be chosen such that the following two conditions are simultaneously satisfied:

$$E\Lambda + G[H|L] \equiv 0 \quad (2.62)$$

$$\bar{C} \bar{\Phi}(t, \tau)(B\Lambda + F[H|L]) \equiv 0 \quad (2.63)$$

where $u_d(\tau) = \Lambda_1 z(\tau) + \Lambda_2 x(\tau)$ and $\Lambda = [\Lambda_1 | \Lambda_2]$. Since $\bar{\Phi}(t, \tau)$ is a non-singular matrix its nullity is zero; therefore, in order to satisfy Equation (2.63), Λ must be chosen such that $\bar{\Phi}(t, \tau)(B\Lambda + F[H|L])$ is in the nullspace of $\bar{C}$.

Equation (2.63) can be re-expressed in the form

$$\bar{C}[\alpha_0(t)I + \alpha_1(t)\bar{A} + \alpha_2(t)\bar{A}^2 + \dots + \alpha_{U-1}(t)\bar{A}^{U-1}] [B\Lambda + F[H|L]] \equiv 0, \quad (2.64)$$

or, without loss of generality,

$$\bar{C} [\alpha_0(t)I + \alpha_1(t)\bar{A} + \dots + \alpha_{n-1}(t)\bar{A}^{n-1}] \hat{B} \equiv 0 \quad (2.65)$$

where $\hat{B} = B\Lambda + F[H|L]$ and the $\alpha_i(t)$ are linearly independent on every positive interval of time. Equation (2.65) is satisfied if, and only if,

$$\bar{C} [\hat{B}|\bar{A}\hat{B}|\bar{A}^2\hat{B}|\dots|\bar{A}^{n-1}\hat{B}] \equiv 0. \quad (2.66)$$

The column range space of $[\hat{B}|\bar{A}\hat{B}|\dots|\bar{A}^{n-1}\hat{B}]$ has been called [17] the "disturbable subspace" of the system characterized by $(\bar{A}, \hat{B})$ and one of the criteria for the controller design is that Λ be such that the disturbable subspace becomes totally unobservable.

If $u_d(t)$ cannot be designed such that

$$\begin{bmatrix} B \\ - \\ E \end{bmatrix} u_d(t) + \begin{bmatrix} F \\ - \\ G \end{bmatrix} w(t) \equiv 0, \quad (2.67)$$

or to satisfy Equations (2.62) and (2.63) simultaneously, then the disturbance effects cannot be completely eliminated from the plant output. The control $u_d(t)$ should then be designed to minimize the disturbance effects in the output, for instance by minimizing the norm

$$\|Eu_d(t) + Gw(t) + \bar{C} \int_{t_0}^t \bar{\Phi}(t, \tau)(Bu_d(\tau) + Fw(\tau))d\tau\|_Q \quad (2.68)$$

which is the contribution of the last two terms on the right-hand side of Equation (2.61). Thus $u_d(t)$ must minimize a norm containing the direct external disturbance residual vector in the output space and the external disturbance residual vector transformed from the state space. For Equation (2.68), u_d must be found in the form $u_d(\tau)$.

Since the matrices are all time-invariant, if the external disturbance is constant expression (2.68) can be written as

$$\|Eu_d + Gw_0 + (\bar{C} \int_0^t e^{\bar{A}(t-\tau)} d\tau)(Bu_d + Fw_0)\|_Q. \quad (2.69)$$

In the steady-state, expression (2.69) will be

$$\begin{aligned} ||Eu_d + Gw_0 - \bar{C} \bar{A}^{-1}(Bu_d + Fw_0)||_Q = \\ ||(E - \bar{C} \bar{A}^{-1}B)u_d + (G - \bar{C} \bar{A}^{-1}F)w_0||_Q \end{aligned} \quad (2.70)$$

and for $Q = I$, u_d can be designed as

$$u_d^* = - (E - \bar{C} \bar{A}^{-1}B)^\dagger (G - \bar{C} \bar{A}^{-1}F)w_0 \quad (2.71)$$

in order to minimize the norm of the external disturbance effect, where u_d^* in Equation (2.71) is itself of minimum norm. The final disturbance contribution in the output will then be given by

$$\begin{aligned} & - (E - \bar{C} \bar{A}^{-1}B)(E - \bar{C} \bar{A}^{-1}B)^\dagger (G - \bar{C} \bar{A}^{-1}F)w_0 + (G - \bar{C} \bar{A}^{-1}F)w_0 \\ & = (I - (E - \bar{C} \bar{A}^{-1}B)(E - \bar{C} \bar{A}^{-1}B)^\dagger)(G - \bar{C} \bar{A}^{-1}F)w_0 . \end{aligned} \quad (2.72)$$

In the event that the plant output is modeled as

$$y(t) = Cx(t), \quad (2.73)$$

the only disturbance effect in the output will be due to the last term on the right-hand side of Equation (2.61). The control objective is then to minimize the distance between the nullspace of C and the external disturbance residual vector contained in the disturbable subspace of $(\bar{A}, \hat{B})$.

For a discussion of techniques for stabilizing linear dynamical systems to arbitrary subspaces one can refer to Section 4.4 and [16].

PART II

DISTURBANCE MINIMIZATION DESIGN TECHNIQUES

CHAPTER III
DIRECT DISTURBANCE MINIMIZATION FOR LINEAR, TIME-
INVARIANT DYNAMICAL SYSTEMS; STATE SET-
POINT REGULATION/STABILIZATION

3.1 Summary of Chapter III

This chapter covers specific controller design techniques which can be used for the direct minimization of disturbance effects in linear time-invariant state set-point regulation/stabilization problems. The design of control laws which can minimize weighted norms of the error vector is discussed. Minimization of disturbance effects on the steady-state error using both allocated and unallocated disturbance control vectors is examined and results compared. Minimization of disturbance effects on the error dynamics using an allocated disturbance control vector is also examined. A synopsis of results from an example problem is presented to illustrate differences in results between minimizing disturbance effects on ϵ_{ss} and on $\dot{\epsilon}(t)$. The notion of disturbance utility in disturbance minimization problems is presented and illustrated. Methods for design of controllers to minimize disturbance effects on critical state variable subsets are presented.

3.2 Linear Time-Invariant Dynamical Models

The plants and disturbances which are considered in the remainder of this dissertation are modeled as linear time-invariant dynamical

systems. This means that none of the matrix elements in the respective state models are functions of time. The pertinent state equations will thus be:

(1) Plant:

$$\dot{x}(t) = Ax(t) + Bu(t) + Fw(t) \quad (3.1)$$

$$y(t) = Cx(t) + Eu(t) + Gw(t)$$

(2) Disturbance Process:

$$w(t) = Hz(t) + Lx(t) \quad (3.2)$$

$$\dot{z}(t) = Dz(t) + Mx(t) + \sigma(t) .$$

The state transition matrix for a system equation of the form $\dot{v} = Pv$ will be expressed in exponential form as

$$\Phi(t, t_0) = e^{P(t-t_0)} \quad (3.3)$$

or, if t_0 is assumed to be zero,

$$\Phi(t) = e^{Pt} . \quad (3.4)$$

Note that P may correspond to either the plant matrix A or a combination of the A matrix and state feedback control and/or disturbance model terms.

3.3 Direct Disturbance Minimization

Direct disturbance minimization is, as the name implies, minimization by the control vector acting directly on the $\dot{x}_i$ expression for the state(s) involved, rather than by influence on an intermediate, directly controllable state x_j . This type of control action includes the maximum partial absorption, critical state variable (direct absorption) and norm minimization methods. As previously explained in Chapter 2, Section 2.2.1, the maximum partial absorption method involves cancellation of the disturbance effects on as many individual state

variables as possible. The critical state variable method involves cancellation or minimization of the disturbance effects on some selected subset of state variables. Norm minimization methods are concerned with minimizing some specified norm of the disturbance effects on an ensemble of states.

Disturbance absorption techniques have been covered in great detail in other sources, for instance [8,9,10,13,17,24], and those are the techniques which one would apply to the determination of the proper control vector to absorb the above mentioned disturbance components if they were amenable to complete absorption.

This chapter will be primarily devoted to methods for minimizing, via direct control action, the effects of those disturbance components which are not completely absorbable. As stated in Section 2.2.2, the metric used for the minimization process will be the norm defined by

$$||Ax - b||_Q^2 = (Ax - b)^T Q (Ax - b), \quad Q > 0. \quad (3.5)$$

Generalized inverses of matrices and several techniques for computing generalized inverses are presented in Appendix A; however, several theorems dealing with a minimizing solution for an expression in the form of Equation (3.5) will be presented here.

Theorem [3]: Let A be an $m \times n$ matrix, b an m vector, and Q a positive definite $m \times m$ matrix. Then $||Ax - b||_Q$ is smallest when $x = Xb$, where X satisfies

$$AXA = A, \quad (QAX)^T = QAX. \quad (3.6)$$

Theorem [3]: Let A be an $m \times n$ matrix, b an m vector and P a positive definite $n \times n$ matrix. If $Ax=b$ has a solution for x , the unique solution for which $||x||_P$ is smallest is given by $x = Xb$, where X satisfies

$$AXA = A, \quad (PXA)^T = PXA \quad \blacksquare \quad (3.7)$$

Corollary [3]: Let A be an $m \times n$ matrix, b an m vector, Q a positive definite $m \times m$ matrix and P a positive definite $n \times n$ matrix. Then, there is a unique matrix X satisfying

$$(QAX)^T = QAX, \quad (PXA)^T = PXA \quad (3.8)$$

Moreover, $\|Ax-b\|_Q$ assumes its minimum value for $x=Xb$, and in the set of vectors x for which the minimum value is assumed, $x=Xb$ is the one for which $\|x\|_P$ is smallest. $\blacksquare$

In the above three theorems, X represents a weighted generalized inverse which provides a minimizing solution, or best approximate solution, for the weighted norm in expression (3.5). By proper choice of the components of the weighting matrix Q the designer can reflect the relative importance attached to the various components of the norm. One should note that, if the weighting matrices Q and P are chosen to be the identity matrix I , i.e., the components of the norm have equal weighting, then expression (3.5) reduces to the classical Euclidean norm given by $\|Ax-b\|^2 = (Ax-b)^T(Ax-b)$ and the properties which X satisfies in the theorems reduce to properties satisfied by the Moore-Penrose generalized inverse.

3.4 Minimization of $\|e_{ss}\|$ by Use of an Unallocated Disturbance Control Component

In this section it is assumed that one is dealing with a state set-point regulator problem with the plant described by Equations (3.1).

The control error vector is defined as

$$e(t) = x_{sp} - x(t) \quad (3.9)$$

and the error dynamics are computed to be

$$\dot{\epsilon}(t) = -\dot{x}(t) = A\epsilon(t) - Ax_{sp} - Fw(t) - Bu(t) \quad (3.10)$$

The control vector $u(t)$ will be allocated into two parts as $u(t) = u_p(t) + u_d(t)$, with $u_p(t) = -K\epsilon(t)$ and $u_d(t)$ not further allocated. The part u_d will be designed to minimize the combined effects of the external and set-point disturbance terms.

If one substitutes $u(t) = -K\epsilon(t) + u_d(t)$ into Equation (3.10) and expands $w(t)$ using Equation (3.2), the error dynamics can be re-expressed as

$$\dot{\epsilon}(t) = (A+FL+BK)\epsilon(t) - ((A+FL)x_{sp} + FHZ(t)) - Bu_d(t). \quad (3.11)$$

The general solution to Equation (3.11) is given by

$$\epsilon(t) = e^{\tilde{A}t}\epsilon(0) - \int_0^t e^{\tilde{A}(t-\tau)}((A+FL)x_{sp} + FHZ(\tau) + Bu(\tau))d\tau \quad (3.12)$$

where $\tilde{A} = A+FL+BK$. We are interested, however, in the steady-state solution for $\epsilon(t)$, i.e., ϵ_{ss} . If one assumes that a unique steady-state solution exists for $\epsilon(t)$, i.e., all λ_i of $\tilde{A}$ have negative real parts and all disturbance terms have a limit as time approaches infinity, then ϵ_{ss} can be found by setting $\dot{\epsilon}(t)$ in Equation (3.11) to zero and solving for the ϵ which satisfies the resulting equation. If this is done, the expression for the steady-state error is found to be

$$\epsilon_{ss} = \tilde{A}^{-1}((A+FL)x_{sp} + FHZ_{\infty} + Bu_d) \quad (3.13)$$

The term $(A+FL+BK)$ is invertible since K was chosen in such a way as to ensure that all eigenvalues of $(A+FL+BK)$ are in the left-half plane.

The objective of $u_d(t)$ is to minimize $\|\epsilon_{ss}\|_Q^2$, i.e., minimize a distance measure between the set-point specified and that actually achieved. Ideally, one would like to have $\epsilon_{ss} = 0$, or

$$\tilde{A}^{-1}Bu_d = -\tilde{A}^{-1}((A+FL)x_{sp} + FHZ_{\infty}), \quad (3.14)$$

i.e., x_{sp} is completely regulatable and the external disturbances are completely absorbable. Since it is assumed that this is not possible, one must instead solve for a $u_d(t)$ which will minimize $||\epsilon_{ss}||_Q$. This can be accomplished by using generalized inverse techniques. If for example, the designer wished to minimize $||\epsilon_{ss}||_I$, then the $u_d(t)$ which will give a steady-state error vector of minimum norm, for ϵ_{ss} as defined by Equation (3.13), and which will itself be of minimum norm is given by

$$u_d^*(t) = -(\tilde{A}^{-1}B)^\dagger \tilde{A}^{-1}((A+FL)x_{sp} + FH z_\infty) \quad (3.15)$$

where $(\cdot)^\dagger$ denotes the Moore-Penrose generalized inverse of $(\cdot)$. Note that in the context of the norm being used in this dissertation, a "minimum norm" solution indicates the solution of least magnitude over all possible solutions for the given problem. If one next substitutes Equation (3.15) into Equation (3.13), the resulting expression yields the minimal norm set-point error ϵ_{ss}^* as

$$\epsilon_{ss}^* = (I - (\tilde{A}^{-1}B)(\tilde{A}^{-1}B)^\dagger) \tilde{A}^{-1}((A+FL)x_{sp} + FH z_\infty) \quad (3.16)$$

In the more general case where Q is not chosen equal to I , the control vector which is of minimum $||u_d||_p$, and which minimizes $||\epsilon_{ss}||_Q$, can be found from expressions (3.8). In the terminology of Equation (3.13), one would first solve for a matrix X satisfying

$$(Q\tilde{A}^{-1}BX)^T = Q\tilde{A}^{-1}BX, \quad (PX\tilde{A}^{-1}B)^T = PX\tilde{A}^{-1}B \quad (3.17)$$

The control $u_d(t)$ is then given by

$$u_d^*(t) = X\tilde{A}^{-1}((A+FL)x_{sp} + FH z_\infty) \quad (3.18)$$

Substitution of Equation (3.18) into Equation (3.13) then yields the weighted minimal norm set-point error ϵ_{ss}^* as

$$\epsilon_{ss}^* = (I - \tilde{A}^{-1}BX)\tilde{A}^{-1}((A+FL)x_{sp} + FH z_\infty) \quad (3.19)$$

3.5 Minimization of $\|\epsilon_{ss}\|$ by Use of an Allocated Disturbance Control Vector

As an alternative approach to the problem stated in Section 3.4, one can set $u_d(t) = u_{ds}(t) + u_{dw}(t)$ such that Equation (3.11) has the form

$$\dot{\epsilon}(t) = \tilde{A}\epsilon(t) - ((A + FL)x_{sp} + Bu_{ds}(t)) - (FHz(t) + Bu_{dw}(t)). \quad (3.20)$$

where $\tilde{A} = (A + FL + BK)$. Then, one can minimize the effects of the last two bracketed terms on the right-hand side of Equation (3.20) independently, i.e., find $u_{ds}(t)$ and $u_{dw}(t)$ which will give minimal norm for the steady-state residuals from each disturbance term, considered separately.

The general solution to Equation (3.20) is given by

$$\epsilon(t) = e^{\tilde{A}t}\epsilon(0) - \int_0^t e^{\tilde{A}(t-\tau)} ((A+FL)x_{sp} + Bu_{ds}(\tau) + FHz(\tau) + Bu_{dw}(\tau)) d\tau. \quad (3.21)$$

However, we are interested in finding the steady-state solution for $\epsilon(t)$, i.e., ϵ_{ss} . If one assumes, as in Section 3.4, that disturbance terms acting on the system have a limit as $t \rightarrow \infty$ and that a unique steady-state solution exists (recall that $u_p(t) = -K\epsilon(t)$ is chosen to stabilize the plant) then ϵ_{ss} is found by setting $\dot{\epsilon}(t) = 0$ in Equation (3.20) and solving for the ϵ which satisfies the resulting equation. The steady-state error is thus found to be

$$\epsilon_{ss} = \tilde{A}^{-1}((A + FL)x_{sp} + Bu_{ds} + FHz_{\infty} + Bu_{dw}). \quad (3.22)$$

We will now proceed to examine each of the disturbance terms, and its associated control vector, shown on the right-hand side of Equation (3.22).

We shall first assume that the set-point disturbance term is zero, in which case Equation (3.22) becomes

$$\epsilon_{ss1} = \tilde{A}^{-1}(\tilde{F}H z_{\infty} + B u_{dw}) . \quad (3.23)$$

Ideally, one would like to have $\epsilon_{ss1} = 0$, i.e., $H z_{\infty}$ is completely absorbable, but this is assumed to be not possible. Therefore, one must instead solve for a $u_{dw}(t)$ which will minimize $\|\epsilon_{ss1}\|_Q$. If Q is chosen to be I , then the minimum norm $u_{dw}(t)$ which will give a steady-state error vector ϵ_{ss1} of minimum Euclidean norm, for ϵ_{ss1} defined by Equation (3.23), will be

$$u_{dw}^* = -(\tilde{A}^{-1}B)^{\dagger} \tilde{A}^{-1} \tilde{F}H z_{\infty} \quad (3.24)$$

and the resulting steady-state error vector will be

$$\epsilon_{ss1}^* = (I - (\tilde{A}^{-1}B)(\tilde{A}^{-1}B)^{\dagger}) \tilde{A}^{-1} \tilde{F}H z_{\infty} . \quad (3.25)$$

If we next assume that the external disturbance term $z(t)$ is zero, then Equation (3.22) can be written as

$$\epsilon_{ss2} = \tilde{A}^{-1}((A+FL)x_{sp} + B u_{ds}) . \quad (3.26)$$

Again, one would like to have $\epsilon_{ss2} = 0$, and again it is assumed that this is not possible. Therefore, one must instead solve for a u_{ds} which will minimize $\|\epsilon_{ss2}\|_Q$. If Q is again chosen as I , then the u_{ds} of minimum norm which will give a steady-state error vector ϵ_{ss2} of minimum Euclidean norm, for ϵ_{ss2} defined by Equation (3.26), is

$$u_{ds}^* = -(\tilde{A}^{-1}B)^{\dagger} \tilde{A}^{-1}(A+FL)x_{sp} , \quad (3.27)$$

and the resulting steady-state error vector will be

$$\epsilon_{ss2}^* = (I - (\tilde{A}^{-1}B)(\tilde{A}^{-1}B)^{\dagger}) \tilde{A}^{-1}(A+FL)x_{sp} . \quad (3.28)$$

If one now compares Equations (3.24) and (3.27) with Equation (3.15), one sees that $u_d^* = u_{dw}^* + u_{ds}^*$. Also, if one compares Equations (3.25) and (3.28) with Equation (3.16), one sees that $\epsilon_{ss}^* = \epsilon_{ss1}^* + \epsilon_{ss2}^*$. Therefore, if the norm being minimized is the Euclidean norm

$||\epsilon_{ss}||_I^2 = \epsilon_{ss}^T \epsilon_{ss}$, the method of Section 3.4 and the method of this section yield the same result.

For the more general case, where Q is not chosen equal to I , the control vector, which is of minimum $||u_{dw}(t)||_{P_1}$ and which minimizes $||\epsilon_{ss1}||_{Q_1}$, can be found from expressions (3.8). For ϵ_{ss1} as defined by Equation (3.23), one would first solve for a matrix X_1 satisfying

$$(Q_1 \tilde{A}^{-1} B X_1)^T = Q_1 \tilde{A}^{-1} B X_1, \quad (P_1 X_1 \tilde{A}^{-1} B)^T = P_1 X_1 \tilde{A}^{-1} B. \quad (3.29)$$

The control $u_{dw}^*(t)$ is then given by

$$u_{dw}^* = X_1 \tilde{A}^{-1} F H z_\infty \quad (3.30)$$

and the resulting steady-state error vector ϵ_{ss1} by

$$\epsilon_{ss1}^* = (I - \tilde{A}^{-1} B X_1) \tilde{A}^{-1} F H z_\infty. \quad (3.31)$$

The control vector, which is of minimum $||u_{ds}(t)||_{P_2}$ and which minimizes $||\epsilon_{ss2}||_{Q_2}$, can also be found from expression (3.8) by solving for a matrix X_2 satisfying

$$(Q_2 \tilde{A}^{-1} B X_2)^T = Q_2 \tilde{A}^{-1} B X_2, \quad (P_2 X_2 \tilde{A}^{-1} B)^T = P_2 X_2 \tilde{A}^{-1} B. \quad (3.32)$$

The control $u_{ds}^*(t)$ is then found as

$$u_{ds}^* = X_2 \tilde{A}^{-1} (A + FL) x_{sp} \quad (3.33)$$

and the resulting steady-state error vector ϵ_{ss2} is

$$\epsilon_{ss2}^* = (I - \tilde{A}^{-1} B X_2) \tilde{A}^{-1} (A + FL) x_{sp}. \quad (3.34)$$

If $P_1 = P_2 = P$ and $Q_1 = Q_2 = Q$, then $X_1 = X_2 = X$ and the method from Section 3.4 which resulted in Equations (3.18) and (3.19) will produce the same results as are obtained by summing Equations (3.30), (3.33) and Equations (3.31), (3.34), respectively. The method presented in this section allows the designer the capability of varying the emphasis to be placed on the minimization of the various disturbance terms by choosing different weighting matrices P_1 , P_2 , Q_1 and Q_2 .

3.6 Minimization of Disturbance Effects on Error Dynamics

A third approach to disturbance minimization, which has been suggested in [19,20], is to require minimization of the norms of each disturbance term prior to integration of the state equations. The expression for the error dynamics, for the developments in this section, will be the same as that given in Section 3.5 by Equation (3.20), i.e.,

$$\dot{\epsilon}(t) = \tilde{A}\epsilon(t) - ((A+FL)x_{sp} + Bu_{ds}(t)) - (FH_z(t) + Bu_{dw}(t)) \quad (3.20)$$

and the procedure to be followed is to minimize the pair

$$\| (A+FL)x_{sp} + Bu_{ds}(t) \|_Q, \quad \| FH_z(t) + Bu_{dw}(t) \|_Q \quad (3.35)$$

prior to solving for the steady-state error vector.

First, let us examine the case when $Q = I$ and $P = I$. In this case, each norm in expression (3.35) will be the Euclidean norm and the Moore-Penrose generalized inverse can be used to solve for $u_{ds}^*(t)$ and $u_{dw}^*(t)$ which will minimize those norms as follows. The minimum norm $u_{ds}(t) = u_{ds}^*(t)$ which will minimize the first of the norms in expressions (3.35) is given by

$$u_{ds}^*(t) = -B^+(A+FL)x_{sp}, \quad (3.36)$$

and the minimum norm $u_{dw}(t) = u_{dw}^*(t)$ which will minimize the second of the norms in expressions (3.35) is given by

$$u_{dw}^*(t) = -B^+FH_z(t). \quad (3.37)$$

If one now substitutes Equations (3.36) and (3.37) into Equation (3.20), the resulting expression for the error dynamics will be

$$\dot{\epsilon}(t) = \tilde{A}\epsilon(t) - (I - BB^+)((A+FL)x_{sp} + FH_z(t)). \quad (3.38)$$

If the assumption is made that disturbance states acting on the system have a steady-state (limiting) value z_∞ , the unique steady-state solution for Equation (3.38) is found to be

$$\epsilon_{ss}^* = \tilde{A}^{-1}(I - BB^+)((A+FL)x_{sp} + FH_z_\infty). \quad (3.39)$$

If one compares Equation (3.39) with the results obtained in Sections 3.4 and 3.5, given by Equation (3.16) for the case where $Q = I$, it is seen that the results are not identical; however, it is not immediately evident which of the methods would give the "best" result, i.e., the smallest error norm. This must be determined in each problem by substituting the appropriate quantities into Equation (3.16) and Equation (3.39) and calculating the steady-state error magnitude.

For a case wherein the designer chooses to use a weighting matrix other than the identity matrix, a solution can be obtained in the following manner. Given the norms in expressions (3.35), the designer can utilize different weighting matrices for the external and set-point disturbance terms in order to vary the relative importance attached to reducing the various disturbance effects. The problem can thus be formulated as: (1) Given the set-point disturbance term norm

$$\| (A+FL)x_{sp} + Bu_{ds} \|_{Q_1}, \quad (3.40)$$

find the control vector u_{ds} which is of minimum $\|u_{ds}\|_{P_1}$ and which will provide a minimum value for the norm of Equation (3.40); (2) Given the external disturbance term norm

$$\| FH_z(t) + Bu_{dw}(t) \|_{Q_2}, \quad (3.41)$$

find the control vector $u_{dw}(t)$ which is of minimum $\|u_{dw}(t)\|_{P_2}$ and which will provide a minimum value for the norm of Equation (3.41).

One can now utilize expressions (3.8) to solve for the required control vectors. From expression (3.8), a $u_{ds}^*(t)$ of minimum $\|u_{ds}(t)\|$, weighted by P_1 , which will minimize the norm of expression (3.40) is found by first solving for a matrix X_1 satisfying

$$(Q_1 B X_1)^T = Q_1 B X_1, \quad (P_1 X_1 B)^T = P_1 X_1 B. \quad (3.42)$$

The required control $u_{ds}^*(t)$ is then given by

$$u_{ds}^*(t) = X_1 (A + FL) x_{sp}. \quad (3.43)$$

A $u_{dw}^*(t)$ of minimum norm, weighted by P_2 , which will minimize the norm of expression (3.41) is found by first solving for a matrix X_2 satisfying

$$(Q_2 B X_2)^T = Q_2 B X_2, \quad (P_2 X_2 B)^T = P_2 X_2 B. \quad (3.44)$$

The required control $u_{dw}^*(t)$ is then given by

$$u_{dw}^*(t) = X_2 F H z(t). \quad (3.45)$$

If one now substitutes Equations (3.43) and (3.45) into Equation (3.20), the error dynamics can be expressed as

$$\dot{\epsilon}(t) = \tilde{A} \epsilon(t) - (I + B X_1)(A + FL) x_{sp} - (I + B X_2) F H z(t). \quad (3.46)$$

An expression for the steady-state error vector can then be found from Equation (3.46) to be

$$\epsilon_{ss}^* = \tilde{A}^{-1} (I + B X_1)(A + FL) x_{sp} + \tilde{A}^{-1} (I + B X_2) F H z_{\infty}. \quad (3.47)$$

If the designer wishes to weight the norms of the disturbance terms without placing any weighting on the control vectors, then a procedure similar to that shown in Equations (3.42) through (3.47) can be followed using expressions (3.6) to find the appropriate control vectors.

3.7 Performance of Norm-Minimization Controllers for a Second-Order

Example (Appendix B)

The disturbance minimization controllers developed in Sections 3.4, 3.5 were identical (for identical weighting matrices) and were different from that of Section 3.6. To demonstrate the performance of these norm-minimization controllers and compare results between the two

different controllers a second-order example was studied in detail. The calculations are fully developed in Appendix B but some selected results are presented in this section.

The example involves a second-order, linear time-invariant state set-point regulator problem. The plant is given by

$$\begin{pmatrix} \dot{x}_1 \\ \dot{x}_2 \end{pmatrix} = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} + \begin{pmatrix} 1 \\ 2 \end{pmatrix} u + \begin{pmatrix} 1 \\ 1 \end{pmatrix} w \quad (\text{B.2})$$

$$y = (1, 0)x \quad (\text{B.3})$$

and the disturbance is piecewise constant and modeled by

$$w = z \quad (\text{B.5})$$

$$\dot{z} = \sigma \quad (\text{B.6})$$

A specified set-point vector x_{sp} is given and the task is to steer the plant state x to x_{sp} in the face of uncertain disturbances w . However, consistent with the developments in the previous sections, the problem is restated in terms of the error vector, $\epsilon(t) = x_{sp} - x(t)$, so that the goal is to minimize the norm of ϵ_{ss} , i.e., $\|\epsilon_{ss}\|_I$.

A solution will first be developed by use of the controller from Section 3.5. Examining the set-point disturbance component given by Equation (3.26), $\epsilon_{ss2} = \tilde{A}^{-1}(Ax_{sp} + Bu_{ds})$, the steady-state error contribution is determined to be

$$\begin{pmatrix} \epsilon_{ss21} \\ \epsilon_{ss22} \end{pmatrix} = \begin{pmatrix} 0.0854 \\ 1.829 \end{pmatrix} x_{sp1} + \begin{pmatrix} -0.305 \\ 0.609 \end{pmatrix} u_{ds} \quad (3.48)$$

This equation, incidentally, provides an example of the critical variable direct absorption technique also. At this point, if it had been decided that $x_1(t)$ was "critical", then u_{ds} could have been chosen

to give $\epsilon_{ss_{21}} = 0$ at the expense of having $\epsilon_{ss_{22}} = 1.829x_{sp1}$ (see Figure 3-1). Also, if $x_2(t)$ were deemed critical, u_{ds} could be chosen to give $\epsilon_{ss_{22}} = 0$ at the expense of having $\epsilon_{ss_{21}} = 0.0854x_{sp1}$. Neither of these two extremes for u_{ds} gives a minimum for $\|\epsilon_{ss}\|_I$, however, as is indicated by Figure 3-2 which graphs ϵ_{ss} as a function of u_{ds} between the values obtained for each of the critical state cases mentioned above. Using generalized inverse techniques on Equation (3.48), that value of $u_{ds} = u_{ds}^*$ which results in a minimum norm for ϵ_{ss} and is itself minimal in norm is found from Equation (3.27), for $x_{sp} = (10., 0.)^T$, to be

$$u_{ds}^* = -(\tilde{A}^{-1}B)^{\dagger}\tilde{A}^{-1}Ax_{sp} = -23.44. \quad (B.17)$$

Using this value for u_{ds}^* in Equation (3.29) results in $\|\epsilon_{ss_2}^*\|_I = 8.944$. This should be the length of the component of the set-point disturbance vector lying in the orthogonal complement of the range space of B. The geometry of the situation is graphed in Figure 3-3, and the length of $\epsilon_{ss_{min}}$ can be calculated to be 8.944.

Assuming next that the external disturbance $w(t)$ is not necessarily zero, a procedure identical to that for the set-point disturbance can be followed to find $u_{dw} = u_{dw}^*$ which will minimize $\|\epsilon_{ss_1}\|_I$ as given by Equation (3.23). The contribution of the external disturbance to the total error vector can be shown to be

$$\begin{pmatrix} \epsilon_{ss_{11}} \\ \epsilon_{ss_{12}} \end{pmatrix} = \begin{pmatrix} -0.1097 \\ 1.219 \end{pmatrix} z + \begin{pmatrix} -0.3048 \\ 0.6094 \end{pmatrix} u_{dw}. \quad (3.49)$$

As with the set-point disturbance, one or the other of the plant states can be made "error-free" in the steady-state, with respect to the external disturbance. Or, generalized inverses can be used to find the u_{dw}^* which will give minimum norm for ϵ_{ss_1} . The effects of the external

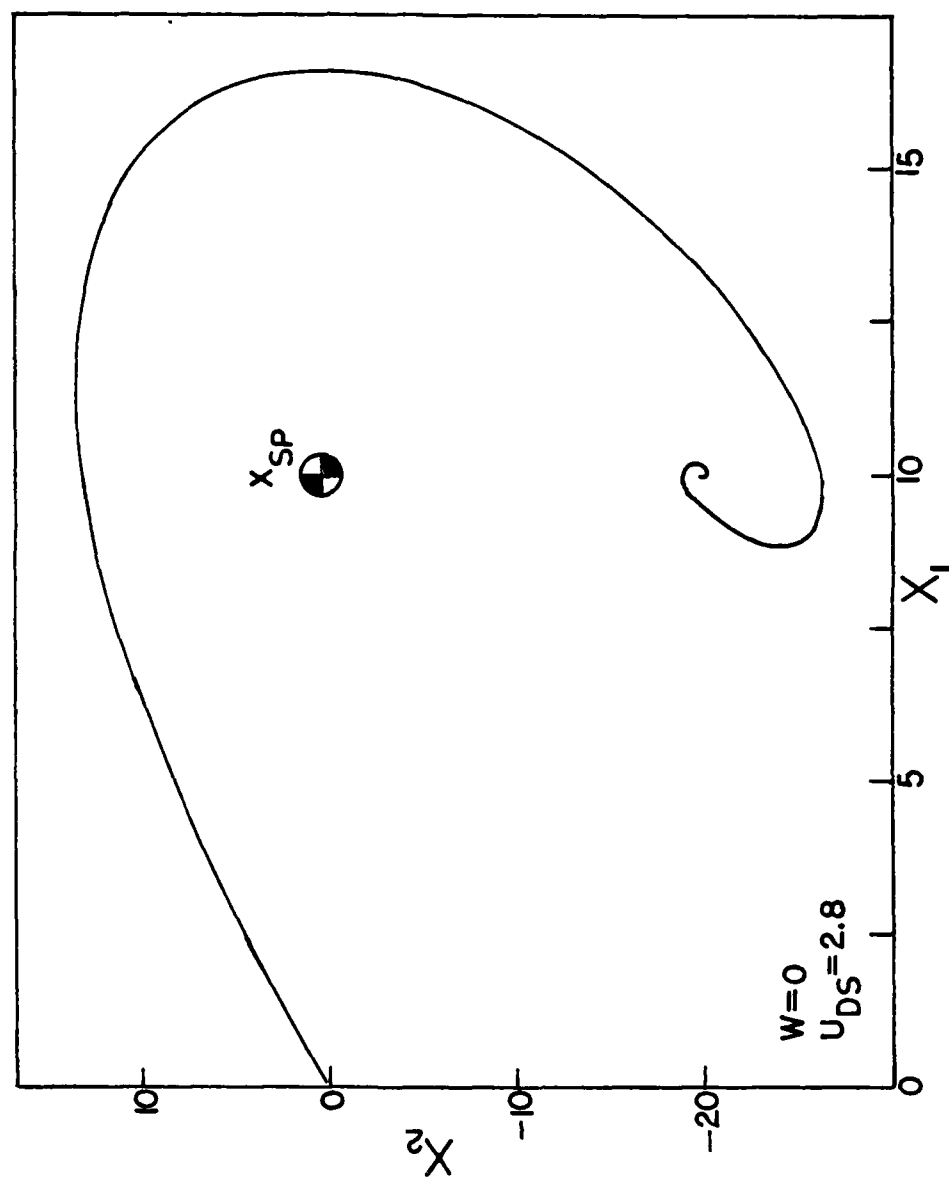


Figure 3-1. Plant state trajectory when x_1 is critical state-variable.

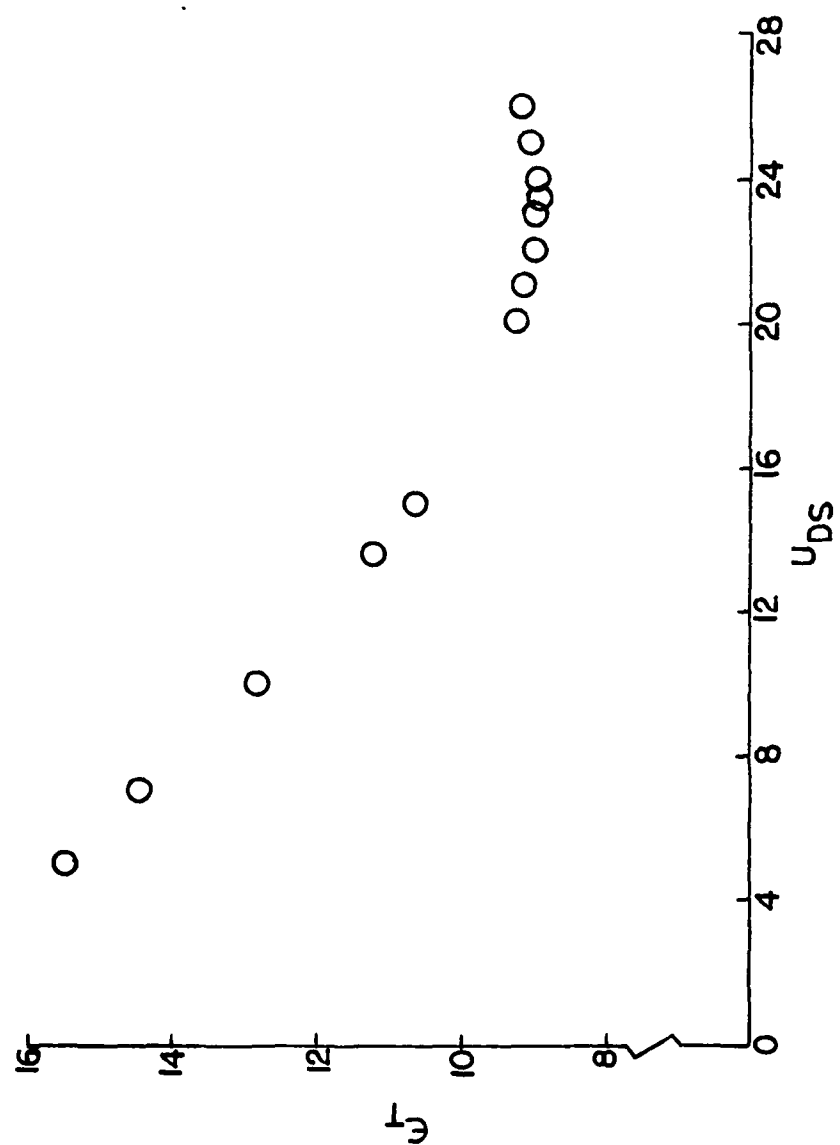


Figure 3-2. Steady-state error as a function of u_{DS} .

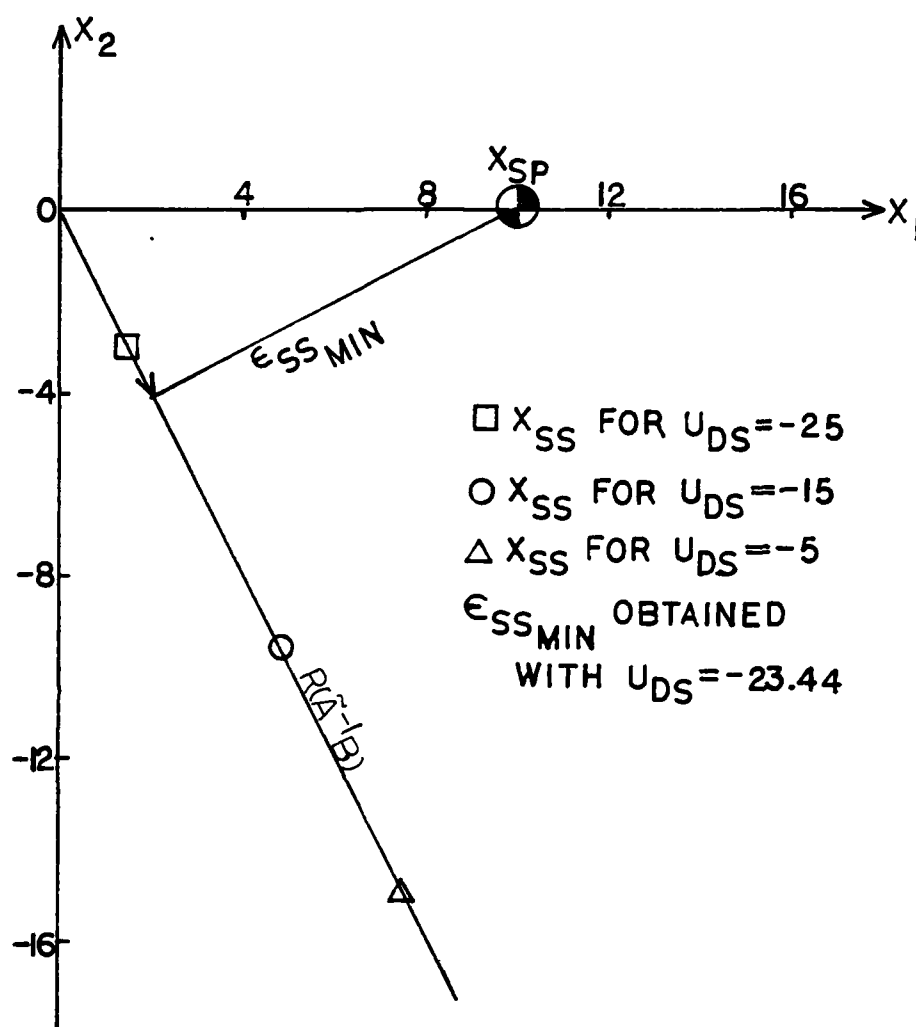


Figure 3-3. Phase-plane geometry for minimum norm error, second-order example (Appendix B).

disturbance on $||\epsilon_{ss}||_I$ for various combinations of $u_{ds}(t)$, $u_{dw}(t)$ and $w(t)$ are tabulated in Table 3-1 and are shown graphically via phase plane plots in Figures 3-4, 3-5 and 3-6.

Table 3-1. Effects of $w(t)$ on $||\epsilon_{ss}||$

u_{ds}	u_{dw}	z	$ \epsilon_{ss} $
2.8	0.	0.	20.
2.8	0.	5.	26.1
2.8	-0.3601z	5.	25.
-30.0	0.	0.	10.
-30.0	0.	5.	12.5
-30.0	-2.0z	5.	11.25
-23.449	0.	0.	8.885
-23.449	0.	5.	12.55
-23.449	-1.672z	5.	11.18

If the control design technique of Section 3.6 is applied to the problem, $u_{ds}(t)$ is first calculated to minimize $||Ax_{sp} + Bu_{ds}||_I$ and is then used in the integration of the state equations. The resulting steady-state norm, with $w = 0$, is found from Equation (3.39) and is about 76 percent larger than was obtained via the previous method. Likewise, choosing $u_{dw}(t)$ to minimize $||FHx_{\infty} + Bu_{dw}||_I$ results in a steady-state error norm about 78 percent larger.

3.8 The Notion of Disturbance Utility as Applied to Minimization of State Set-Point Regulation/Stabilization Problems with Disturbances

The steady-state error given by Equation (3.16), (3.31), (3.34), (3.39) or (3.47) is the residual part of the sum of the set-point and external disturbance vectors which lies in the orthogonal complement of the range space of B and hence is unabsorbable by the control vector.

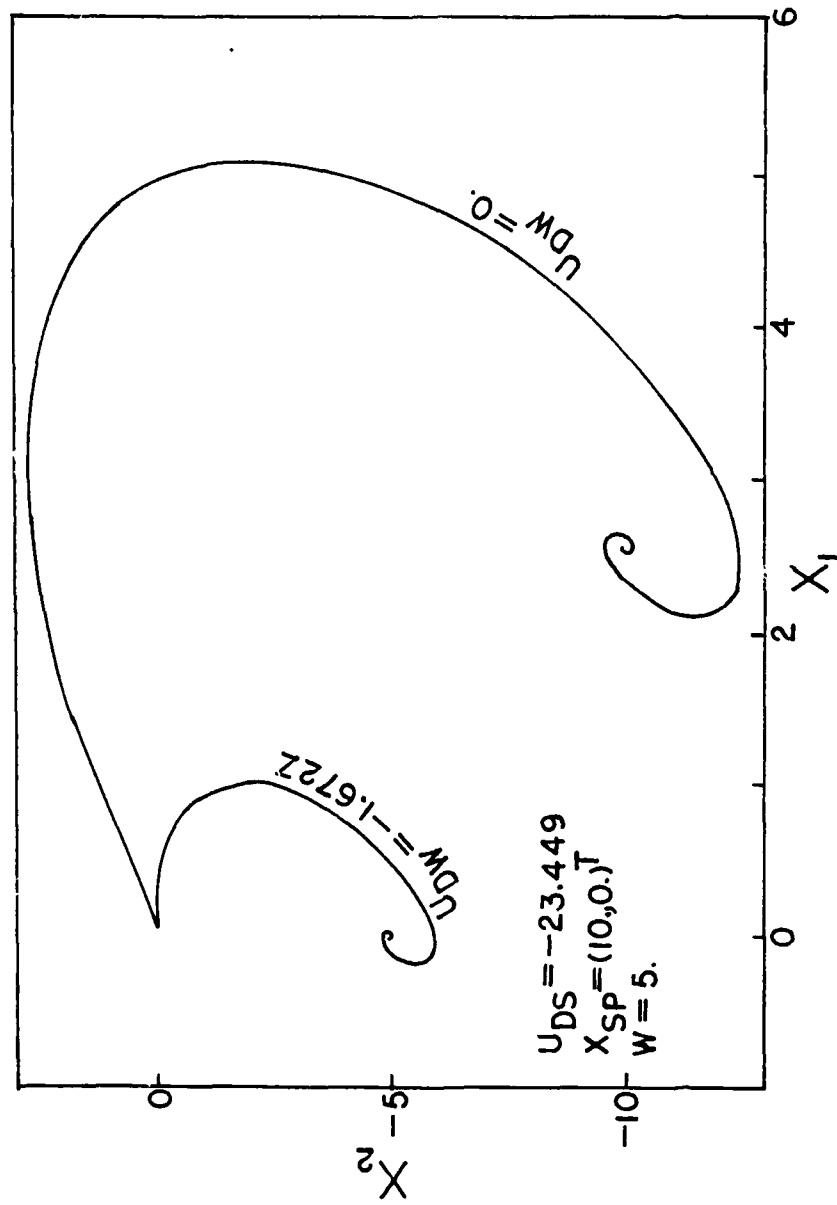


Figure 3-4. Plant state trajectories as a function of u_{ds} .

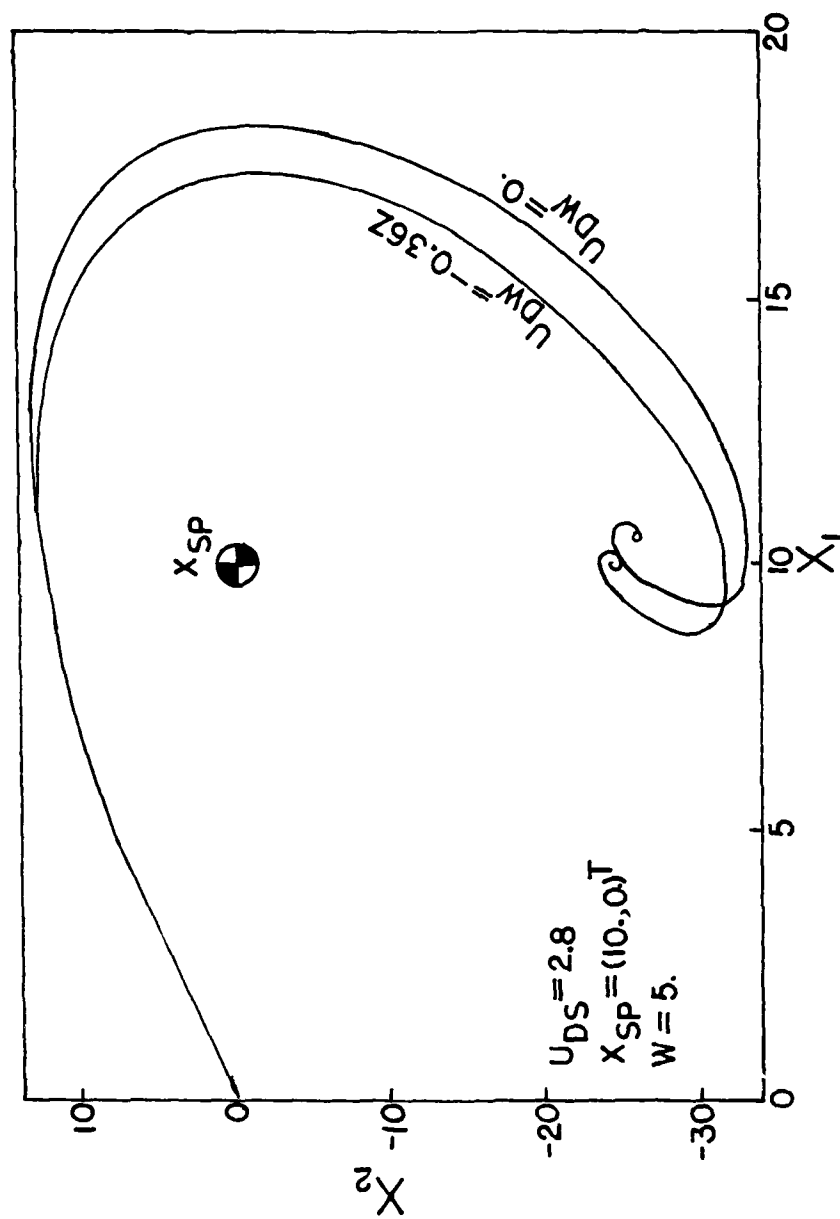


Figure 3-5. Plant state trajectories as a function of u_{ds} .

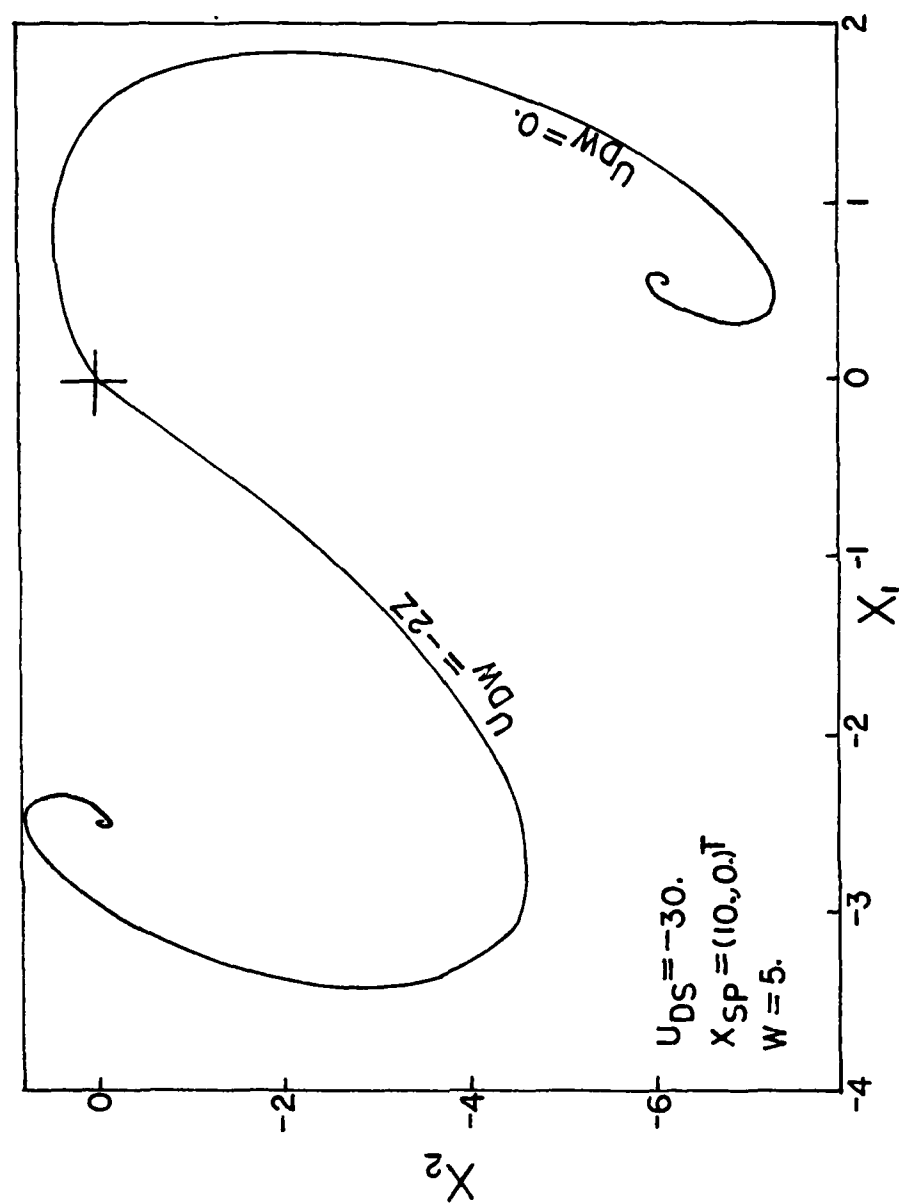


Figure 3-6. Plant state trajectories as a function of u_{ds} .

However, from examination of Equation (3.16) it is seen that ϵ_{ss} may be reduced, independently of the control vector, by a fortuitous combination of the disturbance vectors. For this purpose, it is useful to distinguish between two cases: (1) the set-point x_{sp} is the origin (stabilization) and (2) x_{sp} is any other set-point.

In case (1), there will be no set-point disturbance vector. Hence, if no external disturbance is present, or if external disturbance terms are present but satisfaction of the complete absorption criterion is realized, the task reduces to one of placing the eigenvalues of $(A+BK)$ as desired to obtain asymptotic stability of $x(t)$ to zero. If external disturbances are present, but complete absorption is not possible, and $u_d(t)$ is chosen to give a best approximate solution to $Bu_d = -Fw(t)$, then the resulting error ϵ_{ss} cannot be further reduced in the sense of our defining norm (Section 3.3).

In case (2), if $w(t) = 0$ then the resulting ϵ_{ss} due to that component of Ax_{sp} which is unabsorbable by $u_d(t)$ is not further reducible in the sense of our norm (Section 3.3). However, if $w(t)$ is not zero, then some interesting possibilities occur depending on whether

$$||Ax_{sp} + FH_z(t)|| > ||Ax_{sp}|| \quad (3.50)$$

or
$$||Ax_{sp} + FH_z(t)|| < ||Ax_{sp}|| \quad (3.51)$$

If the given set-point is the origin, then Equation (3.50) will always be obtained. For a non-zero set-point, however, result (3.51) might be obtained, i.e., the external disturbance might act in such a way as to reduce, instead of increase, the final set-point error.

It is convenient to define the "utility" U of the disturbance $w(t)$ as

$$U = ||\epsilon_{ss}||_{w=0} - ||\epsilon_{ss}||_{w \neq 0} \quad (3.52)$$

In other words, if $U > 0$ it means the disturbance can actually aid in further reducing $||\epsilon_{ss}||$. On the other hand, if $U < 0$ the presence of $w(t)$ aggravates (increases) the value of $||\epsilon_{ss}||$.

In order to provide a positive utility U the external disturbance must reduce $||\epsilon_{ss}||$, i.e., it must reduce the magnitudes of at least some of the components of $||\epsilon_{ss}||_{w=0}$. Let

$$a^i = \sum_{j=1}^n a_{ij} x_{spj} \quad (3.53)$$

where a_{ij} is the ij -th element of the matrix A . Also, let

$$h^j = \sum_{k=1}^p h_{jk} z_k \quad (3.54)$$

where h_{jk} is the jk -th element of the matrix H , and

$$f^i = \sum_{j=1}^p f_{ij} h^j \quad (3.55)$$

where f_{ij} is the ij -th element of the matrix F . Then,

$$(Ax_{sp} + FH z(t))_i = (a^i + f^i), \quad i=1, 2, \dots, n. \quad (3.56)$$

In general, some of the $(a^i + f^i)$ will be greater than a^i , some will be less than a^i and some will be equal to a^i . If one rearranges the components of $(a^i + f^i)$ so that for $i=1, \dots, U$, $|a^i + f^i| < |a^i|$; for $i=U+1, \dots, K$, $|a^i + f^i| > |a^i|$; and for $i=K+1, \dots, n$, $|a^i + f^i| = |a^i|$, then

$$(Ax_{sp} + FH z) = \begin{bmatrix} a^1 + f^1 \\ a^U + f^U \\ \frac{a^{U+1} + f^{U+1}}{a^{K+1} + f^{K+1}} \\ a^K + f^K \\ \frac{a^{K+1} + f^{K+1}}{a^n + f^n} \end{bmatrix} = \begin{bmatrix} \gamma^1 \\ \gamma^U \\ \frac{\gamma^{U+1}}{\gamma^{K+1}} \\ \gamma^K \\ \gamma^n \end{bmatrix} \quad (3.57)$$

The set-point disturbance term, and also the external disturbance term, can be represented as the sum of two vectors, one of which lies in $R(B)$ and one of which lies in $R(B)^\perp$. Here, the concern is for the vectors in $R(B)^\perp$ since the control vector cannot absorb them. Therefore, Equation (3.57) will be expressed as

$$\gamma = \gamma_{R(B)} + \gamma_{R(B)^\perp} \quad (3.58)$$

and we will examine

$$\|\gamma_{R(B)^\perp}\|_I^2 = \sum_{i=1}^U (\gamma^i)^2_{R(B)^\perp} + \sum_{i=U+1}^K (\gamma^i)^2_{R(B)^\perp} + \sum_{i=K+1}^n (\gamma^i)^2_{R(B)^\perp}. \quad (3.59)$$

Since the last term on the right-hand side of Equation (3.59) contains those elements of $Ax_{sp} + FHz$ for which $a^i + f^i = a^i$, i.e., $f^i = 0$, they will not affect the difference between $\|\epsilon\|_{w=0}$ and $\|\epsilon\|_{w \neq 0}$; therefore, we will examine the terms

$$\sum_{i=1}^U (\gamma^i)^2_{R(B)^\perp} + \sum_{i=U+1}^K (\gamma^i)^2_{R(B)^\perp} = r^2. \quad (3.60)$$

In order to calculate the utility U of the disturbance w , one must determine the net change in magnitude between $Ax_{sp} + FHz$ and Ax_{sp} . This can be done by subtracting from r^2 the quantity given by the sum of the squares of those components of Ax_{sp} corresponding to the re-ordered terms ($i=1, \dots, K$) in Equation (3.57). Doing this results in the following implications

$$- \left[r^2 - \sum_{i=1}^K (a^i)^2_{R(B)} \right] \begin{cases} < 0 & w(t) \text{ has negative utility } U \\ = 0 & w(t) \text{ has zero utility } U \\ > 0 & w(t) \text{ has positive utility } U. \end{cases} \quad (3.61)$$

At this point, it is interesting to examine the conditions which must be satisfied by the external disturbance vector in order to

provide a positive utility U in achieving the control set-point objectives. According to Equation (3.61), in order for U to be positive one must have

$$\sum_{i=1}^K \frac{(a^i + f^i)^2}{R(B)^\perp} < \sum_{i=1}^K \frac{(a^i)^2}{R(B)^\perp} \quad (3.62)$$

or, if the terms where $|a^i + f^i| = |a^i|$ are included,

$$\sum_{i=1}^n \frac{(a^i + f^i)^2}{R(B)^\perp} < \sum_{i=1}^n \frac{(a^i)^2}{R(B)^\perp} \quad (3.63)$$

If one lets $\bar{a}^i, \bar{f}^i$ represent $\frac{a^i}{R(B)^\perp}$ and $\frac{f^i}{R(B)^\perp}$, respectively, then the inequality in (3.63) can be re-expressed as

$$\sum_{i=1}^n (\bar{a}^i + \bar{f}^i)^2 - \sum_{i=1}^n (\bar{a}^i)^2 < 0 \quad (3.64)$$

Equation (3.64) can then be expanded as

$$(\bar{a}^1 + \bar{f}^1)^2 + (\bar{a}^2 + \bar{f}^2)^2 + \dots + (\bar{a}^n + \bar{f}^n)^2 - (\bar{a}^1)^2 - (\bar{a}^2)^2 - \dots - (\bar{a}^n)^2 < 0$$

or

$$2\bar{a}^1\bar{f}^1 + 2\bar{a}^2\bar{f}^2 + \dots + 2\bar{a}^n\bar{f}^n + (\bar{f}^1)^2 + (\bar{f}^2)^2 + \dots + (\bar{f}^n)^2 < 0 \quad (3.65)$$

Equation (3.65) can be more compactly expressed as

$$\langle 2\bar{a} + \bar{f}, \bar{f} \rangle < 0 \quad (3.66)$$

where $\langle \cdot, \cdot \rangle$ denotes the inner product. By definition of the inner product, Equation (3.66) implies

$$|2\bar{a} + \bar{f}| \cdot |\bar{f}| \cdot \cos\theta < 0, \quad (3.67)$$

where θ is the angle between the two vectors $(2\bar{a} + \bar{f})$ and $(\bar{f})$. From Equation (3.67) it can be seen that a condition for obtaining $U > 0$ is that $\cos\theta$ be negative, i.e., $90^\circ < \theta < 270^\circ$.

There are thus two conditions which must be satisfied by FHz in order to guarantee that U will be positive. One condition is that $90^\circ < \Theta < 270^\circ$ and the other condition is the magnitude inequality given by Equation (3.64). If $\bar{a}$ and $\bar{f}$ are colinear, then from Equation (3.64) and the angle constraint one must have

$$||\bar{f}|| < ||2\bar{a}|| \quad (3.68)$$

with $\Theta = 180^\circ$ in order to obtain $U > 0$. If the magnitude of $\bar{f}$ were greater than or equal to $2\bar{a}$ then Equation (3.64) could not be satisfied regardless of the value for Θ . Thus, the inequality in (3.68) serves to set an upper bound on the permissible magnitude of $FHz_{R(B)^\perp}$ when considering whether or not it is possible to attain $U > 0$.

An Example:

If one refers to the second-order example of Section 3.7, the projection of Ax_{sp} on $R(B)^\perp$ along $R(B)$ is given by

$$(I - BB^\dagger)Ax_{sp} = (8., -4.)^T = \bar{a} \quad (3.69)$$

and similarly for FHz ,

$$(I - BB^\dagger)FHz = (2., -1.)^T = \bar{f}. \quad (3.70)$$

The component vector of $Ax_{sp} + FHz$ lying in $R(B)^\perp$ is thus found, by adding Equations (3.69) and (3.70), to be

$$\gamma_{R(B)^\perp} = (10., -5.)^T. \quad (3.71)$$

From Equation (3.60), r^2 is calculated to be $r^2 = 125$.

Using Equations (3.69) and (3.70) one finds that $||\bar{a}||^2 =$

80., $||\bar{f}||^2 = 5$., therefore, $\bar{f}$ meets the constraint of

(3.68). However, upon examining the two vectors $\bar{a}$ and $\bar{f}$,

it becomes apparent that they are colinear and that θ is

thus $\theta = 0^\circ$. The angle constraint is thus not met and one

should expect that U will not be positive. From Equation

(3.61) one has

$$- \left[r^2 - \sum_{i=1}^2 (\bar{a}^i)^2 \right] = - (125. - 80.) = -45., \quad (3.72)$$

i.e., a negative utility U .

3.9 The Critical-Variable Approach

3.9.1 Introduction. As mentioned in Sections 2.2

and 2.3.4, critical state variables are those plant states

for which the designer is most interested in reducing

disturbance effects. The disturbance effect on a critical

variable can appear directly in the differential equation

for the critical state (direct effect), or it can arise as a

result of coupling of the critical state with other plant states which have direct disturbance inputs (indirect effect). The problem of absorbing the direct disturbance effect on any one critical state variable, or on several critical variables simultaneously, is addressed in [17]. Those methods address the absorption of disturbance effects on the critical state rates. The effects of disturbances on the non-critical states are assumed to be acceptable.

In practice, the situation might arise in which the conditions for complete absorption of disturbances on critical variables are not satisfied. A minimization procedure can then be employed to reduce the disturbance effects on the critical state set, where the procedure to be utilized may depend on whether or not there are constraints on the non-critical set of plant states.

3.9.2 Direct Disturbance Absorption for a Critical State Set.

When one wishes to absorb the direct disturbance effect on a critical state or critical state set, it must first be determined whether or not the absorption will be possible. As shown in [17], if the plant state equations are of the form of Equations (3.1) and if the critical set consists of only one state, say x_i , with a disturbance acting directly on it, it will be possible to cancel the disturbance by choosing $u_d(t)$

as

$$u_d(t) = \left(\frac{-\langle (fh)^i, z(t) \rangle - \langle (fl)^i, x(t) \rangle}{||b^i||^2} \right) b^i \quad (3.73)$$

(provided $b^i \neq 0$), where $(fh)^i$ refers to the i -th row of FH , $(fl)^i$ refers to the i -th row of FL and $\langle . \rangle$ denotes the inner product.

If the critical set contains more than a single state component, for instance if more than one critical variable has a disturbance acting

directly on it, or if some of the plant states which couple indirect disturbance effects to a critical state are included, then, in order to have complete absorption $u_d(t)$ would have to be chosen such that

$$\bar{B}u_d(t) + \bar{F} \bar{H} z(t) + \bar{F} \bar{L} x(t) = 0, \quad (3.74)$$

where $\bar{B} = (b_1, b_2, \dots, b_q)^T$, $\bar{F} \bar{H} = ((fh)^1, (fh)^2, \dots, (fh)^q)^T$, $\bar{F} \bar{L} = ((fl)^1, (fl)^2, \dots, (fl)^q)^T$ and $\{x_1, x_2, \dots, x_q\}$ comprises the set of critical states, with some re-ordering of the original state equations possibly included. The criterion which must be met in order to be able to find a $u_d(t)$ which will satisfy Equation (3.74) is

$$\text{rank} ((\bar{B} | \bar{F} \bar{H} | \bar{F} \bar{L})) = \text{rank} (\bar{B}). \quad (3.75)$$

If expression (3.75) is satisfied, then it will be possible to find a matrix Γ (possibly non-unique) such that

$$\bar{B} \Gamma = [\bar{F} \bar{H} | \bar{F} \bar{L}] , \quad (3.76)$$

and therefore, the direct disturbance effect is absorbed by the control

$$u_d(t) = - \Gamma(z(t) | x(t))^T. \quad (3.77)$$

3.9.3 Direct Disturbance Minimization for a Critical State Set.

If complete absorption of disturbances on critical states is not possible, the designer may then choose to apply a disturbance minimization procedure. For this purpose, one should first reformulate the problem, as in previous sections, in terms of the error vector between the desired set-point and the achieved set-point. Using Equation (3.11) with $u(t)$ not allocated allows one to express the error dynamics as

$$\dot{\epsilon}(t) = (A+FL)\epsilon(t) - (A+FL)x_{sp} - Bu(t) - FH z(t). \quad (3.78)$$

If the pair $((A+FL), B)$ is completely controllable, one can proceed by designing $u_p(t) = -K\epsilon(t)$ such that for the solution of the homogeneous part of Equation (3.78) one obtains

$$\lim_{t \rightarrow \infty} e^{\tilde{A}t} \epsilon(0) \rightarrow 0, \quad (3.79)$$

where $\tilde{A} = A + FL + BK$. Then, assuming disturbances acting on the system have a limiting value as $t \rightarrow \infty$, the unique steady-state solution for Equation (3.78) is given by

$$\epsilon_{ss} = \tilde{A}^{-1}((A+FL)x_{sp} + FH z_{\infty} + Bu_d) \quad (3.80)$$

Next, re-order Equation (3.80) into two parts, one consisting of those states in the critical set, the other containing the remaining states. Thus

$$\begin{pmatrix} \epsilon_{ss}^{(1)} \\ \epsilon_{ss}^{(2)} \end{pmatrix} = \begin{bmatrix} \hat{A}_{11} & \hat{A}_{12} \\ \hat{A}_{21} & \hat{A}_{22} \end{bmatrix} \begin{pmatrix} x_{sp}^{(1)} \\ x_{sp}^{(2)} \end{pmatrix} + \begin{pmatrix} (\tilde{A}^{-1}B)^{(1)} \\ (\tilde{A}^{-1}B)^{(2)} \end{pmatrix} u_d + \begin{pmatrix} (A^{-1}FH)^{(1)} \\ (A^{-1}FH)^{(2)} \end{pmatrix} z_{\infty} \quad (3.81)$$

where $\hat{A} = \tilde{A}^{-1}(A+FL)$ and $\epsilon_{ss}^{(1)}$ represents the critical state set. Expanding Equation (3.81), one obtains the following expressions for the allocation of steady-state error between the critical and non-critical state sets.

$$\begin{aligned} \epsilon_{ss}^{(1)} &= (\hat{A}_{11}x_{sp}^{(1)} + \hat{A}_{12}x_{sp}^{(2)} + (\tilde{A}^{-1}FH)^{(1)}z_{\infty}) + (\tilde{A}^{-1}B)^{(1)}u_d \\ &= \hat{F}_1 + \hat{B}_1u_d \end{aligned} \quad (3.82)$$

$$\begin{aligned} \epsilon_{ss}^{(2)} &= (\hat{A}_{21}x_{sp}^{(1)} + \hat{A}_{22}x_{sp}^{(2)} + (\tilde{A}^{-1}FH)^{(2)}z_{\infty}) + (\tilde{A}^{-1}B)^{(2)}u_d \\ &= \hat{F}_2 + \hat{B}_2u_d \end{aligned} \quad (3.83)$$

The primary objective is to minimize the disturbance effects on $\epsilon_{ss}^{(1)}$, since complete absorption is assumed not possible. As one method of approach, the designer can choose the minimum norm control u_d^* to minimize $\|\epsilon_{ss}^{(1)}\|_I$. This approach will result in a control vector given by

$$u_d^*(t) = -\hat{B}_1^+ \hat{F}_1 \quad (3.84)$$

Substitution of Equation (3.84) into Equations (3.82) and (3.83) yields

$$\epsilon_{ss}^{(1)*} = (I - \hat{B}_1 \hat{B}_1^+) \hat{F}_1 \quad (3.85)$$

and

$$\epsilon_{ss}^{(2)*} = \hat{F}_2 - \hat{B}_2 \hat{B}_1^+ \hat{F}_1 \quad (3.86)$$

Choosing $u_d(t)$ as in Equation (3.84) will give a minimum Euclidean norm of $\epsilon_{ss}^{(1)}$ based upon the components included in $\hat{F}_1$ but it will not necessarily result in the absolute minimum value achievable. The reason for this is that the norm minimization process minimizes the length of a vector based on equal weighting on all the components involved, if the weighting matrix is chosen as I. Thus, if $u_d(t)$ is chosen to minimize $\|\epsilon_{ss}\|_I$, it is possible that the portion corresponding to $\epsilon_{ss}^{(1)}$ will be of smaller magnitude than that obtained from Equations (3.85) and (3.86)

One should note from Equation (3.80) that with a non-zero set-point it is possible for the disturbance to reduce the steady-state error, i.e., exhibit a positive utility.

3.9.4 Disturbance Minimization for a Critical State Set With Restrictions on Non-Critical State Set. In the two previous sections, the disturbance effect on the non-critical states was assumed to be acceptable. Suppose, however, that the excursions of the non-critical states are not completely unrestricted and that it is necessary to trade off critical state disturbance minimization results against non-critical state constraints, with the prime concern still being achievement of the maximum (practical) amount of disturbance minimization on the critical set. One method of approach to this problem is to utilize the weighted generalized inverse (Section A.6.2). This approach minimizes a positive quadratic form in the error,

$$\|\hat{B}u_d - \hat{F}\|_Q^2 = (\hat{B}u_d - \hat{F})^T Q (\hat{B}u_d - \hat{F}) \quad (3.87)$$

where Q is a given positive definite weighting matrix and $\hat{B}$, $\hat{F}$ are as defined in Equation (3.82). By proper choice of Q , the relative

importance of minimizing ϵ_{ssQ} can be traded off against constraints on the magnitude of ϵ_{ssp} . This procedure will probably require several iterations in order to arrive at a suitable Q . The control vector $u_d(t)$ can be found by using condition (3.6), (3.7) or (3.8).

3.9.5 Indirect Disturbance Minimization for a Critical State Set.

3.9.5.1 Introduction. This section is concerned with further investigation of a technique briefly mentioned in Section 2.2.4.2 for indirectly minimizing disturbance effects as applied to the critical state variable approach. Since it is difficult to describe the technique presented in this section in terms of a general recipe, the main ideas will be illustrated by considering a specific example.

3.9.5.2 The Notion of Indirect Minimization. As mentioned in Section 2.2.4.2, if $b_i = 0$ for a particular critical state variable, it will not be possible to apply the control u directly to the state equation for $\dot{x}_i$ in order to achieve direct disturbance absorption. For such cases a technique termed "indirect disturbance absorption" has been proposed [17]. This technique attempts to steer those states x_j which can be directly affected by the control in such a way that their motion, coupled through the plant A matrix, produces the desired result on the critical state x_i .

Assuming that the control objective is to stabilize one specified critical state to the origin, the method presented in [17] involves the following steps: (1) determine a homogeneous differential equation for the critical state by computing higher derivatives of the original plant state equation and by substituting the original state equation in those expressions, (2) design u_d to absorb the disturbance terms

appearing in the resulting homogeneous equation for the critical state, (3) design u_p to supply any "missing" derivatives of the critical state, including appropriate coefficients, such that the closed-loop differential equation governing the critical state will satisfy the Routh-Hurwitz stability criterion. This method is valid for all $w(t)$.

The example given in [17] to illustrate the steps listed above is as follows: let

$$\dot{x}_1 = x_2 + w ; \quad \dot{x}_2 = x_3 ; \quad \dot{x}_3 = x_2 + u \quad (3.88)$$

with the objective being to stabilize $y = x_1$ to zero. The motions of x_2 and x_3 are assumed to be unrestricted. From Section 3.9.5.2, step 1, by differentiating $\dot{x}_1$ and substituting in from $\dot{x}_2$ and $\dot{x}_3$ one obtains

$$\ddot{x}_1 - \dot{x}_1 = u - w + \dot{w} . \quad (3.89)$$

From step 2, in order to absorb the disturbance terms one must have

$$u_d = w - \dot{w} . \quad (3.90)$$

From step 3, in order to stabilize the motion of x_1 one must have

$$u_p = -k_1 x_1 - k_2 \dot{x}_1 - k_3 \ddot{x}_1 . \quad (3.91)$$

Thus, the closed-loop differential equation for x_1 is

$$\ddot{x}_1 + k_3 \dot{x}_1 + (k_2 + 1) \dot{x}_1 + k_1 x_1 = 0 , \quad k_i > 0 , \quad (3.92)$$

and k_1, k_2, k_3 are chosen such that the Routh-Hurwitz stability criterion is satisfied.

Since the output equation was given as $y = x_1$, this means $x_2, x_3, w, \dot{w}$ are not directly measureable and hence will have to be reconstructed using a composite observer. Since $w = Hz$ for this example, an expression for $\dot{w}$ can be found as follows,

$$\begin{aligned} \dot{w} &= H\dot{z} = HDz \\ \dot{w} &= HD^2 z . \end{aligned} \quad (3.93)$$

A schematic diagram for this example is shown in Figure 3-7. Since, in

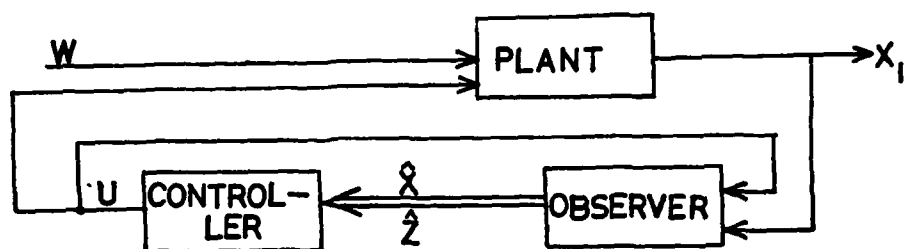


Figure 3-7. System schematic.

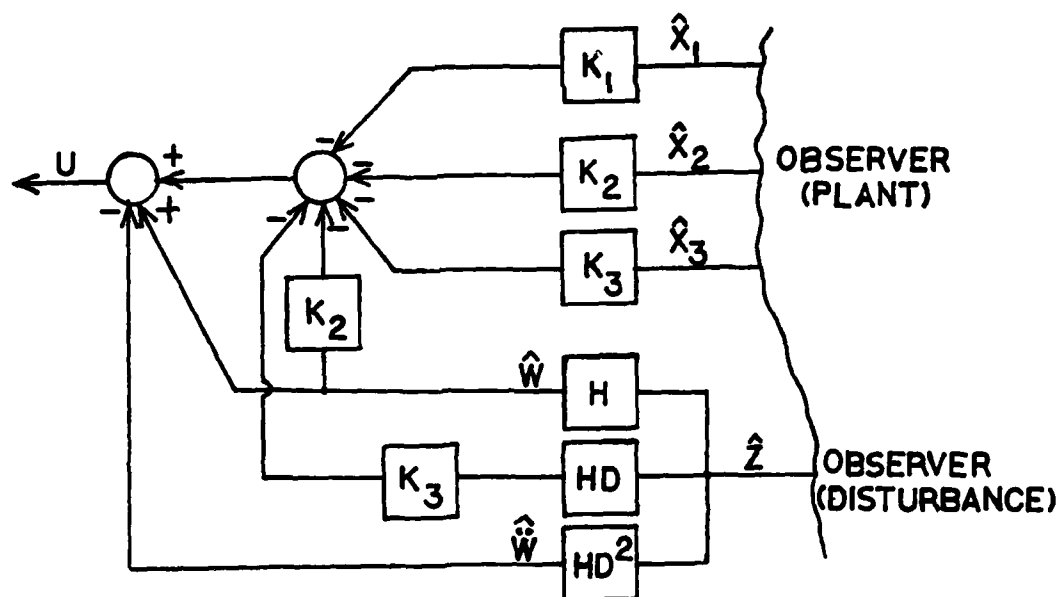


Figure 3-8. Controller.

this example, $L \equiv E \equiv M \equiv 0$, the observer would be modeled as [14]:

$$\begin{pmatrix} \dot{\hat{x}} \\ \dot{\hat{z}} \end{pmatrix} = \begin{bmatrix} A+K_1C & FH \\ K_2C & D \end{bmatrix} \begin{pmatrix} \hat{x} \\ \hat{z} \end{pmatrix} - \begin{bmatrix} K_1 \\ K_2 \end{bmatrix} x_1 + \begin{bmatrix} B \\ 0 \end{bmatrix} u \quad (3.94)$$

and the controller section would be as shown in Figure 3-8.

3.9.5.3 Alternate Solution for Cases With Constant Disturbances.

For cases where the external disturbance is piecewise constant $w=Hz$, z =piecewise constant, and where the pair (A,B) is completely state controllable, thereby permitting pole placement techniques to be used, the following alternative procedure can be used to design u_d . If one sets $\dot{e}(t)=0$ in Equation (3.11) and finds the steady-state error ϵ_{ss} defined by the resulting equation, the expression for ϵ_{ss} is

$$\epsilon_{ss} = \tilde{A}^{-1}(Bu_d + FH z + Ax_{sp}), \quad z=\text{constant} \quad (3.95)$$

where $\tilde{A}$ is defined as $\tilde{A}=A+BK$ and u_p has been designed as $u_p = -K\epsilon(t)$ such that the system is stabilized, i.e., the characteristic values of $\tilde{A}$ are all in the left-half plane and are placed so as to provide the desired system transient response.

If Equation (3.95) is now re-ordered by grouping the components of ϵ_{ss} into two sets, one containing the critical states and the other containing the non-critical states, then u_d^* can be designed as in Equation (3.84) to minimize the norm of that part of ϵ_{ss} corresponding to the critical state set.

Using the example of Section 3.9.5.2 to illustrate the indirect method described in this section, the plant model can be expressed as

$$\begin{pmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \end{pmatrix} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} + \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} u + \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} w \quad (3.96)$$

$$y = (1, 0, 0)x$$

and the piecewise constant disturbance model

$$w = z, \quad z = \sigma(t), \quad (3.97)$$

and suppose the objective is to stabilize the critical state variable x_1 to zero in the face of all $x_1(0)$ and all external disturbances $w=z$.

The state controllability matrix for the plant is calculated as

$$P = [B | AB | A^2B] = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix} \quad (3.98)$$

and it is found that $\det(P) = -1$, which implies that $\text{rank}(P) = 3 = n$, hence, the plant is completely state controllable. The characteristic values of $\tilde{A} = A+BK$ will be placed at $\lambda_1 = -1$, $\lambda_2 = -2$, $\lambda_3 = -3$. In order to obtain these values it can be determined that the gain matrix K must be

$$K = (-6, -12, -6) \quad (3.99)$$

Using this value of K to calculate $\tilde{A}$ then permits one to obtain $\tilde{A}^{-1}$ as

$$\tilde{A}^{-1} = \begin{bmatrix} -11/6 & -1 & -1/6 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix} \quad (3.100)$$

and Equation (3.95) becomes

$$\epsilon_{ss} = ((-11/6)z - (1/6)u_d, z, 0)^T \quad (3.101)$$

From Equation (3.101) it can readily be seen that for ϵ_{ss1} to be zero will require that $u_d = -11z$. It can also be seen that ϵ_{ss2} will be equal to z and ϵ_{ss3} will be equal to zero.

This example was simulated on a digital computer and results for x_1 for various conditions are shown on Figure 3-9. From the data in

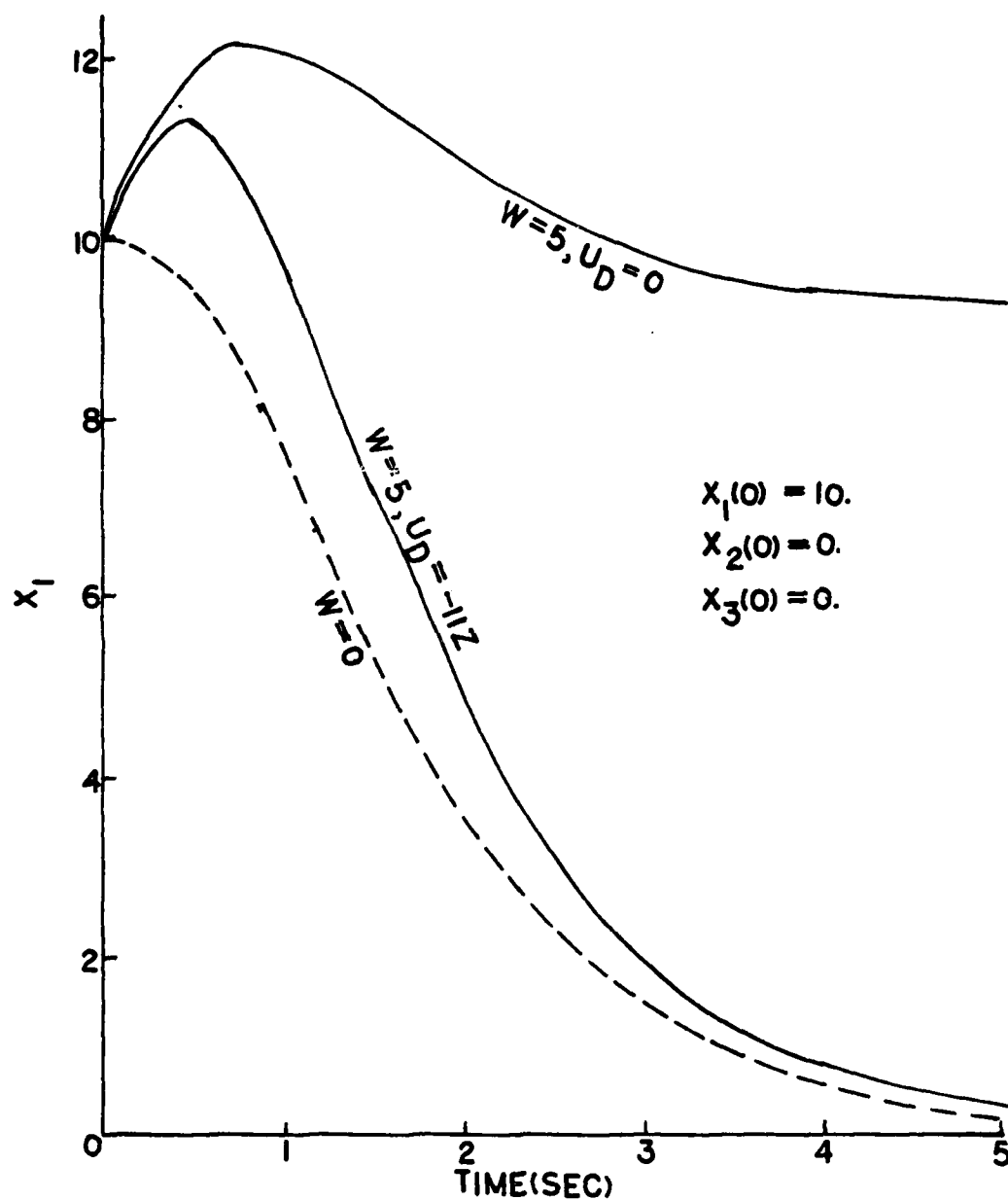


Figure 3-9. x_1 versus time, indirect disturbance minimization.

Table 3.2. Simulation results, indirect disturbance absorption

Time (sec)	Case 1			Case 2			Case 3		
	x_1	x_2	x_3	x_1	x_2	x_3	x_1	x_2	x_3
0.0	10.0	0.0	0.0	10.0	0.0	0.0	10.0	0.0	0.0
0.125	9.98	-0.36	-5.1	10.6	-0.37	-5.3	10.59	-0.7	-10.0
0.25	9.89	-1.14	-6.94	11.14	-1.2	-7.5	11.04	-2.25	-13.9
0.50	9.39	-2.82	-5.88	11.85	-3.13	-7.3	11.3	-5.72	-12.7
0.75	8.53	-3.95	-3.11	12.11	-4.7	-5.1	10.8	-8.3	-7.94
1.0	---	---	---	12.05	-5.7	-2.9	9.73	-9.72	-3.56
1.125	6.91	-4.44	0.19	---	---	---	---	---	---
1.5	5.31	-4.04	1.73	11.48	-6.38	-0.29	7.18	-10.1	1.29
2.0	3.53	-3.03	2.08	10.8	-6.26	0.57	4.87	-9.04	2.4
3.0	1.42	-1.35	1.2	9.86	-5.64	0.53	1.99	-6.87	1.64
4.0	0.54	-0.53	0.51	9.43	-5.26	0.24	0.76	-5.74	0.71
5.0	0.2	-0.2	0.2	9.27	-5.1	0.1	0.28	-5.28	0.28
10.0	0.0	0.0	0.0	9.17	-5.0	0.0	0.0	-5.0	0.0

Notes:

(a) Case 1, $w = 0$ (b) Case 2, $w = 5.$, $u_d = 0.$ (c) Case 3, $w = 5.$, $u_d = -11z$

Table 3.2, it can be seen that ϵ_{ss2} and ϵ_{ss3} do indeed attain the values predicted by Equation (3.101).

CHAPTER IV

DIRECT DISTURBANCE MINIMIZATION FOR LINEAR, TIME-
INVARIANT DYNAMICAL SYSTEMS; OUTPUT SET-
POINT REGULATION/STABILIZATION

4.1 Summary of Chapter IV

This chapter introduces the linear algebra associated with the various conditions which can apply in an output set-point problem, depending on the given set-point and the output matrix C . The four basic cases which arise are examined in detail as applies to minimization of the output set-point error. Those cases are: (1) $y_{sp} = 0$, C non-singular, (2) $y_{sp} = 0$, C non-invertible, (3) $y_{sp} \neq 0$, C non-singular, (4) $y_{sp} \neq 0$, C non-invertible.

4.2 Linear Algebra of the Output Set-Point Problem

In output set-point regulation/stabilization problems the control objective is to steer $y(t)$ to y_{sp} in the face of all admissible plant initial conditions $x(t_0)$ and all external disturbances $w(t)$. The basic plant model considered in this chapter will be taken to be

$$\dot{x}(t) = Ax(t) + Bu(t) + Fw(t) \quad (4.1a)$$

$$y(t) = Cx(t) \quad (4.1b)$$

The matrix C in Equation (4.1b) represents a linear mapping between the state space R^n and the output space R^m . The matrix C can be either an $n \times n$ non-singular matrix, an $n \times n$ singular matrix or an $m \times n$ rectangular matrix, $m < n$. In the case where C is non-singular, $\text{rank}(C) = \rho(C) = n$ and the nullity of C is $\nu(C) = 0$. Thus, the nullspace of C in

that case consists of the one point $x=0$ ($N(C)=0$) and the only solution to $Cx = 0$ will be the trivial case $x = 0$. When C is non-singular its inverse C^{-1} exists. Therefore, given any y in R^m there is an x in R^n such that $y = Cx$ and that x will be uniquely given by $x = C^{-1}y$.

If C is non-square ($m < n$), or an $n \times n$ singular matrix, then $\rho(C) = r < n$ and $\nu(C) = n - r > 0$. Thus, the column range space of C , $R(C)$, will be a subspace of R^m and, since the dimension of the nullspace of C is greater than 0, $Cx = 0$ has a subspace of solutions other than the trivial case $x = 0$. If C has maximal rank $\rho(C) = m$, where m is the dimension of the output space, then C represents an onto mapping from R^n to R^m and it will be possible to attain any point in R^m with some appropriate x in R^n . If $\rho(C) < m$, the mapping from R^n to R^m represented by C is not an onto mapping and $R(C)$ will be a subset of R^m instead of the entire output space R^m . It will not be possible to find an x in R^n which will map to a point in R^m which is not contained within $R(C)$.

For the case of a non-singular matrix C , $x = C^{-1}y$ and thus if $y_{sp} = 0$ and $w(t) = 0$, then the design objective is to create a control $u(t)$ such that $y \rightarrow y_{sp} = 0$, i.e., such that $x \rightarrow 0$. If y_{sp} is not equal to zero, but $w = 0$, then the objective is to design $u(t)$ such that $y \rightarrow y_{sp}$, i.e., such that $x \rightarrow C^{-1}y_{sp}$. If $w(t)$ is not zero, and total absorption of disturbances is not possible it will not be possible to achieve these design goals. In that event, the revised objective is one of utilizing a minimization method to design u such that $\|\epsilon_{ss}\|_Q$ is minimized, where the error vector is defined as $\epsilon(t) = y_{sp} - y(t)$.

If C is singular, and the output set-point and external disturbances are both zero, the design objective will be to create a u such that $x \rightarrow N(C)$, i.e., $y \rightarrow 0$. If the external disturbance is not zero then

it might not be possible to steer x to $N(C)$ or keep it there. In that event the revised objective again becomes the design of u such that $\|e_{ss}\|_Q$ is minimized. If $y_{sp} \neq 0$ and $\rho(C) < m$, then y_{sp} might not lie in $R(C)$ in which case it will not be an attainable set-point even if $w = 0$. In this case, one must utilize a minimization method to design u such that $\|e_{ss}\|_Q$ is minimized.

For those cases where C is a singular matrix, it would be helpful for visualization purposes to identify a set of basis vectors for the nullspace of C and for the column range space of C . To determine the basis set for $N(C)$, one can set $Cx = 0$ and solve the resulting set of equations simultaneously. There should be $(n-r)$ arbitrary constants involved in the solution and the result should be $(n-r)$ linearly independent vectors for the basis set.

If C is $m \times n$, where $m < n$, and $\rho(C) = m$, then C represents an onto transformation, $R(C) = R^m$, and the natural basis set for R^m can be chosen as the basis set for $R(C)$. If $m > n$, or if $m < n$ and $\rho(C) = r < m$, then the transformation is not onto. In this case, one choice for a basis set would be the first r columns of the unit matrix, I_m [3]. If $\rho(C) > 0$, a basis set for $N(C)$ will span the subspace of R^n to which the state vector x should be steered if the objective is to have $y_{sp} = 0$. For the case where $m < n$ and $y_{sp} \neq 0$, the expression $y_{sp} = Cx$ can be used to solve for the set of points in R^n to which x must be controlled in order to give $y = y_{sp}$.

The basic plant model (4.1) could be slightly generalized to read

$$\begin{aligned}\dot{x}(t) &= Ax(t) + Bu(t) + Fw(t) \\ y(t) &= Cx(t) + Eu(t) + Gw(t).\end{aligned}\tag{4.2}$$

If $u(t)$ is allocated as $u = u_p + u_d$, with $u_p = Kx$, then Equations (4.2) can be re-written as

$$\begin{aligned}\dot{x}(t) &= (A+BK)x(t) + Bu_d(t) + Fw(t) = \tilde{A}x(t) + Bu_d(t) + Fw(t) \\ y(t) &= (C+EK)x(t) + Eu_d(t) + Gw(t) = \tilde{C}x(t) + Eu_d(t) + Gw(t).\end{aligned}\quad (4.3)$$

The transformation between R^n and R^m is now represented by $\tilde{C}$ instead of C . Figure 4-1 illustrates this transformation where $y_t = \tilde{C}x$, and $m < n$. The vector $Eu_d + Gw$ represents the direct effect of v and w on the output in addition to the indirect effects embedded in the transformation of x .

The state space R^n and output space R^m can also be illustrated as in Figure 4-2 with $\tilde{C}$ shown as the transformation between them. The state space R^n is decomposed into $N(\tilde{C})$ and $N(\tilde{C})^\perp$ while the output space R^m is decomposed into $R(\tilde{C})$ and $R(\tilde{C})^\perp$. The dimension of $N(\tilde{C})$ is equal to $\dim(N(\tilde{C})) = n - \rho(\tilde{C})$ and a set of $n - \rho(\tilde{C})$ basis vectors can be determined for $N(\tilde{C})$ from $\tilde{C}x = 0$. If $y_{sp} = 0$, then these basis vectors will define $N(\tilde{C})$ to which x must be steered to give $y=0$ (in the absence of disturbances). For a given output set-point y_{sp} the set of equations $y_{sp} = \tilde{C}x$ can be used to solve for the set of points in R^n which transform to y_{sp} .

Another concept which should be addressed at this point is that of "miss distance" between a point P and a linear subspace S . The meaning of "miss distance" between two points is unambiguous; however, when "miss distance" is specified as being between a point and a linear subspace, clarification is required as to just which of the many possible distances is meant. In this dissertation, "miss distance" between P and S refers to the norm of that vector which, out of all possible vectors which can be drawn between P and S , has minimum norm.

4.3 Output Set-Point Stabilization; $y_{sp}=0$, C Non-Singular

For the type of problem to be considered in this section, C is assumed non-singular. Hence, $\rho(C)$ is n and $\nu(C)$ is zero and $y(t)$ will be zero if and only if $x = 0$. In the absence of external disturbances, achieving $y \rightarrow y_{sp} = 0$ requires that x be controlled to the origin of the state space, i.e., creation of an asymptotically stable system. With external disturbances present, achieving this control objective will require total absorption of disturbance effects. Failing this,

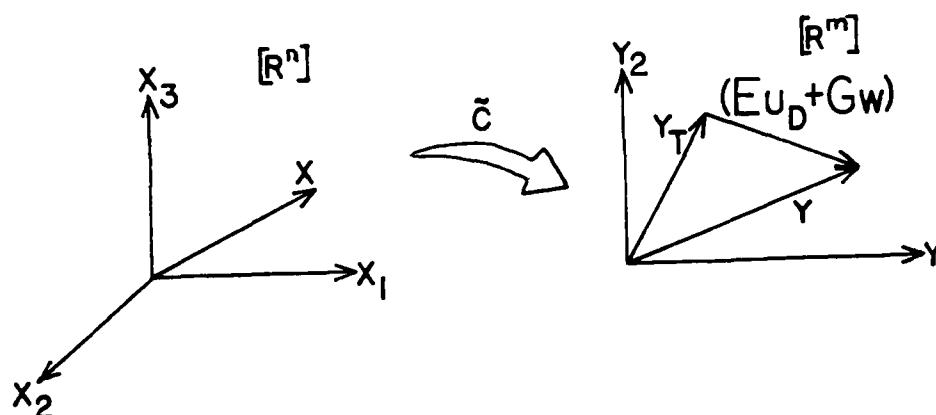


Figure 4-1. State space/output space.

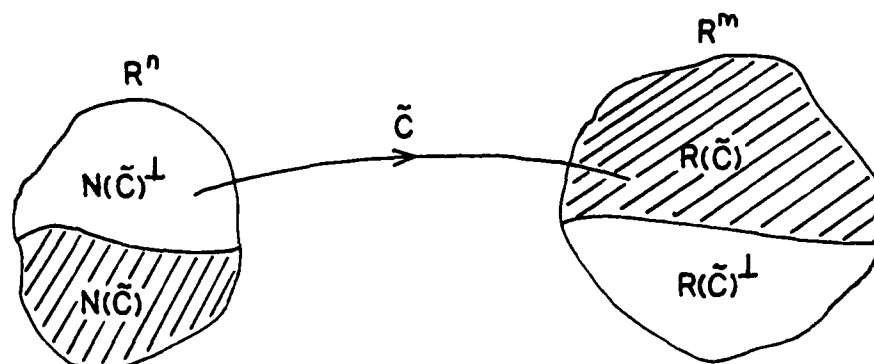


Figure 4-2. State space/output space decomposition.

one alternative would be to minimize the error between the attainable plant output and the specified output set-point y_{sp} , i.e., minimize $\|\epsilon_{ss}\|_Q$. The procedures for performing this minimization can be developed in several ways. For instance, an equation can be derived for the error dynamics and, if a steady-state solution exists, can be solved for

$$\epsilon_{ss} = f(A, B, K, C, F, z_\infty, u_d). \quad (4.4)$$

Equation (4.4) can then be solved for the appropriate u_d via generalized inverse techniques. Alternatively, one can solve the first of Equations (4.3) to obtain

$$x_{ss} = g(A, B, K, F, z_\infty, u_d) \quad (4.5)$$

which can then be solved for a u_d which will minimize $\|x_{ss}\|_Q$. The steady-state error is then given by

$$\epsilon_{ss} = Cx_{ss}. \quad (4.6)$$

Note that Equation (4.6) might not give an absolute minimum for $\|\epsilon_{ss}\|_Q$ since the final result depends on the characteristics of the transformation represented by the matrix C .

For the output stabilization problem, the error vector $\epsilon(t)$ is defined as

$$\epsilon(t) = -y(t) = -Cx(t), \quad (4.7)$$

therefore, the evolution equation for the error dynamics will be given by

$$\dot{\epsilon}(t) = -C\dot{x}(t) = -C(A+FL)x(t) - CBu(t) - CFHz(t) \quad (4.8)$$

where $w(t) = Hz(t) + Lx(t)$. Since C is assumed non-singular C^{-1} exists, hence, from Equation (4.1b), one can obtain

$$x(t) = C^{-1}y(t) = -C^{-1}\epsilon(t). \quad (4.9)$$

Substitution of Equation (4.9) into Equation (4.8) then yields

$$\dot{\epsilon}(t) = C(A+FL)C^{-1}\epsilon(t) - CBu(t) - CFHz(t). \quad (4.10)$$

The previous practice of splitting u into $u_p + u_d$ is again followed, with

$u_p(t) = -K\varepsilon(t)$, and results in Equation (4.10) being expressible as

$$\begin{aligned}\dot{\varepsilon}(t) &= (C(A+FL)C^{-1} + CBK)\varepsilon(t) - CBu_d(t) - CFHz(t) \\ &= \hat{C}\varepsilon(t) - C(Bu_d(t) + FH_z(t)) .\end{aligned}\quad (4.11)$$

If the pair $(C(A+FL)C^{-1}, CB)$ is completely controllable, then it will be possible to design K such that $\lim_{t \rightarrow \infty} \phi_{\varepsilon}(t, t_0)\varepsilon(t_0) \rightarrow 0$ where $\phi_{\varepsilon}(t, t_0)$ is the transition matrix for $\hat{C}$. Thus, in order to make $\varepsilon \rightarrow 0$, it would be necessary to have $C(Bu_d(t) + FH_z(t)) = 0$, i.e., complete absorption of the external disturbances must be possible.

If complete absorption is not possible, then the control objective is to minimize an appropriate norm of the error vector,

$\|\varepsilon\|_Q$. The general solution to Equation (4.11) is given by

$$\varepsilon(t) = \phi_{\varepsilon}(t, t_0)\varepsilon(t_0) - \int_{t_0}^t \phi_{\varepsilon}(t, \tau)C(Bu_d(\tau) + FH_z(\tau))d\tau, \quad (4.12)$$

however, we are interested in the steady-state solution ε_{ss} . Assuming disturbance terms acting on the system have a limit as $t \rightarrow \infty$ and that a unique steady-state solution exists, ε_{ss} is found by setting $\dot{\varepsilon} = 0$ in Equation (4.11) and solving for the ε which satisfies the resulting equation. If this is done, the steady-state error is found to be

$$\varepsilon_{ss} = \hat{C}^{-1}C(Bu_d + FH_{z\infty}) . \quad (4.13)$$

Ideally, it is desired to have $\varepsilon_{ss} = 0$, or

$$\hat{C}^{-1}CBu_d = -\hat{C}^{-1}CFH_{z\infty} . \quad (4.14)$$

Since condition (4.14) is assumed not achievable, one must instead find a u_d which will minimize $\|\varepsilon_{ss}\|_Q$. If one chooses, for instance, to let $Q \equiv I$, then the u_d which has minimum norm and which gives a minimum norm for $\|\varepsilon_{ss}\|_I$ is given by

$$u_d^* = -(\hat{C}^{-1}CB)^{\dagger} \hat{C}^{-1}CFH_{z\infty} \quad (4.15)$$

Substitution of Equation (4.15) into Equation (4.13) then gives the

expression for the minimum norm steady-state error vector as

$$\epsilon_{ss}^* = (I - \hat{C}^{-1}CB(\hat{C}^{-1}CB)^\dagger)\hat{C}^{-1}CFHz_\infty. \quad (4.16)$$

If the designer wishes to choose Q to be other than the identity matrix, in order to weight the various components of ϵ_{ss} relative to each other in some manner other than equally, then expressions (3.6), (3.7) and (3.8) can be used to solve for the required u_d^* .

This output set-point stabilization problem can also be approached in an alternative fashion by working directly with the state vector instead of with an error vector defined in the output space. For this purpose, one can use the first of Equations (4.1) with $u_p(t) = Kx(t)$ to obtain

$$\dot{x}(t) = \hat{A}x(t) + Bu_d(t) + FHz(t) \quad (4.17)$$

where $\hat{A} = A + FL + BK$. If the pair $(A + FL, B)$ is completely controllable then K can be designed such that $\lim_{t \rightarrow \infty} e^{\hat{A}t}x(0) \rightarrow 0$ and, under the same assumptions given below Equation (4.12), a unique solution x_{ss} is found as

$$x_{ss} = \hat{A}^{-1}(Bu_d + FHz_\infty). \quad (4.18)$$

Ideally, one would like to have $x_{ss} = 0$ but this has been assumed to be not achievable. Therefore, one must design u_d to minimize $\|x_{ss}\|_Q$. If $Q=I$, then the minimum norm u_d which will minimize $\|x_{ss}\|_I$ is given by

$$u_d^* = -(\hat{A}^{-1}B)^\dagger \hat{A}^{-1}FHz_\infty. \quad (4.19)$$

Using Equation (4.19) in Equation (4.18) yields the minimum norm steady-state vector

$$x_{ss}^* = (I - \hat{A}^{-1}B(\hat{A}^{-1}B)^\dagger)\hat{A}^{-1}FHz_\infty. \quad (4.20)$$

The output steady-state error, defined as $\epsilon_{ss} = -y_{ss}$, would thus be

$$\epsilon_{ss} = -Cx_{ss}^*. \quad (4.21)$$

If Q is not chosen to be the identity matrix, then expressions (3.6) to (3.8) can be used to determine the required u_d^* .

An Example

A second-order example of output stabilization utilizing the techniques developed in this section is worked-out in detail in Appendix D. Some selected results from that example are presented here. The example is a second-order, linear, time-invariant output stabilization problem with the plant modeled by

$$\begin{aligned} \begin{pmatrix} \dot{x}_1 \\ \dot{x}_2 \end{pmatrix} &= \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} + \begin{pmatrix} 1 \\ 2 \end{pmatrix} u + \begin{pmatrix} 1 \\ 1 \end{pmatrix} w \\ \begin{pmatrix} y_1 \\ y_2 \end{pmatrix} &= \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \end{aligned} \quad (D.1)$$

and the "constant" external disturbance modeled by

$$w = z \quad ; \quad \dot{z} = \sigma(t) . \quad (D.2)$$

The control objective is to stabilize y to zero in the face of uncertain disturbances w . The problem is developed in terms of the error vector $\epsilon = y_{sp} - y$, and the objective is re-stated as, minimize $\|\epsilon_{ss}\|_I$. The error dynamics are found from Equations (4.11), (D.1) and (D.2) to be

$$\dot{\epsilon}(t) = (A + BK)\epsilon(t) - Bu_d(t) - FH\dot{z} . \quad (4.22)$$

The plant A and B matrices in Equations (D.1) are identical to those used in Appendix B (Equation B.2) and it is shown in Appendix B that the pair (A,B) is completely controllable. The same K matrix used in Appendix B, i.e., $(-3., -0.36)$ is used in the present example.

Equation (4.15), the minimum norm control u_d^* which minimizes $\|\epsilon_{ss}\|_I$ is found as

$$u_d^* = -(\hat{C}^{-1}CB)^+ \hat{C}^{-1}CFH\dot{z} = -2.9\dot{z} \quad (D.12)$$

and the steady-state error which results from use of Equation (D.12) is found from Equation (4.16) to be

$$\epsilon_{ss}^* = (I - \hat{C}^{-1}CB(\hat{C}^{-1}CB)^{\dagger})\hat{C}^{-1}CFHz = (0.4, 0.2)^T z. \quad (D.14)$$

The "miss distance", i.e., $||\epsilon_{ss}^*||$, will thus be

$$||\epsilon_{ss}^*|| = 0.4472z. \quad (D.15)$$

Recall that the utility U of the external disturbance was defined by Equation (3.52) as $U = ||\epsilon_{ss}||_{w=0} - ||\epsilon_{ss}||_{w \neq 0}$. From Equation (D.15) it can be seen that, for the output set-point stabilization example considered here, a non-zero external disturbance $w=c \neq 0$ will always exhibit a negative utility.

An expression for u_d^* in terms of general elements of the gain matrix K can be derived. The resulting expression is

$$u_d^* = (-0.4 + 0.4k_1 + 0.2k_2)z \quad (D.20)$$

where $K = (k_1, k_2)$. By using Equation (D.20) in Equation (4.16) it is found that the minimum norm steady-state error vector is still given by

$$\epsilon_{ss}^* = (0.4, 0.2)^T z. \quad (D.22)$$

This example was programmed on a digital computer and two sets of runs were made; Case 1 used a gain matrix $K = (-6., -0.5)$ and Case 2 used $K = (-15., 2.)$. Figure 4-3 illustrates the values obtained for $||\epsilon_{ss}||_I$ for Case 1 and Figure 4-4 illustrates the values for Case 2. Each figure contains three curves; curve 1 is for a run with $w = 0.$, curve 2 is for a run with $w = 2.$, $u_d = 0.$, and curve 3 is for a run with $w = 2.$, $u_d^* = -2.9z$. As can be seen, the minimizing controller u_d^* does reduce the steady-state error and examination of Figures 4-3 and 4-4 shows that $||\epsilon_{ss}||_I \approx 0.9$ for run 3 in each case. From Equation (D.15), for $w = 2$, $||\epsilon_{ss}^*||_I = 0.4472z = 0.8944$. The controller u_d^* therefore produced the minimum norm steady-state error vector given by (D.14).

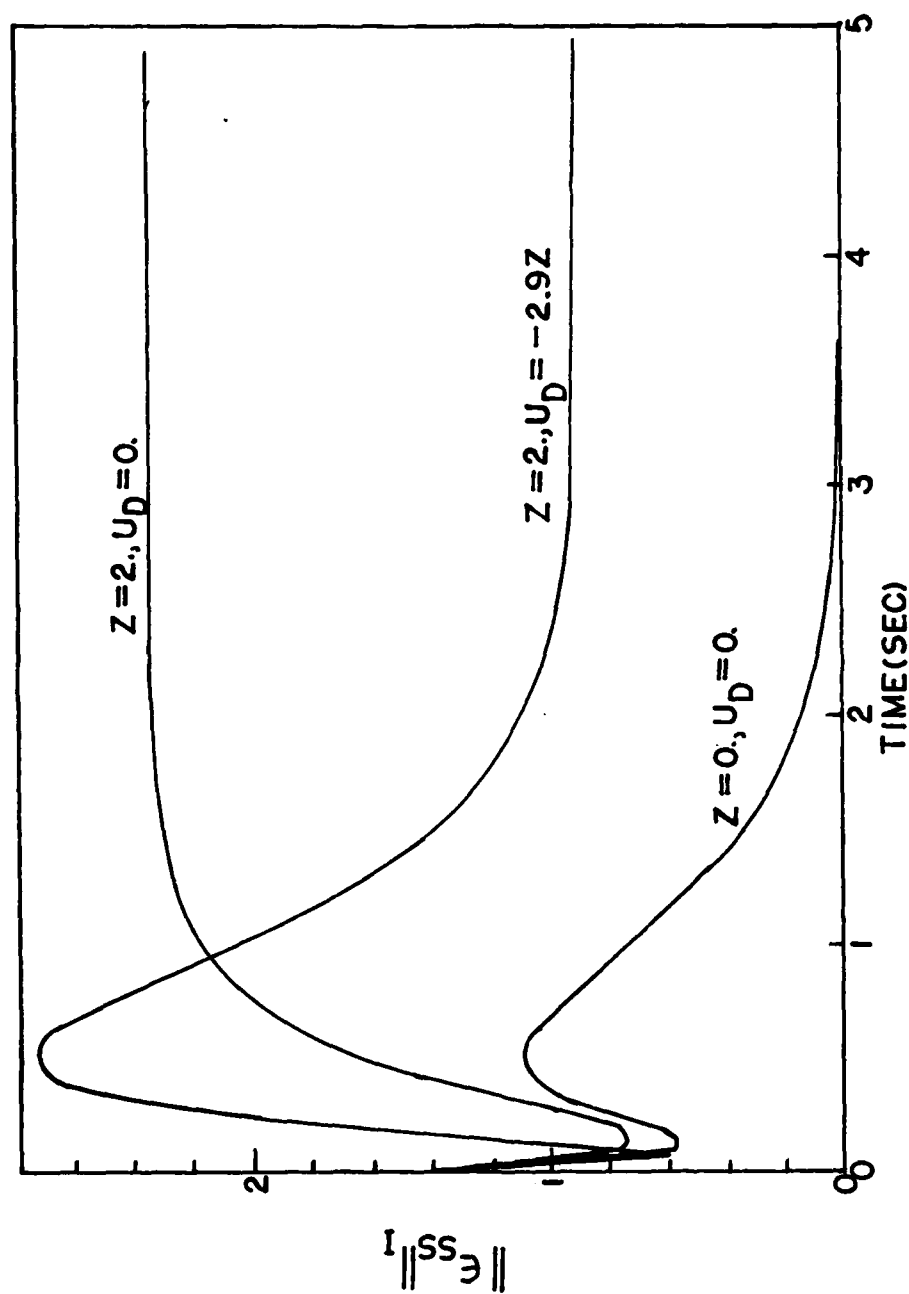


Figure 4-3. Norm of ϵ_{ss} versus time, $K = (-6., -0.5)$.

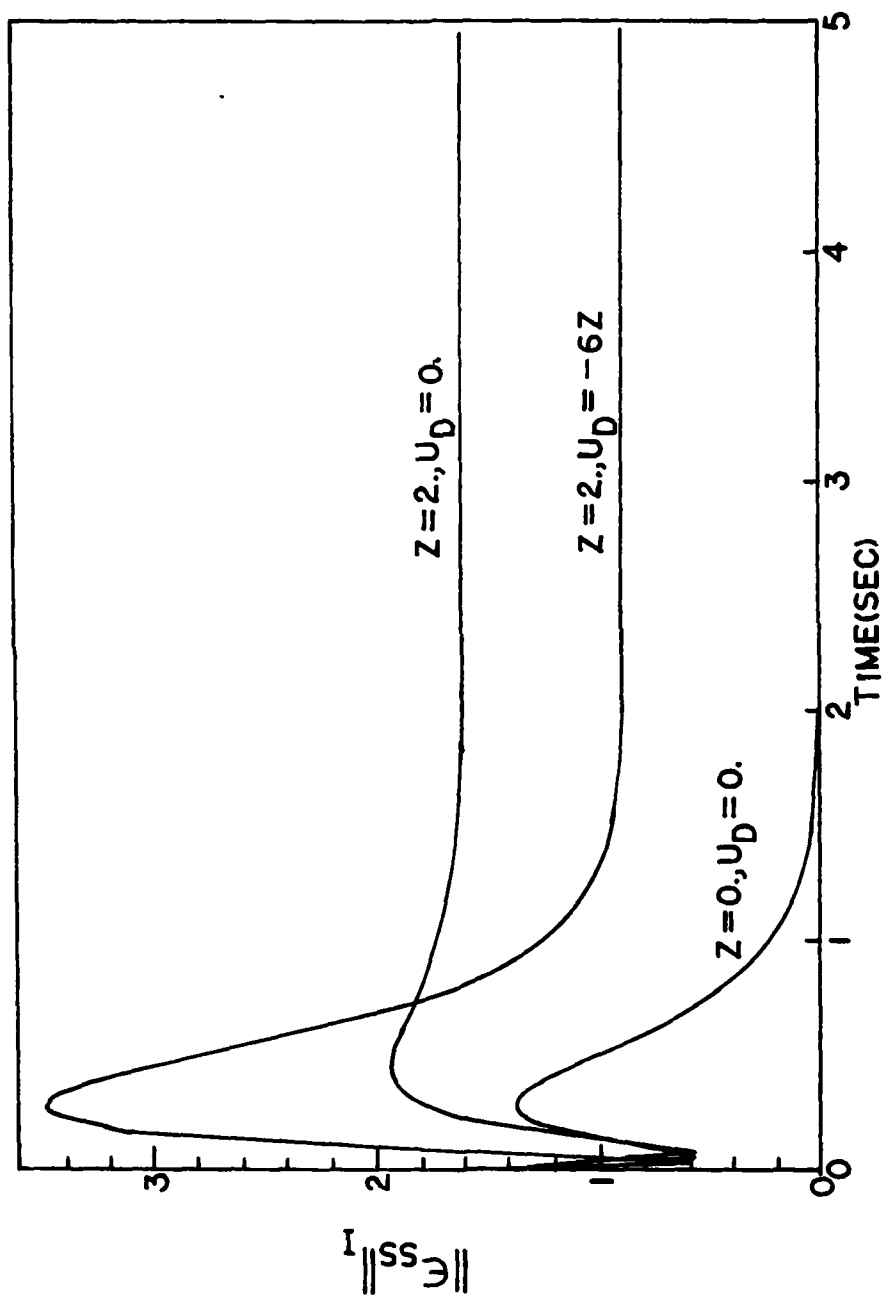


Figure 4-4. Norm of ϵ_{ss} versus time, $K = (-15., 2.)$.

4.4 Output Set-Point Stabilization; $y_{sp} = 0$, C Non-Invertible

For the output set-point stabilization problem considered in this section the output matrix C is assumed rectangular or square singular and is hence non-invertible. As a consequence of this the null-space of C is of dimension greater than zero. In order to steer the output $y(t)$ to zero and keep it there it will be necessary to steer $x(t)$ to $N(C)$ and keep it there. Disturbance effects may prevent this if total absorption is not possible.

The state and output spaces relevant to the problem in this section are illustrated in Figure 4-1 with the transformation represented by C instead of $\tilde{C}$. The state space R^n is decomposed into the direct sum of $N(C)$ and $N(C)^\perp$ (see Appendix A), represented as $N(C) \oplus N(C)^\perp$, and any vector x in R^n can be expressed as the sum of two components, v_1 in $N(C)^\perp$ and v_2 in $N(C)$. Thus

$$x = v_1 + v_2 \quad (4.23)$$

where v_2 is that component of x which will transform, via C , to zero in the output space and v_1 is that component of x which will result in $y \neq 0$ when transformed via C to the output space.

It is assumed that total absorption of disturbance effects is not possible, so the revised design objective is to minimize the "miss distance" between x and $N(C)$, i.e., minimize the norm of v_1 . If x could be controlled to $N(C)$, then $v_1 = 0$ and y would thus be made zero. Otherwise, the error between y and $y_{sp} = 0$ will be

$$\epsilon = y_{sp} - y = -y = -C(v_1 + v_2) = -Cv_1. \quad (4.24)$$

One requirement in obtaining a solution to the output set-point stabilization problem is that x be stabilized to a linear subspace of $N(C)$. A procedure for accomplishing this has been detailed in [16] for systems

modeled as $\dot{x} = Ax + Bu$ and is presented, with modifications, in the following.

Let the nullspace of C be a p -dimensional subspace spanned by a set of p n -dimensional basis vectors, $\{m_1(t), m_2(t), \dots, m_p(t)\}$. The problem is to find a feedback control $u_p = Kx$ such that all solutions x of the plant asymptotically approach $N(C)$ as t approaches infinity, or, if this is not possible, approach "as close as possible" to $N(C)$ as t approaches infinity.

Let $M = [m_1 | m_2 | \dots | m_p]$ be the $n \times p$ matrix composed of the basis vectors for $N(C)$. Let P be an $(n-p) \times n$ matrix, with $\rho(P) = n-p$, whose rows form a basis for $N(C)^\perp$ such that $PM = 0$ for all $t > t_0$. Consider the linear transformation

$$x = [P^\dagger | M]v \quad (4.25)$$

where $P^\dagger = P^T(PP^T)^{-1}$ is the $n \times (n-p)$ right inverse of P . Note that the transformation is non-singular since $[P^\dagger | M]$ is an $n \times n$ matrix of rank n . The inverse of the transformation shown in Equation (4.25) is given as

$$v = (v_1, v_2)^T = (P, M^\dagger)^T x \quad (4.26)$$

where $v_1 = n-p$ vector, $v_2 = p$ -vector and $M^\dagger = (M^T M)^{-1} M^T$ is the left inverse of M .

If x is contained in $N(C)$, then it can be expressed as

$$x = \sum_{i=1}^p c_i m_i = M(c_1, c_2, \dots, c_p)^T. \quad (4.27)$$

Since

$$v_1 = Px = PM(c_1, c_2, \dots, c_p)^T \quad (4.28)$$

and since $PM = 0$, then x contained in $N(C)$ implies that $v_1 = 0$. The problem now is to find a matrix K such that $v_1 = 0$ is an asymptotically stable solution of the resulting transformed equation

$$\dot{v}_1 = P((A + BK)P^\dagger)v_1 + P((A + BK)M)v_2. \quad (4.29)$$

Expression (4.29) is found by differentiating Equation (4.25), using $\dot{x} = Ax + Bu_p$ with $u_p = Kx$, and substituting Equation (4.25). Upon solving for $\dot{v}$, Equation (4.29) is one of the resulting terms. Stabilization of Equation (4.29) is possible if and only if a K can be found which will simultaneously give

$$P(A + BK)M = 0 \quad (4.30)$$

and cause

$$\dot{v}_1 = P(A + BK)P^\dagger v_1 \quad (4.31)$$

to be asymptotically stable to $v_1 = 0$. If such a K can be found, then Equation (4.29) reduces to

$$\dot{v}_1 = ((PAP^\dagger) + PBL)v_1 = (A_1 + B_1L)v_1 \quad (4.32)$$

where $L = WP^\dagger$, W is an arbitrary rxn real matrix, $A_1 = PAP^\dagger$ and $B_1 = PB$.

For the time-invariant case we are considering here, one has the following sharpened result.

Theorem 5[16]: Suppose (A_1, B_1) are constant. Let $\mathcal{C}$ denote the ρ -dimensional column range space of the composite matrix

$$[B_1 \mid A_1B_1 \mid A_1^2B_1 \mid \text{-----} \mid A_1^{(n-p-1)}B_1],$$

and let N be any real $(n-p) \times (n-p-\rho)$ matrix whose columns span $\mathcal{C}^\perp$. Then, there exists a real matrix L such that

$$\dot{v}_1 = (PAP^\dagger + PBL)v_1$$

is stabilized to $v_1 = 0$ if, and only if, all eigenvalues of $(N^TN)^{-1}N^TA_1N$ have negative real parts. Moreover, in that case, L may be chosen as a constant matrix. ■

Hence, $v_{1ss} = 0$ as $t \rightarrow \infty$ implies that $x \rightarrow N(C)$ which implies that $y \rightarrow 0$ as $t \rightarrow \infty$, which is the desired result. Note, however, that this development is for $w(t) = 0$.

If $w(t)$ is not zero, but the total absorption of external disturbances is possible (that is, there exists a u_d such that $Bu_d + Fw = 0$) then the development just presented is unchanged. However, suppose total absorption of $w(t)$ is not possible, where the plant state vector dynamics is expressed as

$$\dot{x} = Ax + Bu_p + Bu_d + FHx + FLx, \quad (4.33)$$

and the disturbance model is $w = [L|H](x, z)^T$. With $u_p = Kx$, Equation (4.33) can be re-written in the form

$$\dot{x} = (A + FL + BK)x + Bu_d + FHx. \quad (4.34)$$

Another expression for $\dot{x}$ can be obtained by differentiating Equation (4.25) with respect to time to obtain

$$\dot{x} = [P^+ | M] \dot{v}. \quad (4.35)$$

Upon equating Equations (4.34) and (4.35) one obtains

$$(A + FL + BK)x + FHx + Bu_d = [P^+ | M] \dot{v}. \quad (4.36)$$

If one now lets $\bar{A} = A + FL + BK$ and substitutes Equation (4.25) into Equation (4.36) one then obtains

$$\bar{A}[P^+ | M]v + FHx + Bu_d = [P^+ | M] \dot{v}$$

which can be rearranged to the form

$$[P^+ | M] \dot{v} = [\bar{A}P^+ | \bar{A}M]v + (FHx + Bu_d). \quad (4.37)$$

If one solves Equation (4.37) for $\dot{v}$, the resulting equation is

$$\dot{v} = \begin{bmatrix} P \\ - \\ M^+ \end{bmatrix} [\bar{A}P^+ | \bar{A}M]v + \begin{bmatrix} P \\ - \\ M^+ \end{bmatrix} (FHx + Bu_d),$$

or, in expanded form

$$\begin{pmatrix} \dot{v}_1 \\ \dot{v}_2 \end{pmatrix} = \begin{bmatrix} \bar{A}P^+ & \bar{A}M \\ M^+ \bar{A}P^+ & M^+ \bar{A}M \end{bmatrix} \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} + \begin{bmatrix} P(FHx + Bu_d) \\ M^+(FHx + Bu_d) \end{bmatrix} \quad (4.38)$$

According to Lemma 1 of [16]: a matrix K solves the subspace stabilization problem if and only if

(a) $\bar{P}\bar{A}M = 0$, $t \geq t_0$, and

(b) the reduced system $\dot{\tilde{v}}_1 = \bar{P}\bar{A}P^\dagger \tilde{v}_1$ is asymptotically stable to the equilibrium point $\tilde{v}_1 = 0$.

For the case of non-zero disturbances, this result must be modified as follows.

Lemma 1A: A matrix K solves the subspace stabilization problem, in the absence of external disturbances, or when total disturbance absorption is possible, if, and only if (a), (b), and if u_d is chosen such that $Bu_d + FHZ \equiv 0$. Moreover, if $w(t)$ is not zero and total disturbance absorption is not possible, the norm of the minimum distance between $x(t)$ and $N(C)$, given in [16] as $||v_1(t)||$, is minimized if K is chosen such that (a) and

(c) the homogeneous part of the reduced system

$$\dot{\tilde{v}}_1 = \bar{P}\bar{A}P^\dagger \tilde{v}_1 + P(FHZ + Bu_d)$$

is asymptotically stable to the equilibrium point $\tilde{v}_1 = 0$ and

(d) u_d is chosen to minimize the norm of v_1 in the solution of the reduced system. $\blacksquare$

An equivalent procedure when a steady-state solution exists is to choose K as in (b) or (c) and choose u_d to minimize $||v_{1ss}||$, v_{1ss} obtained from

$$\dot{v}_1 = \bar{P}\bar{A}P^\dagger v_1 + \bar{P}\bar{A}Mv_2 + P(FHZ + PBu_d).$$

Following through the development in [16] of the set $\mathcal{K}$ of all possible solutions K of (a), (b) and substituting into Equation (4.38) yields

$$\begin{pmatrix} \dot{v}_1 \\ \dot{v}_2 \end{pmatrix} = \begin{bmatrix} P(A+FL)P^\dagger + PB\hat{L} & 0 \\ M^\dagger(A+FL+BW)P^\dagger & A_2+B_2Z_1 \end{bmatrix} \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} + \begin{bmatrix} P(FHZ+Bu_d) \\ M^\dagger(FHZ+Bu) \end{bmatrix} \quad (4.39)$$

where

$$A_2 = M^\dagger (A + FL - B(PB)^\dagger P(A + FL)M) ,$$

$$B_2 = M^\dagger B(I - (PB)^\dagger PB) ,$$

$$L = WP^\dagger ,$$

W is an arbitrary rxn real parameter matrix, and

Z is an arbitrary rxp real parameter matrix.

Using Equation (4.39), one can modify Theorem 1 in [16] as follows.

Theorem 1 [16] (modified):

In the presence of external disturbances and incomplete disturbance absorption, the plant state vector can be stabilized to within a minimum distance of $N(C)$ if, and only if:

(a) $R(-P(A + FL)M)$ is contained within $R(PB)$,

(b) there exists a real matrix $\hat{L}(t)$ such that the system $\dot{\tilde{v}}_1 = (A_1 + B_1 \hat{L})\tilde{v}_1$ is asymptotically stable to the equilibrium point $\tilde{v}_1 = 0$, where $A_1 = P(A + FL)P^\dagger$, $B_1 = PB$, $(A_1 + B_1 \hat{L})$ is an $(n-p) \times (n-p)$ non-singular matrix, and

(c) the disturbance control component u_d is chosen to minimize the norm of $\tilde{v}_1$, where $\tilde{v}_1$ is the solution of the equation

$$\dot{\tilde{v}}_1 = (A_1 + B_1 \hat{L})\tilde{v}_1 + P(Bu_d + FHZ) . \quad (4.40)$$

If a unique steady-state solution existed for Equation (4.40) it would be given by

$$\tilde{v}_{1ss} = -(A_1 + B_1 \hat{L})^{-1} P(Bu_d + FHZ_\infty) . \quad (4.41)$$

If conditions (a), (b), (c) are met, then the norm of $\tilde{v}_{1ss}$ will be minimized and this will be as close as it is possible to steer the state vector to $N(C)$. The final set-point error from $y_{sp} = 0$ must then be found by transforming $\tilde{v}_1$ back into the state-space via Equation (4.25),

$$x = (P^\dagger | M)(\tilde{v}_{1ss}, 0)^T \quad (4.42)$$

(where it is assumed that $\tilde{v}_2$ has been controlled to zero as $t \rightarrow \infty$) and

then forming

$$y = Cx = CP^{\dagger} \tilde{v}_{1ss} . \quad (4.43)$$

The set-point error can then be found as

$$\epsilon_{sp} = -y = -CP^{\dagger} \tilde{v}_{1ss} . \quad (4.44)$$

An alternative approach to this problem is as follows. Since the columns of M are a basis for $N(C)$, the projection of x onto $N(C)$ along $N(C)^{\perp}$ is given by (see Appendix A)

$$MM^{\dagger}x = M(M^T M)^{-1} M^T x . \quad (4.45)$$

Let $x = v_1 + v_2$ where v_1 is contained in $N(C)^{\perp}$ and v_2 is contained in $N(C)$, as before. Using Equation (4.45) to decompose x into its components results in

$$v_2 = MM^{\dagger}x \quad (4.46)$$

$$v_1 = (I - MM^{\dagger})x . \quad (4.47)$$

If it is possible to asymptotically stabilize x to $N(C)$, then $v_1 \rightarrow 0$ as $x \rightarrow N(C)$. If such stabilization is not possible, then it is desired to control x "as close as possible" to $N(C)$, i.e., minimize $\|v_1\|$ as $t \rightarrow \infty$.

Thus, if one assumes that a unique steady-state solution exists for x , then for $y_{ss} = Cx_{ss}$ the steady-state error is

$$\epsilon_{ss} = y_{sp} - y_{ss} = -y_{ss} = -Cx_{ss} = -Cv_{ss} = -Cv_{1ss} . \quad (4.48)$$

and the object is to design u such that

$$\|\epsilon_{ss}\| = \|-Cv_{1ss}\|$$

is minimized.

To proceed, one has the fact that for plants modeled as in Equation (4.34) with solutions

$$x(t) = e^{\hat{A}t}x(0) + \int_0^t e^{\hat{A}(t-\tau)} (FHz(\tau) + Bu_d(\tau))d\tau , \quad (4.49)$$

it is well-known that if $((A+FL), B)$ is completely controllable there

exists a K such that $\lim_{t \rightarrow \infty} e^{\hat{A}t} x(0) \rightarrow 0$. The steady-state solution for x can

be found, using Equation (4.34), to be

$$x_{ss} = -\hat{A}^{-1}(FHz_{\infty} + Bu_d) . \quad (4.50)$$

The component of x_{ss} lying in the orthogonal complement of $N(C)$ is then given by

$$x_{ss_{N(C)^{\perp}}} = v_{lss} = -(I - MM^{\dagger})\hat{A}^{-1}(FHz_{\infty} + Bu_d) \quad (4.51)$$

and the steady-state set-point error ϵ_{ss} from $y_{sp}=0$ will thus be (from Equation (4.48))

$$\epsilon_{ss} = C(I - MM^{\dagger})\hat{A}^{-1}(FHz_{\infty} + Bu_d) . \quad (4.52)$$

To minimize $\|\epsilon_{ss}\|_I$, one can choose u_d to be (see Appendix A),

$$u_d^* = (C(MM^{\dagger} - I)\hat{A}^{-1}B)^{\dagger}(C(MM^{\dagger} - I)\hat{A}^{-1}FHz_{\infty}) . \quad (4.53)$$

The resulting steady-state error will then be found by substituting Equation (4.53) into Equation (4.52) to obtain

$$\epsilon_{ss}^* = (I + \hat{C}_1\hat{C}_1^{\dagger})\hat{C}_2 z_{\infty} \quad (4.54)$$

where

$$\begin{aligned} \hat{C}_1 &= C(MM^{\dagger} - I)\hat{A}^{-1}B \\ \hat{C}_2 &= C(MM^{\dagger} - I)\hat{A}^{-1}FH . \end{aligned} \quad (4.55)$$

To minimize $\|\epsilon_{ss}\|_Q$, $Q \neq I$, expressions (3.6) to (3.8) can be used to calculate the required u_d^* .

4.5 Output Set-Point Regulation; $y_{sp} \neq 0$, C Non-Singular

The output set-point regulator problem with $y = Cx$ and C invertible can be easily solved by first computing the associated (unique) x_{sp} from the equation

$$x_{sp} = C^{-1}y_{sp} \quad (4.56)$$

and then treating the problem as a state set-point regulator problem as solved in Sections 3.4, 3.5 and 3.6. Another method of solution, similar to that presented in Section 4.3, is as follows. Define $\epsilon(t)$ as

$$\epsilon(t) = y_{sp} - y(t) = y_{sp} - Cx(t) \quad (4.57)$$

If one differentiates Equations (4.57) and substitutes the plant dynamics given by Equation (4.1), with full disturbance model, the error dynamics can be expressed as

$$\dot{\epsilon}(t) = -C((A+FL)x(t) + Bu(t) + FH_z(t)). \quad (4.58)$$

Since $Cx(t) = y_{sp} - \epsilon(t)$ (from Equation (4.57)), it is possible to solve for $x(t)$ as

$$x(t) = x_{sp} - C^{-1}\epsilon(t) \quad (4.59)$$

and substitution of Equation (4.59) into Equation (4.58) then results in

$$\dot{\epsilon}(t) = -C((A+FL)C^{-1}y_{sp} - (A+FL)C^{-1}\epsilon(t) + Bu(t) + FH_z(t)). \quad (4.60)$$

If one next splits u into $u_p + u_d$, where $u_p = -K\epsilon$, Equation (4.60) can be rewritten as

$$\dot{\epsilon}(t) = \hat{C}\epsilon(t) - C(A+FL)C^{-1}y_{sp} - CFH_z(t) - CBu_d(t) \quad (4.61)$$

where $\hat{C} = C(A+FL)C^{-1} + CBK$. To minimize the disturbance effects, K is designed such that, for the homogeneous part of Equation (4.61),

$\lim_{t \rightarrow \infty} e^{\hat{C}t}\epsilon(0) \rightarrow 0$, and u_d is designed to minimize $\|\epsilon\|_Q$, where ϵ is the solution of Equation (4.61). If one assumes that a unique steady-state

solution ϵ_{ss} exists for Equation (4.61) it can be found to be

$$\epsilon_{ss} = \hat{C}^{-1}(C(A+FL)C^{-1}y_{sp} + CFH_{z\infty} + CBu_d). \quad (4.62)$$

The minimum norm control u_d^* which will result in a minimum norm for

$\|\epsilon_{ss}\|_I$ can then be found as

$$u_d^* = -(\hat{C}^{-1}CB)^\dagger(\hat{C}^{-1}C(A+FL)C^{-1}y_{sp} + \hat{C}^{-1}CFHz_\infty). \quad (4.63)$$

Upon substitution of Equation (4.63) into Equation (4.62) the minimum norm steady-state error is found to be

$$\epsilon_{ss}^* = (I - \hat{C}^{-1}CB(\hat{C}^{-1}CB)^\dagger)(\hat{C}^{-1}(A+FL)C^{-1}y_{sp} + \hat{C}^{-1}CFHz_\infty). \quad (4.64)$$

It can be seen from Equation (4.64) that it is possible for the disturbance to exhibit a positive utility; that is, to aid in the minimization of the steady-state error. To see this, denote the expression $(I - \hat{C}^{-1}CB(\hat{C}^{-1}CB)^\dagger)\hat{C}^{-1}C$ in Equation (4.64) by C_1 and re-express Equation (4.64) as

$$\epsilon_{ss}^* = C_1(A+FL)C^{-1}y_{sp} + C_1FHz_\infty. \quad (4.65)$$

From Equation (4.65) it is seen that

$$||C_1(A+FL)C^{-1}y_{sp} + C_1FHz_\infty||_I \quad (4.66)$$

can be greater than, less than, or equal to

$$||C_1(A+FL)C^{-1}y_{sp}||_I, \quad (4.67)$$

depending upon the magnitude and direction of the component of $C^{-1}CFHz_\infty \in R(C^{-1}CB)^\perp$ relative to the magnitude and direction of $C^{-1}(C(A+FL))C^{-1}y_{sp} \in R(C^{-1}CB)^\perp$ (see Section 3.8).

An Example:

In order to illustrate these results, we again use the example in Appendix B, with $C=I$ and $y_{sp}=x_{sp}$. The control u_d and steady-state error should agree with the results given in Section 3.4. If $L=0$ in the disturbance model then u_d^* , from Equation (4.63), will be

$$u_d^* = -(\hat{C}^{-1}CB)^\dagger(\hat{C}^{-1}CAC^{-1}y_{sp} + \hat{C}^{-1}CFHz_\infty). \quad (4.68)$$

Since

$$\hat{C} = C(A+FL)C^{-1} + CBK = A + BK = \tilde{A}$$

with $C = I$, Equation (4.68) then becomes

$$u_d^* = -(\tilde{A}^{-1}B)^{\dagger}(\tilde{A}^{-1}Ax_{sp} + \tilde{A}^{-1}FHz_{\infty}) , \quad (4.69)$$

which is identical to Equation (3.15). Moreover, the steady-state error becomes

$$e_{ss}^* = (I - \tilde{A}^{-1}B(\tilde{A}^{-1}B)^{\dagger})(\tilde{A}^{-1}Ax_{sp} + \tilde{A}^{-1}FHz_{\infty}) , \quad (4.70)$$

which is identical to Equation (3.16).

4.6 Output Set-Point Regulation; $y_{sp} \neq 0$, C Non-Invertible

4.6.1 Introduction. In this section, we again consider linear dynamical systems modeled as in Equations (4.1). In the development of this section, C is considered to be singular or rectangular, hence, C^{-1} does not exist. However, if C is of maximal rank m , a linear transformation can be found which maps x -space to ξ -space and specifies the output vector to be a state sub-vector in ξ -space. In effect this makes the output space a subspace of ξ -space. The critical state techniques of Section 3.9 can then be used to minimize the error components in the required subspace. If C is not of maximal rank, then one should refer to Section 2.3.3 for an alternate method of solution.

4.6.2 Transformation of the Output Vector to a State Sub-Vector.

The solution of output set-point regulation type problems using the method whereby the output vector is transformed to a state sub-vector in a new coordinate system has been presented in Sections 1.5.2.1 and 2.3.3. Expressions were given for the transformed state vector, Equation (1.21), for the dynamics of the error vector, Equation (1.28), and for the

dynamics of the transformed vector, Equation (1.25), and for the steady-state error, Equation (2.49). This section will illustrate how the linear transformation matrix for the transformation from x -space to ξ -space is obtained, will develop a control u_d^* which will minimize the steady-state error vector and will apply the critical state variable approach to obtain a solution for a minimum norm error.

Linear transformations were represented by the symbol K in previous sections. In this section, they will be denoted as T in order to avoid confusion with the gain matrix K used with u_p . As stated, the objective is to find a linear transformation T such that

$$x = T\xi \quad (4.71)$$

and

$$\xi = T^{-1}x = (\bar{T}_1^T | \bar{T}_2^T)^T x \quad (4.72)$$

where it is required that

$$\bar{T}_1 x = y = Cx \quad (4.73)$$

Since T is a non-singular matrix, its inverse T^{-1} exists and

$$T^{-1}T = TT^{-1} = I \quad (4.74)$$

One can define $T = (T_1 | T_2)$, $T^{-1} = (\bar{T}_1^T | \bar{T}_2^T)^T$ and re-write Equation (4.74) as

$$(T_1 | T_2)(\bar{T}_1^T | \bar{T}_2^T)^T = T_1\bar{T}_1^T + T_2\bar{T}_2^T = I \quad (4.75)$$

and

$$\begin{bmatrix} \bar{T}_1^T \\ \bar{T}_2^T \end{bmatrix} [T_1 | T_2] = \begin{bmatrix} \bar{T}_1^T T_1 & \bar{T}_1^T T_2 \\ \bar{T}_2^T T_1 & \bar{T}_2^T T_2 \end{bmatrix} = \begin{bmatrix} I & 0 \\ 0 & I \end{bmatrix} \quad (4.76)$$

Since it is required that $\bar{T}_1 = C$ (Equation (4.73)) then from Equation (4.76) one has that

$$\bar{T}_1 T_2 = CT_2 = 0 \quad (4.77)$$

and

$$\bar{T}_1 T_1 = C T_1 = I. \quad (4.78)$$

From Equation (4.78) it is seen that T_1 is a right inverse of C (see Appendix A), i.e.,

$$T_1 = C^T (C C^T)^{-1}. \quad (4.79)$$

Also, from Equation (4.76) one has that $\bar{T}_2 T_2 = I$, which implies that $\bar{T}_2$ is a left inverse of T_2 (see Appendix A), i.e.,

$$\bar{T}_2 = (T_2^T T_2)^{-1} T_2^T. \quad (4.80)$$

One can now express T and T^{-1} as

$$T = (T_1 | T_2) = (C^T (C C^T)^{-1} | T_2) \quad (4.81)$$

and

$$T^{-1} = \begin{bmatrix} \bar{T}_1 \\ \bar{T}_2 \end{bmatrix} = \begin{bmatrix} C \\ (T_2^T T_2)^{-1} T_2^T \end{bmatrix}. \quad (4.82)$$

From Equation (1.24), with K now denoted as T , one has

$$\dot{\xi} = (T^{-1} A T) \xi + T^{-1} B u + T^{-1} F w. \quad (4.83)$$

Upon expanding Equation (4.83), using Equations (4.81) and (4.82), the final result of performing the transformation from x -space to ξ -space will be

$$\begin{aligned} \begin{pmatrix} \dot{y} \\ \dot{\xi}_2 \end{pmatrix} &= \begin{bmatrix} C(A+FL)C^T(CC^T)^{-1} & C(A+FL)T_2 \\ (T_2^T T_2)^{-1} T_2^T (A+FL)C^T(CC^T)^{-1} & (T_2^T T_2)^{-1} T_2^T (A+FL)T_2 \end{bmatrix} \begin{pmatrix} y \\ \xi_2 \end{pmatrix} \\ &+ \begin{bmatrix} CB \\ (T_2^T T_2)^{-1} T_2^T B \end{bmatrix} u + \begin{bmatrix} CF \\ (T_2^T T_2)^{-1} T_2^T F \end{bmatrix} Hz \end{aligned} \quad (4.84)$$

$$= \begin{bmatrix} \tilde{A}_{11} & \tilde{A}_{12} \\ \tilde{A}_{21} & \tilde{A}_{22} \end{bmatrix} \begin{pmatrix} y \\ \xi_2 \end{pmatrix} + \begin{bmatrix} \tilde{B}_1 \\ \tilde{B}_2 \end{bmatrix} u + \begin{bmatrix} \tilde{F}_1 \\ \tilde{F}_2 \end{bmatrix} Hz. \quad (4.85)$$

Use of the linear transformation method requires that C be of full rank m , hence $R(C)$ is the entire output space R^m . In that case, for a given y_{sp} at least one corresponding x_{sp} can always be found in

the state space such that $y_{sp} = Cx_{sp}$. Using such an x_{sp} , a set-point ξ_{sp} can then be defined in ξ -space as (Equation (4.72))

$$\xi_{sp} = (y_{sp}, \xi_{2sp})^T = T^{-1}x_{sp} = \left[\frac{C}{(T_2^T T_2)^{-1} T_2^T} \right] x_{sp}. \quad (4.86)$$

The corresponding error vector in ξ -space can be defined as

$$\epsilon_{\xi}(t) = \xi_{sp} - \xi(t) \quad (4.87)$$

or, in expanded form (using the fact that y is a component of ξ)

$$\epsilon_{\xi} = \begin{pmatrix} \epsilon_y \\ \epsilon_{\xi} \end{pmatrix} = \begin{pmatrix} y_{sp} \\ \xi_{2sp} \end{pmatrix} - \begin{pmatrix} y \\ \xi_2 \end{pmatrix}. \quad (4.88)$$

The evolution equation for the error dynamics is found by first differentiating Equation (4.87) to obtain

$$\dot{\epsilon}_{\xi}(t) = -\dot{\xi}(t). \quad (4.89)$$

Expanding this expression, using Equation (4.85) then results in

$$\dot{\epsilon}_{\xi} = \tilde{A}\epsilon_{\xi} - \tilde{A}\xi_{sp} - \tilde{B}u_p - \tilde{B}u_d - \tilde{F}Hz. \quad (4.90)$$

Since T is a non-singular transformation, if the plant is completely state controllable in x -space, it will also be completely controllable in ξ -space. Hence, a gain matrix K can be designed such that for the homogeneous part of Equation (4.90), $\lim_{t \rightarrow \infty} \epsilon_{\xi}(t) \rightarrow 0$. The control part u_d must then be designed to minimize $\|\epsilon_{\xi}\|_Q$. If one assumes a unique steady-state solution exists for ϵ_{ξ} , with $\lim_{t \rightarrow \infty} w = z_{\infty}$, one can proceed to design u_d to minimize $\|\epsilon_{\xi ss}\|_Q$. The steady-state value of ϵ_{ξ} is found from Equation (4.90) to be

$$\epsilon_{\xi ss} = \hat{A}^{-1}(\tilde{A}\xi_{sp} + \tilde{F}Hz_{\infty} + \tilde{B}u_d). \quad (4.91)$$

The disturbance control of minimum norm which minimizes $\|\epsilon_{\xi ss}\|_I$ is given by

$$u_d^* = -(\hat{A}^{-1}\tilde{B})^{\dagger}\hat{A}^{-1}(\tilde{A}\xi_{sp} + \tilde{F}Hz_{\infty}). \quad (4.92)$$

However, due to the interplay between the various components of the

error vector, if u_d is chosen so as to arrive at an overall minimum norm of $\epsilon_{\xi ss}$, the component with which we are most interested, ϵ_y , will not necessarily be minimized. Therefore, the components of ϵ_{ξ} which compose ϵ_y should be considered as critical state variables and u_d then designed as in Section 3.9.3 to minimize $\|\epsilon_y\|_I$. For instance, upon expanding Equation (4.91) one has

$$\begin{pmatrix} \epsilon_{\xi 1ss} \\ \epsilon_{\xi 2ss} \end{pmatrix} = \begin{bmatrix} \hat{A}_{11}^{-1} & \hat{A}_{12}^{-1} \\ \hat{A}_{21}^{-1} & \hat{A}_{22}^{-1} \end{bmatrix} \left\{ \begin{bmatrix} \tilde{A}_1 \\ \tilde{A}_2 \end{bmatrix} \xi_{sp} + \begin{bmatrix} \tilde{F}_1 \\ \tilde{F}_2 \end{bmatrix} Hz_{\infty} + \begin{bmatrix} \tilde{B}_1 \\ \tilde{B}_2 \end{bmatrix} u_d \right\}. \quad (4.93)$$

If $\epsilon_{\xi 1ss}$ is selected as the critical state set, it can be represented from Equation (4.93) as

$$\epsilon_{\xi 1ss} = \hat{A}_{11}^{-1} \tilde{A}_1 \xi_{sp} + \hat{A}_{12}^{-1} \tilde{A}_2 \xi_{sp} + (\hat{A}_{11}^{-1} \tilde{F}_1 + \hat{A}_{12}^{-1} \tilde{F}_2) Hz_{\infty} + (\hat{A}_{11}^{-1} \tilde{B}_1 + \hat{A}_{12}^{-1} \tilde{B}_2) u_d. \quad (4.94)$$

The control u_d of minimum norm which would result in a minimum

$\|\epsilon_{\xi 1ss}\|_I$ is (see Appendix A)

$$u_d^* = -(\hat{A}_{11}^{-1} \tilde{B}_1 + \hat{A}_{12}^{-1} \tilde{B}_2)^{\dagger} ((\hat{A}_{11}^{-1} \tilde{A}_1 \xi_{sp} + \hat{A}_{12}^{-1} \tilde{A}_2 \xi_{sp}) + (\hat{A}_{11}^{-1} \tilde{F}_1 + \hat{A}_{12}^{-1} \tilde{F}_2) Hz_{\infty}). \quad (4.95)$$

In specific applications, the results obtained via use of Equation (4.95) should be compared to the norm of ϵ_{yss} resulting from use of Equation (4.92) to determine which choice gives the best practical results.

A further practical consideration which should not be lost sight of is that $\|\epsilon_{\xi 2ss}\|_Q$ must remain bounded for bounded inputs. The total steady-state error obtained via use of Equation (4.92) will be

$$\epsilon_{\xi ss}^* = (I - (\hat{A}^{-1} \tilde{B})(\hat{A}^{-1} \tilde{B})^{\dagger}) \hat{A}^{-1} (\tilde{A} \xi_{sp} + \tilde{F} Hz_{\infty}) \quad (4.96)$$

and from use of Equation (4.95) can be expressed as

$$\epsilon_{\xi ss}^* = \begin{bmatrix} \hat{A}_{11}^{-1} & \hat{A}_{12}^{-1} \\ \hat{A}_{21}^{-1} & \hat{A}_{22}^{-1} \end{bmatrix} \left\{ \begin{pmatrix} \tilde{A}_1 \\ \tilde{A}_2 \end{pmatrix} \xi_{sp} + \begin{pmatrix} \tilde{F}_1 \\ \tilde{F}_2 \end{pmatrix} Hz_{\infty} + \begin{pmatrix} \tilde{B}_1 \\ \tilde{B}_2 \end{pmatrix} (\hat{A}_{11}^{-1} \tilde{B}_1 + \hat{A}_{12}^{-1} \tilde{B}_2)^{\dagger} (-(\hat{A}_{11}^{-1} \tilde{A}_1 \xi_{sp} + \hat{A}_{12}^{-1} \tilde{A}_2 \xi_{sp}) - (\hat{A}_{11}^{-1} \tilde{F}_1 + \hat{A}_{12}^{-1} \tilde{F}_2) Hz_{\infty}) \right\}. \quad (4.97)$$

If the function chosen for u_d does not satisfy stability requirements upon the norm of the non-critical set of states, then one can employ

the alternate procedure used in Section A.6.2 whereby different relative weightings can be applied to the magnitudes of the error components, to arrive at a suitable tradeoff expression for a stabilizing function u_d .

Finally, the total control, $u = u_p + u_d$, which accomplishes the required objective must be transformed back into the plant state space in order to implement the control with respect to the actual plant.

4.6.3 Alternative Approach in the Plant State Space. An alternative approach to that presented in Section 4.6.2 will be presented in this section. This alternative approach is valid whether or not C is of rank m . If $\rho(C)$ is less than m , the column range space of C is not the entire output space R^m . Thus, a given set-point y_{sp} may or may not lie in $R(C)$.

In general, a set-point y_{sp} can be expressed as the sum of two component vectors,

$$y_{sp} = v_1 + v_2 \quad (4.98)$$

where v_1 lies in $R(C)$ and v_2 lies in $R(C)^\perp$. In order to minimize the error between y and y_{sp} , it is necessary to have v_1 equal to the orthogonal projection of y_{sp} onto $R(C)$ along $R(C)^\perp$, i.e., v_1 must be given by (see Appendix A),

$$v_1 = CC^+ y_{sp} . \quad (4.99)$$

The total set-point error will then be given by v_2 , i.e., by the orthogonal projection of y_{sp} onto $R(C)^\perp$ along $R(C)$. Using Equation (4.99) in Equation (4.98), the set-point y_{sp} can be expressed as

$$y_{sp} = v_1 + v_2 = CC^+ y_{sp} + (I - CC^+) y_{sp} \quad (4.100)$$

where

$$v_2 = (I - CC^+) y_{sp} = \epsilon . \quad (4.101)$$

The control objective is to have $y = v_1$, or be as "close" to v_1 as possible in the face of disturbances, in order to minimize v_2 .

Since v_1 represents an attainable set-point, i.e., a vector in the output space which lies in $R(C)$, knowledge of y_{sp} , and hence of v_1 , permits calculation of a (possibly non-unique) x_{sp} such that $v_1 = Cx_{sp}$. This x_{sp} can be expressed as the sum of two components in x -space

$$x_{sp} = v_3 + v_4, \quad (4.102)$$

where v_3 lies in $N(C)^\perp$ and v_4 lies in $N(C)$, with respect to the natural basis set for R^n . Under the transformation C between R^n and R^m , v_4 will transform to zero and v_3 will transform to v_1 . The control objective can now be re-stated as: steer $x(t)$ to x_{sp} such that $Cx_{sp} = v_1$.

The error vector in R^m can be expressed as a function of an error vector in R^n as

$$\epsilon_y = y_{sp} - y = Cx_{sp} - Cx = C\epsilon_x. \quad (4.103)$$

If one assumes that a unique steady-state solution exists, then from Equation (3.13), the steady-state solution for ϵ_x can be found as

$$\epsilon_{xss} = \tilde{A}^{-1}((A+FL)x_{sp} + FH\tilde{z}_\infty + Bu_d). \quad (4.104)$$

The control $u_d = u_d^*$ of minimum norm which will yield a minimum norm for ϵ_{xss} is

$$u_d^* = -(\tilde{A}^{-1}B)^\dagger \tilde{A}^{-1}((A+FL)x_{sp} + FH\tilde{z}_\infty). \quad (4.105)$$

The control function (4.105) results in a steady-state output error given by (see Equations (4.104), (4.103))

$$\epsilon_{yss} = C\epsilon_{xss} = C(I - (\tilde{A}^{-1}B)(\tilde{A}^{-1}B)^\dagger) \tilde{A}^{-1}((A+FL)x_{sp} + FH\tilde{z}_\infty). \quad (4.106)$$

As can be seen from Equation (4.106), when $x_{sp} \neq 0$ the possibility exists for the disturbance to contribute a positive utility in accomplishing

the control task of minimizing disturbance effects. Since x_{sp} may be non-unique, the magnitude of the error will also be dependent upon the value chosen for x_{sp} (which may be restricted due to control energy or plant limitations).

4.6.4 Some Example Results (Appendix E). Two third-order examples illustrating the application of the techniques in Section 4.6 to the design of disturbance minimizing controllers for output set-point regulation type problems (when C is non-invertible) are worked-out in detail in Appendix E. Some selected results from those examples are presented here. The plant is a third-order, linear time-invariant plant modeled as

$$\begin{pmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \end{pmatrix} = \begin{bmatrix} -1 & -2 & -2 \\ 0 & -1 & 1 \\ 1 & 0 & -1 \end{bmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} + \begin{pmatrix} 2 \\ 0 \\ 1 \end{pmatrix} u + \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} w \quad (\text{E.1a})$$

$$y = (1, 1, 0)x \quad (\text{E.1b})$$

and the piecewise constant external disturbance is modeled as

$$w = z ; \quad \dot{z} = \sigma(t) . \quad (\text{E.2})$$

The control objective is to steer $y(t)$ to a given $y_{sp} \neq 0$ in the face of uncertain disturbances $w(t)$. The first example utilizes the method whereby the plant output is elevated to the status of a state sub-vector in a new coordinate system. In the second example a solution is developed in the original plant state space.

Example 1:

For the first example, it can be seen from Equation (E.1b) that $\rho(C) = 1 = m$, therefore, it will be possible to use the technique of Section 4.6.2. A linear transformation matrix T such that $x = T\xi$ is found, using Equations (4.71) to (4.82) to be

$$T = [T_1 \mid T_2] = \left[\begin{array}{c|cc} 0.5 & 1 & 0 \\ 0.5 & -1 & 0 \\ \hline 0 & 0 & 1 \end{array} \right] \quad (E.9)$$

If the decision is made to minimize the norm of the total steady-state error vector $\epsilon_{\xi ss}$ given by Equation (4.91), the resulting minimum norm $\|\epsilon_{\xi ss}\|_I$, given by Equation (4.96), is found as

$$\epsilon_{\xi ss}^* = \begin{bmatrix} 0.5528 & -0.2203 & -0.4426 \\ -0.2206 & 0.8881 & -0.2234 \\ -0.4434 & -0.2222 & 0.5547 \end{bmatrix} \begin{pmatrix} \xi_{sp1} \\ \xi_{sp2} \\ \xi_{sp3} \end{pmatrix} + \begin{pmatrix} -0.4863 \\ 0.3541 \\ 0.3102 \end{pmatrix} z_{\infty} \quad (E.19)$$

For $y_{sp} = 5$, one x_{sp} which would give $y_{sp} = 5$ is found from $y_{sp} = Cx_{sp}$ to be $x_{sp} = (2.5, 2.5, 0.)^T$. If this x_{sp} is transformed to ξ -space and substituted into Equation (E.19), the result is

$$\epsilon_{\xi ss}^* = \begin{pmatrix} 2.764 \\ -1.103 \\ -2.217 \end{pmatrix} + \begin{pmatrix} -0.4863 \\ 0.3541 \\ 0.3102 \end{pmatrix} z_{\infty} \quad (E.23)$$

and it can be seen that, even if $z=0$, the resulting steady-state error is large.

Since $\epsilon_{\xi lss}$ corresponds to the steady-state error in the output space, if it is chosen as a critical state variable and if u_d is designed to minimize $\|\epsilon_{\xi lss}\|_I$, then $\epsilon_{\xi lss}^*$ is found to be (assuming $z=0$)

$$\epsilon_{\xi lss}^* = 1.4167. \quad (E.26)$$

The error on the component of interest has thus been reduced from 2.764 to 1.4167 in ξ -space, with $z=0$. If the total error vector obtained via use of the critical state variable approach, i.e.,

$$\epsilon_{\xi ss} = (1.4167, -1.773, -3.56)^T, \quad (E.27)$$

is transformed back into x -space and the corresponding x_{ss} is found and transformed to the output space to give y_{ss} , then the resulting steady-state error in y -space is

$$\epsilon_{yss} = 0.53. \quad (E.31)$$

Example 2:

In the second example, a solution is sought in the original state space. Since C is of maximal rank and, hence, $y_{sp}=5$ lies in $R(C)$, the control objective is to steer $x(t)$ to some x_{sp} such that $y = Cx_{sp} = x_1 + x_2 = y_{sp}$. It is seen that any set-point located on a plane perpendicular to the plane defined by the x_1 and x_2 axes and intersecting it along a line defined by $x_1+x_2=5$ will suffice as x_{sp} . We shall choose $x_{sp} = (2.5, 2.5, 0.)^T$ to correspond to the choice of state set-point in the first example. If a u_d^* is chosen to yield a minimum norm for $\|\epsilon_{xss}\|_I$, where ϵ_{xss} is defined by Equation (4.104), the resulting steady-state output error is found from Equation (4.106) to be

$$\epsilon_{yss} = 2.778 + 0.6664z_{\infty} . \quad (E.36)$$

For comparison with the first example, if the steady-state error vector in ξ -space given by Equation (E.23) is mapped back into the output space the result is $\epsilon_{yss} = 2.764 + 0.708z_{\infty}$, where ϵ_{yss} refers to the same distance in both cases.

CHAPTER V
DISTURBANCE MINIMIZATION FOR LINEAR, TIME-INVARIANT
SERVO-TRACKING PROBLEMS

5.1 Summary of Chapter V

This chapter will deal with techniques available for the minimization of disturbance effects as applied to linear, time-invariant servo-tracking problems. The system and error equations are developed and "trackability" is defined. The norm minimization, maximum partial absorption and critical-state variable minimization techniques are applied to the servo-tracking problem and examples are provided to illustrate the application of several of the techniques.

5.2 System Equations

In this chapter, consideration will be given to the problems of minimizing disturbance effects on servo-tracking problems in which the control task is to have $y(t)$ track an input servo-command $y_c(t)$ with high fidelity. In other words, the objective is to design a control $u(t)$ such that the norm of the instantaneous value of the error vector, defined by

$$\epsilon(t) = y_c(t) - y(t) \quad , \quad (5.1)$$

is minimized in the face of all external disturbances, initial conditions and input servo-commands.

The plants to be considered in this Chapter are linear, time-invariant dynamical systems of the form

$$\begin{aligned}\dot{x}(t) &= Ax(t) + Bu(t) + Fw(t) \\ y(t) &= Cx(t)\end{aligned}\tag{5.2}$$

with external disturbances modeled as

$$\begin{aligned}w(t) &= Hz(t) + Lx(t) \\ \dot{z}(t) &= Dz(t) + Mx(t) + \sigma(t) .\end{aligned}\tag{5.3}$$

As mentioned in Section 2.3.2, the input servo-commands are assumed to be modeled by a linear, time-invariant dynamic process similar to the disturbance model and are represented as

$$\begin{aligned}y_c(t) &= Rc(t) \\ \dot{c}(t) &= Sc(t) + \mu(t)\end{aligned}\tag{5.4}$$

where $c(t)$ is the servo-command state vector and $\mu(t)$ serves the same function in the servo-command model as does $\sigma(t)$ in the external disturbance model.

5.3 Trackability and the Linear Algebra of the Output Servo-Tracking Problem

As stated in Section 4.2, the plant output matrix C in Equations (5.2) represents a linear mapping between the state space x and the output space y . Moreover, C can be a square nonsingular matrix, i.e., $\rho(C)=n$, or a square singular or rectangular matrix, in which case $\rho(C)<n$. If $\rho(C) = n$, then C represents a one-to-one, onto linear transformation and the column range space of C is the entire output space y . In this case, any servo-command input modeled as in Equations (5.4) will be completely contained within $R(C)$, i.e.,

$$R(R) \subseteq R(C)\tag{5.5}$$

and it will be possible to express the matrix R as some linear combination of the columns of C as

$$R \equiv C\Theta \quad (5.6)$$

for some (possibly nonunique) matrix Θ . Condition (5.5) or (5.6) constitutes the requirement which must be satisfied in order for theoretically exact servo-tracking to be possible, i.e., the "exact trackability" condition. Satisfaction of the exact trackability condition implies that for some $x(t)$ it will be possible to have

$$\varepsilon(t) = y_c(t) - y(t) = C(\Theta c(t) - x(t)) \equiv 0. \quad (5.7)$$

If it is theoretically possible to satisfy Equation (5.7), the question which must next be answered is: Does there exist an admissible control u such that the required $x(t)$ can be achieved and maintained? For the case where C is an invertible matrix, the required $x(t)$ is uniquely found as

$$x(t) = C^{-1}y_c(t) = \Theta c(t). \quad (5.8)$$

If $\rho(C) = m < n$, where m is the dimension of the output space, then C represents an onto linear transformation from R^n to R^m and it will be possible to attain any point in R^m with some appropriate $x(t)$ in R^n , that is, the column range space of C still encompasses the entire output space. In this case it will still be possible to satisfy the exact trackability condition for some $x(t)$ in R^n ; however, since $N(C)$ is now of dimension greater than zero, the $x(t)$ required to satisfy Equation (5.7) may not be unique.

Finally, if $\rho(C) < m$ the linear transformation represented by C is not an onto mapping and the column range space of C will be a subspace of the output space rather than the entire output space R^m . In this case the possibility exists that the input servo-command will not be completely contained within the column range space of C and thus, that the exact trackability condition will not be satisfied. Under these

conditions, one must attempt to steer $x(t)$ such that the distance between $(\theta_c(t) - x(t))$ and $N(C)$ is minimized.

The above considerations assume that the disturbances are zero. If the disturbances are not zero, then it might not be possible to satisfy Equation (5.7) even if the exact trackability condition is satisfied, unless all disturbances are completely absorbable. Since it is assumed in this dissertation that the disturbances are not completely absorbable, a disturbance minimizing control must then be designed which will minimize those disturbance effects which make $\varepsilon(t)$ nonzero. If condition (5.6) is not satisfied, or if it is satisfied but it is not possible to achieve the $x(t)$ necessary to satisfy Equation (5.7) with an admissible control $u(t)$, then the control task is to minimize the disturbance effects and simultaneously steer $x(t)$ so as to minimize the distance between $(\theta_c(t) - x(t))$ and $N(C)$ [10].

5.4 The Error Equations

The control task for the servo-command problem is to attain satisfaction of Equation (5.7), if it is possible to do so and otherwise to minimize, in some sense, the instantaneous value of the error vector defined by Equation (5.1). There are several methods of approach to solving such a problem. For instance, if C is a square singular or rectangular matrix of maximal rank m , a linear transformation could be found which would transform $y(t)$ to a sub-state vector in a new coordinate system (see Sections 1.5.2.1, 1.5.2.2, 2.3.4, 4.6.2). An expression for the error dynamics could then be developed and one of the minimization techniques presented in Chapter 2 could be applied; the particular technique to be used would

depend on the problem structure and requirements.

If the method whereby $y(t)$ is transformed to a state sub-vector is to be used, the error dynamics can be developed as follows. One begins with Equation (1.24), re-written as

$$\dot{\xi}(t) = T^{-1}(A+FL)T\xi(t) + T^{-1}Bu_p(t) + T^{-1}Bu_d(t) + T^{-1}FHz(t), \quad (5.9)$$

and designs u_p as $u_p(t) = -K\xi(t)$ in order to stabilize the linear, homogeneous part of (5.9), i.e.,

$$\dot{\xi}(t) = (T^{-1}(A+FL)T - T^{-1}BK)\xi(t) = \tilde{A}\xi(t). \quad (5.10)$$

Then Equation (5.9) becomes

$$\dot{\xi}(t) = \tilde{A}\xi(t) + \tilde{B}u_d(t) + \tilde{F}z(t). \quad (5.11)$$

Since $\dot{\xi}_1(t) = \dot{y}(t)$ and $\dot{\epsilon}(t) = \dot{y}_c(t) - \dot{y}(t)$, the equation for $\dot{\epsilon}(t)$ is found to be

$$\begin{aligned} \dot{\epsilon}(t) &= RSc(t) - \tilde{A}_{11}Rc(t) + \tilde{A}_{11}\epsilon(t) - \tilde{A}_{12}\xi_2(t) - \tilde{B}_1u_d(t) - \tilde{F}_1z(t) \\ &= \tilde{A}_{11}\epsilon(t) + \tilde{R}c(t) - \tilde{A}_{12}\xi_2(t) - \tilde{B}_1u_d(t) - \tilde{F}_1z(t). \end{aligned} \quad (5.12)$$

Equation (5.11) can now be re-expressed in the form

$$\begin{pmatrix} \dot{\epsilon} \\ \dot{\xi}_2 \end{pmatrix} = \begin{bmatrix} \tilde{A}_{11} & -\tilde{A}_{12} \\ -\tilde{A}_{21} & \tilde{A}_{22} \end{bmatrix} \begin{pmatrix} \epsilon \\ \xi_2 \end{pmatrix} + \begin{bmatrix} -\tilde{B}_1 \\ \tilde{B}_2 \end{bmatrix} u_d(t) + \begin{bmatrix} -\tilde{F}_1 \\ \tilde{F}_2 \end{bmatrix} z(t) + \begin{bmatrix} \tilde{R} \\ \tilde{A}_{21} \end{bmatrix} c(t) \quad (5.13)$$

where the term $\tilde{F}_1z(t)$ represents the external disturbance, the term $(R\tilde{S} - \tilde{A}_{11}R)c(t) = \tilde{R}c(t)$ represents the servo-command disturbance, and in the equation for $\dot{\epsilon}(t)$, the term $-\tilde{A}_{12}\xi_2(t)$ constitutes a "coupling" disturbance.

Another approach to the problem is presented in [10]. This latter approach utilizes Equation (5.7) and a new variable e_{ss} defined as

$$e_{ss}(t) = \Theta c(t) - x(t). \quad (5.14)$$

In this way Equation (5.7) is re-written as

$$\dot{\epsilon}(t) = Ce_{ss}(t). \quad (5.15)$$

If Equation (5.14) is differentiated and the appropriate terms from Equations (5.2) and (5.4) are included, the following linear differential equation describing the dynamics of e_{ss} is obtained,

$$\dot{e}_{ss} = \Theta S c(t) + \Theta u(t) - A x(t) - B u(t) - F w(t) \quad (5.16)$$

Using the fact that $x(t) = \Theta c(t) - e_{ss}(t)$ (Equation (5.14)) along with the disturbance model given by Equation (5.3) one can re-express Equation (5.16) in the form

$$\begin{aligned} \dot{e}_{ss}(t) &= (A+FL)e_{ss}(t) - ((A+FL)\Theta - \Theta S)c(t) - FHz(t) - Bu(t) \\ &= \hat{A}e_{ss}(t) - \hat{\Theta}c(t) - FHz(t) - Bu_d(t) \end{aligned} \quad (5.17)$$

where $u = u_d - Ke_{ss}$ and $\hat{A} = A + FL + BK$.

In minimizing the error $\epsilon(t)$ given by Equation (5.15) there are two situations which must be considered in order to determine how to approach the minimization problem. The first situation arises when C is either a contractive mapping or an isometry. If C is contractive then $||\epsilon(t)||_Q < ||e_{ss}(t)||_Q$. If C is an isometry then $||\epsilon(t)||_Q = ||e_{ss}(t)||_Q$. In either of these two cases, designing a control such that $||e_{ss}(t)||_Q$ is minimized will result in a norm for $\epsilon(t)$ which is at least as small as that for $e_{ss}(t)$. However, if C is neither contractive nor isometric, then minimization of the norm of $e_{ss}(t)$ may not result in the desired result for $\epsilon(t)$ since $||\epsilon(t)||_Q > ||e_{ss}(t)||_Q$. For this latter circumstance, a better approach is to design $u(t)$ to minimize $||Ce_{ss}||_Q$.

Given the error equations (5.11), (5.15) and (5.17), if the designer is interested in minimizing the tracking error for a few states which are of major interest, then the critical-state variable technique (Section 2.2.4) can be used to design the control vector. If complete absorption is possible on some subset of the disturbance

states then the maximum partial absorption technique (Section 2.2.3) can be used. The norm minimization technique (Section 2.2.5) can also be applied. The following sections of this chapter will cover the application of each of these techniques to the solution of the servo-tracking disturbance minimization problem.

5.5 Norm Minimization Techniques for Servo-Tracking Problems

This section will examine the application of the norm minimization techniques from Section 2.2.5 to servo-tracking type problems. One way in which norm minimization can be used is as follows. If one desires to minimize the disturbance effects on the error dynamics $\dot{e}_{ss}$, with e_{ss} as given by Equation (5.17), u_d can be chosen to minimize

$$\|((A+FL)\theta - \theta S)c(t) + FH_z(t) + Bu_d(t)\|_Q. \quad (5.18)$$

Alternatively, one can allocate u_d between the external and servo-command disturbances and u_{dw} , u_{dc} can be designed to minimize

$$\|FH_z(t) + Bu_{dw}(t)\|_Q \quad (5.19)$$

$$\|(A+FL)\theta - \theta S)c(t) + Bu_{dc}(t)\|_Q, \quad (5.20)$$

respectively. The minimum norm control vectors which minimize the norms in Equations (5.18) through (5.20), for $Q=I$, are

$$u_d^*(t) = -B^+(((A+FL)\theta - \theta S)c(t) + FH_z(t)) \quad (5.21)$$

$$u_{dw}^*(t) = -B^+FH_z(t) \quad (5.22)$$

$$u_{dc}^*(t) = -B^+((A+FL)\theta - \theta S)c(t), \quad (5.23)$$

respectively. If the weighting matrix Q is chosen to be other than the identity matrix, then the criteria in Section 3.3 can be used to calculate the necessary control vectors. The error $e_{ss}(t)$ is found by substituting Equation (5.21), or Equations (5.22) and (5.23), into Equation (5.17) and solving the resulting expression to obtain

$$e_{ss}^*(t) = \phi_e(t, t_0) e_{ss}(t_0) - \int_{t_0}^t \phi_e(t, \tau) (I - BB^\dagger) (((A + FL)\theta - \theta S)c(\tau) + FHz(\tau)) d\tau \quad (5.24)$$

Now, assume that the design method outlined in Sections 1.5.2.1, 1.5.2.2 is to be used and that the output vector given by $y = Cx$ is to be transformed to a new coordinate system ξ in which it will be a state sub-vector. The error dynamics $\dot{e}(t)$ are given by Equation (5.12) (see Section 5.4). The transformed state vector dynamics $\dot{\xi}(t)$, with $\dot{\xi}_1(t)$ re-expressed in terms of $e(t)$, is given in matrix form by Equation (5.13) where this equation will now be expressed in the simplified notation

$$\dot{\bar{\xi}} = \bar{A}\bar{\xi} + \bar{B}u_d + \bar{F}z + \bar{R}c \quad (5.25)$$

Applying the norm minimization technique in order to minimize the disturbance effects on $\bar{\xi}(t)$, one can solve for the $u_d^*(t)$ of minimum norm which will give a minimum norm for

$$\|\bar{F}z + \bar{R}c + \bar{B}u_d\|_Q \quad (5.26)$$

The resulting minimum norm control vector, for $Q \equiv I$, is found to be

$$u_d^* = -\bar{B}^\dagger(\bar{F}z + \bar{R}c) \quad (5.27)$$

If this u_d^* is substituted into Equation (5.25) the result is

$$\dot{\bar{\xi}} = \bar{A}\bar{\xi} + (I - \bar{B}\bar{B}^\dagger)(\bar{R}c + \bar{F}z) \quad (5.28)$$

From Equation (5.28), $\bar{\xi}(t)$ can then be found as

$$\bar{\xi}(t) = \bar{\Phi}(t, t_0)\bar{\xi}(t_0) + \int_{t_0}^t \bar{\Phi}(t, \tau)(I - \bar{B}\bar{B}^\dagger)(\bar{R}c(\tau) + \bar{F}z(\tau)) d\tau, \quad (5.29)$$

with $\bar{\Phi}(t, t_0)$ representing the state transition matrix associated with $\bar{A}$.

If the designer wishes to assign different levels of importance to the external and the servo-command disturbances, $u_d(t)$ can be

allocated as $u_d = u_{dw} + u_{dc}$ and u_{dw} , u_{dc} can be designed to minimize the following norms, respectively,

$$\|\bar{F}z(t) + \bar{B}u_{dw}(t)\|_{Q_1} \quad (5.30)$$

$$\|\bar{R}c(t) + \bar{B}u_{dc}(t)\|_{Q_2} \quad (5.31)$$

The relative amount of control energy to be expended on each disturbance can also be assigned by specifying that the control vectors be designed by finding the control vectors of minimum norm $\|u_{dw}(t)\|_{P_1}$ and $\|u_{dc}(t)\|_{P_2}$ which minimize Equations (5.30) and (5.31), respectively (see Section 3.3).

If one designs u_d as in Equation (5.27), or by using Equations (5.30) and (5.31), the result will be a minimization of the external and servo-command disturbance residual norms on $\dot{e}(t)$ and $\bar{\xi}_2(t)$. The effect of the coupling disturbance $\tilde{A}_{12}\bar{\xi}_2(t)$ on $\dot{e}(t)$ will be determined by the performance of $u_p(t)$ since u_d does not directly access this disturbance term.

Another approach to the problem is to design $u_d(t)$ by considering only the equation for $\dot{e}(t)$, i.e., Equation (5.12). In this case, u_d is designed to minimize

$$\|\tilde{R}c(t) - \tilde{A}_{12}\bar{\xi}_2(t) - \bar{F}_1z(t) - \bar{B}_1u_d(t)\|_Q \quad (5.32)$$

or it can be allocated as $u_d(t) = u_{dw}(t) + u_{dc}(t) + u_{dcp}(t)$ and each part designed to minimize

$$\|\bar{F}_1z(t) + \bar{B}_1u_{dw}(t)\|_{Q_1} \quad (5.33)$$

$$\|\bar{R}c(t) - \bar{B}_1u_{dc}(t)\|_{Q_2} \quad (5.34)$$

$$\|\tilde{A}_{12}\bar{\xi}_2(t) + \bar{B}_1u_{dcp}(t)\|_{Q_3} \quad (5.35)$$

respectively. Again, different weightings can be assigned to each control part when solving for the minimum norm control vector which will minimize each of the norms in Equations (5.33) through (5.35). In expression (5.32), for instance, if Q is taken to be $Q \equiv I$ then $u_d^*(t)$ is

$$u_d^*(t) = -\bar{B}_1^+ (\bar{F}_1 z(t) + \tilde{A}_{12} \xi_2(t) - \tilde{R}c(t)) \quad (5.36)$$

and $\dot{\epsilon}(t)$ will be given by

$$\dot{\epsilon}(t) = \tilde{A}_{11} \epsilon(t) + (I - \bar{B}_1 \bar{B}_1^+) (\tilde{R}c(t) - \tilde{A}_{12} \xi_2(t) - \tilde{F}_1 z(t)) \quad (5.37)$$

5.6 Maximum Partial Absorption Technique

This section will develop application of the maximum partial absorption technique of Section 2.2.3 to the servo-tracking problem. Recall that the maximum partial absorption technique gives a disturbance control vector which results in the complete absorption of all those disturbance components for which the complete absorption criterion (see Equation (1.35) or (1.36)) is satisfied.

In Equation (5.17) it is seen that the disturbance terms are the external disturbance $FHz(t)$ and the servo-command disturbance $((A+FL)\Theta - \Theta S)c(t) = \hat{\Theta}c(t)$. In order to apply the maximum partial absorption technique one proceeds as follows. For the external disturbance, F is column partitioned as in Equation (2.3) and each column f_i is checked for satisfaction of the absorbability condition $f_i \in R(B)$. Those columns which satisfy the absorbability condition are then grouped in a subset as in expression (2.8) with the remaining columns grouped as in expression (2.9). A control vector $u_{dw}(t)$ can then be designed as

$$u_{dw}(t) = - \left(\sum_{j=1}^q \zeta_j h_j \right) z(t) + \bar{u}_{dw}(t) \quad (5.38)$$

If those columns of F which did not satisfy the complete absorbability

criterion are denoted as $\bar{F}$, then the unabsorbable external disturbance terms can be expressed as

$$\bar{F}Hz(t) \quad (5.39)$$

and $\bar{u}_{dw}(t)$ can be designed, for instance, to minimize

$$\|B\bar{u}_{dw}(t) + \bar{F}Hz(t)\|_Q. \quad (5.40)$$

If $Q \equiv I$, the $\bar{u}_{dw}^*(t)$ which will give a minimum norm for expression (5.40) is

$$\bar{u}_{dw}^*(t) = -B^+ \bar{F}Hz(t). \quad (5.41)$$

For the servo-command disturbance term, $\hat{\theta}$ is column partitioned as

$$\hat{\theta} = [\hat{\theta}_1 | \hat{\theta}_2 | \dots | \hat{\theta}_v] \quad (5.42)$$

and each column is checked for satisfaction of $\hat{\theta}_i \in R(B)$. Those columns which satisfy this condition are grouped into a subset and re-ordered as

$$\{\hat{\theta}_1, \hat{\theta}_2, \dots, \hat{\theta}_r\}, \quad (5.43)$$

$r \leq v$, and those which do not are grouped as

$$\{\hat{\theta}_{r+1}, \dots, \hat{\theta}_v\} = \bar{\theta}. \quad (5.44)$$

For the columns $\hat{\theta}_1, \dots, \hat{\theta}_r$ the implication is that there exists a γ_i such that

$$\hat{\theta}_i = B\gamma_i, \quad i=1,2,\dots,r \quad (5.45)$$

and a control $u_{dc}(t)$ can thus be designed as

$$u_{dc}(t) = - \left(\sum_{k=1}^r \gamma_k \right) c(t) + \bar{u}_{dc}(t) \quad (5.46)$$

with $\bar{u}_{dc}(t)$ designed to minimize, for instance,

$$\|B\bar{u}_{dc}(t) + \bar{\theta}c(t)\|_Q. \quad (5.47)$$

For $Q \equiv I$, the control $u_{dc}^*(t)$ which minimizes expression (5.47) is

$$u_{dc}^*(t) = -B^+ \bar{\theta}c(t). \quad (5.48)$$

The total disturbance control vector u_d associated with Equation (5.17) will thus be

$$u_d(t) = -\left(\sum_{j=1}^q \zeta_j h_j\right)z(t) - \left(\sum_{k=1}^r \gamma_k\right)c(t) + \bar{u}_{dw}(t) + \bar{u}_{dc}(t) . \quad (5.49)$$

Substituting Equation (5.49) into (5.17), with $\bar{u}_{dw}$ from Equation (5.41) and $\bar{u}_{dc}$ from Equation (5.48), will give the resultant error dynamics as

$$\dot{e}_{ss}(t) = \hat{A}e_{ss}(t) - (I - BB^+) (\bar{F}Hz(t) + \bar{\Theta}c(t)) . \quad (5.50)$$

If one uses the approach which results in Equation (5.12), the maximum partial absorption technique can still be applied with $u_d(t)$ allocated as $u_d(t) = u_{dw}(t) + u_{dc}(t) + u_{dcp}(t)$. The parts $u_{dw}(t)$ and $u_{dc}(t)$ are associated with the external and servo-command disturbance terms, respectively. The part $u_{dcp}(t)$ is associated with the "coupling" disturbance term. The part $u_{dw}(t)$ is calculated as in Equations (5.38) and (5.41) after the columns of $\tilde{F}_1$ are checked for satisfaction of the absorbability condition. Likewise, if one lets $(RS - \tilde{A}_{11}R)c(t) = \tilde{R}c(t)$ and checks the columns of $\tilde{R}$ for complete absorbability, $u_{dc}(t)$ can be designed as in Equations (5.45) to (5.48) with $\hat{\Theta}$, $\bar{\Theta}$ replaced by $\tilde{R}$, $\tilde{\bar{R}}$. The coupling term $\tilde{A}_{12}\xi_2(t)$ can be designed in the same fashion as

$$u_{dcp}(t) = -\left(\sum_{i=1}^s \alpha_i\right)\xi_2(t) + \bar{u}_{dcp}(t) \quad (5.51)$$

where $\tilde{a}_{12_i} = B\alpha_i$, $i=1,2,\dots,s$ and $\tilde{a}_{12_i}$ is the i -th column of the subset of columns of $\tilde{A}_{12}$ which satisfy the absorbability condition. The part $\bar{u}_{dcp}(t)$ can be designed to minimize, for instance

$$\|B\bar{u}_{dcp}(t) + \tilde{\bar{A}}_{12}\xi_2(t)\|_Q . \quad (5.52)$$

The total disturbance control vector associated with Equation (5.8) is thus given by

$$u_d(t) = -\left(\sum_{j=1}^q \zeta_j\right)z(t) - \left(\sum_{k=1}^r \gamma_k\right)c(t) - \left(\sum_{i=1}^s \alpha_i\right)\xi_2(t) + \bar{u}_{dw}(t) + \bar{u}_{dc}(t) + \bar{u}_{dcp}(t) \quad (5.53)$$

and if $\bar{u}_{dw}$, $\bar{u}_{dc}$ and $\bar{u}_{dcp}$ were designed to minimize the norms of the respective unabsorbable disturbances, with $Q \equiv I$, substitution of Equation (5.53) into (5.12) would yield

$$\dot{\epsilon}(t) = \bar{A}_{11}\epsilon(t) - (I - \bar{B}_1\bar{B}_1^\dagger)(\bar{F}_1z(t) - \bar{R}c(t) + \bar{A}_{12}\xi_2(t)) \quad (5.54)$$

5.7 The Critical State-Variable Technique

In this section the critical state-variable technique, from Section 2.2.4.1, is used as a means of minimizing disturbance effects. Recall that critical state-variables are those variables which the designer considers to be of major concern in the design of the controller.

To begin the development, expand Equation (5.25) to obtain

$$\begin{pmatrix} \bar{\xi}_1 \\ \bar{\xi}_2 \\ \vdots \\ \bar{\xi}_n \end{pmatrix} = \begin{bmatrix} \bar{a}_1 \\ \bar{a}_2 \\ \vdots \\ \bar{a}_n \end{bmatrix} \bar{\zeta} + \begin{bmatrix} \bar{f}_1 \\ \bar{f}_2 \\ \vdots \\ \bar{f}_n \end{bmatrix} z + \begin{bmatrix} \bar{r}_1 \\ \bar{r}_2 \\ \vdots \\ \bar{r}_n \end{bmatrix} c + \begin{bmatrix} \bar{b}_1 \\ \bar{b}_2 \\ \vdots \\ \bar{b}_n \end{bmatrix} u_d \quad (5.55)$$

where $\bar{a}_i$, $\bar{f}_i$, $\bar{r}_i$ and $\bar{b}_i$ denote the i -th rows of $\bar{A}$, $\bar{F}$, $\bar{R}$ and $\bar{B}$, respectively. The direct disturbance effects on any given $\bar{\xi}_i$ are seen from Equation (5.55) to be

$$\bar{\delta}_i = \bar{f}_i z + \bar{r}_i c \quad (5.56)$$

To completely absorb $\bar{\delta}_i$ there must exist a u_d such that

$$\bar{b}_i u_d = -\bar{\delta}_i \quad (5.57)$$

Provided that $\bar{b}_i$ is not a zero row of $\bar{B}$, Equation (5.57) can always be

satisfied for any one critical variable $\bar{\xi}_i$.

If the critical variable set consists of more than one variable, then one can proceed as follows. First, re-order Equation (5.55) so that the critical states are grouped as

$$\{ \bar{\xi}_1, \bar{\xi}_2, \dots, \bar{\xi}_j \}, \quad (5.58)$$

$j \leq n$, so that the disturbances acting on the critical-state variables are

$$\bar{\delta}_i = \bar{f}_i z + \bar{r}_i c, \quad i=1, \dots, j. \quad (5.59)$$

For complete absorption of the direct disturbance effects on the set of states in expression (5.58), the set (5.59) must satisfy the complete absorption criterion, i.e.,

$$\text{rank}[\hat{B}|\hat{F}|\hat{R}] = \text{rank}[\hat{B}] \quad (5.60)$$

where $\hat{B} = [\bar{b}_1 | \bar{b}_2 | \dots | \bar{b}_j]^T$, $\hat{F} = [\bar{f}_1 | \bar{f}_2 | \dots | \bar{f}_j]^T$ and $\hat{R} = [\bar{r}_1 | \bar{r}_2 | \dots | \bar{r}_j]^T$. If condition (5.60) is satisfied, there exists a (possibly non-unique) matrix Γ such that

$$[\hat{F}|\hat{R}] = \hat{B}\Gamma \quad (5.61)$$

and u_d can be designed as

$$u_d = -\Gamma(z|c)^T. \quad (5.62)$$

If one expresses the re-ordered dynamics as

$$\begin{pmatrix} \dot{\xi}^{(i)} \\ \dot{\xi}^{(k)} \end{pmatrix} = \begin{bmatrix} A^{(1)} & A^{(2)} \\ A^{(3)} & A^{(4)} \end{bmatrix} \begin{pmatrix} \xi^{(i)} \\ \xi^{(k)} \end{pmatrix} + \begin{bmatrix} F^{(i)} \\ F^{(k)} \end{bmatrix} z + \begin{bmatrix} R^{(i)} \\ R^{(k)} \end{bmatrix} c + \begin{bmatrix} B^{(i)} \\ B^{(k)} \end{bmatrix} u_d \quad (5.63)$$

where $\xi^{(i)}$ represents the critical-state set given by expression (5.58)

and $\xi^{(k)}$ represents the non-critical-state set, and u_d from Equation

(5.62) is substituted into Equation (5.63), the result will be as

follows.

$$\dot{\xi}^{(i)} = A^{(1)} \xi^{(i)} + A^{(2)} \xi^{(k)} \quad (5.64)$$

$$\dot{\xi}^{(k)} = A^{(3)} \xi^{(i)} + A^{(4)} \xi^{(k)} + [F^{(k)} - B^{(k)} \Gamma | R^{(k)} - B^{(k)} \Gamma] (z|c)^T. \quad (5.65)$$

If condition (5.60) is not satisfied, the designer can design u_d by applying the norm minimization technique of Section 5.5 to the equation for the critical-state variables, i.e.,

$$\xi^{(i)} = A^{(1)}\xi^{(i)} + A^{(2)}\xi^{(k)} + F^{(i)}z + R^{(i)}c + B^{(i)}u_d. \quad (5.66)$$

In this case, u_d is designed to minimize, for instance,

$$\|F^{(i)}z + R^{(i)}c + B^{(i)}u_d\|_Q \quad (5.67)$$

where, if $Q \equiv I$, u_d^* is given by

$$u_d^* = -B^{(i)\dagger}(F^{(i)}z + R^{(i)}c). \quad (5.68)$$

5.8 Isobasis Control Design Technique

In the previous three sections of this chapter, no a priori assumptions were made about the form of the disturbance minimizing control vectors prior to their calculation. However, given the form of the error dynamics equations in each case it was reasonable to expect that the control vectors would be composed of some combination involving the disturbances. This turns out to have been the case; see for instance Equations (5.21) to (5.23), (5.27), (5.49), (5.53), and (5.68). Suppose, therefore, that the assumption is made a priori that the control vectors will be some function of the disturbances. For instance, if $w(t)$ is piecewise constant, $w=c_0$, then one might assume that $u_{dw} = \bar{b}w$. If $w(t)$ were a ramp, $w(t) = \alpha t$, then one might assume that $u_{dw} = \bar{b}\alpha t = \bar{b}w$, etc. This method of approach will be referred to as the isobasis disturbance minimizing control design technique.

If one assumes a parametric form for the control vectors, one can solve for $e_{ss}(t)$ from Equation (5.17). Then, using the

preferred minimization technique (norm minimization, critical-state variable or maximum partial absorption) one can determine the optimal parameter values for each control vector, i.e., determine an expression for the "best" value of $\bar{b}$. This procedure will permit the parameters of the minimizing controller to be designed to minimize the disturbance effects on the error vector instead of on the error vector rate.

An Example

An example illustrating application of the techniques of this and the previous three sections to an output servo-tracking problem is presented in detail in Appendix F. A review of the results obtained in Appendix F will be presented here.

The example uses a second-order plant modeled as

$$\begin{pmatrix} \dot{x}_1 \\ \dot{x}_2 \end{pmatrix} = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} + \begin{pmatrix} 1 \\ 2 \end{pmatrix} u + \begin{pmatrix} 1 \\ 1 \end{pmatrix} w \quad (\text{F.1})$$

$$y = (1, 0)x. \quad (\text{F.2})$$

The external disturbance is given as

$$w(t) = \alpha t, \quad (\text{F.3})$$

with α an arbitrary, unknown constant, and with state model

$$w = (1, 0)z \quad ; \quad \dot{z} = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} z + \sigma. \quad (\text{F.4})$$

The output servo-command is also assumed to be a ramp,

$$y_c(t) = \beta t, \quad (\text{F.5})$$

with β an arbitrary constant, and with state model

$$y_c = (1, 0)c \quad (\text{F.6})$$

$$\dot{c} = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} c + \mu. \quad (\text{F.7})$$

The control task is to design a control $u(t)$ such that the error given by $e(t) = y_c(t) - y(t)$ is minimized, in some sense, in the face of all

initial conditions, external disturbances and input servo-commands.

For this problem, the exact trackability criterion is satisfied and Θ in Equation (5.6) is found to be

$$\Theta = 1. \quad (\text{F.8})$$

The problem is solved using three different methods. The first solution applies the norm minimization technique from Section 5.5 to Equation (5.17) to minimize the norms given by Equations (5.19) and (5.20) with the weighting matrix in expression (5.20) taken to be $Q \equiv I$ and the weighting matrix in expression (5.19) given as

$$Q = \begin{bmatrix} q_{11} & 0 \\ 0 & q_{22} \end{bmatrix}. \quad (\text{F.15})$$

The control parts u_{dc}^* and u_{dw}^* were calculated to be

$$u_{dc}^* = -0.2y_c - 0.4\dot{y}_c \quad (\text{F.14})$$

$$u_{dw}^* = - \left(\frac{q_{11} + 2q_{22}}{q_{11} + 4q_{22}} \right) w. \quad (\text{F.20})$$

Using the results of Equations (F.14) and (F.20), along with $u_p(t) = (-20., 3.5)e_{ss}(t)$, $e_{ss}(0) = 0$, $w = 2t$, $y_c = 2t$ and $q_{11} = q_{22} = 10$, permits $\varepsilon(t) = Ce_{ss}(t) = e_{ss1}(t)$ to be found as shown in Figure 5-1. As a comparison against the results obtained, values are also shown on Figure 5-1 for a case with $u_d = 0$. After examining Figure 5-1, recall that u_d was chosen to minimize the norms of the various disturbance residuals in the expression for $\dot{e}_{ss}(t)$ and not to specifically minimize the norm of $e_{ss1}(t)$. Examination of Figure 5-2 indicates that, for the above two cases, u_d did what it was designed to do, at the expense of an increase in $e_{ss1}(t)$.

The second solution examined three different variations on the critical-state variable technique (or the maximum partial absorption

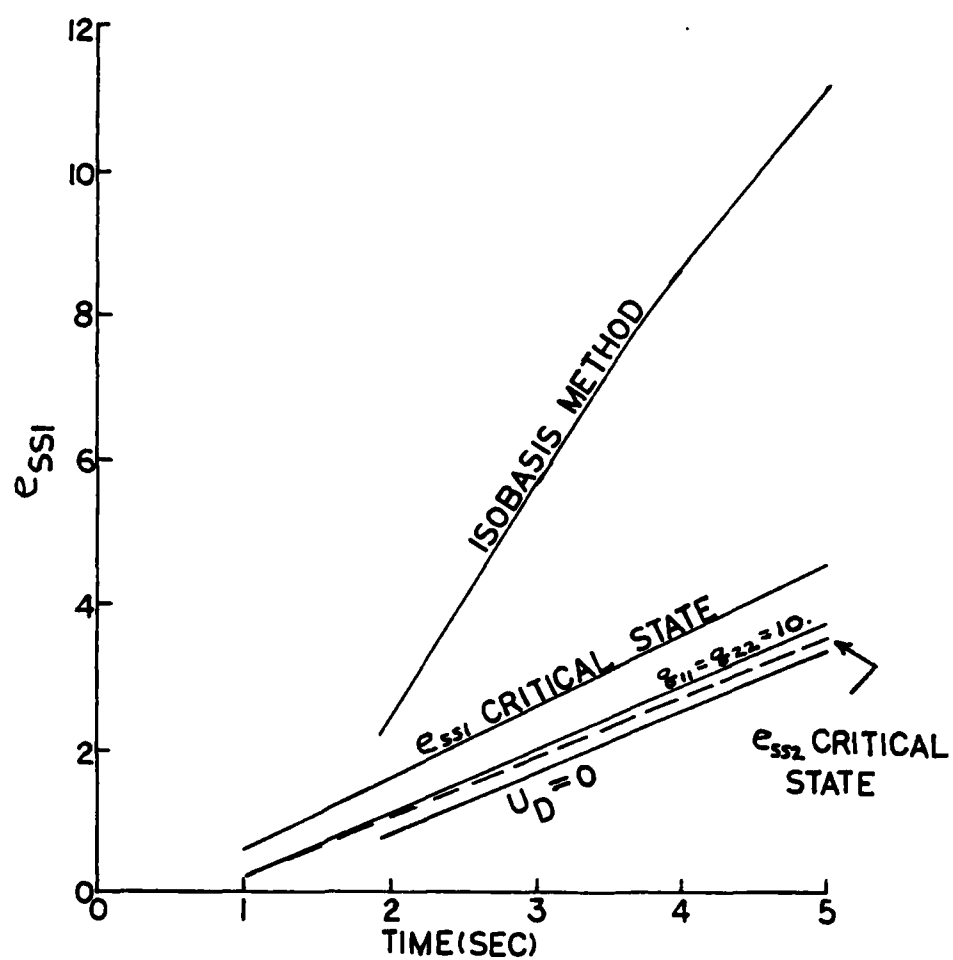


Figure 5-1. e_{ss1} versus time.

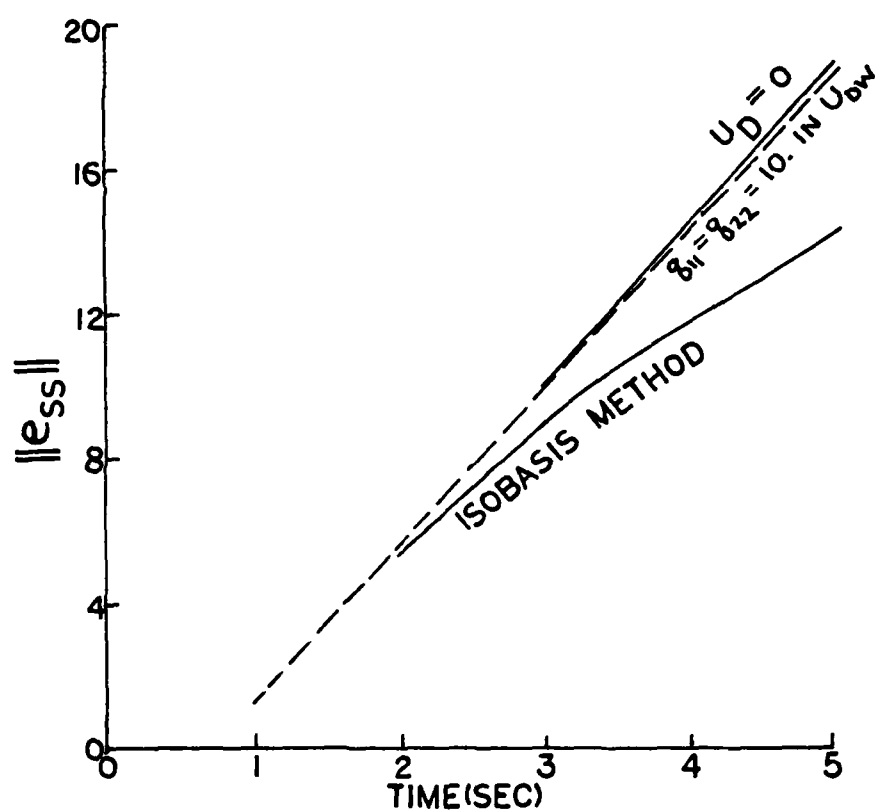


Figure 5-2. Norm of e_{ss} versus time.

technique, which is also applicable to this problem) by assuming first that e_{ss1} was the critical-state, then that e_{ss2} was the critical-state and thirdly that e_{ss1} was the critical-state with the component of e_{ss2} which is included in the dynamics of $\dot{e}_{ss1}$ considered as a "coupling disturbance".

The control parts which result from the above three variations are as follows:

$$(a) \quad u_d(t) = -w(t) - y_c(t) \quad (F.34)$$

$$(b) \quad u_d(t) = -\frac{1}{2}(w(t) + \dot{y}_c(t)) \quad (F.39)$$

$$(c) \quad u_d(t) = -w(t) - y_c(t) + 4.5e_{ss2}(t) \quad (F.43)$$

The values obtained for $e_{ss1}(t)$ using Equations (F.34) and (F.39) are also shown on Figure 5-1. Use of Equation (F.43) results in $e_{ss1}(t) = 0$ after transients settle out.

The first two solution approaches minimize the disturbance effects on $\dot{e}_{ss}(t)$ (not on $e_{ss}(t)$ directly) and, except for the results obtained with Equation (F.43), do not result in improvements in $\epsilon(t)$ over the case with $u_d = 0$. The third approach utilizes the isobasis design technique. The assumption is made that $u_{dc}(t)$ has the form $u_{dc}(t) = \bar{b}\beta t$ and that $u_{dw}(t)$ has the form $u_{dw}(t) = \bar{a}\alpha t$. These assumed forms for the control parts are next substituted into Equation (5.17) and the resulting equation is solved for $e_{ss}(t)$. For convenience, the exponential terms are dropped, i.e., the solution is considered for the time frame after the transients have settled out. Expressions can then be found for the $\bar{b}^*$ and $\bar{a}^*$ which minimize the norms of the error residuals associated with the servo-command and external disturbances, respectively, in the expression for $e_{ss}(t)$. The resulting expression for $\bar{b}^*$ is

$$\bar{b}^* = \frac{-0.1301\dot{y}_c^2 + 0.2254\dot{y}_c y_c - 0.0918y_c^2}{0.0104\dot{y}_c^2 - 0.0126\dot{y}_c y_c + 0.0064y_c^2} \quad (F.55)$$

and that for $\bar{a}^*$ is given as

$$\bar{a}^* = \frac{-0.0348\dot{w}^2 + 0.0979\dot{w}w - 0.0491w^2}{0.0104\dot{w}^2 - 0.0126\dot{w}w + 0.0064w^2} \quad (F.57)$$

The results obtained for $\epsilon(t)$ and $||e_{ss}(t)||$ by using these expressions for $\bar{b}^*$ and $\bar{a}^*$ to determine u_{dc} and u_{dw} are shown on Figures 5-1 and 5-2, respectively.

One can also use the maximum partial absorption technique to design $\bar{b}$ and $\bar{a}$ such that $e_{ss1}(t) = 0$. The results in this case are found to be

$$\bar{b} = \frac{-0.3087\dot{y}_c + 0.2857y_c}{0.0218\dot{y}_c + 0.0357y_c} \quad (F.58)$$

$$\bar{a} = \frac{-0.0849\dot{w} + 0.125w}{0.0218\dot{w} + 0.0357w} \quad (F.59)$$

5.9 State Servo-Tracking Problems

The techniques so far presented in this chapter have all dealt with output servo-tracking type problems. However, if one has a state servo-tracking problem which requires disturbance minimization, it can be considered as a special case of the class of output servo-tracking problems where $C \equiv I_{n \times n}$. The state servo-command model can still be as given by Equations (5.4), with $y_c(t)$ replaced by $x_c(t)$ and the error vector $\epsilon(t)$ given by Equation (5.1) then becomes

$$\epsilon(t) = x_c(t) - x(t) \quad (5.69)$$

Since the input state servo-command vector is contained in the state

space, there will exist some $x(t)$ such that

$$\epsilon(t) = x_c(t) - x(t) \equiv 0, \quad (5.70)$$

so theoretically exact servo-tracking is possible for any $x_c(t)$ (assuming all disturbances are zero). Since $C=I$ is an invertible matrix of rank n , the $x(t)$ required to satisfy Equation (5.70) is uniquely found from Equation (5.8) to be $x(t) = \Theta c(t) = Rc(t)$ where, from Equation (5.6), $R \equiv \Theta$. However, the question still arises: Can an admissible control $u(t)$ be designed such that the required $x(t)$ is achieved?

For state servo-tracking problems, the linear transformation matrix used to transform $y(t)$ to a sub-state vector in ξ -space can be considered to be $T \equiv I$, therefore, Equation (5.9) reduces to the usual equation for the state dynamics, i.e.,

$$\dot{\xi}(t) = \dot{x}(t) = (A+FL)x(t) + Bu_p(t) + Bu_d(t) + FH_z(t). \quad (5.71)$$

Differentiating Equation (5.69) and substituting the resulting expression from Equations (5.4) (re-expressed in terms of x_c) and (5.71) gives

$$\begin{aligned} \dot{\epsilon}(t) &= (A+FL+BK)\epsilon(t) - ((A+FL)R-RS)c(t) - FH_z(t) - Bu_d(t) \\ &= \bar{A}\epsilon(t) - \bar{R}c(t) - \bar{F}z - Bu_d \end{aligned} \quad (5.72)$$

which is similar to Equation (5.25). The disturbance minimization techniques developed in Sections 5.5, 5.6 and 5.7 for application to Equation (5.25) can also be applied to Equation (5.72) in order to minimize the disturbance effects on the tracking error in state servo-tracking problems. Since $C \equiv I$, Equation (5.15) simplifies to $\epsilon(t) = e_{ss}(t)$ and Equation (5.17) is seen to be identical to Equation (5.72), with $\bar{R} = \hat{\Theta}$. The isobasis design technique can also be utilized. Appendix G contains an example illustrating the application of disturbance minimizing techniques to a state servo-tracking problem. Some of the results are presented in the following.

An Example

An example illustrating the application of the techniques of this chapter to a state servo-command problem is presented in detail in Appendix G. A review of some of the results will be presented here. The example uses a second-order plant and an external disturbance modeled as shown in Section 5.8, Equations (F.1) to (F.4). The input state servo-command vector is assumed to be composed of a ramp for each state and is modeled as

$$x_c(t) = \beta t = (\beta_1, \beta_2)^T t, \quad (G.4)$$

β_1 and β_2 arbitrary constants, with state model

$$\begin{aligned} \begin{pmatrix} x_{c1} \\ x_{c2} \end{pmatrix} &= \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix} c \\ \dot{c} &= \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix} c + \begin{pmatrix} 0 \\ \mu_1 \\ 0 \\ \mu_2 \end{pmatrix}. \end{aligned} \quad (G.5)$$

The control task is to minimize the error

$$\epsilon(t) = x_c(t) - x(t) \quad (G.6)$$

in the face of all initial conditions, input state servo-commands and external disturbances.

Three methods of solution were used to work the problem. In the first method, the norm minimization technique of Section 5.5 was used to obtain control parts u_{dc}^* and u_{dw}^* which would minimize the norms of the disturbance residuals given by Equations (5.20) (with Θ replaced by R) and (5.19), respectively. The resulting control parts are given as

$$u_{dc}^* = 0.2(\dot{x}_{c1} - x_{c1}) - 0.2(2\dot{x}_{c2} - 3x_{c2}) \quad (G.9)$$

$$u_{dw}^* = -0.6w. \quad (G.8)$$

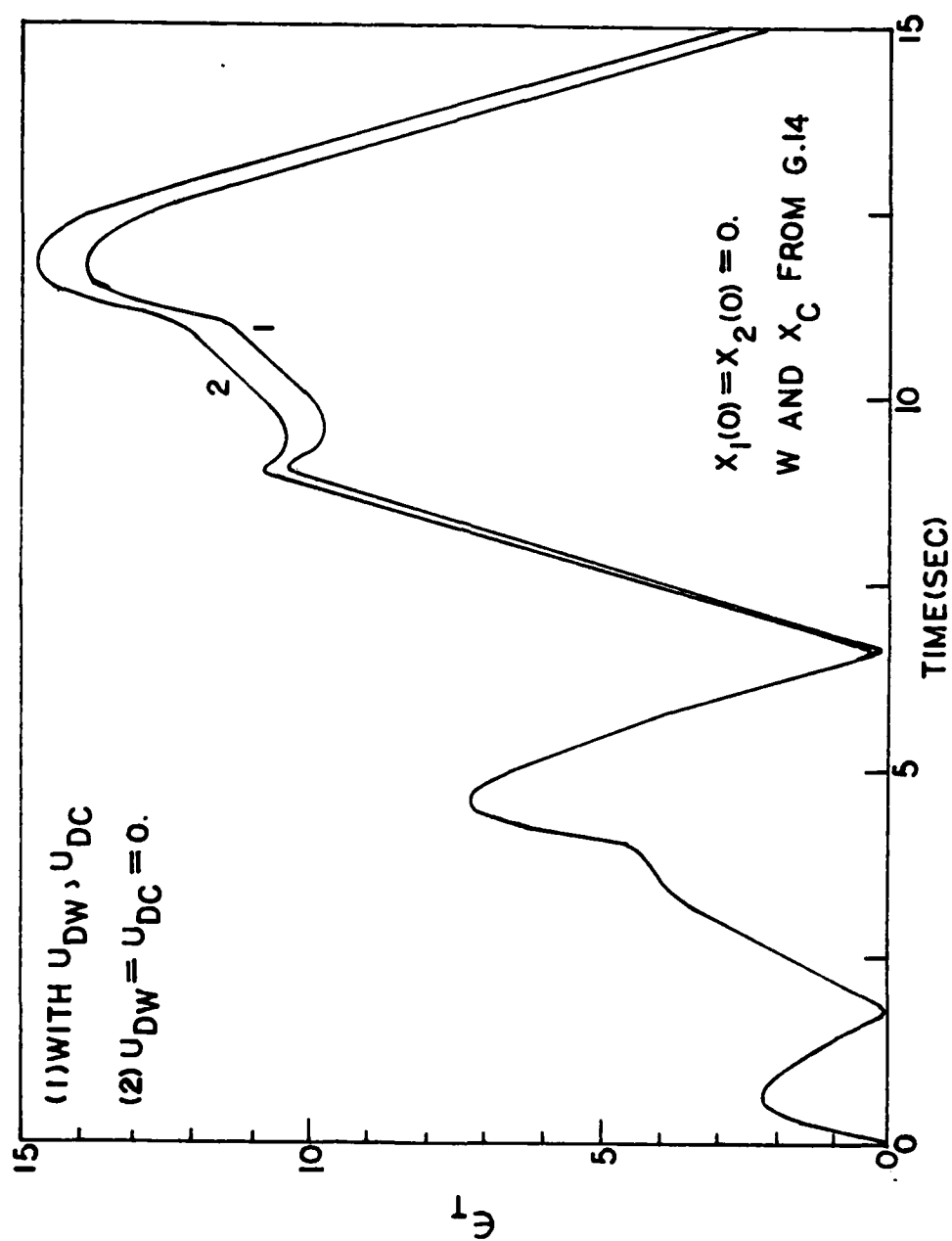
Using these control parts, the error dynamics are expressed as

$$\dot{\epsilon}(t) = \begin{bmatrix} -19. & 4.5 \\ -40. & 8. \end{bmatrix} \epsilon(t) - \begin{pmatrix} 0.4 \\ -0.2 \end{pmatrix} w(t) - \begin{bmatrix} 0.8 & -0.8 & 0.4 & 0.4 \\ -0.4 & 0.4 & -0.2 & -0.2 \end{bmatrix} (x_{c1}, \dot{x}_{c1}, x_{c2}, \dot{x}_{c2})^T \quad (G.10)$$

The problem was programmed on a digital computer and cases were run for various combinations of values of α , β_1 and β_2 . Sample results are shown in Figure 5-3. The coefficients used for w and x_c are as shown in Equations (G.14) and Figure 5-3 has two cases plotted, one with u_{dc}^* , u_{dw}^* as shown in Equations (G.9) and (G.8) and one with $u_{dc} = u_{dw} = 0$. As can be seen, the disturbance minimizing control vector did decrease the overall error vector.

For the second method, the isobasis design technique, whereby the form of u_{dw} and u_{dc} are assumed a priori to be the same as that of w and x_c , respectively, is utilized to obtain a solution. The control part $u_{dw}(t)$ is assumed to have the form $u_{dw}(t) = \bar{a} \dot{w} t$ and the control part $u_{dc}(t)$ is assumed to have the form $u_{dc}(t) = \bar{b} \dot{x}_c t$. These assumed forms are substituted into Equation (5.72) and the resulting equation is solved for $\epsilon(t)$. Expressions can then be found for the $\bar{a}^*$ and $\bar{b}^*$ which will result in a minimum norm for the external and servo-command residual, respectively, in the expression for $\epsilon(t)$. The resulting expressions for these two parameters are found to be

$$\bar{a}^* = \frac{-0.0491\dot{w}^2 + 0.0974\dot{w}\ddot{w} - 0.0345\ddot{w}^2}{0.006374\dot{w}^2 - 0.01256\dot{w}\ddot{w} + 0.01028\ddot{w}^2} \quad (G.15)$$

Figure 5-3. ϵ_T versus time.

$$\bar{b}^* = \left(\frac{-0.0918y_c^2 + 0.2743\dot{y}_c y_c - 0.198\dot{y}_c}{0.006374y_c^2 - 0.01256\dot{y}_c y_c + 0.01028\dot{y}_c^2} , \right. \\ \left. \frac{-0.0581y_c^2 + 0.0548\dot{y}_c y_c + 0.03617\dot{y}_c^2}{0.006374y_c^2 - 0.01256\dot{y}_c y_c + 0.01028\dot{y}_c^2} \right) . \quad (G.17)$$

This solution was also programmed on a digital computer and several cases were run. Figure 5-4 shows the results which were obtained when Equations (G.15) and (G.17) were used to calculate the control parts u_{dw}^* and u_{dc}^* , for run conditions corresponding to the data in Figure 5-3. As can be seen by comparing Figures 5-3 and 5-4, the procedure of Section G.4 gives better results.

The third solution utilized the critical state-variable technique for disturbance minimization. The error dynamics $\dot{\epsilon}(t)$ can be expressed in the form (Equation (5.72)),

$$\dot{\epsilon}(t) = \begin{bmatrix} -19. & 4.5 \\ -40. & 8. \end{bmatrix} \epsilon(t) - \begin{pmatrix} 1 \\ 2 \end{pmatrix} u_d(t) - \begin{pmatrix} 1 \\ 1 \end{pmatrix} w(t) - \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} x_c + \dot{x}_c . \quad (G.20)$$

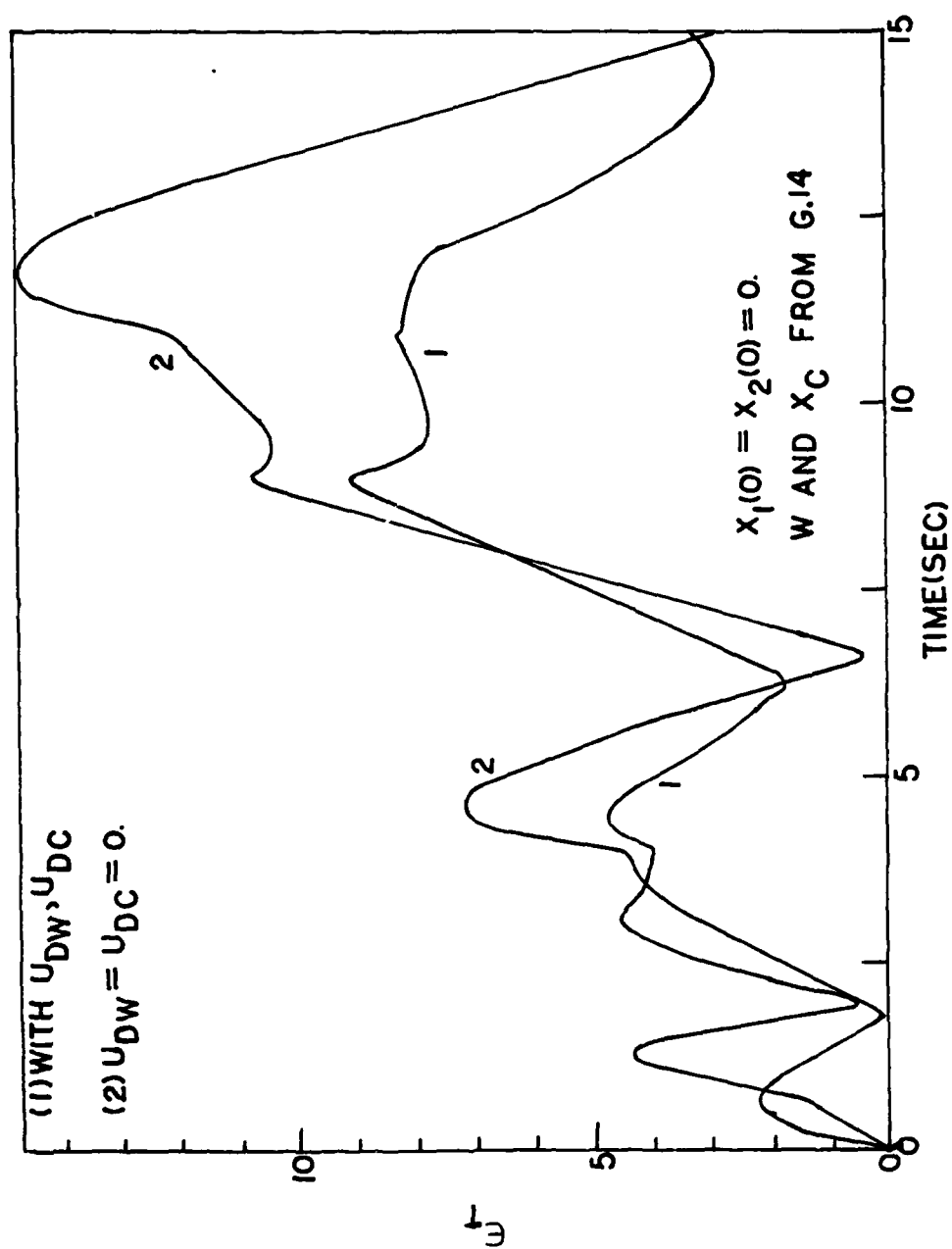
From Equation (G.20), it can be seen that if ϵ_1 is chosen to be the critical state-variable then a u_d of

$$u_d(t) = -w(t) - x_{c1}(t) - x_{c2}(t) + \dot{x}_{c1}(t) \quad (G.21)$$

will completely absorb the direct disturbance effects on $\dot{\epsilon}_1(t)$. If, on the other hand, $\epsilon_2(t)$ is chosen as the critical state-variable then a u_d of

$$u_d(t) = 0.5(-w(t) - x_{c2}(t) + \dot{x}_{c2}(t)) \quad (G.23)$$

will completely absorb the direct disturbance effects on $\dot{\epsilon}_2(t)$. This example was also simulated on a digital computer and runs were made

Figure 5-4. ϵ_T versus time.

for various cases. Figures 5-5 and 5-6 show the data for $\epsilon_T(t)$ for a case with $u_d(t)$ as in Equation (G.21) and Equation (G.23), respectively, with $x_c(t) = 0.$, α as shown in expressions (G.19), and with an initial condition of $x(0) = 1.$

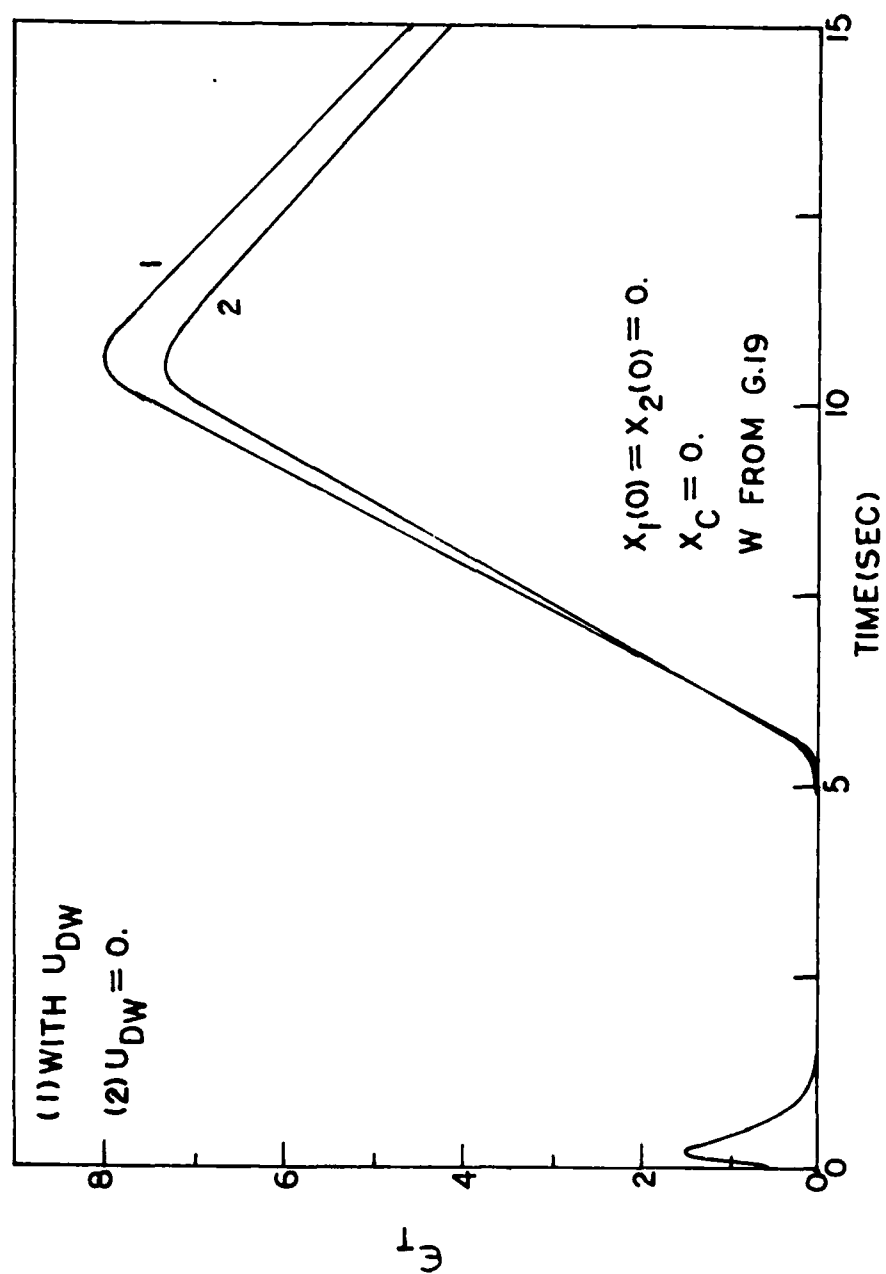


Figure 5-5. U versus time with ϵ_1 as critical state variable.

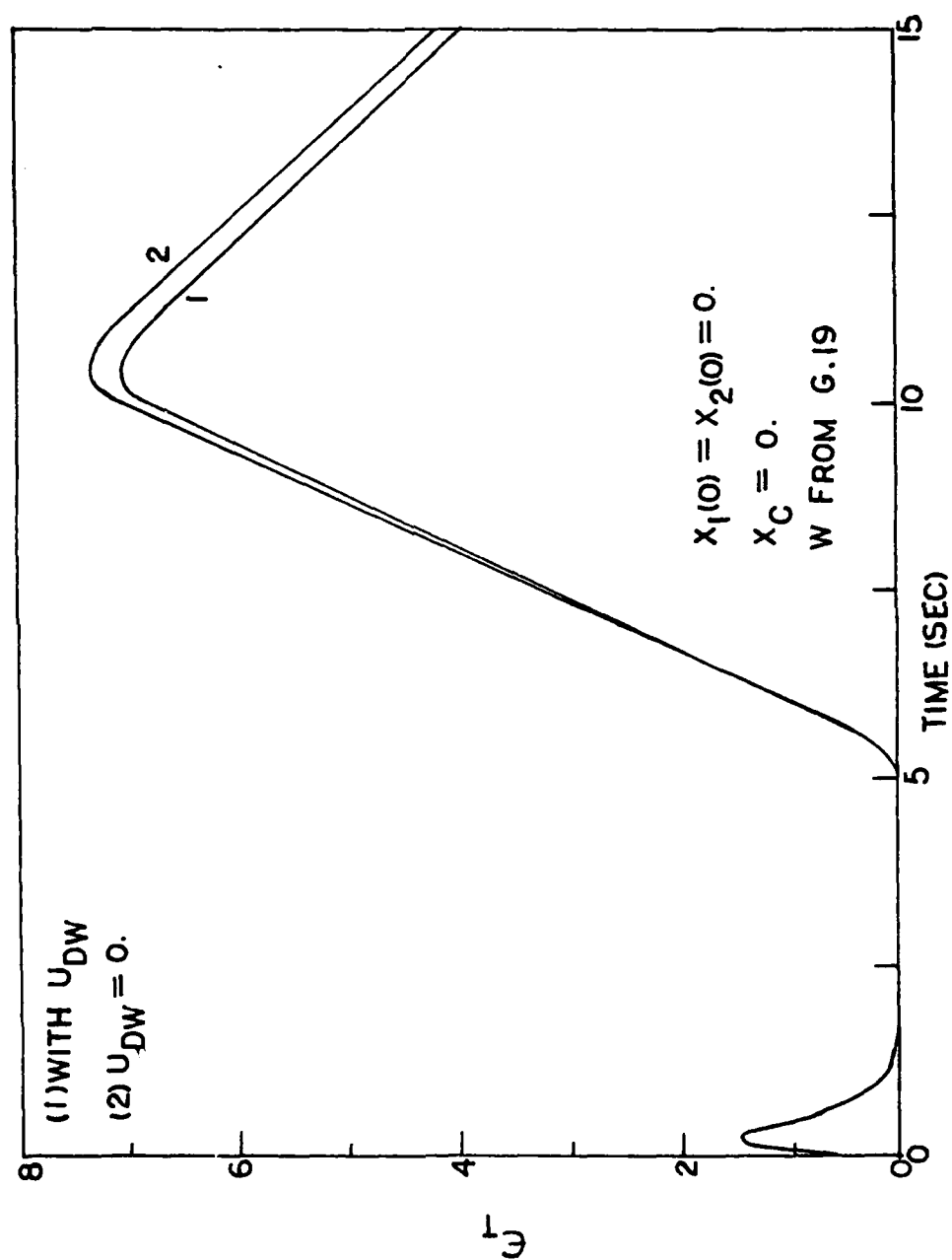


Figure 5-6. ϵ_T versus time with ϵ_2 as critical state-variable.

PART III

CONCLUSIONS AND RECOMMENDATIONS

CHAPTER VI
CONCLUSIONS AND RECOMMENDATIONS
FOR FURTHER WORK

6.1 Introduction

The work described in this dissertation constitutes the first concerted effort to collect, unify, extend and illustrate (through examples) the theory of disturbance minimizing control. This chapter presents the conclusions of this research and offers recommendations for further work.

6.2 Conclusions

This research has demonstrated that system performance degradation, due to disturbance inputs which are not amenable to complete cancellation, can be reduced by application of disturbance minimization design techniques. It has also been shown that the properties of the matrix generalized inverse, combined with the use of a weighted quadratic metric in the disturbance residual and allocation of the disturbance control vector, provide the designer the versatility needed to vary the relative importance of each disturbance input and corresponding control part in obtaining an overall reduction of disturbance effects.

The idea that the external disturbance can aid in the accomplishment of the control task (previously developed in conjunction with Disturbance Utilizing Control Theory) has been extended to disturbance minimizing control theory. It has been shown that it is possible for the external disturbance to aid in the reduction of set-point and servo errors

and the conditions under which this will occur have been presented.

The application of disturbance minimizing control techniques to state and output set-point regulator and servo-tracking type problems has been studied. For state set-point regulation it was shown that allocated and unallocated control parts will produce identical results if the weighting matrices in the metric are identical. Expressions were given for the control parts and the resulting steady-state set-point error for the case when a steady-state solution exists. For the set-point disturbance term only, and pole placement such that the closed-loop system is stable, a steady-state solution can always be found if one wishes to work the problem in that manner. In the case of external disturbances w , the technique of assuming a priori that the control part has the same form as w can be used to find a u_{dw}^* which will minimize the norm of the error resulting from the w input to the system. It has been shown that a simpler technique, which minimizes the disturbance effects on the error dynamics, is to minimize the norm of the disturbance residuals for terms included in the equations for the error (or plant state) rates.

The critical-variable approach, whereby one or more states can be designated as being of prime importance, was studied. It was shown that, if the complete absorption criterion is satisfied for the states in the critical set, it is possible to design a control part which will absorb the direct disturbance terms on those critical states. If complete absorption is not possible, a control part can be designed to

minimize the norm of the direct disturbance residuals on the critical set.

A second-order state set-point example was presented (Appendix B) to illustrate the application of these techniques and the performance of the resulting controllers. It was shown that the set-point error associated with either state could be made to be zero when each was considered to be a critical-variable. It was also shown that designing the control parts as in Section 3.6 resulted in steady-state errors which were 76 percent and 78 percent larger for the set-point and external disturbance terms, respectively, than were obtained when the control parts were designed as in Section 3.5. It was also shown that the external disturbance in the example contributed a negative utility to the system, i.e., would never aid in reducing the set-point error.

For output set-point regulators, the disturbance minimizing techniques were applied to four separate possible cases. These were: (1) $y_{sp}=0$, C invertible, (2) $y_{sp}=0$, C non-invertible, (3) $y_{sp}\neq 0$, C invertible and (4) $y_{sp}\neq 0$, C non-invertible. In case (1), since C is invertible, hence non-singular, the only x which will yield $y=0$ is $x=0$. This problem can therefore be solved as a state set-point problem using the techniques of Chapter 3. An expression for the error dynamics associated with the error as defined in output space was also developed and used to obtain a solution. A second-order example illustrating this case was presented in Appendix D. In problems of this type, with $y_{sp}=0$, the external disturbance w will always exhibit a negative utility.

In case (2), since C is singular, the nullspace of C is of dimension greater than zero. Thus, any x contained in $N(C)$ will yield $y=0$. The control objective must be to steer x as close as possible to $N(C)$,

i.e., minimize the disturbance effects which are acting to prevent x from being stabilized to $N(C)$. A technique for stabilizing x to an arbitrary subspace [16] was modified in order to accomplish this and permit one to minimize the "miss distance" between x and $N(C)$.

In case (3), since C^{-1} exists, for any given y_{sp} a corresponding, unique $x_{sp} = C^{-1}y_{sp}$ can be found and the techniques for state set-point regulators can be applied to the problem. Alternatively, again since C^{-1} exists, an expression was derived for the error dynamics associated with the set-point error as defined in output space and disturbance minimization techniques were applied directly to this expression in order to minimize disturbance effects.

In case (4) there are two sub-cases, (a) C is of maximal rank m , (b) C is of rank less than m . When C is of maximal rank the approach whereby y is transformed to a state sub-vector in another coordinate system (ξ -space) is applied in order to obtain a solution to the problem. Since C is of maximal rank, at least one x_{sp} can be found such that $y_{sp} = Cx_{sp}$ and this x_{sp} permits one to further define a ξ_{sp} in ξ -space such that an error vector, and its associated dynamics, can be defined in ξ -space. The norm minimization, critical state-variable or maximum partial absorption technique can then be applied, where possible and as desired, in order to obtain a solution. An alternate approach which can be used, whether or not C is of maximal rank, is to start by decomposing y_{sp} into a component in $R(C)$ and one in $R(C)^\perp$. In order to minimize the output set-point error it is necessary to steer x such that the component of y lying in $R(C)$ is as close as possible to the orthogonal projection of y_{sp} onto $R(C)$ along $R(C)^\perp$. It is possible to find an x_{sp} which will transform under C to the desired component of y in $R(C)$

and design the control parts to steer x as close as possible to this x_{sp} in order to produce the desired output in $R(C)$. A third-order output set-point example was presented for case (4) and solutions were found using the procedures in sub-cases (a) and (b) (Appendix E). The results were similar in both cases.

For output servo-tracking problems, the exact trackability condition must be satisfied in order to have the output servo-tracking error be zero (in the absence of disturbances). If it is theoretically possible to have zero tracking error, the disturbances will prevent this from being achieved unless they are completely absorbable. One method of approaching a solution with this type problem if C is of maximal rank is to transform y to a state sub-vector in another coordinate system as in the output set-point problem. The various minimization techniques can then be applied to the resulting equations. Another approach [10] is to use the auxiliary variable $e_{ss} = \Theta c - x$ and minimize the disturbance effects on e_{ss} . This latter approach will not necessarily result in the minimum achievable tracking error ϵ , however, since the C matrix intervenes between e_{ss} and ϵ .

Another approach which was presented is the isobasis technique which assumes a priori that the control parts will have the same form as the disturbances. The error dynamics can then be solved and the necessary final form for each control part can be determined. A second-order example was presented (Appendix F) to illustrate the application of disturbance minimizing techniques to output servo-tracking type problems. It was shown that the isobasis technique produced a smaller norm of e_{ss} than did the methods from Sections 5.5 and 5.7, however, it produced the largest value for the tracking error.

State servo-tracking problems can be considered as a special class of output servo-tracking problems with $C \equiv I$. All the techniques which were applied to the class of output problems can thus be applied to the state problems. An example second-order problem is worked (Appendix G) to illustrate a state servo-tracking problem. The techniques of Sections 5.5, 5.7 and 5.8 were applied to determine the relative merits of each approach. For cases with the same input parameters the method of Section 5.8 produced better results than did the method of Section 5.5. It also produced better results on the norm of ϵ (for a comparable run) than did the method of Section 5.7.

In summary, it has been shown that for those cases wherein disturbances cannot be completely absorbed it is possible to effect an improvement in system performance by utilizing Disturbance-Minimizing Control Design Theory. The amount of improvement obtained will vary from technique to technique and so, for a given system, several of the design options might be investigated to determine which one will provide the most acceptable results. If ease of design is considered most important then designing u_d^* to minimize the norm of the disturbance residuals in the expressions for the state or error dynamics is the best technique. For cases in which one variable can be considered to be a critical state-variable, designing u_d to absorb (assuming $b^i \neq 0$) the direct disturbance effects is also straightforward.

6.3 Recommendations

There are several areas in which further efforts in the study of Disturbance-Minimizing Control Theory could profitably be expended. Those areas may be described and listed as follows:

(a) The application of optimal control techniques to the problem of minimizing disturbance effects should be investigated. The use of the quadratic norm in the disturbance residual is one suggestion for inclusion in the performance index, i.e., if one lets $\delta = ||Bu_d + \tilde{F}||$, where $\tilde{F}$ represents a disturbance term, then the performance index J can be expressed as

$$J = \frac{1}{2} \delta^T(T) S \delta(T) + \frac{1}{2} \int_{t_0}^T [\delta^T(t) Q(t) \delta(t) + u^T(t) R u(t)] dt ;$$

with $S = S^T \geq 0$, Q a given positive definite matrix and R a given symmetric, positive definite matrix.

(b) All of the developments in this dissertation utilized the weighted quadratic norm as the metric. The application of norms other than this to the disturbance minimization problem should be investigated and conditions under which each particular norm would give the "best" results should be enumerated.

(c) For the isobasis design technique, the control vector is assumed a priori to have the same form as the disturbance, i.e., the control and disturbance are both linear combinations of the same set of basis functions. The results of using a more robust set of basis functions to define the control vector should be investigated. For instance, if w is a ramp, what would be the benefits of assuming that u_{dw} is a combination of a ramp and a constant instead of just a ramp, etc.

(d) The examples presented in this dissertation assumed that there were no limitations on the control vector. It would be interesting to see what effect occurs on the results when limits are imposed on the various control parts, i.e., $||u_{dw}||_{p_1}$, $||u_{ds}||_{p_2}$, etc.

(e) Control vectors which result from application of optimal control techniques to the disturbance minimization problem should be compared to the control vectors obtained from use of matrix generalized inverse techniques to see what, if any, correlation exists.

(f) The application of the isobasis design technique to state and output set-point type problems in order to derive an expression for the control part associated with minimizing the external disturbance residual is an area which should be further developed.

(g) The application of disturbance minimization techniques to systems which have the output expressed as

$$y = Cx + Eu + Gw$$

should be further explored.

(h) Since any practical implementation of disturbance minimizing controllers will require use of state reconstructors, the interaction of the reconstructor dynamics with the disturbance minimizing control dynamics should be examined to see if any significant degradation of results may occur. If the reconstructor dynamics do adversely affect the performance of the disturbance minimizing control terms, it may prove necessary to consider the error between the reconstructor estimate and the true state value as an additional disturbance source which must be minimized.

(i) Another subject area for further study is the application of disturbance minimizing control techniques to systems which may be time-varying, i.e., to systems of the form

$$\begin{aligned}\dot{x}(t) &= A(t)x(t) + B(t)u(t) + F(t)w(t) \\ y(t) &= C(t)x(t) + E(t)u(t) + G(t)w(t).\end{aligned}$$

This topic would necessarily include the mathematics of time-varying

matrix generalized inverses.

(j) The application of disturbance-minimizing control theory to realistic problems in order to ascertain what benefits accrue in a realistic environment is a task which should be undertaken immediately.

(k) The application of the isobasis design technique to the solution of indirect disturbance minimization type problems for cases in which the disturbance terms are not constant should also be examined.

(l) The geometry of areas of positive and negative utility in n -dimensional spaces should be more fully explicated.

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PART IV

APPENDICES

APPENDIX A

APPENDIX A

GENERALIZED INVERSES

A.1 Introduction

Since matrix generalized inverses play a major role in the developments in this dissertation, a short summary of the theory of generalized inverses is presented in this Appendix. Some of the terminology useful to an understanding of the concept of a generalized inverse will be introduced. The generalized inverse will then be discussed and several different generalized inverses which are used in this dissertation will be defined. Several norms which can be used in connection with generalized inverses will be listed, but the one which will be used in this dissertation is the weighted quadratic norm defined in Section 2.2.2. Several methods for calculating generalized inverses will be presented and this Appendix will conclude with examples illustrating the application of these methods.

A.2 Terminology

Prior to describing the generalized inverse, it will prove beneficial to define some of the nomenclature which is connected with the algebra of the topic (see for instance [1, 3, 4, 5, 6, 7, 22, 23, 25, 30, 31, 34]):

- (1) Normed Linear Space: A normed linear space is a linear

space V with a real-valued function, denoted by $||\cdot||$, which satisfies the following axioms for all $x, y \in V$ and all α in F :

- (a) $||x|| \geq 0$, with equality holding if, and only if $x = 0$.
- (b) $||x + y|| \leq ||x|| + ||y||$
- (c) $||\alpha x|| = |\alpha| ||x||$, α a constant.

(2) Inner Product Space: An inner product space is a linear space V with a scalar-valued function, denoted by $(\cdot, \cdot)$ and called an inner product, which satisfies the following axioms for all x, y, z in V and for all α, β in the field over which V is defined:

- (a) $(x, y) = \overline{(y, x)}$ where the overbar denotes the conjugate
- (b) $(\alpha x + \beta y, z) = \alpha (x, z) + \beta (y, z)$
- (c) $(x, x) \geq 0$ with equality if, and only if, $x = 0$.

An inner product space is also a normed linear space with $||x|| = (x, x)^{1/2}$.

(3) Distance: In a normed linear space, the Euclidean distance between two points x, y in the space is defined as

$$d(x, y) = ||x - y|| = (x - y, x - y)^{1/2}.$$

(4) Hilbert Space: A Hilbert space H is a complete inner product space. By complete is meant that every Cauchy sequence in H converges to some point of H . The finite dimensional space R^n is a Hilbert space under the inner product $(x, y) = (x^T y)^{1/2}$.

(5) Orthogonal Complement: If W is a subspace of an inner

product space V , its orthogonal complement $W^\perp$ is defined as

$$W^\perp = \{y \in V : x \perp y \text{ for all } x \in W\},$$

where $x \perp y$ implies $(x, y) = 0$.

(6) Linear Transformations: If U, V are vector spaces over the same field F then a linear transformation L from U into V is a correspondence which associates to every vector in U a unique vector in V and $L(\alpha x + \beta y) = \alpha L(x) + \beta L(y)$ for all x, y in U and for all α, β in F .

(7) $L(U, V)$: Let $L(U, V)$ be the set of all bounded linear transformations from U to V , where bounded implies that if $A \in L(U, V)$ then $\|A\| = \sup\{\|Ax\|_V : \|x\|_U = 1\}$ is finite.

(8) Column Range Space: If A is a matrix representing a bounded linear transformation from U to V then the column range space of A is

$$R(A) = \{y \in V : y = Ax \text{ for some } x \in U\}.$$

The column range space of A is a linear subspace of V .

(9) Nullspace: If A is a matrix representing a bounded linear transformation from U to V then the nullspace of A is

$$N(A) = \{x \in U : Ax = 0\}.$$

The nullspace of A is a linear subspace of U . A Hilbert space can be expressed as the direct sum of a closed subspace and its orthogonal complement. Since the nullspace is always closed in the topology of U then $N(A)^\perp$ is a closed subspace of U also. Therefore, H is equal to the direct sum of $N(A)$ and $N(A)^\perp$ and any vector in H can be written

as $x = x_1 + x_2$ where x_1 is an element of $N(A)$ and x_2 is an element of $N(A)^\perp$.

(10) **Projector:** A homogeneous linear operator E is a projector if, and only if, E is idempotent, i.e., $E^2 = E$. If E is the projector on a subspace M along the subspace N then $(I - E)$ is the projector on N along M . Associated with the definition of a projector is the

Theorem [3]: For every idempotent matrix E contained in $R^{n \times n}$, $R(E)$ and $N(E)$ are complementary subspaces with E the projector on $R(E)$ along $N(E)$. Conversely, if M and N are complementary subspaces, there is a unique idempotent E such that $R(E) = M$ and $N(E) = N$. $\blacksquare$

If a vector space V is the direct sum of M and N so that every x in V may be written uniquely in the form $x = x_1 + x_2$ with x_1 an element of M and x_2 an element of N , the projection on M along N is the linear transformation defined by $E x = x_1$. If E is the projection on M along N , then M and N are, respectively, the sets of all solutions of the equations $E x = x$, and $E x = 0$, i.e., $M = R(E)$, $N = N(E)$.

(11) **Orthogonal Projector:** A homogeneous linear operator E is an orthogonal projector if, and only if,

$$(a) \quad E^2 = E \text{ and}$$

$$(b) \quad E^T = E.$$

A projector on M along N is called an orthogonal projector on M if N is $M^\perp$ in the vector space.

(12) Orthogonal Projection: For a vector x in R^n and a subspace M of R^n , there is in M a unique vector y^0 that is closest to x in the sense that the distance $||x - y^0||$ is smaller for $y = y^0$ than for any other y in M . The vector y^0 is uniquely characterized by the fact that $x - y^0$ is orthogonal to y^0 . The vector y^0 is called the orthogonal projection of x on M .

(13) Convex Subset: A subset C of a normed linear space is called convex if

$$t x + (1 - t) y \in C$$

for all x, y in C and for all t in $[0, 1]$.

In addition to the preceding definitions, the following theorem and corollary [5] are also important for what follows:

Theorem: A closed convex subset of a Hilbert space contains a unique vector of smallest norm. **|**

Corollary: If C is a closed convex subset of a Hilbert space H , then for each u in H there is a unique x in C such that $||u - x|| = \inf \{ ||u - y|| : y \text{ is in } C \}$. **|**

A.3 Generalized Inverse

Matrices represent linear transformations. As stated in Section A.2 a linear transformation σ satisfies the relationship

$$\sigma (\alpha x + \beta y) = \alpha \sigma (x) + \beta \sigma (y) \quad (A.1)$$

for all x, y in U and for all α, β in F . The image of σ is defined as

$$\text{Im } (\sigma) = \sigma (U) = \{\sigma (\alpha) : \alpha \text{ is in } U\} \quad (\text{A.2})$$

and $\sigma (U)$ is a subspace of V . If σ is one to one, then whenever $\sigma (x) = \sigma (y)$ it implies that $x = y$. If σ is onto, then $\text{Im } (\sigma)$ is the entire vector space V . If σ is both one to one and onto, i.e., an isomorphism, then for every $\bar{x}$ in V there is a unique x in U such that $\sigma (x) = \bar{x}$, i.e., the linear transformation has an inverse.

The rank $\rho (\sigma)$ of a linear transformation σ is defined to be the dimension of the image of σ ,

$$\rho (\sigma) = \text{Dim Im } (\sigma) \leq \text{Dim } V . \quad (\text{A.3})$$

The nullity $\nu (\sigma)$ of a linear transformation σ is defined to be the dimension of the nullspace of σ ,

$$\nu (\sigma) = \text{Dim } N (\sigma) . \quad (\text{A.4})$$

For the linear transformation denoted by $\sigma: U \rightarrow V$, where $\text{Dim } U = n$, $\text{Dim } V = m$, one has that

$$\rho (\sigma) + \nu (\sigma) = n . \quad (\text{A.5})$$

Suppose, however, that n is less than m . In that case, since $\nu (\sigma) \geq 0$, one has that

$$\rho (\sigma) = n - \nu (\sigma) \leq n < m = \text{Dim } V , \quad (\text{A.6})$$

Thus, σ could not be an onto transformation. If, on the other hand, n were greater than m then one would have

$$\nu (\sigma) = n - \rho (\sigma) \quad (\text{A.7})$$

and since $\rho(\sigma) \leq \dim V = m$, then

$$\nu(\sigma) = n - m > 0 \quad (\text{A.8})$$

and σ could not be a one to one transformation. Therefore, in order for σ to be an isomorphism it must be true that $n = m$, i.e., U and V must be of the same dimension.

The linear transformation $\sigma: U \rightarrow V$, where $\dim U = n$, $\dim V = m$, is represented by an $m \times n$ matrix, say A , with respect to a given basis in U and a given basis in V . Thus, an isomorphism is represented by an $n \times n$ matrix A . Since only isomorphisms have inverses then only square matrices can have inverses. The inverse of σ is represented as $\sigma^{-1}(\sigma(x)) = x$. With A representing σ , A^{-1} represents σ^{-1} and since $\sigma\sigma^{-1} = \sigma^{-1}\sigma = I$ then $AA^{-1} = A^{-1}A = I$. If A^{-1} exists then A is invertible (non-singular). This follows from the fact that $\rho(A) = \text{number of linearly independent columns in } A$ and for an isomorphism $\nu(A) \neq 0$ which implies that $\rho(A) = n$. Thus, with n linearly independent columns the determinant of A is not zero and A is non-singular.

For those cases where the matrix A is either singular or rectangular the notion of a partial inverse has been developed. This inverse is called a generalized or pseudo-inverse. The concept of a generalized matrix inverse was apparently first noted by E. H. Moore in the early 1900's. In the 1950's, certain least-squares properties of some generalized inverses rekindled an interest and in 1955 Penrose [28] showed that for a given matrix the Moore inverse is a unique matrix which satisfies certain conditions. This matrix has come to be called the Moore-Penrose inverse.

Some might tend to equate the terms "generalized inverse" and "Moore-Penrose inverse". In fact, however, the Moore-Penrose inverse is a generalized inverse which satisfies a certain set of conditions. There are many "generalized inverses" which have been developed. For a tabular listing of some of these generalized inverses and their purposes, see [30].

Moore developed the generalized inverse concept for an arbitrary $m \times n$ complex matrix in the context of linear transformations from C^m to C^n (complex vector spaces with Euclidean norm). His definition of generalized inverse was [30]: G is the $n \times m$ generalized inverse of A ($m \times n$) if $A G = P_A$, where P_A is the projector from C^m onto $R(A)$, and if $G A = P_G$, where P_G is the projector from C^n onto $R(G)$.

Penrose later defined a generalized inverse whose definition, as applied to real vector spaces is: G is the generalized inverse of A if

$$\begin{aligned} \text{(a)} \quad & A G A = A \\ \text{(b)} \quad & (A G)^T = A G \\ \text{(c)} \quad & G A G = G \\ \text{(d)} \quad & (G A)^T = G A \end{aligned} \tag{A.9}$$

This generalized inverse is the same as Moore's inverse if the norm, defined on R^m and R^n , is the Euclidean norm. This inverse is known as the Moore-Penrose generalized inverse and is generally symbolized as $(\cdot)^\dagger$.

Definitions for several of the generalized inverses which are pertinent to the work in this dissertation are as follows: [30]

- (1) A generalized inverse for a minimum norm solution of

$Ax = y$ (consistent): In a problem of this type the objective is to find a generalized inverse G for A such that the solution $Gy = x$ has the smallest norm over all such solutions of $Ax = y$. The solution is found from the

Theorem: Let G be a generalized inverse of A such that Gy is a minimum norm solution of $Ax = y$ for any y in $R(A)$. Then it is necessary and sufficient that

$$AGA = A, \text{ and } (GA)^T = GA, \quad (\text{A.10})$$

and the minimum norm solution is unique. ■

(2) A generalized inverse for a least-squares solution of $Ax = y$ (inconsistent): In a problem of this type, a least-squares solution of $Ax = y$ is $\hat{x}$ such that

$$\|A\hat{x} - y\| = \inf_x \|Ax - y\|.$$

Theorem: Let G be a matrix such that Gy is a least-squares solution of $Ax = y$ for any y in R^m . Then, it is necessary and sufficient that

$$AGA = A, \text{ and } (AG)^T = AG. \quad (\text{A.11})$$

The $\min \|Ax - y\|$ is unique, but a least-squares solution may not be.

(3) A generalized inverse for a minimum norm least-squares solution of $Ax = y$ (inconsistent). In this type problem a unique solution is found from the

Theorem: Let there exist a matrix G such that Gy is a minimum norm least-squares solution of $Ax = y$. Then it is necessary and sufficient that

$$AGA = A, (AG)^T = AG, GAG = G, (GA)^T = GA, \quad (A.12)$$

i.e., the Moore-Penrose generalized inverse gives the minimum norm least-squares solution.

(4) Right inverse [30]: If A is an $m \times n$ matrix of rank m , then AA^T is an $m \times m$ matrix of rank m . Therefore, the inverse $(AA^T)^{-1}$ exists,

$$I = (AA^T)(AA^T)^{-1} = A \left[A^T (AA^T)^{-1} \right], \quad (A.13)$$

and $A^T (AA^T)^{-1}$ is called the right inverse of A .

(5) Left inverse [30]: If A is an $m \times n$ matrix of rank n , then $A^T A$ is an $n \times n$ matrix of rank n , therefore, the inverse $(A^T A)^{-1}$ exists,

$$I = (A^T A)^{-1} (A^T A) = ((A^T A)^{-1} A^T) A \quad (A.14)$$

and $(A^T A)^{-1} A^T$ is called the left inverse of A .

If n is not equal to m only one of the inverses, right or left, can be defined for a given A and if the rank of A is not equal to n or m neither of these two inverses exist.

Another theorem which will prove useful is the following:

Theorem [30]: If A is of order $m \times n$ and G is any generalized inverse of A , then:

(a) A general solution of the homogeneous equation

$Ax = 0$ is

$$x = (I - GA) z \quad (A.15)$$

where z is an arbitrary vector, and

(b) A general solution of a consistent non-homogeneous equation $Ax = y$ is

$$x = Gy + (I - GA) z \quad (A.16)$$

where z is an arbitrary vector. ■

A.4 Norms

As mentioned in Section A.2, a vector norm is a real-valued function, $|| \cdot ||$ on R^n , which satisfies the axioms

- (a) $||x|| \geq 0$, with equality if, and only if $x = 0$
- (b) $||\alpha x|| = |\alpha| ||x||$, α a scalar in R (A.17)
- (c) $||x+y|| \leq ||x|| + ||y||$ for all x, y in R^n .

An L_p -norm is a norm defined on R^n by

$$||x||_p = \left(\sum_{j=1}^n |x_j|^p \right)^{1/p}, \quad p \geq 1. \quad (A.18)$$

The most popular L_p norms are for $p = 1, 2, \infty$ and they are defined as

- (a) $||x||_1 = \sum_{j=1}^n |x_j|$ (L_1 - norm)
- (b) $||x||_2 = \left(\sum_{j=1}^n |x_j|^2 \right)^{1/2} = (x^T x)^{1/2}$ (L_2 or Euclidean norm) (A.19)

$$(c) \quad ||x||_{\infty} = \max \{ |x_j| : j = 1, 2, \dots, n \}$$

(L_{∞} or Tchebycheff norm).

The solutions for minimum-norm, least-squares and minimum norm least-squares given by the theorems in Section A.3 are defined using the Euclidean norm. A solution utilizing the L_{∞} -norm is given in [3]. Generalized inverses can also be used to give solutions with norms other than the Euclidean or Tchebycheff. Some of these can be found in [3]. The control processes to be developed in this dissertation will use the generalized inverse theory developed with respect to the L_2 (Euclidean) norm.

A.5 Methods For Calculating Matrix Generalized Inverses

This section will present several methods for calculating a generalized inverse of a rectangular or square-singular matrix.

A.5.1 Full Rank Matrix. A vector u_d is a least-squares solution of $B u_d = \delta$ if, and only if, u_d is a solution of the "normal equation",

$$B^T B u_d = B^T \delta. \quad (A.20)$$

The vector u_d will be given by

$$u_d = (B^T B)^{-1} B^T \delta \quad (A.21)$$

Since $B^T B$ is an $n \times n$ matrix and B is of full rank then $B^T B$ is not singular, thus $(B^T B)^{-1}$ exists. The generalized inverse of B is then represented as

$$B^- = (B^T B)^{-1} B^T. \quad (A.22)$$

A.5.2 Full-Rank Factorization Method [3,30]. If B is a non-null matrix and is not of full rank, it can be expressed as the product of two other matrices, one of which is of full column rank and the other of full row rank. Adopting the notation used in [3], if $B \in R_r^{m \times n}$, $r > 0$ (i.e., if B is a member of the class of $m \times n$ matrices of rank r), then there exist matrices $P \in R_r^{m \times r}$, $Q \in R_r^{r \times n}$ such that $B = PQ$. Given this factorization, then B^+ is determined as

$$B^+ = Q^T (P^T B Q^T)^{-1} P^T. \quad (A.23)$$

In determining P and Q the following method may be used. In general, P is a matrix whose columns are a basis for $R(B)$ or whose columns can be chosen as any maximal linearly independent set of columns of B . Also, if B is transformed to Hermite normal form the first r rows form a basis for the space spanned by the rows of B and these r rows may be used for Q . Given that P or Q is found from the above considerations, the other can be determined from the relationship $B = P Q$.

A.5.3 Q R Factorization Method [3]. The Q R method is another factorization method which can be used to calculate u_d and also the minimum values of $\|B u_d - \delta\|^2$. For the procedure given below, it is assumed that B is of full column rank.

If $B \in R_r^{m \times n}$, a factorization for B is given by

$$B = Q R = \tilde{Q} \tilde{R} \quad (A.24)$$

where

$Q \in R^{m \times m}$ is unitary ($Q^T Q = I$)

$R = \begin{bmatrix} \tilde{R} \\ 0 \end{bmatrix}$ where $\tilde{R}$ is an $n \times n$ upper triangular,
non-singular matrix, and

$\tilde{Q}$ consists of the first n columns of Q .

The columns of Q form an orthogonal basis for R^m and the columns of $\tilde{Q}$ and $\tilde{R}$ may be obtained by orthonormalizing the columns of B . Given that these matrices are found, u_d can be determined from

$$\tilde{R} u_d = \tilde{Q}^T \delta \quad (A.25)$$

which is equivalent to the normal equation. If one expresses $Q^T \delta$ as

$$Q^T \delta = [c_i], \quad i = 1, 2, \dots, m,$$

then the minimum value of $\|B u_d - \delta\|^2$ is given by

$$\sum_{i=n+1}^m |c_i|^2. \quad (A.26)$$

A.5.4 Eigenvector Method [32]. For the Eigenvector method we have the

Theorem: Let B be an $m \times n$ matrix of rank r , where $r \leq m$, $r \leq n$, and let g_i be an orthonormal eigenvector of $B^T B$. Let f_i be an orthonormal eigenvector of BB^T and let ρ_i^2 be the nonzero eigenvalues of $B^T B$. Under these conditions, B can be written as

$$B = \sum_{i=1}^r \rho_i f_i g_i^T. \quad (A.27)$$

The generalized inverse of B , symbolized by B^- , is given by the

Definition: The generalized inverse B^- of the $m \times n$ real matrix B is the $n \times m$ real matrix

$$B^- = \sum_{i=1}^r \rho_i^{-1} g_i f_i^T \quad (A.28)$$

and this generalized inverse will give a minimum norm solution for the equation $Bx = y$.

A.6 Other Areas of Interest

There are several additional topics pertaining to generalized inverses which involve a minimization process when various constraints are imposed. These topics will be presented in this section.

A.6.1 Control Constraints [3]. Suppose that one is given the problem of minimizing $\|B u_d - \delta\|$ subject to the restriction that $\|u_d\| = p$, i.e., with limitations on the magnitude of the control vector which is to be used to accomplish the objective. In this type problem, the unique solution for u_d is given by

$$u_d = (B^T B + \alpha^2 I)^{-1} B^T \delta \quad (A.29)$$

with α uniquely determined from

$$\|(B^T B + \alpha^2 I)^{-1} B^T \delta\| = p. \quad (A.30)$$

In the more general case where $\|u_d\| \leq p$, if $\|B^+ \delta\| \leq \|p\|$ then $u_d = B^+ \delta$ is a solution and, if the equality holds it is the unique solution. However, if $\|B^+ \delta\| \not\leq \|p\|$ the unique solution is given by Equation (A.29).

A.6.2 Weighted Generalized Inverses [3]. In a manner similar to that used in optimal control whereby "weights" are applied to the components of vectors of interest such as control, velocity, position at terminal time, etc., the minimization problem can be formulated such that the minimization of a given positive quadratic form in the residual is the control objective, i.e., minimize

$$||\Delta||_Q^2 = ||B u_d - \delta||_Q^2 = (B u_d - \delta)^T Q (B u_d - \delta) \quad (A.31)$$

with the kernel Q being a given positive definite matrix.

In a problem of this type, $||\Delta||_Q$ is smallest when

$$u_d = X \delta \quad (A.32)$$

where X satisfies

$$(a) \quad BXB = B \quad (A.33)$$

$$(b) \quad (QBX)^T = QBX.$$

If B is of full column rank a unique solution for u_d exists; otherwise, a solution may be chosen from the class of generalized least-square solutions as the one for which $||u_d||_U^2 = u_d^T U u_d$ is smallest, where U is a given positive definite matrix.

As an alternate means of finding the u_d which minimizes Equation (A.31), u_d minimizes $||\Delta||_Q^2$ if, and only if, u_d is a solution of

$$B^T Q B u_d = B^T Q \delta \quad (A.34)$$

A.6.3 Minimization Subject to Second Condition on u_d [3].

Suppose a problem were given in which the control objective was to minimize $\|\tilde{B}_1 u_d - \delta_1\|_Q$ subject to a second constraint that $\tilde{B}_2 u_d = \delta_2$, where

$$\tilde{B}_1 \in R^{m_1 \times n}, \delta_1 \in R^{m_1}, \tilde{B}_2 \in R^{m_2 \times n}, \delta_2 \in R^{m_2}.$$

This situation might be encountered in output control problems wherein there are interactions between ξ_1 and ξ_2 and where the error vector is to be minimized. A vector u_d will minimize $\|\tilde{B}_1 u_d - \delta_1\|_Q$ subject to the second constraint if, and only if, there is a vector y in R^{m_2} such that $\{u_d, y\}$ is a solution of

$$\begin{bmatrix} \tilde{B}_1^T Q \tilde{B}_1 & \tilde{B}_2^T \\ \tilde{B}_2 & 0 \end{bmatrix} \begin{pmatrix} u_d \\ y \end{pmatrix} = \begin{bmatrix} \tilde{B}_1^T Q \delta_1 \\ \delta_2 \end{bmatrix}. \quad (A.35)$$

A.7 Examples

This section contains examples which illustrate the application of the methods given in Section A.6 for finding matrix generalized inverses.

A.7.1 Matrix of Full Column Rank. For this example, let $B = (1, 2)^T$. The rank of B is one, hence B is of full rank and, from Equation (A.22),

$$B^+ = (B^T B)^{-1} B^T = \left[(1, 2) \begin{pmatrix} 1 \\ 2 \end{pmatrix} \right]^{-1} (1, 2) = (0.2, 0.4). \quad (A.36)$$

A.7.2 Full-Rank Factorization. For this example, let

$$B = \begin{bmatrix} 1 & 2 \\ 1.5 & 3 \end{bmatrix}.$$

The determinant of B is zero, hence the rank of B is less than two, i.e., B is not of full column rank. Therefore, using the procedure of Section A.5.2, one should find $P \in R_r^{m \times r}$, $Q \in R_r^{r \times n}$ such that $B = PQ$.

(a) Using the first method below Equation (A.23) P can be chosen as $P = (1, 1.5)^T$. A matrix Q can then be found which satisfies the relationship $B = PQ$ as

$$\begin{bmatrix} 1 & 2 \\ 1.5 & 3 \end{bmatrix} = \begin{pmatrix} 1 \\ 1.5 \end{pmatrix} (q_1, q_2) \Rightarrow Q = (1, 2).$$

The matrix B can thus be expressed as

$$B = \begin{pmatrix} 1 \\ 1.5 \end{pmatrix} (1, 2). \quad (\text{A.37})$$

(b) Using the second method below Equation (A.23), the Hermite normal form of B is first obtained as

$$\begin{bmatrix} 1 & 2 \\ 0 & 0 \end{bmatrix}.$$

This implies that the rank of B is one, so Q can be chosen as $Q = (1, 2)$ and a matrix P is then found which will satisfy $B = PQ$. The result is $P = (1, 1.5)^T$.

A generalized inverse for B can now be found from Equation (A.23) to be

$$\begin{aligned}
 B^{\dagger} &= \begin{pmatrix} 1 \\ 2 \end{pmatrix} \left[(1, 1.5) \begin{bmatrix} 1 & 2 \\ 1.5 & 3 \end{bmatrix} \begin{pmatrix} 1 \\ 2 \end{pmatrix} \right]^{-1} (1, 1.5) \\
 &= \frac{1}{65} \begin{bmatrix} 4 & 6 \\ 8 & 12 \end{bmatrix} .
 \end{aligned}
 \tag{A.38}$$

A.7.3 QR Factorization Method. For this example let $B = (1, 2)^T \in \mathbb{R}_1^{2 \times 1}$. From Equation (A.24) it can be determined that one must have $Q \in \mathbb{R}^{2 \times 2}$ and $\tilde{R} \in \mathbb{R}^{1 \times 1}$. This will result in $R = (\tilde{r}, 0)^T$. If the first column of Q is calculated as $B/||B||$, where $\tilde{r} = ||B|| = \sqrt{5}$, then from Equation (A.24) one has that

$$\begin{pmatrix} 1 \\ 2 \end{pmatrix} = \begin{bmatrix} 1/\sqrt{5} & q_{12} \\ 2/\sqrt{5} & q_{22} \end{bmatrix} \begin{pmatrix} \sqrt{5} \\ 0 \end{pmatrix} .
 \tag{A.39}$$

The second column of Q can be determined from Equation (A.39) and the requirements that

$$\begin{pmatrix} q_{12} \\ q_{22} \end{pmatrix} \perp \begin{pmatrix} 1/\sqrt{5} \\ 2/\sqrt{5} \end{pmatrix} \text{ and } Q^T Q = I .
 \tag{A.40}$$

Since two vectors are perpendicular if their inner product is zero, the first of the requirements in (A.40) results in

$$\frac{q_{12}}{\sqrt{5}} + \frac{2q_{22}}{\sqrt{5}} = 0 \Rightarrow q_{12} = -2q_{22} .
 \tag{A.41}$$

The second requirement of (A.40) results in

$$q_{12}^2 + q_{22}^2 = 1 . \quad (\text{A.42})$$

If Equation (A.41) is squared and the square is substituted into Equation (A.42) the result will be

$$q_{22} = 1/\sqrt{5}, \quad q_{12} = -2/\sqrt{5} . \quad (\text{A.43})$$

The matrix B can thus be factored as

$$B = \begin{pmatrix} 1 \\ 2 \end{pmatrix} = \frac{1}{\sqrt{5}} \begin{bmatrix} 1 & -2 \\ 2 & 1 \end{bmatrix} \begin{pmatrix} \sqrt{5} \\ 0 \end{pmatrix} = QR . \quad (\text{A.44})$$

From Equation (A.25), with $\delta = Ax_{sp}$, one has

$$\tilde{R} u_{ds} = \tilde{Q}^T Ax_{sp} \quad (\text{A.45})$$

and, upon substituting appropriate terms, Equation (A.45) yields

$$u_{ds} = 0.2 x_{sp1} . \quad (\text{A.46})$$

If δ is set equal to $\delta = FH_z$ in Equation (A.25) the resulting control vector u_{dw} will be

$$u_{dw} = 0.6 z . \quad (\text{A.47})$$

The minimum value of $\|B u_{ds} - Ax_{sp}\|^2$ can be calculated from Equation (A.26), with

$$Q^T A x_{sp} = \begin{pmatrix} 1/\sqrt{5} \\ -2/\sqrt{5} \end{pmatrix} x_{sp1} = \begin{pmatrix} c_1 \\ c_2 \end{pmatrix} \quad (A.48)$$

as

$$\sum_{i=2}^2 |c_i|^2 = \left| \frac{-2x_{sp1}}{\sqrt{5}} \right|^2 = 0.8 x_{sp1}^2. \quad (A.48)$$

A.7.4 Control Vector Constraints. illustrate the method of Section A.6.1, assume that the problem is to minimize $\|B u_{ds} - A x_{sp}\|$ subject to $\|u_{ds}\| = 2$ where $B = (1, 2)^T$. Upon substituting these parameters into Equation (A.30) to calculate α the result is

$$\alpha^2 = \frac{1}{2} x_{sp1} - 5. \quad (A.49)$$

The control u_{ds} is then calculated from Equation (A.29) to be

$$u_{ds} = (5 + \frac{1}{2} x_{sp1} - 5)^{-1} x_{sp1} = 2. \quad (A.50)$$

As another example, let $B = \begin{bmatrix} 1 & 2 \\ 1.5 & 3 \end{bmatrix}$, $p = 2$.

In this case Equation (A.30) yields

$$\alpha^2 = \frac{1}{8} (\sqrt{80} x_{sp1} - 130). \quad (A.51)$$

Equation (A.29) will then give the control u_{ds} as

$$u_{ds} = \frac{1}{\sqrt{80}} \begin{pmatrix} 8 \\ 16 \end{pmatrix} \quad (A.52)$$

and it is seen that

$$||u_{ds}|| = (320/80)^{1/2} = 2 .$$

APPENDIX B

APPENDIX B

AN EXAMPLE MINIMIZATION PROBLEM FOR A SECOND-ORDER SET-POINT REGULATOR; SECTIONS 3.5 AND 3.6

B.1 Introduction

This appendix contains a detailed example for a second-order state set-point regulator problem with the design objective being minimization of the distance between the attainable and desired set-points. Here distance is defined by the Euclidean norm, $d^2 = \|\epsilon\|_I^2 = \epsilon^T \epsilon$, of the error vector between x_{sp} and the plant state $x(t)$ with $\epsilon(t)$ defined as $\epsilon(t) = x_{sp} - x(t)$. The example will be developed in terms of the error vector $\epsilon(t)$ consistent with the development in Sections 3.4 through 3.6.

B.2 Plant and Disturbance Models

The plant state and output equations for the set-point regulator of this example will be of the form

$$\begin{aligned}\dot{x}(t) &= Ax(t) + Bu(t) + Fw(t) \\ y(t) &= Cx(t) \quad .\end{aligned}\tag{B.1}$$

In particular, the model will be

$$\begin{pmatrix} \dot{x}_1 \\ \dot{x}_2 \end{pmatrix} = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} + \begin{pmatrix} 1 \\ 2 \end{pmatrix} u + \begin{pmatrix} 1 \\ 1 \end{pmatrix} w\tag{B.2}$$

$$y = (1, 0)x(t) \quad .\tag{B.3}$$

The disturbance model will be of the form

$$\begin{aligned} w(t) &= Hz(t) \\ \dot{z}(t) &= Dz(t) + \sigma(t) \end{aligned} \quad (B.4)$$

where for the specific example considered the model will be

$$w = z \quad (B.5)$$

$$\dot{z} = \sigma(t) \quad (B.6)$$

i.e., the disturbance will be constant between $\sigma(t)$ arrivals. The target state set-point vector is given as

$$x_{sp} = (x_{sp,1}, 0)^T \quad (B.7)$$

and the control objective is to have $x(t) \rightarrow x_{sp}$.

The plant is first checked to see if it is completely controllable. The controllability matrix for the given plant is

$$S = (B \mid AB) = \begin{bmatrix} 1 & 3 \\ 2 & 2 \end{bmatrix}.$$

The determinant of S is not zero, hence the rank of S is equal to two ($=n$) and thus the plant is completely state controllable. This

means that $u_p(t) = -Kx(t)$ or $u_p(t) = -K\hat{x}(t)$ can be designed as required to settle out the transient portion of the plant response.

Next, the condition specified by Equation (1.45) is used to check the constructability of the system. Doing this, one finds that

$$T = (C^T \mid A^T C^T) = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}.$$

Since the determinant of T is not zero, the rank of T is also equal to n , which is two. Therefore, the system is completely constructible, i.e., completely observable. A composite state reconstructor can thus be designed to provide estimates of the plant and disturbance states for the controller. For the purposes of this example, which is not to show how well a state reconstructor works, it will be assumed that all necessary state information is available from an "ideal" reconstructor.

Before proceeding with the example, let us discuss the linear

algebra associated with it. First of all, since this is a second-order plant, it is in R^2 state-space. Since B is a 2×1 matrix of rank 1 B does not span the state space, hence $Bu_d(t)$ will have a limited set of attainable points. In this example, the set of attainable points consists of the straight line in R^2 which passes through the origin and has slope $m = -2$. This line, which is $R(\tilde{A}^{-1}B)$, is a subspace L of the vector space R^2 since, for two vectors r and s in L and two real numbers α and β , $\alpha r + \beta s = x$ where x is a vector which is contained in L .

The space R^2 is a Hilbert space and thus can be represented as the direct sum of $N(B)$ and its orthogonal complement, $R(B)$, since $N(B)$ is closed in a Hilbert space. Therefore, any vector which is in R^2 , but not contained entirely in L , can be represented as $x = y + \bar{z}$, with y in L and $\bar{z} = (x - y)$ in the orthogonal complement of L .

B.3 Error Evolution Equations

This example is to be developed in terms of the error between $x(t)$ and x_{sp} since this error is the quantity to be directly minimized. The error vector is defined to be

$$\epsilon(t) = x_{sp} - x(t) \quad (B.8)$$

and the error dynamics are given by Equation (3.10) as

$$\dot{\epsilon}(t) = A\epsilon(t) - Ax_{sp} - Bu(t) - Fw(t). \quad (B.9)$$

The term Ax_{sp} is the set-point disturbance and the term $Fw(t)$ is the external disturbance.

The example will be solved first by using the technique illustrated in Section 3.5. According to the control allocation procedure of Section 3.5, $u(t)$ is allocated as

$$u(t) = u_p(t) + u_{ds}(t) + u_{dw}(t) \quad (3.10)$$

where $u_p(t)$ is designed to stabilize the plant, $u_{ds}(t)$ is to be designed to minimize the set-point disturbance effects and $u_{dw}(t)$ is to be designed to minimize the external disturbance effects. For this case, if Equation (B.10) is substituted into Equation (B.9) the error dynamics can be re-expressed as

$$\dot{\epsilon}(t) = (A+BK)\epsilon(t) - (Ax_{sp} + Bu_{ds}(t)) - (FH_z(t) + Bu_{dw}(t)) \quad (B.11)$$

The method of solution illustrated in Section 3.6 will next be applied to this example. By making the control allocation as in Section 3.6, one also has $u(t)$ allocated as

$$u(t) = u_p(t) + u_{ds}(t) + u_{dw}(t) \quad (B.12)$$

where $u_p(t)$ is to be designed to stabilize the plant and u_{ds} , u_{dw} are to be designed to minimize the combined disturbance effects. If Equation (B.12) is substituted into Equation (B.9), the error dynamics can be expressed in the same form as was obtained in Equation (B.11),

$$\dot{\epsilon}(t) = (A+BK)\epsilon(t) - (Ax_{sp} + Bu_{ds}(t)) - (FH_z(t) + Bu_{dw}(t)) \quad (B.13)$$

The general solution to Equation (B.11) is given by Equation (3.21) as

$$\epsilon(t) = e^{\tilde{A}t} \epsilon(0) - \int_0^t e^{\tilde{A}(t-\tau)} (Ax_{sp} + Bu_{ds} + FH_z + Bu_{dw}) d\tau \quad (B.14)$$

and the general solution to Equation (B.13) is also found to be given as

$$\epsilon(t) = e^{\tilde{A}t} \epsilon(0) - \int_0^t e^{\tilde{A}(t-\tau)} (Ax_{sp} + Bu_{ds} + FH_z + Bu_{dw}) d\tau \quad (B.15)$$

with $\tilde{A} = A+BK$. We are interested, however, in the steady-state solution for $\epsilon(t)$, i.e., ϵ_{ss} , in each case. The solutions for these will be developed in the following sections.

For the solution of the present example, the value of the stabilizing gain K was chosen to be $K=(-3., -0.36)$. If one uses this gain

matrix K in the definition of $\tilde{A} = A + BK$, the result is

$$\tilde{A} = \begin{bmatrix} -2. & 0.64 \\ -6. & 0.28 \end{bmatrix} .$$

Let us assume that the target set-point is $x_{sp} = (10., 0.)^T$ and that the external disturbance is $z = 5$. By using these values, plus the plant model given in Equations (B.2) and (B.3) and the matrix calculated for $\tilde{A}$, one can proceed to calculate the following terms which will be needed in the following sections to solve for ϵ_{ss} .

$$\tilde{A}^{-1} = \begin{bmatrix} 0.0854 & -0.1951 \\ 1.829 & -0.6098 \end{bmatrix}$$

$$\tilde{A}^{-1}(Ax_{sp} + FHz) = (0.3055, 24.386)^T$$

$$\tilde{A}^{-1}B = (-0.3048, 0.6094)^T$$

$$(\tilde{A}^{-1}B)^{\dagger} = (-0.6565, 1.3126)$$

$$B^{\dagger} = (0.2, 0.4)$$

$$BB^{\dagger} = \begin{bmatrix} 0.2 & 0.4 \\ 0.4 & 0.8 \end{bmatrix} .$$

B.4 Solution for ϵ_{ss} (Section 3.5), Assuming $w = 0$

In this section the method of solution given in Section 3.5 will be utilized. The external disturbance will be assumed to be zero so that $u_d(t) = u_{ds}(t)$ and minimization of the set-point disturbance can be examined. The steady-state solution for $\epsilon(t)$ in this case is found from Equation (3.26) as

$$\epsilon_{ss2} = \tilde{A}^{-1}(Ax_{sp} + Bu_{ds}) = \begin{pmatrix} 0.0854 \\ 1.829 \end{pmatrix} x_{sp1} + \begin{pmatrix} -0.305 \\ 0.609 \end{pmatrix} u_{ds} .$$

(B.16)

A minimum norm control vector u_{ds}^* which will give a minimum norm steady-state error $||\epsilon_{ss2}^*||_I$ is found from Equation (3.27) as

$$u_{ds}^* = -2.3447x_{sp,1} . \quad (B.17)$$

For $x_{sp,1} = 10.$, the control will thus be

$$u_{ds}^* = -23.447 . \quad (B.18)$$

The value for ϵ_{ss2}^* is next found from Equation (3.28) as

$$\epsilon_{ss2}^* = (8., 4.)^T . \quad (B.19)$$

Upon examining Equation (B.16), it is evident that if one wishes to steer ϵ_{ss21} to zero, with $x_{sp,1}=10.$, then $u_{ds}=2.8$. Using this value for u_{ds} in a digital simulation of this example resulted in the trajectory shown in Figure B-1 where it can be seen that $x_1 \rightarrow x_{sp,1}$, and therefore, $\epsilon_{ss21} \rightarrow 0$. In a similar manner, if one wishes to steer ϵ_{ss22} to zero, with $x_{sp,1}=10.$, then a value of $u_{ds}=-30.$ will be required. As shown in Figure B-2, $u_{ds}=-30.$ essentially steers ϵ_{ss22} to zero. Choosing u_{ds} in this manner to zero out the error for one or the other of the two states is an example of control design for critical state-variables. However, neither $u_{ds} = 2.8$ nor $u_{ds} = -30.$ gives a minimum for $||\epsilon_{ss2}||_I$. The two steady-state points achieved with $u_{ds} = 2.8$ and $u_{ds} = -30.$ serve to define a straight line along which all the "achievable set-points" lie for the given B matrix and for $w = 0$.

Let us next examine the values obtained for $||\epsilon_{ss2}||_I$ for a variety of choices for u_{ds} bounded by the above values of 2.8 and -30. Table B-1 tabulates the results, which are graphed in Figure B-3. It is evident from the curve in Figure B-3 that there is a definite minimum for $||\epsilon_{ss2}||_I$ lying between the boundary values associated with the two individual "critical" state cases. In order to check this

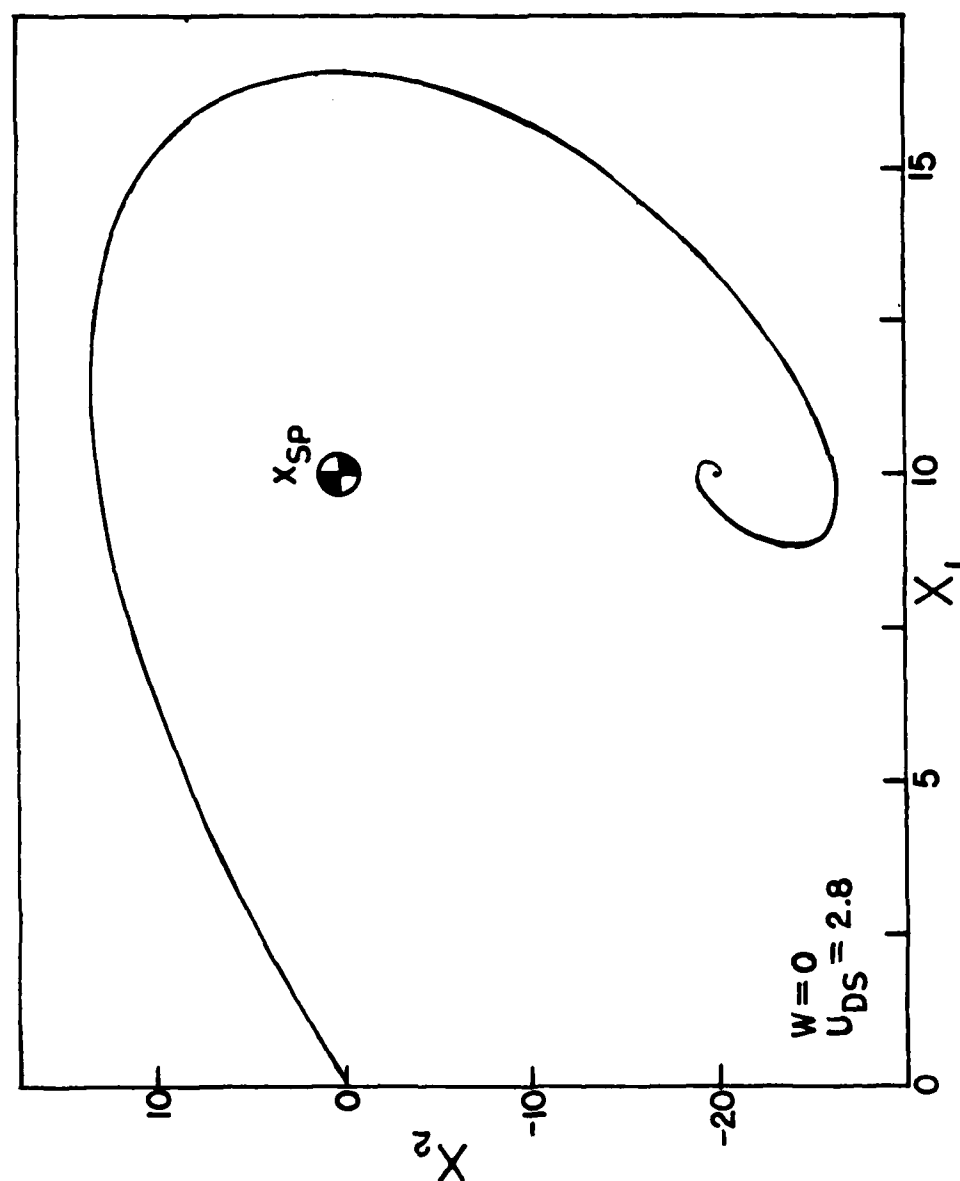


Figure B-1. Plant state trajectory when x_1 is the critical state-variable.

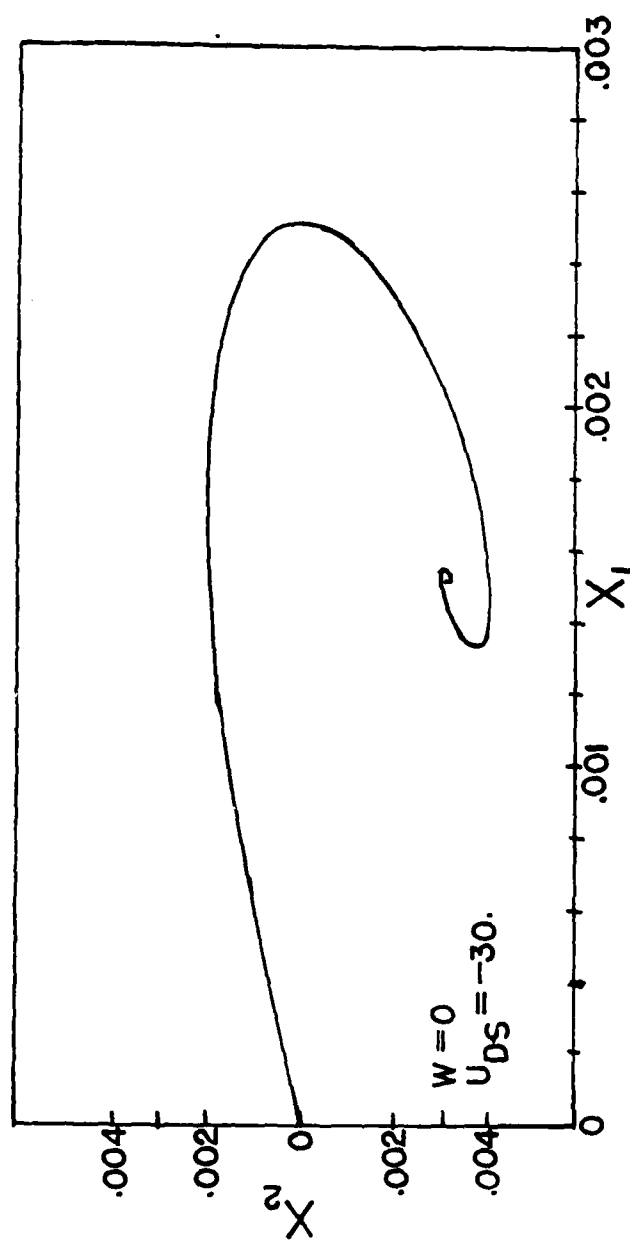


Figure B-2. Plant state trajectory when x_2 is the critical state-variable.

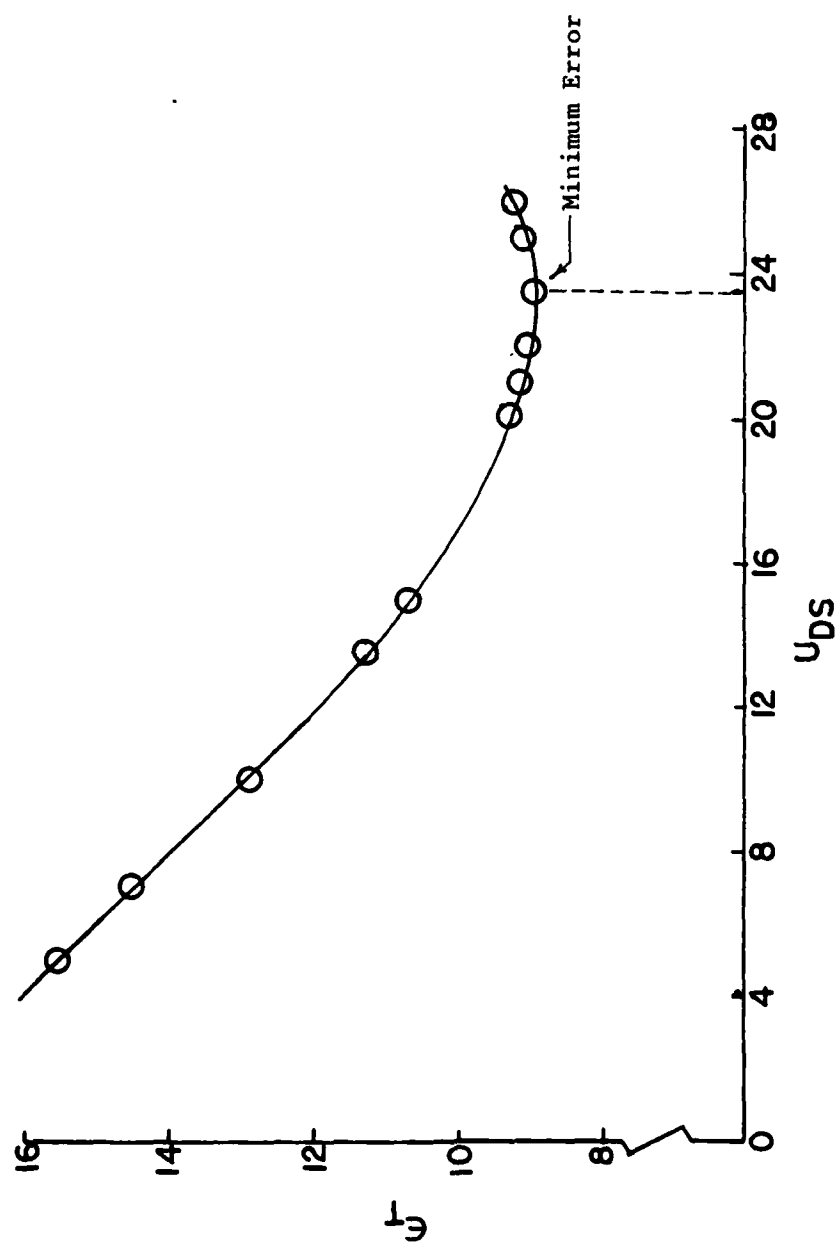


Figure B-3. Steady-state error as a function of u_{DS} .

Table B-1. u_{ds} vs $||\epsilon_{ss2}||_I$

u_{ds}	$ \epsilon_{ss2} _I$
2.8	20.
0.0	18.42
- 1.11	17.74
- 7.14	14.34
-10.0	12.81
-13.6	11.20
-20.0	9.25
-22.0	9.0
-24.0	9.0
-30.0	9.999

minimum and verify the result obtained in Equation (B.18) let us use elementary calculus to find a minimum for the equation $\epsilon_T = ||\epsilon_{ss2}||_I$ using Equation (B.16) to define ϵ_{ss2} . We need to solve $\partial\epsilon_T/\partial u_{ds} = 0.$, where

$\epsilon_T = (\epsilon_{ss21}^2 + \epsilon_{ss22}^2)^{\frac{1}{2}} = ((0.0854x_{sp,1} - 0.305u_{ds})^2 + (1.829x_{sp,1} + 0.609u_{ds})^2)^{\frac{1}{2}}$ is the magnitude of the total steady-state error vector in terms of x_{sp} and u_{ds} . To proceed, we take

$$\partial\epsilon_T/\partial u_{ds} = 0 = \frac{1}{2}(\cdot)^{-\frac{1}{2}}(2.1756x_{sp,1} + 0.9278u_{ds})$$

and solve for u_{ds} in terms of x_{sp} as

$$u_{ds} = -2.1756x_{sp,1}/0.9278.$$

For $x_{sp,1} = 10.$, the resultant control is thus

$$u_{ds} = -23.449 \quad (B.20)$$

which is the control value corresponding to a minimum ϵ_T . This result is in agreement with that obtained in Equation (B.18) using generalized inverses.

It has thus been shown that a value of $u_{ds} = -23.449$ will give a minimum norm solution, with the norm weighted by the identity matrix, for the steady-state error vector in the absence of external

disturbances. The value obtained for $||\epsilon_{ss2}||_I$ with $u_{ds} = -23.449$ is (using Equation B.19)

$$\epsilon_T = ||\epsilon_{ss2}||_I = 8.944, \quad (B.21)$$

and this should be the length of the vector drawn from $x_{sp} = (10., 0.)^T$ perpendicular to the line representing the set of achievable set-points for the given $R(B)$. Figure B-4 defines the line of achievable set-points for several values of u_{ds} (the reader should note the unequal axis scales in Figure B-4 before trying to draw a perpendicular from x_{sp} to $R(\tilde{A}^{-1}B)$).

B.5 Solution for ϵ_{ss} With $w \neq 0$

In the previous section only the effects due to the set-point disturbance term were considered. In this section the additional effects on the plant due to the presence of piecewise constant external disturbances (B.5), (B.6) will be considered. For this purpose, the additional term which must be included in the solution for a total steady-state error vector is given by Equation (3.23) as

$$\epsilon_{ss1} = \tilde{A}^{-1}(FHz + Bu_{dw}) = \begin{pmatrix} -0.1097 \\ 1.2192 \end{pmatrix} z + \begin{pmatrix} -0.3048 \\ 0.6094 \end{pmatrix} u_{dw}. \quad (B.22)$$

In a manner similar to that for the set-point disturbance case, it can be seen from Equation (B.22) that u_{dw} can be chosen so that one or the other of the error components due to external disturbances can be reduced to zero. On the other hand a minimum norm $u_{dw}^* = u_{dw}^*$, which will result in a minimum norm solution for the total external disturbance effect can be found from Equation (3.24) as

$$u_{dw}^* = -(\tilde{A}^{-1}B)^+ \tilde{A}^{-1}FHz = -1.6723z. \quad (B.23)$$

For the chosen value of $z = 5$, therefore,

$$u_{dw}^* = -8.3615. \quad (B.24)$$

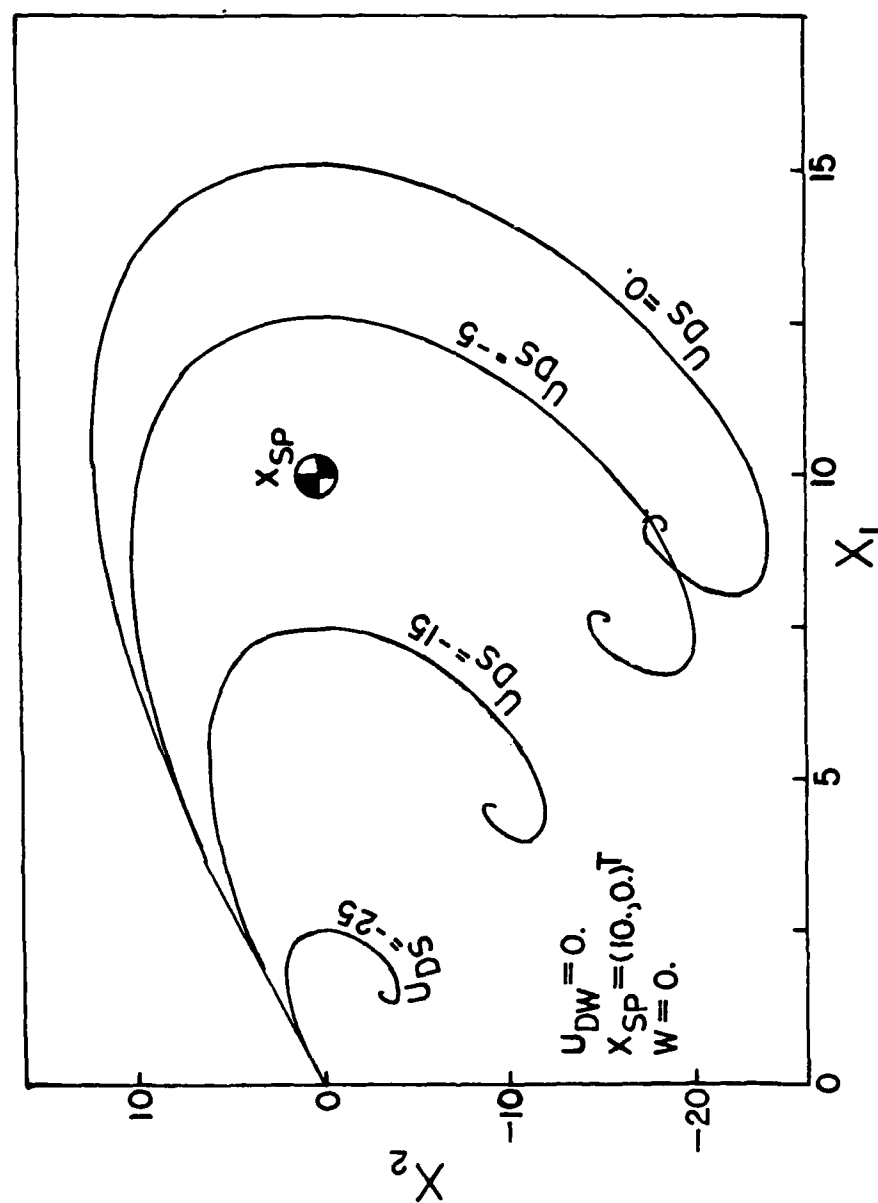


Figure B-4. Plant state trajectories as a function of u_{ds} , with $w=0$.

The value of ϵ_{ss1}^* is then found from Equation (3.25) as

$$\epsilon_{ss1}^* = (I - (\tilde{A}^{-1}B)(\tilde{A}^{-1}B)^{\dagger})\tilde{A}^{-1}FHz = (0.4, 0.2)^T z, \quad (B.25)$$

and for $z = 5$,

$$\epsilon_{ss1}^* = (2., 1.)^T. \quad (B.26)$$

The total error vector ϵ_{ss}^* can now be found as

$$\epsilon_{ss}^* = \epsilon_{ss1}^* + \epsilon_{ss2}^* = (10., 5.)^T \quad (B.27)$$

and the length of ϵ_{ss}^* is

$$||\epsilon_{ss}^*||_I = 11.18. \quad (B.28)$$

Figures B-5, B-6 and B-7 give several trajectories obtained from the digital simulation of this example, with $z = 5$, for various combinations of u_{ds} and u_{dw} . The run conditions and results are presented in Table B-2.

Table B-2. Effects of z on $||\epsilon_{ss}||$

u_{ds}	u_{dw}	z	$ \epsilon_{ss} $
2.8	0.	0.	20.
2.8	0.	5.	26.1
2.8	$-0.3601z$	5.	25.
-30.0	0.	0.	10.
-30.0	0.	5.	12.5
-30.0	$-2.0z$	5.	11.25
-23.449	0.	0.	8.885
-23.449	0.	5.	12.55
-23.449	$-1.672z$	5.	11.18

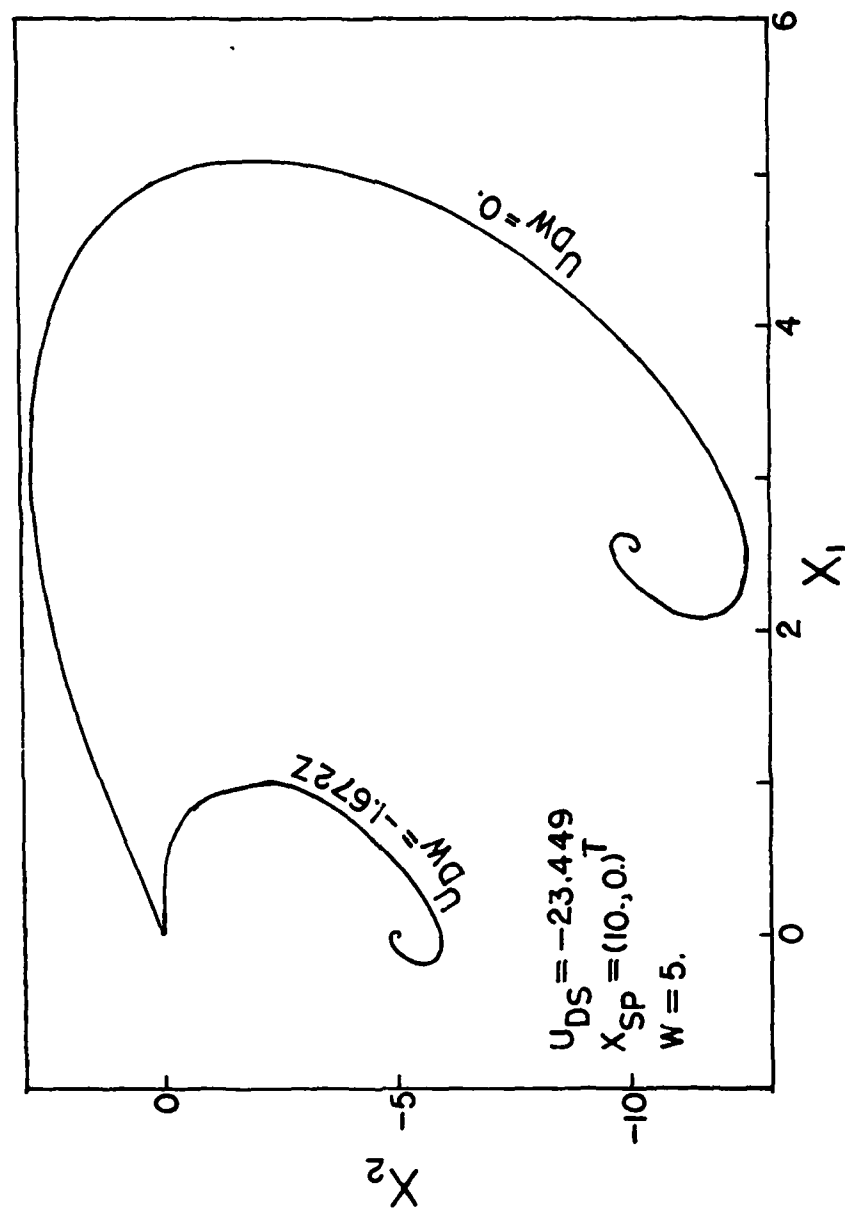


Figure B-5. Plant state trajectories as a function of u_{dw} , with $w=5.$, $u_{ds}=-23.449.$

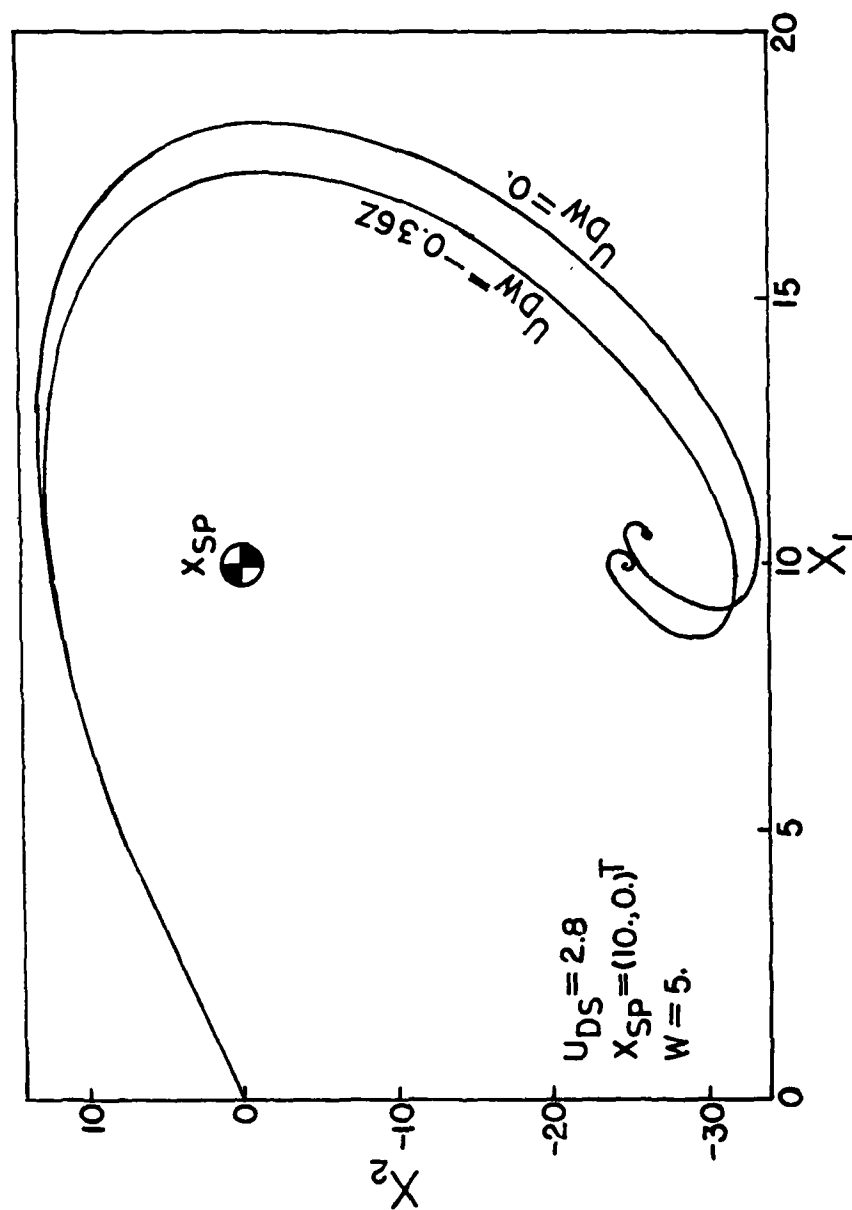


Figure B-6. Plant state trajectories as a function of u_{dw} , with $w=5.$, $u_{ds}=2.8.$

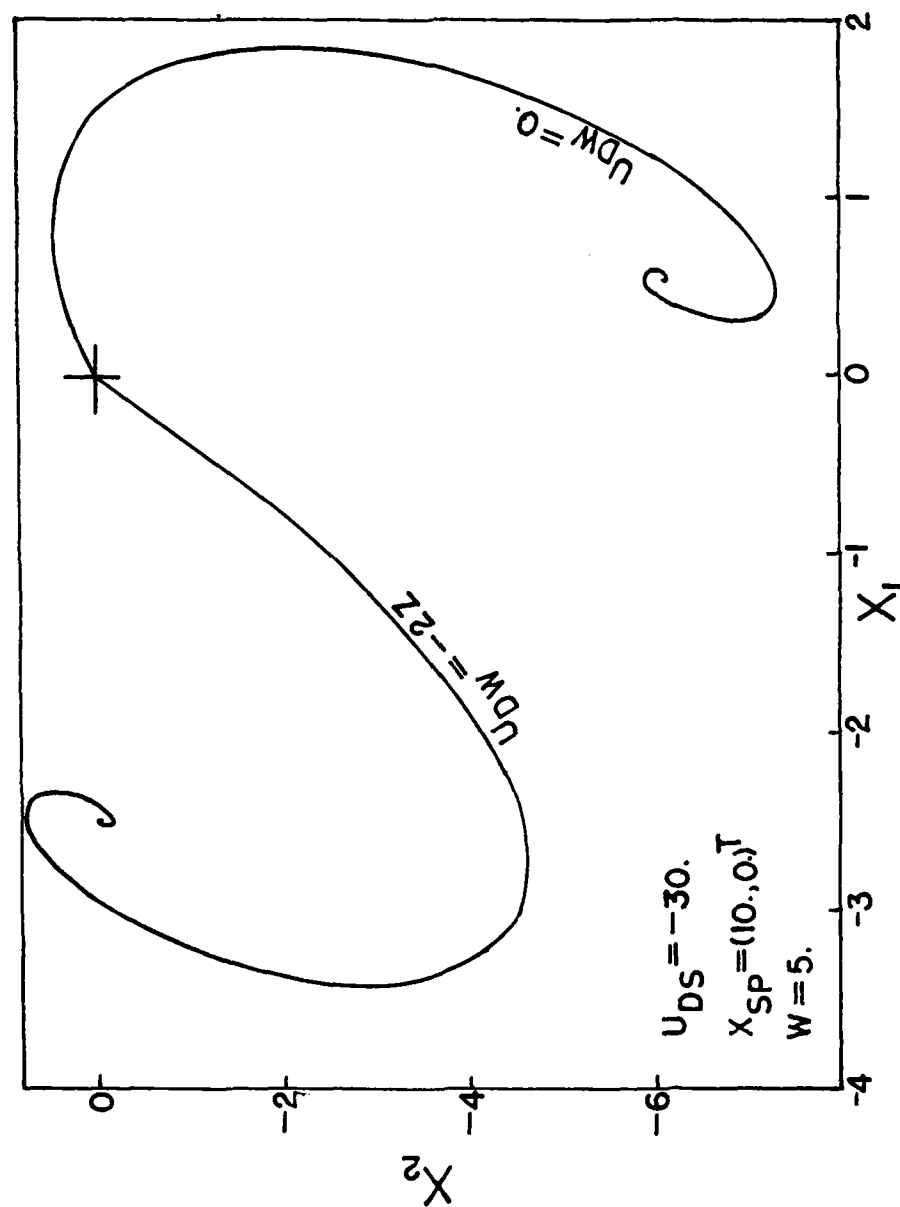


Figure B-7. Plant state trajectories as a function of u_{dw} , with $w=5.$, $u_{ds}=-30.$

B.6 Solution for ϵ_{ss} Using Minimization Technique of Section 3.6

In this section, u_{ds} will be chosen to give a minimum norm for $\|Ax_{sp} + Bu_{ds}\|_I$ and u_{dw} will be chosen to give a minimum norm for $\|FH_z + Bu_{dw}\|_I$, i.e., the disturbance effects on the error dynamics will be minimized instead of the disturbance effects on the steady-state error.

From Equation (3.36), a minimum norm $u_{ds} = u_{ds}^*$ which will minimize $\|Ax_{sp} + Bu_{ds}\|_I$ is found to be

$$u_{ds}^* = -B^+ Ax_{sp} = -0.2x_{sp1} = -2. \quad (B.29)$$

and a minimum norm $u_{dw} = u_{dw}^*$ which will minimize $\|FH_z + Bu_{dw}\|_I$ is found from Equation (3.37) as

$$u_{dw}^* = -B^+ FH_z = -0.6z = -3. \quad (B.30)$$

For the disturbance control vectors shown in Equations (B.29) and (B.30), the resulting steady-state error vector is found from Equation (3.39) to be

$$\epsilon_{ss}^* = \tilde{A}^{-1}(I - BB^+)(Ax_{sp} + FH_z) = (1.83, 21.339)^T. \quad (B.31)$$

The magnitude of ϵ_{ss}^* is thus

$$\|\epsilon_{ss}^*\|_I = 21.42 \quad (B.32)$$

versus the 11.18 obtained using the technique of Section 3.5. As a matter of interest, if z is set to zero in Equation (B.31), the miss-distance due to the set-point disturbance is found to be

$$\epsilon_{ss}^* = (1.464, 17.071)^T, \quad (B.33)$$

so that

$$\|\epsilon_{ss}^*\| = 17.134 \quad (B.34)$$

as compared to 8.944 from Equation (B.19). If x_{sp} is set to zero in Equation (B.31), the miss-distance due to the external disturbance term is found to be

$$\epsilon_{ss}^* = (0.366, 4.2675)^T \quad (B.35)$$

so that

$$||\epsilon_{ss}^*||_I = 4.283 \quad (B.36)$$

as compared to 2.236 from Equation (B.26).

B.7 Effect of the Feedback Gain Matrix K

The gain matrix K is included in $\tilde{A}$ and will, therefore, directly affect the value of the steady-state error. For control design it is useful to know the effect of K on the Euclidean distance between a given x_{sp} and an x_{ss} attained by the plant, with $z=0$. This question was investigated for the present example with $u_p = -(k_1, k_2)\epsilon$. A digital simulation of the system was used in each instance to generate data. Figure B-8 shows constant contours of distance ϑ as a function of (k_1, k_2) with the disturbance control designed as in Section 3.6 and Figure B-9 shows the contours for the case where the disturbance control is designed as in Section 3.5.

In the first case, the transfer function between x_1 and $x_{sp,1}$ is

$$\frac{x_1}{x_{sp,1}} = \frac{-(k_1+0.2)s - (k_1-0.2k_2)}{s^2 - (2+k_1+2k_2)s + (1-k_1+2k_2)} \quad (B.37)$$

where $u_{ds} = -0.2x_{sp,1}$ and $u_{dw} = -0.6z$. From the Routh-Hurwitz criterion, for a stable system one must have

$$-(2 + k_1 + 2k_2) > 0, \quad \text{and} \quad (1 - k_1 + 2k_2) > 0.$$

Thus, if $k_1 = -2$ then k_2 must be $-1.5 < k_2 < 0$. If $k_1 = -4$, then k_2 must be $-2.5 < k_2 < 1$. This serves to establish stability boundaries for the plant as a function of k_1, k_2 .

In the second case, the transfer function between x_1 and $x_{sp,1}$ is

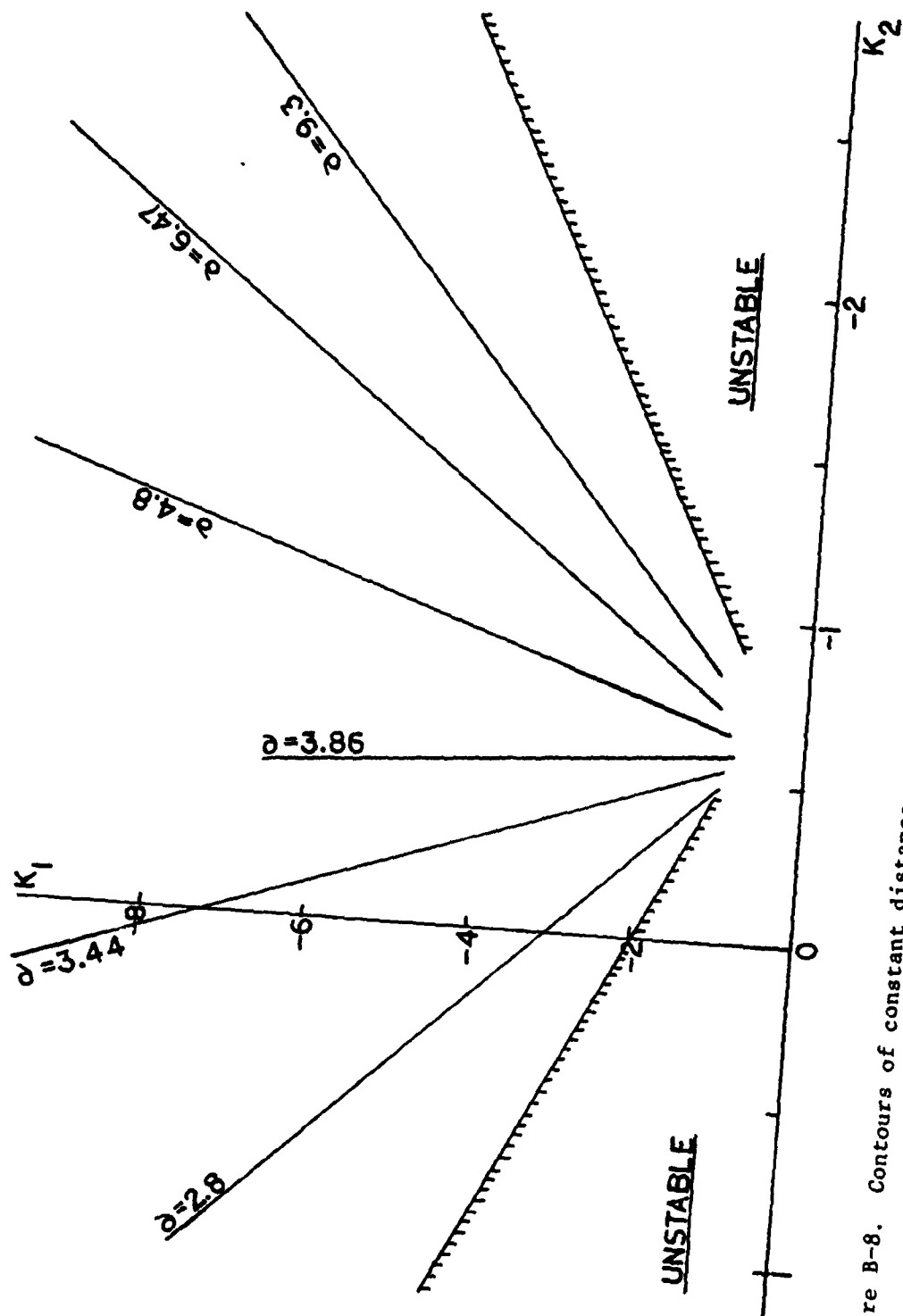


Figure B-8. Contours of constant distance versus gain components, Section 3.6, $w = 0$, $x_{sp} = (2, 0)^T$.

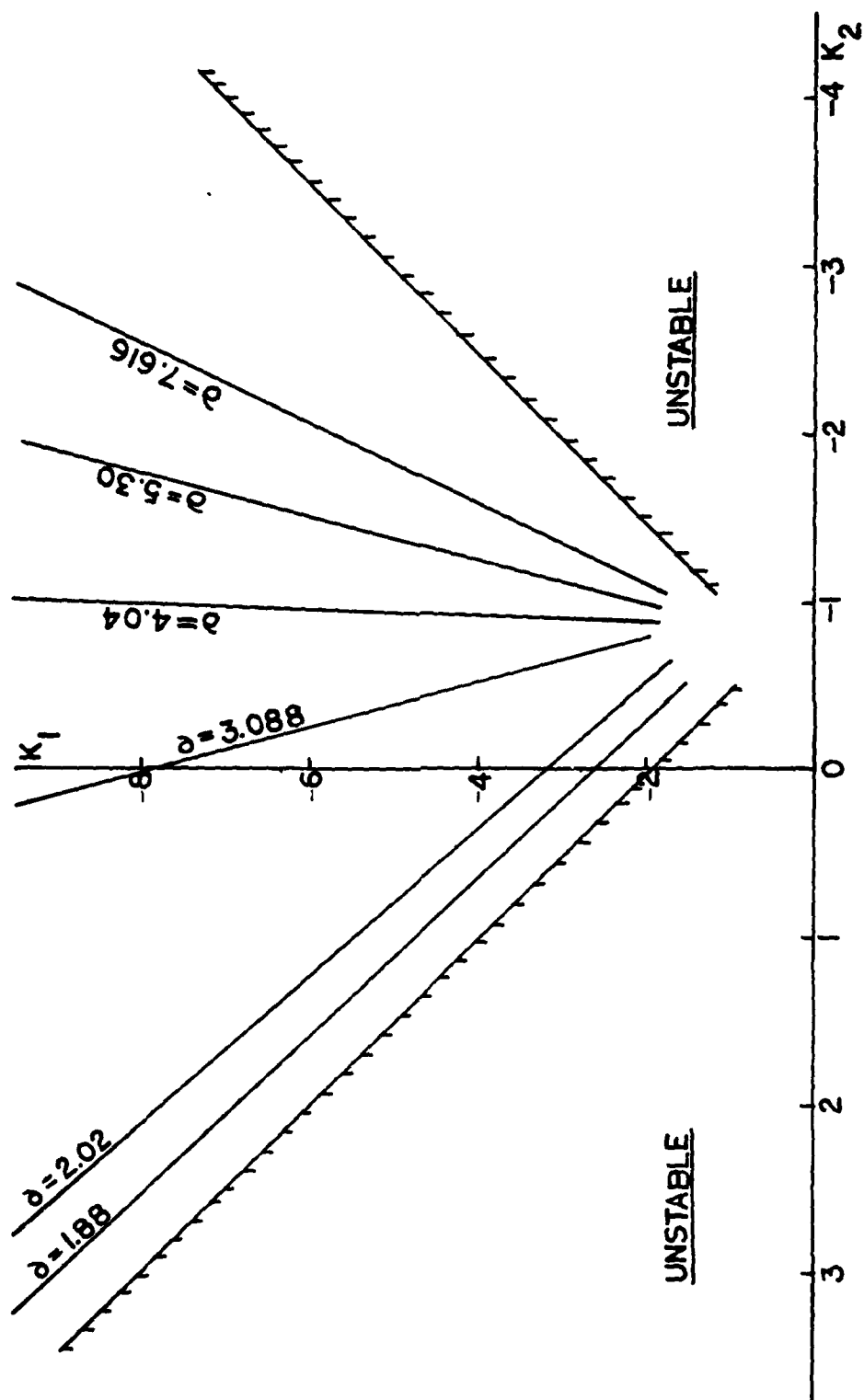


Figure B-9. Contours of constant distance versus gain components, Section 3.5, $w=0$, $x_{sp}=(2,0)^T$.

$$\frac{x_1}{x_{sp,1}} = \frac{c_1(s+1)}{s^2 - (2+k_1+2k_2)s + (1-k_1+2k_2)} \quad (B.38)$$

where c_1 is given by

$$c_1 = - \frac{4k_1^3 + 12k_1^2 + 24k_1k_2 + 9k_1 - 2k_2 + 16k_1^2k_2 + 16k_1k_2^2 - 1}{(2k_1 + 4k_2)^2 + 8k_1 + 16k_2 + 5} \quad (B.39)$$

and u_d is

$$u_d = \frac{4k_1^2 + 4k_1 - 2k_2 + 8k_1k_2 - 1}{(2k_1 + 4k_2)^2 + 8k_1 + 16k_2 + 5} \quad (B.40)$$

The characteristic equation, and thus the stability boundaries, are the same as for the first case.

APPENDIX C

APPENDIX C

EXAMPLE PROBLEMS FOR DIRECT DISTURBANCE MINIMIZATION,
THIRD-ORDER STATE SET-POINT STABILIZATION CRITICAL
VARIABLE PROBLEM, SECTION 3.9

C.1 Introduction

This appendix contains several examples using a third-order plant to illustrate the methods from Section 3.9 for direct disturbance minimization in critical state variable type problems. The examples are formulated in terms of the error vector between the state set-point and the state vector consistent with the developments in Section 3.9.

C.2 Plant and Disturbance Models

The plant and disturbance models for the examples in this Appendix are given by

$$\begin{pmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \end{pmatrix} = \begin{bmatrix} -1. & -2. & -2. \\ 0. & -1. & 1. \\ 1. & 0. & -1. \end{bmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} + \begin{pmatrix} 1. \\ 1. \\ 0. \end{pmatrix} u + \begin{pmatrix} 2. \\ 2. \\ 1. \end{pmatrix} w \quad (C.1)$$

and

$$w = z, \quad \dot{z} = \sigma(t). \quad (C.2)$$

The control task is to stabilize a given critical state or state set to zero.

Upon checking the controllability of the plant one finds that

$$\text{rank} [B | AB | A^2 B] = \text{rank} \begin{bmatrix} 1. & -3. & -1. \\ 1. & -1. & 2. \\ 0. & 1. & -2. \end{bmatrix} = 3 = n ;$$

therefore, the plant is completely state controllable. A control vector of the form $u_p = -K\epsilon$ can thus be designed such that $\lim_{t \rightarrow \infty} e^{\tilde{A}t} \epsilon(0) \rightarrow 0$. For the examples to be presented in this Appendix, K is taken to be $K = (-1./7.)(11., 10., 22.)$ and the resulting characteristic values of $\tilde{A} = A+BK$ are $\lambda_1 = -1., \lambda_2 = -2., \lambda_3 = -3.$

C.3 Solutions

C.3.1 Case I. Case I will correspond to Section 3.9.2, where the procedure for accomplishing direct disturbance absorption on critical state variables was covered. In the present example x_1 is considered to be the critical state. It is obvious from examination of Equation (C.1) that a choice of $u_d = -2z$ will cancel the direct disturbance input to x_1 , but what effect will this choice have on the critical variable at steady-state? The unique steady-state plant state vector, using $u_d = -2z$, $u_p = -Kx$, and with the plant state equation written as

$$\dot{x} = \tilde{A}x + Bu_d + FHz, \quad (C.3)$$

under the assumption that the disturbance $z(t)$ has a limiting value z_∞ as $t \rightarrow \infty$, is found by setting $\dot{x}(t)=0$ in Equation (C.3) and solving for an x_{ss} which will satisfy the resulting equation. The result is given by

$$x_{ss} = -\tilde{A}^{-1}Bu_d - \tilde{A}^{-1}FHz. \quad (C.4)$$

The matrix $\tilde{A}$ is calculated to be

$$\tilde{A} = \begin{bmatrix} -18/7 & -24/7 & -36/7 \\ -11/7 & -17/7 & -15/7 \\ 1 & 0 & -1 \end{bmatrix} \quad (C.5)$$

and $\tilde{A}^{-1}$ is found to be

$$\tilde{A}^{-1} = \begin{bmatrix} -0.4048 & 0.5714 & 0.8571 \\ 0.619 & -1.2857 & -0.4286 \\ -0.2619 & 0.5714 & -0.1429 \end{bmatrix}. \quad (C.6)$$

The resultant expression for x_{ss} is then obtained as

$$x_{ss} = -\tilde{A}^{-1}(-2B+FH)z_{\infty} = (-0.8571, 0.4286, 0.1429)^T z_{\infty}. \quad (C.7)$$

For the critical variable x_1 , therefore, the error from set-point would be $\epsilon_{ss1} = 0.8571z_{\infty}$.

If the problem is reformulated in terms of the error vector, with $\dot{\epsilon}(t)$ given by

$$\dot{\epsilon}(t) = \tilde{A}\epsilon(t) - Bu_d - FH z \quad (C.8)$$

then the steady-state solution ϵ_{ss} can be found to be

$$\epsilon_{ss} = \tilde{A}^{-1}(Bu_d + FH z_{\infty}). \quad (C.9)$$

If one substitutes the appropriate matrix terms into Equation (C.9), the expression for ϵ_{ss1} can be obtained as

$$\epsilon_{ss1} = 0.1666u_d + 1.1903z_{\infty}. \quad (C.10)$$

From Equation (C.10) one finds that for $u_d = -7.1447z_{\infty}$, $\epsilon_{ss1} = 0$. Thus, for this value of u_d the critical state x_1 will have been stabilized to zero.

C.3.2 Case II. The second example will illustrate the developments in Section 3.9.3 for direct disturbance minimization on critical state variables. The critical state set for this example is assumed to be $\{x_1, x_3\}$ and x_{sp} is taken to be the origin.

Using Equation (3.75) to see if complete absorption of the disturbance components on the critical state set is possible, one finds that

$$\text{rank}[B|FH|FL] = \text{rank} \begin{bmatrix} 1. & 2. \\ 0. & 1. \end{bmatrix} = 2 \neq \text{rank}(B) = 1; \text{ therefore,}$$

complete absorption of the direct disturbance components on the critical state set is not possible and a minimization technique must be applied.

The basic equation for the error dynamics in this example is given by Equation (3.78) as

$$\dot{\epsilon}(t) = (A + FL)\epsilon(t) - (A + FL)x_{sp} - Bu - FH z. \quad (3.78)$$

For a constant external disturbance z and with $L=0$, the steady-state

error is found from Equation (3.81) where the steady-state error has been re-ordered into two parts, one consisting of the critical states ($\epsilon_{ss}^{(1)}$) and the other of the remaining states ($\epsilon_{ss}^{(2)}$). If one substitutes the appropriate matrices into Equation (3.81) the expression for ϵ_{ss} is found as

$$\begin{pmatrix} \epsilon_{ss1} \\ \epsilon_{ss2} \\ \epsilon_{ss3} \end{pmatrix} = \left[\begin{array}{cc|c} -0.4048 & 0.8571 & 0.5714 \\ 0.619 & -0.4286 & -1.2857 \\ \hline -0.2619 & -0.1429 & 0.5714 \end{array} \right] \begin{pmatrix} 1. \\ 0. \\ 1. \end{pmatrix} u_d + \begin{pmatrix} 2. \\ 1. \\ 2. \end{pmatrix} z \quad (C.11)$$

From the definitions in Equation (3.82) for the critical state set, i.e.,

$\epsilon_{ss}^{(1)} = \hat{F}_1 + \hat{B}_1 u_d$, one finds $\hat{F}_1$ and $\hat{B}_1$ to be

$$\hat{F}_1 = \begin{bmatrix} -0.4048 & 0.8571 & 0.5714 \\ 0.619 & -0.4286 & -1.2857 \end{bmatrix} \begin{pmatrix} 2. \\ 1. \\ 2. \end{pmatrix} z = \begin{pmatrix} 1.1903 \\ -1.762 \end{pmatrix} z \quad (C.12)$$

$$\hat{B}_1 = \begin{bmatrix} -0.4048 & 0.8571 & 0.5714 \\ 0.619 & -0.4286 & -1.2857 \end{bmatrix} \begin{pmatrix} 1. \\ 0. \\ 1. \end{pmatrix} = \begin{pmatrix} 0.1666 \\ -0.6667 \end{pmatrix} \quad (C.13)$$

Since $\hat{B}_1$ is of full column rank, its generalized inverse can be found from the "normal equation", Equation (A.22) as

$$\hat{B}_1^+ = (0.3528, -1.4118) \quad (C.14)$$

Therefore, from Equation (3.84) the minimum norm control u_d^* which will give a minimum norm for $\epsilon_{ss}^{(1)}$ is calculated to be

$$u_d^* = -\hat{B}_1^+ \hat{F}_1 = -2.9075z \quad (C.15)$$

The steady-state error for the critical state set can now be found from Equation (3.85) as

$$\epsilon_{ss}^{(1)*} = \begin{bmatrix} 0.9412 & 0.2352 \\ -0.2352 & 0.0588 \end{bmatrix} \begin{pmatrix} 1.1903z \\ -1.762z \end{pmatrix} = \begin{pmatrix} 0.7059 \\ 0.1764 \end{pmatrix} z \quad (C.16)$$

For the non-critical state set given by Equation (3.83), i.e., $\epsilon_{ss}^{(2)} =$

$\hat{F}_2 + \hat{B}_2 u_d^*$, with

$$\hat{F}_2 = (-0.2619, -0.1429, 0.5714)(2., 1., 2.)^T z = -0.4761z \quad (C.17)$$

$$\hat{B}_2 = (-0.2619, -0.1429, 0.5714)(1., 0., 1.)^T z = 0.3095, \quad (C.18)$$

one finds from Equation (3.86) that the steady-state error for the non-critical state set with u_d^* from Equation (C.15) is

$$\epsilon_{ss}^{(2)} = (0.4761 - 0.8999)z = -0.4238z. \quad (C.19)$$

Thus, the steady-state error magnitudes for the critical and non-critical state sets are found to be

$$||\epsilon_{ss}^{(1)*}||_I = 0.728z, \quad ||\epsilon_{ss}^{(2)}||_I = 0.4238z. \quad (C.20)$$

If the control u_d^* were chosen to minimize $||\epsilon_{ss}||$ rather than $||\epsilon_{ss}^{(1)}||$, it would be given as

$$u_d^* = -\hat{B}^+ \hat{F} = (-0.2933, 1.1737, -0.5449) \begin{pmatrix} 1.1903 \\ -1.762 \\ 0.4761 \end{pmatrix} z = -2.6766z. \quad (C.21)$$

The steady-state error in this case is then found from $\epsilon_{ss}^* = (I - \hat{B}\hat{B}^+) \hat{F}$ to be

$$\epsilon_{ss}^* = \begin{bmatrix} 0.9511 & 0.1955 & -0.0908 \\ 0.1955 & 0.2175 & 0.3633 \\ -0.0908 & 0.3633 & 0.8314 \end{bmatrix} \begin{pmatrix} 1.1903 \\ -1.762 \\ 0.4761 \end{pmatrix} z = \begin{pmatrix} 0.7444 \\ 0.0224 \\ -0.3524 \end{pmatrix} z. \quad (C.22)$$

For the first two components of ϵ_{ss} as given by Equation (C.22) the norm is found to be $0.745z$ which is larger than the comparable norm given in Equations (C.20). This shows that some reduction is possible for the critical state set by following the procedures in Section 3.9.3. Comparing the last component of ϵ_{ss} in Equation (C.20) with that in Equation (C.22) also indicates the growth in $\epsilon_{ss}^{(2)}$ when u_d is calculated to achieve a minimum error norm on the critical set.

APPENDIX D

APPENDIX D

AN EXAMPLE PROBLEM FOR DIRECT DISTURBANCE MINIMIZATION; SECOND-ORDER OUTPUT STABILIZATION, SECTION 4.3

D.1 Introduction

This Appendix contains an example problem illustrating the methods developed in Section 4.3 for minimizing the disturbance effects for an output stabilization type problem when C is a non-singular matrix.

D.2 Plant and Disturbance Models

The plant and disturbance models used in this example are

$$\begin{pmatrix} \dot{x}_1 \\ \dot{x}_2 \end{pmatrix} = \begin{bmatrix} 1. & 1. \\ 0. & 1. \end{bmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} + \begin{pmatrix} 1. \\ 2. \end{pmatrix} u + \begin{pmatrix} 1. \\ 1. \end{pmatrix} w \quad (D.1)$$

$$\begin{pmatrix} y_1 \\ y_2 \end{pmatrix} = \begin{bmatrix} 1. & 0. \\ 0. & 1. \end{bmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} ,$$

$$w = z$$

$$\dot{z} = \sigma ,$$

(D.2)

and the control task is to stabilize $y(t)$ to the origin.

D.3 Solutions

Developing the example in terms of an error vector $\epsilon(t)$, consistent with Section 4.3, the error dynamics are given by Equation (4.11) as

$$\dot{\epsilon}(t) = (CAC^{-1} + CBK)\epsilon(t) - CBu_d - CFw = \hat{C}\epsilon(t) - CBu_d - CFw . \quad (D.3)$$

If one substitutes the appropriate matrices from Equations (D.1) and (D.2) into Equation (D.3), $\hat{C}$ can be expressed as

$$\hat{C} = \begin{bmatrix} 1.+k_1 & 1.+k_2 \\ 2k_1 & 1.+2k_2 \end{bmatrix}. \quad (D.4)$$

The characteristic equation for the homogeneous part of Equation (D.3) is given by

$$|\hat{C}-\lambda I| = \lambda^2 - (k_1+2k_2+2)\lambda + (1+2k_2-k_1). \quad (D.5)$$

In designing a gain matrix K for $u_p = -K\epsilon$, one wishes to have the characteristic equation from Equation (D.5) in the form

$$\lambda^2 - (\lambda_1+\lambda_2)\lambda + \lambda_1\lambda_2. \quad (D.6)$$

Equating terms between Equations (D.5) and (D.6) results in the following equivalencies,

$$\begin{aligned} \lambda_1+\lambda_2 &= k_1+2k_2+2 \\ \lambda_1\lambda_2 &= 1+2k_2-k_1. \end{aligned} \quad (D.7)$$

In order to obtain $\lambda_1=-2$, $\lambda_2=-3$ (which are the values to be used in this example), the gain components must be $k_1=-6$, $k_2=-0.5$. Upon substituting these gain values into Equation (D.4) the expression for $\hat{C}$ becomes

$$\hat{C} = \begin{bmatrix} -5. & 0.5 \\ -12. & 0. \end{bmatrix}. \quad (D.8)$$

The inverse of $\hat{C}$ is next computed to be

$$\hat{C}^{-1} = \begin{bmatrix} 0. & -1./12. \\ 2. & -5./6. \end{bmatrix}. \quad (D.9)$$

From Equation (4.15) the minimum norm control u_d^* which will provide a minimum norm for the steady-state error solution of Equation (D.3) can be found as

$$u_d^* = -(\hat{C}^{-1}CB)^{\dagger}\hat{C}^{-1}CFHz_{\infty}. \quad (D.10)$$

Substitution of the appropriate matrices into Equation (D.10) yields

$$\hat{C}^{-1}CB = (-1./6., 1./3.)^T$$

and the generalized inverse of this term is found to be

$$(\hat{C}^{-1}CB)^{\dagger} = (-6./5., 12./5.) . \quad (D.11)$$

The control u_d^* can now be calculated as

$$u_d^* = -2.9z_{\infty} . \quad (D.12)$$

Using this value for u_d^* in Equation (4.16) the steady-state error can be found to be

$$\epsilon_{ss}^* = (I - \hat{C}^{-1}B(\hat{C}^{-1}B)^{\dagger})\hat{C}^{-1}FHz_{\infty} = (0.4, 0.2)^T z_{\infty} , \quad (D.14)$$

and the "miss distance" is

$$||\epsilon_{ss}^*||_I = (0.16 + 0.04)^{\frac{1}{2}} z_{\infty} = 0.4472z_{\infty} . \quad (D.15)$$

In general, for the example presented in this Appendix with $\hat{C}$ given by Equation (D.4) as

$$\hat{C} = \begin{bmatrix} 1 + k_1 & 1 + k_2 \\ 2k_1 & 1 + 2k_2 \end{bmatrix} ,$$

the inverse of $\hat{C}$ will be

$$\hat{C}^{-1} = \frac{1}{1 + 2k_2 - k_1} \begin{bmatrix} 1+2k_2 & -(1+k_2) \\ -2k_1 & 1+k_1 \end{bmatrix} . \quad (D.16)$$

To determine u_d^* from Equation (D.10) for this general case, one first calculates $\hat{C}^{-1}CB$ as

$$\hat{C}^{-1}CB = (-1/(1+2k_2-k_1), 2/(1+2k_2-k_1))^T . \quad (D.17)$$

Now, re-express Equation (D.17) in simpler form by letting $a = (1+2k_2-k_1)$ so that $\hat{C}^{-1}CB = (-1/a, 2/a)^T$. The generalized inverse of $\hat{C}^{-1}CB$ then becomes

$$(\hat{C}^{-1}CB)^{\dagger} = (-a/5, 2a/5) . \quad (D.18)$$

Equation (4.15) will yield an expression for u_d^* as follows,

$$u_d^* = (-0.4+0.2k_2+0.4k_1)z_{\infty} , \quad (D.19)$$

with the stability constraints (see Section B.7),

$$\begin{aligned} k_1 + 2k_2 + 2 &< 0 \\ -k_1 + 2k_2 + 1 &> 0 \end{aligned} \quad (D.20)$$

From Equation (4.16), a general expression for the steady-state error ϵ_{ss}^* resulting from use of Equation (D.19) is found as

$$\epsilon_{ss}^* = (0.4, 0.2)^T z_\infty \quad (D.21)$$

with the "miss distance" given by

$$||\epsilon_{ss}^*||_I = 0.4472 z_\infty. \quad (D.22)$$

The steady-state error is thus dependent only upon z and not upon z and K . This might be expected since the error is due entirely to a disturbance vector component which cannot be completely absorbed, i.e., a disturbance vector component in $R(B)^\perp$. Thus, the closest approach to the origin in output space will be obtained by minimizing that component, i.e., by having Bu_d equal to the orthogonal projection of FHz on $R(B)$ along $R(B)^\perp$. The dependence of ϵ_{ss} only upon z is also consistent with the fact that use of the Moore-Penrose generalized inverse will give a unique minimum norm least-squares solution for

$$\epsilon_{ss} = \hat{C}^{-1} C(Bu_d + FHz) \quad (4.13)$$

with a non-unique u_d , i.e., varying k_1 and k_2 will change the value of u_d but the minimum norm value for ϵ_{ss} will not change.

The example in this Appendix was simulated on a digital computer and sample results are presented in Figures D-1 through D-6. Two cases were simulated, Case 1 with $K = (-6, -0.5)$ and Case 2 with $K = (-15, 2)$. Figures D-1 and D-4 are plots of y_1 versus time for Cases 1 and 2, respectively. Figures D-2 and D-5 are plots of y_2 versus time for Cases 1 and 2, respectively and Figures D-3 and D-6 are plots of $||\epsilon_{ss}||_I$ versus time for Cases 1 and 2, respectively.

Three runs were made for each case and correspond to the curves labelled 1, 2 and 3 on each figure. Curve 1 is for a run with $(x_1(0), x_2(0)) = (1., 1.)$, $z = 0$, and $u_d = 0$. This run indicates that the system is stabilized to the origin in the absence of disturbances. Curve 2 is for a run with $(x_1(0), x_2(0)) = (1., 1.)$, $z = 2$, $u_d = 0$. This run indicates the effects on the plant of an external disturbance with no disturbance control present. Curve 3 is for a run with $(x_1(0), x_2(0)) = (1., 1.)$, $z = 2$ and $u_d = -2.9z$ (Case 1) or $u_d = -6z$ (Case 2). This run illustrates the effect of the minimizing disturbance control upon the system. As can be seen from Figures D-3 and D-6, the steady-state error was the same in each of the two cases even though different gain values K were used in the computation of u_d .

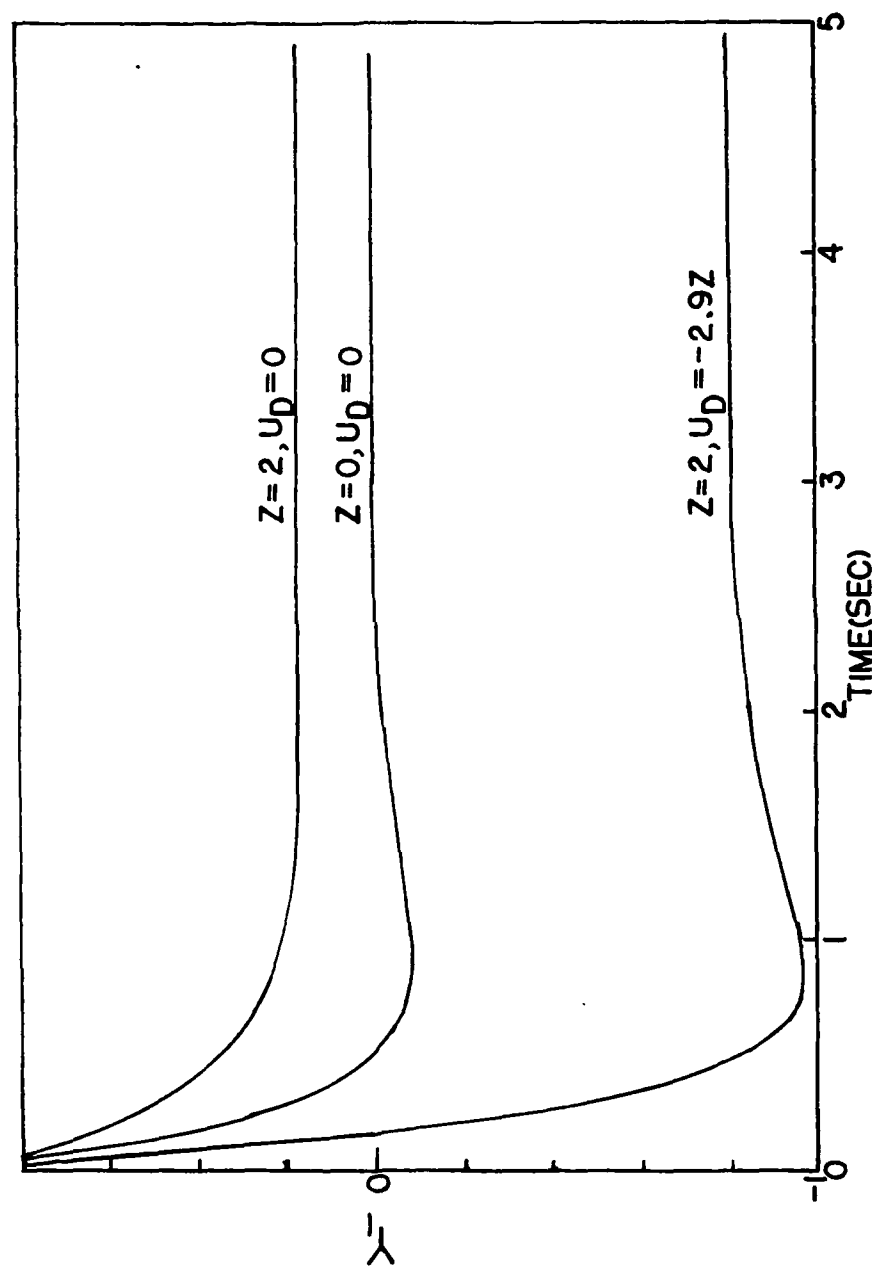


Figure D-1. Y_1 versus time, $K = (-6., -0.5)$.

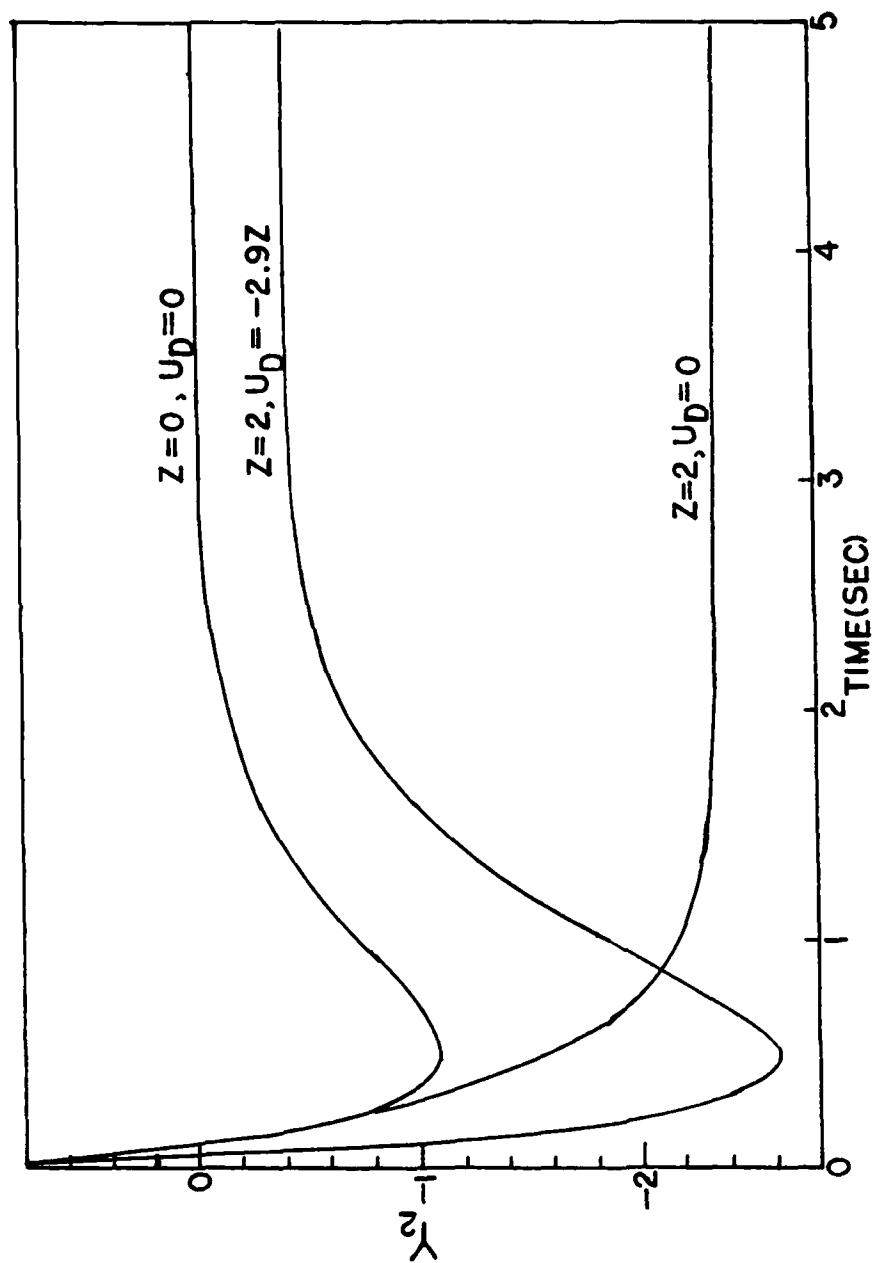


Figure D-2. Y_2 versus time, $K = (-6., -0.5)$.

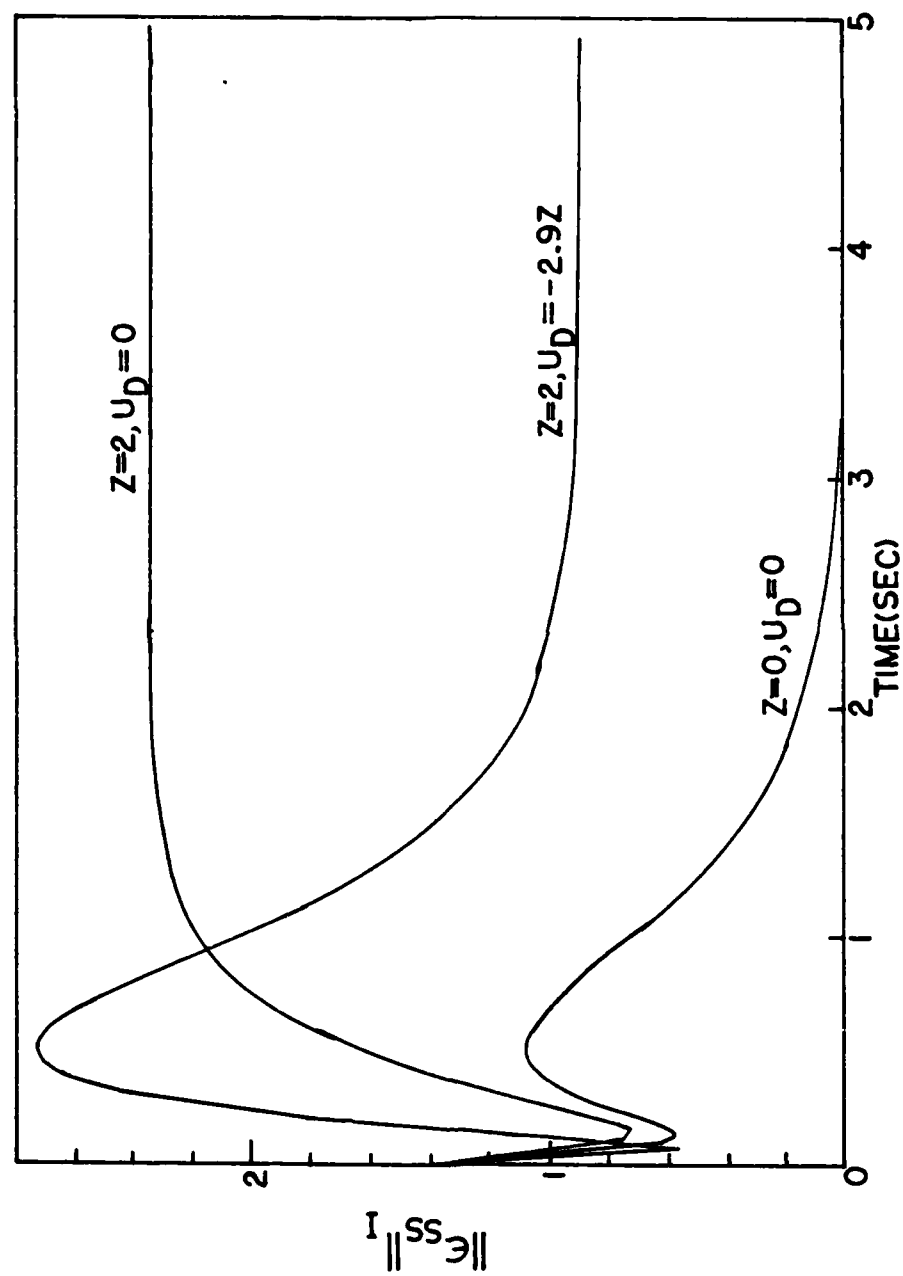


Figure D-3. Norm of ϵ_{ss} versus time, $K = (-6., -0.5)$.

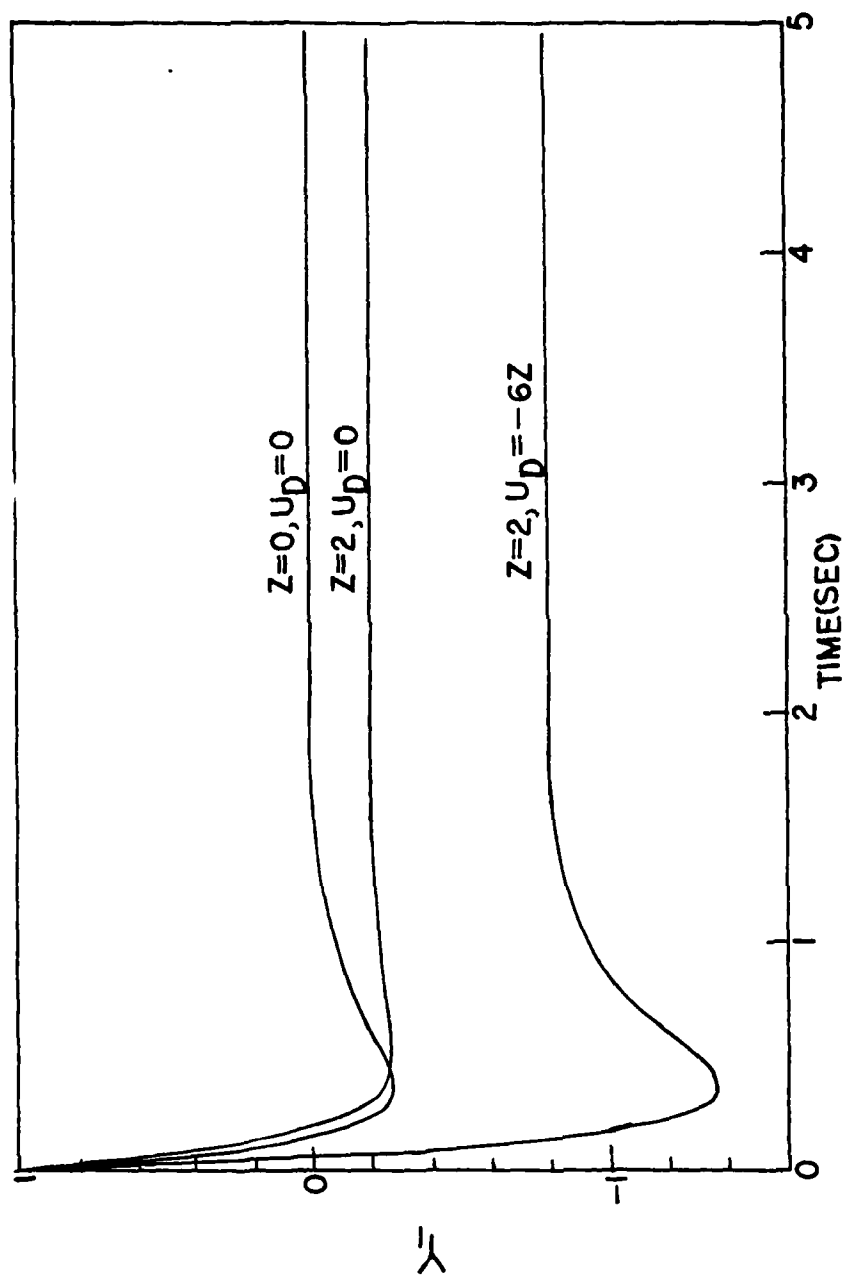


Figure D-4. Y_1 versus time, $K = (-15., 2.)$.

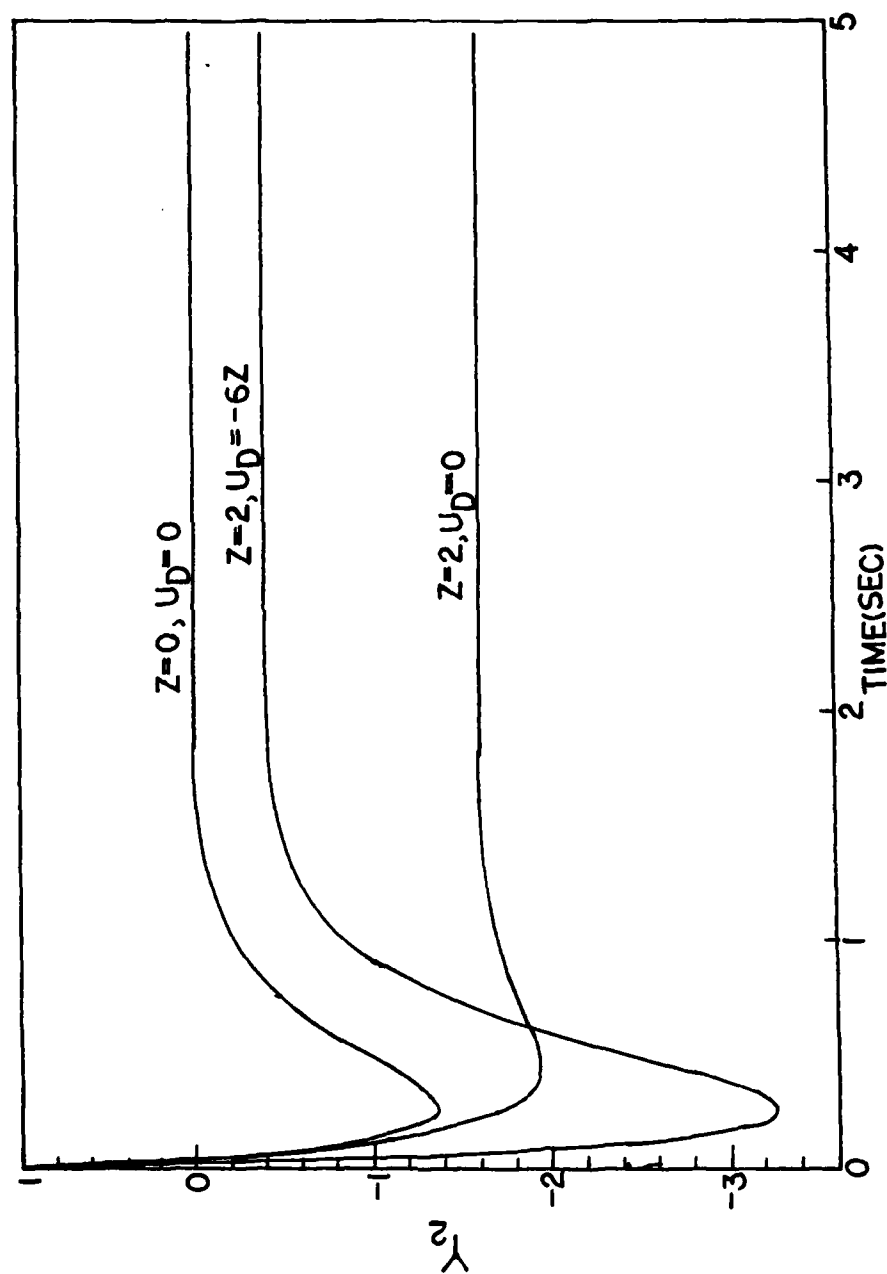


Figure D-5. Y_2 versus time, $K = (-15., 2.)$.

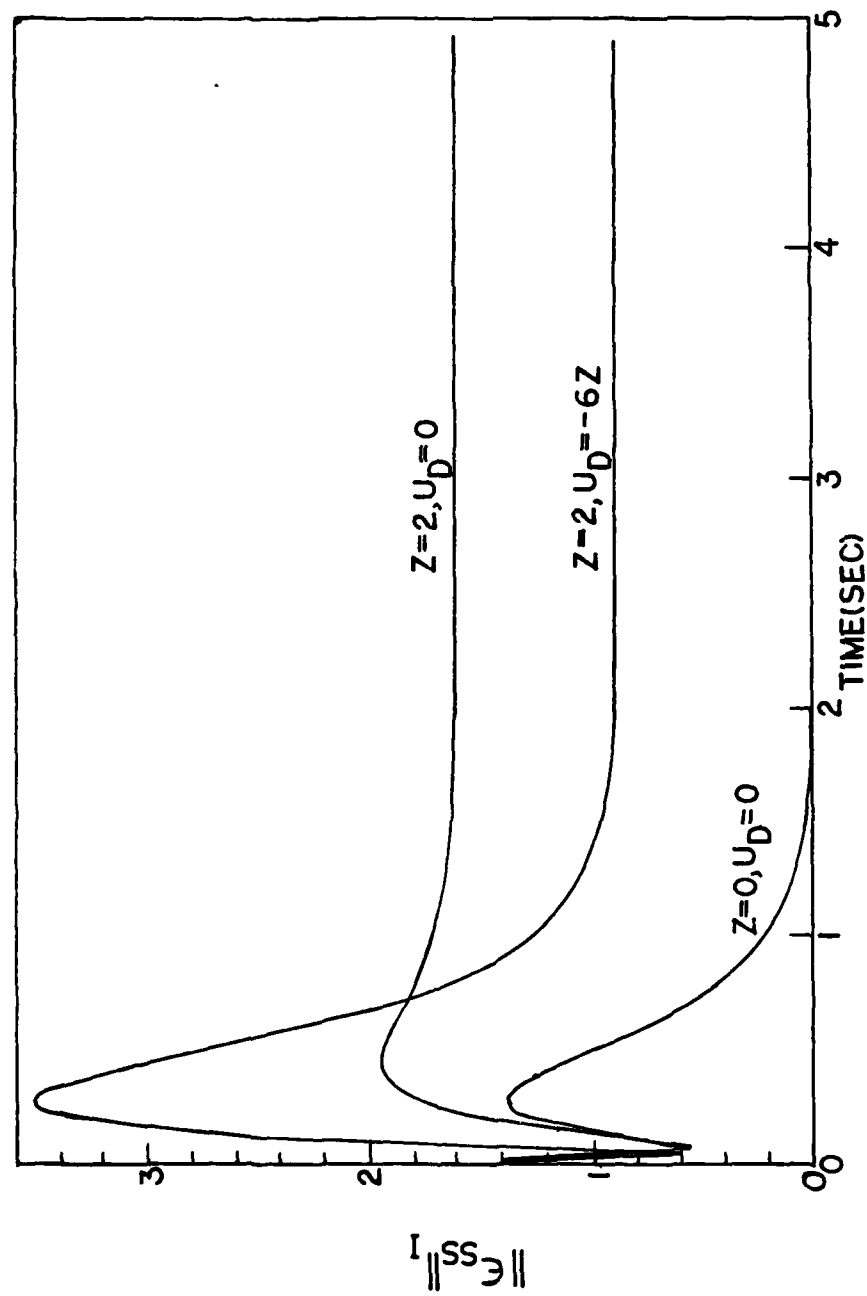


Figure D-6. Norm of ϵ_{ss} versus time, $K = (-15., 2.)$.

APPENDIX E

APPENDIX E

AN EXAMPLE PROBLEM FOR DIRECT DISTURBANCE MINIMIZATION; THIRD-ORDER OUTPUT SET-POINT REGULATION, SECTION 4.6

E.1 Introduction

This Appendix contains two examples illustrating the methods developed in Section 4.6 for regulation of a plant output to a non-zero set-point when the matrix C is a non-invertible matrix. The first example will illustrate the method (Section 4.6.2) whereby $y(t)$ is elevated to the status of a state sub-vector in a new coordinate system. The second example will illustrate a development in the plant state space (Section 4.6.3).

E.2 Plant and Disturbance Models

The plant and disturbance models for the examples in this Appendix are as follows.

$$\begin{pmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \end{pmatrix} = \begin{bmatrix} -1. & -2. & -2. \\ 0. & -1. & 1. \\ 1. & 0. & -1. \end{bmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} + \begin{pmatrix} 2. \\ 0. \\ 1. \end{pmatrix} u + \begin{pmatrix} 1. \\ 1. \\ 1. \end{pmatrix} w \quad (\text{E.1})$$

$$y = (1., 1., 0.)x$$

$$w = z, \quad \dot{z} = \sigma. \quad (\text{E.2})$$

The control task is to steer $y(t)$ to a given $y_{sp} \neq 0$. Using Equation (E.1) one can check for the possibility of complete disturbance absorption, Equation (1.35), as

$$\text{rank}[B|FH] = \text{rank} \begin{bmatrix} 2 & 1 \\ 0 & 1 \\ 1 & 1 \end{bmatrix} = 2 \neq \text{rank}[B] = 1$$

and find that complete disturbance absorption is not possible. Thus, a disturbance minimization method must be applied. From the state controllability matrix [27],

$$P = [B|AB|A^2B] = \begin{bmatrix} 2 & -4 & 4 \\ 0 & 1 & 0 \\ 1 & 1 & -5 \end{bmatrix}, \quad (\text{E.3})$$

it is seen that $\rho(P) = 3 = n$; therefore, the system is completely state controllable. From the output controllability matrix [27],

$$Q = [CB|CAB|CA^2B] = (2, -3, 0), \quad (\text{E.4})$$

it is seen that $\rho(Q) = 1 = m$ hence the system is completely output controllable.

E.3 Development of Solution, Method 1

Since $C = (1, 1, 0)$ is of full rank, a linear transformation T can be found such that $x = T\xi$. The task then is to express the error dynamics in the form given by Equation (4.90) and then design the control vectors as required in order to achieve the control objective of regulating $y(t)$ to y_{sp} . Following the procedure developed in Section 4.6.2, the transformation matrix T is calculated as follows.

From Equation (4.73) one obtains

$$\bar{T}_1 = C = (1, 1, 0), \quad (\text{E.5})$$

and from Equation (4.79) one has that

$$T_1 = C^T(CC^T)^{-1} = (0.5, 0.5, 0)^T. \quad (\text{E.6})$$

Equation (4.77) provides the condition

$$CT_2 = 0 \quad \text{or} \quad (1, 1, 0) \begin{bmatrix} t_1 & t_2 \\ t_3 & t_4 \\ t_5 & t_6 \end{bmatrix} = (0, 0)$$

which can be satisfied by letting

$$(t_1, t_3, t_5)^T = (1, -1, 0)^T$$

and

$$(t_2, t_4, t_6)^T = (0, 0, 1)^T, \text{ i.e.,}$$

$$T_2 = \begin{bmatrix} 1 & 0 \\ -1 & 0 \\ 0 & 1 \end{bmatrix}. \quad (\text{E.7})$$

Given the matrix T_2 , Equation (4.80) can then be utilized to solve for

$\bar{T}_2$ and yields

$$\bar{T}_2 = (T_2^T T_2)^{-1} T_2^T = \begin{bmatrix} 0.5 & -0.5 & 0 \\ 0 & 0 & 1 \end{bmatrix}. \quad (\text{E.8})$$

The transformation matrix T has, therefore, been determined to be

$$T = [T_1 | T_2] = \begin{bmatrix} 0.5 & 1 & 0 \\ 0.5 & -1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \quad (\text{E.9})$$

and its inverse is

$$T^{-1} = [\bar{T}_1 | \bar{T}_2]^T = \begin{bmatrix} 1 & 1 & 0 \\ 0.5 & -0.5 & 0 \\ 0 & 0 & 1 \end{bmatrix}. \quad (\text{E.10})$$

The result of applying the transformation is expressed in Equations (4.84) and (4.85) with terms defined as follows.

$$\tilde{A}_{11} = CAC^T(CC^T)^{-1} = -2.$$

$$\tilde{A}_{12} = CAT_2 = (2, -1)$$

$$\tilde{A}_{21} = (T_2^T T_2)^{-1} T_2^T AC^T(CC^T)^{-1} = (-0.5, 0.5)^T$$

$$\tilde{A}_{22} = (T_2^T T_2)^{-1} T_2^T AT_2 = \begin{bmatrix} 0 & -1.5 \\ 1 & -1 \end{bmatrix}$$

$$\tilde{B}_1 = CB = 2$$

$$\tilde{B}_2 = (T_2^T T_2)^{-1} T_2^T B = (1, 1)^T$$

$$\tilde{F}_1 = CF = 2$$

$$\tilde{F}_2 = (T_2^T T_2)^{-1} T_2^T F = (0, 1)^T.$$

The error dynamics can now be expressed as in Equation (4.90) by

$$\begin{pmatrix} \dot{\epsilon}_{\xi 1} \\ \dot{\epsilon}_{\xi 2} \\ \dot{\epsilon}_{\xi 3} \end{pmatrix} = \begin{bmatrix} -2 & -2 & 1 \\ \frac{1}{2} & 0 & -1.5 \\ -\frac{1}{2} & 1 & -1 \end{bmatrix} \begin{pmatrix} \epsilon_{\xi 1} \\ \epsilon_{\xi 2} \\ \epsilon_{\xi 3} \end{pmatrix} - \begin{bmatrix} -2 & -2 & 1 \\ \frac{1}{2} & 0 & -1.5 \\ -\frac{1}{2} & 1 & -1 \end{bmatrix} \begin{pmatrix} \epsilon_{sp1} \\ \epsilon_{sp2} \\ \epsilon_{sp3} \end{pmatrix} \\
 - \begin{pmatrix} 2 \\ 1 \\ 1 \end{pmatrix} u_p - \begin{pmatrix} 2 \\ 1 \\ 1 \end{pmatrix} u_d - \begin{pmatrix} 2 \\ 0 \\ 1 \end{pmatrix} z. \quad (E.11)$$

The control u_p is designed as $u_p = -K\epsilon_{\xi}$. The homogeneous portion of Equation (E.11) can then be expanded as

$$\dot{\epsilon}_{\xi} = (\tilde{A} + \tilde{B}K)\epsilon_{\xi} = \hat{A}\epsilon_{\xi} = \begin{bmatrix} -2+2k_1 & -2+2k_2 & 1+2k_3 \\ \frac{1}{2} + k_1 & k_2 & -1.5+k_3 \\ -\frac{1}{2}+k_1 & 1 + k_2 & -1 + k_3 \end{bmatrix} \epsilon_{\xi}, \quad (E.12)$$

and the characteristic equation for A is found to be

$$\lambda^3 - (-3+2k_1+k_2+k_3)\lambda^2 - (-5+k_1+2.5k_2+2k_3)\lambda - (-5+5k_1+2.5k_2+5k_3) = 0. \quad (E.13)$$

The desired form of the characteristic equation for use in placing the characteristic values as desired is as follows,

$$\lambda^3 - (\lambda_1 + \lambda_2 + \lambda_3)\lambda^2 + (\lambda_1\lambda_2 + \lambda_1\lambda_3 + \lambda_2\lambda_3)\lambda - \lambda_1\lambda_2\lambda_3 = 0. \quad (E.14)$$

If one equates coefficients of like terms between Equations (E.13) and (E.14), the following expressions are obtained for the components of K ,

$$\begin{aligned} k_1 &= -.25(\lambda_1\lambda_2 + \lambda_1\lambda_3 + \lambda_2\lambda_3) + 0.75(\lambda_1 + \lambda_2 + \lambda_3 + 1.) - 0.05\lambda_1\lambda_2\lambda_3 \\ k_2 &= -.4(\lambda_1\lambda_2\lambda_3 - 5.(\lambda_1 + \lambda_2 + \lambda_3 - k_1) - 10.) \\ k_3 &= \lambda_1 + \lambda_2 + \lambda_3 + 3. - 2.k_1 - k_2 \end{aligned} \quad (E.15)$$

For this example the characteristic values will be placed as

$$\lambda_1 = -3, \quad \lambda_2 = -4, \quad \lambda_3 = -5,$$

and K is thus found from Equations (E.15) to be

$$K = (6.5, -9., -13.) . \quad (E.16)$$

With K as given in (E.16), $\hat{A}$ becomes

$$\hat{A} = \tilde{A} + \tilde{B}K = \begin{bmatrix} 11. & -20. & -25. \\ 7. & -9. & -14.5 \\ 6. & -8. & -14. \end{bmatrix} \quad (\text{E.17})$$

and $\hat{A}^{-1}$ is calculated to be

$$\hat{A}^{-1} = \begin{bmatrix} -0.1667 & 1.333 & -1.0833 \\ -0.1833 & 0.0667 & 0.2583 \\ 0.0333 & 0.5333 & -0.6833 \end{bmatrix}. \quad (\text{E.18})$$

If u_d is chosen to minimize the norm of the total error vector ϵ_ξ then the resulting steady-state error will be given by Equation (4.96) with

$$\begin{aligned} \hat{A}^{-1}\tilde{B} &= (-0.0837, -0.416, -0.0834)^T \\ (\hat{A}^{-1}\tilde{B})^\dagger &= (-5.33, -2.65, -5.315). \end{aligned}$$

The steady-state error vector is calculated to be

$$\epsilon_{\xi ss}^* = \begin{bmatrix} 0.5528 & -0.2203 & -0.4426 \\ -0.2206 & 0.8881 & -0.2234 \\ -0.4434 & -0.2222 & 0.5547 \end{bmatrix} \begin{pmatrix} \xi_{sp1} \\ \xi_{sp2} \\ \xi_{sp3} \end{pmatrix} + \begin{pmatrix} -0.4863 \\ 0.3541 \\ 0.3102 \end{pmatrix} z. \quad (\text{E.19})$$

For a specific example, assume that $y_{sp} = 5$. An x_{sp} which would give $y_{sp} = 5$ in the output space can be found from Equation (E.1), i.e.,

$$y_{sp} = Cx_{sp} = (1, 1, 0)x_{sp}. \quad (\text{E.20})$$

If x_{sp} is chosen to be

$$x_{sp} = (2.5, 2.5, 0.)^T, \quad (\text{E.21})$$

then $y_{sp} = 5$ and transforming x_{sp} to ξ -space results in

$$\xi_{sp} = T^{-1}x_{sp} = (5., 0., 0.)^T. \quad (\text{E.22})$$

If u_d^* is chosen to minimize the length of the overall error vector in ξ -space, then substitution of Equation (E.22) into Equation (E.19) will give $\epsilon_{\xi ss}^*$ as

$$\epsilon_{\xi ss}^* = \begin{pmatrix} 2.764 \\ -1.103 \\ -2.217 \end{pmatrix} + \begin{pmatrix} -0.4863 \\ 0.3541 \\ 0.3102 \end{pmatrix} z \quad (E.23)$$

As can be seen from Equation (E.23), the resulting error is large even if z is zero.

The critical state variable approach will now be used to minimize the error component of interest, i.e., $\epsilon_{\xi ss1}$. If $\epsilon_{\xi ss1}$ is treated as a critical state variable and u_d^* is designed (using Equation (4.95)) to minimize $\|\epsilon_{\xi ss1}\|_I$, the resulting steady-state error for ϵ_{ξ} will be given by Equation (4.93) as

$$\epsilon_{\xi ss} = \begin{bmatrix} -0.1667 & 1.333 & -1.0833 \\ -0.1833 & 0.0667 & 0.2583 \\ 0.0333 & 0.5333 & -0.6833 \end{bmatrix} \begin{pmatrix} -10. \\ 2.5 \\ -2.5 \end{pmatrix} + \begin{pmatrix} 2z \\ 0 \\ z \end{pmatrix} + \begin{pmatrix} 2 \\ 1 \\ 1 \end{pmatrix} u_d \quad (E.24)$$

with u_d^* given by Equation (4.95) as

$$\begin{aligned} u_d^* &= -(\hat{A}_{11}^{-1} \hat{B}_1 + \hat{A}_{12}^{-1} \hat{B}_2)^{\dagger} ((\hat{A}_{11}^{-1} \hat{A}_1 \xi_{sp} + \hat{A}_{12}^{-1} \hat{A}_2 \xi_{sp}) + (\hat{A}_{11}^{-1} \hat{F}_1 + \hat{A}_{12}^{-1} \hat{F}_2)z) \\ &= 11.9474 (7.7078 - 1.4167) = 75.162 \quad (E.25) \end{aligned}$$

Upon substituting Equation (E.25) into Equation (E.24), one can find the steady-state error component $\epsilon_{\xi ss1}^*$ to be (assuming $z=0$),

$$\epsilon_{\xi ss1}^* = 1.4167 \quad (E.26)$$

The steady-state error associated with the non-critical states is found to be

$$\begin{pmatrix} \epsilon_{\xi ss2} \\ \epsilon_{\xi ss3} \end{pmatrix} = \begin{pmatrix} -1.773 \\ -3.56 \end{pmatrix} \quad (E.27)$$

If one compares Equation (E.26) with $\epsilon_{\xi ss1}$ from Equation (E.23), with $z = 0$, it is seen that the error on the component of interest has been reduced from 2.764 to 1.417 in ξ -space. If the error vector

given by Equations (E.26) and (E.27) is transformed back to x -space, the resulting error will be

$$\epsilon_{xss} = T\epsilon_{\xi ss} = \begin{bmatrix} 0.5 & 1. & 0. \\ 0.5 & -0.5 & 0. \\ 0. & 0. & 1. \end{bmatrix} \begin{pmatrix} 1.4167 \\ -1.773 \\ -3.56 \end{pmatrix} = \begin{pmatrix} -1.065 \\ 1.595 \\ -3.56 \end{pmatrix} \quad (E.28)$$

and, from $x_{ss} = x_{sp} - \epsilon_{xss}$, the steady-state plant state vector is found to be

$$x_{ss} = (3.565, 0.905, 3.56)^T. \quad (E.29)$$

The steady-state output vector y_{ss} can now be calculated to be

$$y_{ss} = Cx_{ss} = 4.47, \quad (E.30)$$

therefore, the error in the output space is

$$\epsilon_{yss} = y_{sp} - y_{ss} = 0.53. \quad (E.31)$$

E.4 Development of Solution, Method 2

For the example to be developed in this section, y_{sp} will again be taken to be $y_{sp} = 5$. Since C is of maximal rank y_{sp} is contained within $R(C)$ and is an attainable setpoint. From Equation (4.98) y_{sp} can be expressed as the sum of two components, one in $R(C)$ and one in $R(C)^\perp$ as $y_{sp} = v_1 + v_2$. For the example of this section, therefore, $v_1 = 5$ and $v_2 = 0$. From the plant output equation (Equation (E.1)) one has that

$$y = Cx = x_1 + x_2. \quad (E.32)$$

The control task is thus to steer $x(t)$ to some x_{sp} such that $x_1 + x_2 = y_{sp}$, with x_3 arbitrary. Therefore, any x_{sp} located on a plane perpendicular to the (x_1, x_2) -plane and intersecting the (x_1, x_2) -plane along a line defined by $x_1 + x_2 = 5$ will satisfy the output equation.

Since the plant A matrix in this example is identical to that in Equation (C.1), the gain matrix used in Section C.2 will be used here also, i.e.,

$$K = -1./7.(11., 10., 22.) , \quad (E.33)$$

therefore, the matrix $\tilde{A} = A+BK$ is given by

$$\tilde{A} = (1./7.) \begin{bmatrix} -18. & -24. & -36. \\ -11. & -17. & -15. \\ 1. & 0. & -1. \end{bmatrix} \quad (E.34)$$

and its inverse $\tilde{A}^{-1}$ by

$$\tilde{A}^{-1} = \begin{bmatrix} -0.4048 & 0.5714 & 0.8571 \\ 0.619 & -1.2857 & -0.4286 \\ -0.2619 & 0.5714 & -0.1429 \end{bmatrix} . \quad (E.35)$$

The minimum norm disturbance control vector u_d^* which will result in a minimum norm for ϵ_{xss} is given by Equation (4.105), ϵ_{xss}^* is given by Equation (4.104) and the steady-state error in the output space ϵ_{yss} is found from Equation (4.106).

Using Equation (E.35) and the B matrix from Equations (E.1) gives the terms

$$\begin{aligned} (\tilde{A}^{-1}B) &= (0.0475, 0.8094, -0.6667)^T \\ (\tilde{A}^{-1}B)^\dagger &= (0.04311, 0.7346, -0.6051) , \end{aligned}$$

and substitution of these values into Equation (4.106) results in

$$\begin{aligned} \epsilon_{yss} &= (1,1,0) \begin{bmatrix} -0.4331 & 0.6315 & 0.8662 \\ 0.1368 & -0.2613 & -0.2737 \\ 0.1353 & -0.2724 & -0.2706 \end{bmatrix} \begin{pmatrix} -7.5+z \\ -2.5+z \\ 2.5+z \end{pmatrix} \\ &= 2.778 + 0.6664z \end{aligned} \quad (E.36)$$

where $x_{sp} = (2.5, 2.5, 0.)^T$ was used to correspond to the choice in the

previous section, i.e., Equation (E.21). If the results given by Equation (E.23) are transformed back into the output space the result is $\epsilon_{yss} = 2.764 + 0.708z$, which agrees closely with the results in Equation (E.36).

APPENDIX F

APPENDIX F
AN EXAMPLE FOR DISTURBANCE MINIMIZATION ON A
SECOND-ORDER OUTPUT SERVO-COMMAND PROBLEM,
CHAPTER V

F.1 Introduction

This Appendix describes an example problem illustrating the methods developed in Chapter V for minimization of disturbance effects in a time-invariant output servo-command type problem. The example will first be solved by using the procedure which minimizes disturbance effects on $e_{ss}(t)$. The same example will then be solved by minimizing the disturbance effects on $\epsilon(t)$ using the critical state-variable method.

F.2 Plant, Disturbance and Servo-Command Models

The plant model for this example is given by

$$\begin{pmatrix} \dot{x}_1 \\ \dot{x}_2 \end{pmatrix} = \begin{bmatrix} 1. & 1. \\ 0. & 1. \end{bmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} + \begin{pmatrix} 1. \\ 2. \end{pmatrix} u + \begin{pmatrix} 1. \\ 1. \end{pmatrix} w \quad (F.1)$$

$$y = (1., 0.)x. \quad (F.2)$$

The external disturbance will be considered to be a ramp and is given as

$$w = \alpha t, \quad (F.3)$$

with α an arbitrary, unknown constant, and with state model

$$w = (1., 0.)z; \quad \dot{z} = \begin{bmatrix} 0. & 1. \\ 0. & 0. \end{bmatrix} z + \sigma. \quad (F.4)$$

The output servo-command will also be considered to be a ramp, given by

$$y_c = \beta t, \quad (F.5)$$

with β an arbitrary constant, and with state model

$$y_c = (1., 0.)c \quad (\text{F.6})$$

$$\begin{pmatrix} \dot{c}_1 \\ \dot{c}_2 \end{pmatrix} = \begin{bmatrix} 0. & 1. \\ 0. & 0. \end{bmatrix} \begin{pmatrix} c_1 \\ c_2 \end{pmatrix} + \mu . \quad (\text{F.7})$$

The control task is to design a control u such that the error given by $\varepsilon(t) = y_c(t) - y(t)$ is minimized, in some sense, in the face of all initial conditions, input servo-commands and external disturbances. Checking for satisfaction of the complete absorption criterion, one finds that

$$\text{rank}[B|FH] = \text{rank} \begin{bmatrix} 1. & 1. \\ 2. & 1. \end{bmatrix} = 2 \neq \text{rank}[B] = 1 ;$$

therefore, complete absorption of the external disturbance is not possible. Checking for satisfaction of the exact trackability condition, one finds that

$$\text{rank}[C|R] = \text{rank}[1 \ 0 | 1 \ 0] = 1 = \text{rank}[C] ;$$

therefore, exact trackability is theoretically possible. From Equation (5.6), i.e.,

$$R \equiv C\Theta, \quad (\text{5.6})$$

one finds that the matrix Θ is

$$\Theta = 1. \quad (\text{F.8})$$

F.3 Solution I

In this section, a solution which minimizes disturbance effects on $\dot{e}_{ss}(t)$ by use of the norm minimization technique will be developed. From Equation (5.17) the dynamics of e_{ss} can be expressed in the form

$$\dot{e}_{ss}(t) = \hat{A}e_{ss}(t) - \hat{\Theta}c(t) - Fw(t) - Bu_d(t) . \quad (\text{5.17})$$

The control part $u_p(t) = -Ke_{ss}(t)$ is designed such that transients settle out rapidly, i.e., such that $\hat{A} = A+BK$ has eigenvalues in the left-half plane. Using pole placement techniques, with the eigenvalues of $\hat{A}$ placed at $\lambda_1 = -4$, $\lambda_2 = -7$, K was determined to be $K = (-20., 3.5)$. This value for K results in the following expression for $\hat{A}$,

$$\hat{A} = \begin{bmatrix} -19. & 4.5 \\ -40. & 8. \end{bmatrix}, \quad (F.9)$$

with $\hat{A}^{-1}$ calculated to be

$$\hat{A}^{-1} = \begin{bmatrix} 0.2857 & -0.1607 \\ 1.4286 & -0.6786 \end{bmatrix}. \quad (F.10)$$

From Equations (F.6) to (F.8) one has that

$$\hat{\Theta} = A\Theta - \Theta S = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \quad (F.11)$$

and that

$$c(t) = (\beta t, \beta)^T. \quad (F.12)$$

For this solution procedure, the control part $u_d(t)$ will be allocated as $u_d(t) = u_{dw}(t) + u_{dc}(t)$ and Equations (5.22) and (5.23) will be used to calculate u_{dw}^* and u_{dc}^* , respectively. We wish to find the $u_{dw}^*(t)$ which will minimize $\|Fw(t) + Bu_{dw}(t)\|_Q$ and the $u_{dc}^*(t)$ which will minimize $\|\hat{\Theta}c(t) + Bu_{dc}(t)\|_I$.

The control part $u_{dc}^*(t)$ can be found directly from Equation (5.23) to be

$$u_{dc}^*(t) = -B^{\dagger}(A\Theta - \Theta S)c(t) = -B^{\dagger}\hat{\Theta}c(t) \quad (5.23)$$

where $B^{\dagger}$ is given by

$$B^{\dagger} = (B^T B)^{-1} B^T = 0.2(1., 2.) = (0.2, 0.4). \quad (F.13)$$

Therefore, $u_{dc}^*(t)$ is found as

$$u_{dc}^*(t) = -(0.2, 0.4)I(\beta t, \beta)^T = -0.2\beta(t+2) . \quad (F.14)$$

To calculate $u_{dw}^*(t)$, assume that the weighting matrix Q is given

as

$$Q = \begin{bmatrix} q_{11} & 0 \\ 0 & q_{22} \end{bmatrix} \quad (F.15)$$

and use the procedure shown in Section 3.3, Equation (3.6). One must find a matrix X such that

$$BXB = B ; \quad (QBX)^T = QBX . \quad (3.6)$$

Proceeding to solve for X , one has from the first of Equations (3.6) that

$$\begin{pmatrix} x_1 + 2x_2 \\ 2x_1 + 4x_2 \end{pmatrix} = \begin{pmatrix} 1 \\ 2 \end{pmatrix} ,$$

which provides the condition that

$$x_1 + 2x_2 = 1 . \quad (F.16)$$

From the second of Equations (3.6) one finds that

$$\begin{bmatrix} q_{11}x_1 & 2q_{22}x_1 \\ q_{11}x_2 & 2q_{22}x_2 \end{bmatrix} = \begin{bmatrix} q_{11}x_1 & q_{11}x_2 \\ 2q_{22}x_1 & 2q_{22}x_2 \end{bmatrix}$$

therefore, one must have

$$q_{11}x_2 = 2q_{22}x_1 . \quad (F.17)$$

From Equation (F.16), one has that $x_1 = 1 - 2x_2$ and substituting this in Equation (F.17) results in

$$x_2 = \frac{2q_{22}}{q_{11} + 4q_{22}} , \quad (F.18)$$

and

$$x_1 = \frac{q_{11}}{q_{11} + 4q_{22}} \quad (F.19)$$

with q_{11} , q_{22} arbitrary weighting constants. The control $u_{dw}^*(t)$ which minimizes $\|Fw(t) + Bu_{dw}(t)\|_Q$ is, therefore, found as

$$u_{dw}^*(t) = -X F w(t) = -\left(\frac{q_{11}}{q_{11} + 4q_{22}}, \frac{2q_{22}}{q_{11} + 4q_{22}} \right) (\alpha t, \alpha t)^T$$

or,

$$u_{dw}^*(t) = \frac{-\alpha t}{q_{11} + 4q_{22}} (q_{11} + 2q_{22}) . \quad (F.20)$$

If $u_{dc}^*(t)$ from Equation (F.14) and $u_{dw}^*(t)$ from Equation (F.20) are substituted into Equation (5.17), the resulting expression for $\dot{e}_{ss}(t)$ is

$$\dot{e}_{ss}(t) = \begin{bmatrix} -19. & 4.5 \\ -40. & 8. \end{bmatrix} \begin{pmatrix} e_{ss1} \\ e_{ss2} \end{pmatrix} - \begin{pmatrix} (0.8\beta + \frac{2q_{22}\alpha}{q_{11}+4q_{22}})t - 0.4\beta \\ (-0.4\beta - \frac{q_{11}\alpha}{q_{11}+4q_{22}})t + 0.2\beta \end{pmatrix} \quad (F.21)$$

The solution for $e_{ss}(t)$ can now be found by evaluating the expression

$$e_{ss}(t) = e^{\hat{A}t} e_{ss}(0) - \int_0^t e^{\hat{A}(t-\tau)} \begin{pmatrix} c_1 \tau - 0.4\beta \\ c_2 \tau + 0.2\beta \end{pmatrix} d\tau . \quad (F.22)$$

where $c_1 = 0.8\beta + \frac{2q_{22}\alpha}{q_{11}+4q_{22}}$, $c_2 = -0.4\beta - \frac{q_{11}\alpha}{q_{11}+4q_{22}}$. If one calculates

$e^{\hat{A}t}$ using [27]:

$$e^{\hat{A}t} = \mathcal{L}^{-1}\{(sI - \hat{A})^{-1}\} ,$$

the result is found to be

$$e^{\hat{A}t} = \begin{bmatrix} -4e^{-4t} + 5e^{-7t} & 1.5e^{-4t} - 1.5e^{-7t} \\ (-40/3)(e^{-4t} - e^{-7t}) & 5e^{-4t} - 4e^{-7t} \end{bmatrix} . \quad (F.23)$$

The integral on the right-hand side of Equation (F.22) can thus be evaluated as

$$\begin{aligned}
& - \int_0^t e^{\hat{A}(t-\tau)} \left[\begin{pmatrix} c_1^T \\ c_2^T \end{pmatrix} + \begin{pmatrix} -0.4\beta \\ 0.2\beta \end{pmatrix} \right] d\tau \\
& = - (\hat{A}^{-2}(e^{\hat{A}t} - I) - \hat{A}^{-1}t)(c_1, c_2)^T - \hat{A}^{-1}(e^{\hat{A}t} - I)(-0.4, 0.2)^T.
\end{aligned} \tag{F.24}$$

The term $\hat{A}^{-1}$ is given by Equation (F.10) and $\hat{A}^{-2}$ is thus

$$\hat{A}^{-2} = \begin{bmatrix} -0.148 & 0.0631 \\ -0.5613 & 0.2309 \end{bmatrix}, \tag{F.25}$$

and Equation (F.24) can be expanded to give the following two components:

$$\begin{aligned}
& (0.3973c_1 - 0.0935c_2 + 0.5892)e^{-4t} + (-0.1013c_1 + 0.0304c_2 - 0.3286)e^{-7t} + \\
& (-0.148c_1 + 0.0631c_2 - 0.1464) + (0.2857c_1 - 0.1607c_2)t
\end{aligned} \tag{F.26}$$

$$\begin{aligned}
& (1.3948c_1 - 0.312c_2 + 2.1544)e^{-4t} + (-2.722c_1 + 0.082c_2 - 0.8763)e^{-7t} + \\
& (-0.5613c_1 + 0.2309c_2 - 0.7071) + (1.4286c_1 - 0.6786c_2)t.
\end{aligned}$$

To examine the performance of the disturbance minimizing control vector, assume first that $e_{ss}(0) = 0.$, $w(t) = 2t$, $y_c(t) = 2t$ and that $q_{11} = q_{22} = 10$. Given these values, c_1 and c_2 can be calculated to be $c_1 = 2.4$, $c_2 = -1.2$. Substituting these values into Equations (F.26), (F.22) permits $e_{ss}(t)$ to be evaluated as

$$e_{ss}(t) = \begin{pmatrix} 2.2441e^{-4t} - 0.9368e^{-7t} + 0.8785t - 0.7237 \\ 8.0307e^{-4t} - 8.3838e^{-7t} + 4.243t - 3.0384 \end{pmatrix}. \tag{F.27}$$

Since $\epsilon(t) = Ce_{ss}(t)$, where $C = (1, 0)$, then $\epsilon(t) = e_{ss1}(t)$. Using $e_{ss1}(t)$ from Equation (F.27), Figure F-1 shows a plot of $\epsilon(t)$ versus time for this case.

If $u_d(t)$ is taken to be zero in Equation (5.17), i.e., if no disturbance minimizing control is used, then the expression for $\dot{e}_{ss}(t)$ becomes

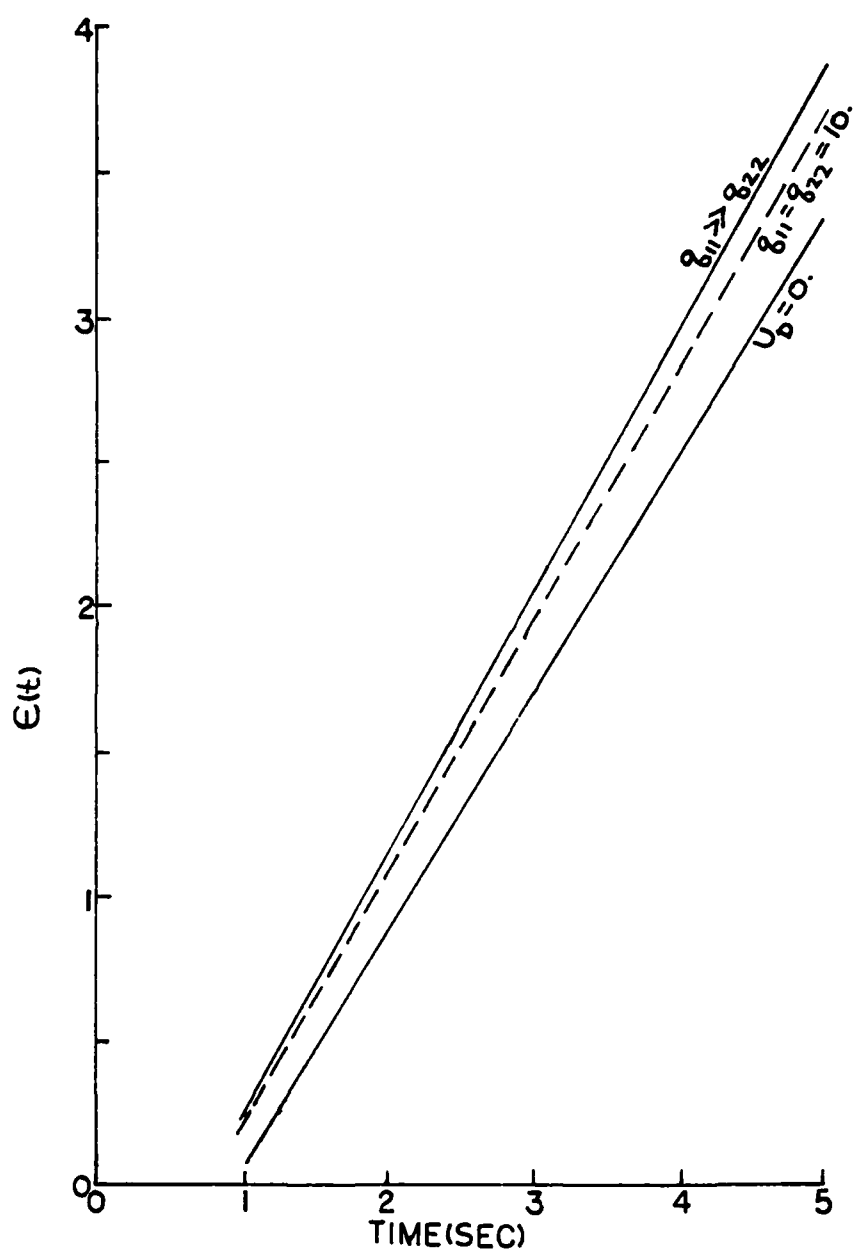


Figure F-1. ϵ versus time, norm minimization technique.

$$\dot{e}_{ss}(t) = \begin{bmatrix} -19. & 4.5 \\ -40. & 8. \end{bmatrix} \begin{pmatrix} e_{ss1} \\ e_{ss2} \end{pmatrix} - \begin{pmatrix} \beta t + \alpha t \\ \beta + \alpha t \end{pmatrix} \quad (F.28)$$

and $e_{ss}(t)$ is found to be as follows,

$$e_{ss1}(t) = (0.7723\beta + 0.3038\alpha)e^{-4t} - (0.3155\beta + 0.0709\alpha)e^{-7t} \\ - (0.3087\beta + 0.0849\alpha) + (0.2857\beta + 0.125\alpha)t \quad (F.29)$$

$$e_{ss2}(t) = (2.6449\beta + 1.0828\alpha)e^{-4t} - (3.2935\beta + 2.64\alpha)e^{-7t} \\ - (1.2399\beta + 0.3304\alpha) + (1.4286\beta + 0.75\alpha)t. \quad (F.30)$$

A plot of $e_{ss1}(t)$ versus time is also plotted on Figure F-1 for the case with $u_d(t) = 0$. As can be seen from Figure F-1, the case with $u_d = 0$ has a smaller value for $e(t)$ than does the case with $u_d \neq 0$. Recall, however, that $u_d^*(t)$ was chosen to minimize $\|e_{ss}(t)\|$. One must therefore examine the difference between the norm of $e_{ss}(t)$ with and without u_d^* . Figure F-2 shows the norm of $e_{ss}(t)$ for the same two cases plotted in Figure F-1 and it is shown that the norm of $e_{ss}(t)$ is smaller for the case when $u_d^* \neq 0$, even though the difference is not large. The disturbance minimizing controller reduced the magnitude of e_{ss2} and increased the magnitude of e_{ss1} , from the values with $u_d = 0$, in order to decrease the norm of $e_{ss}(t)$.

In order to see what effect the weighting matrix will have on the results, assume that $q_{11} \gg q_{22}$. In this case, one will have $c_1 \rightarrow 0.8\beta$, $c_2 \rightarrow (-0.4\beta - \alpha)$, or for $\alpha = \beta = 2$, $c_1 \rightarrow 1.6$ and $c_2 \rightarrow -2.8$. The vector $e_{ss}(t)$ thus becomes

$$e_{ss}(t) = \begin{pmatrix} 2.0759e^{-4t} - 0.9044e^{-7t} + 0.9071t - 0.7063 \\ 7.4141e^{-4t} - 6.3374e^{-7t} + 4.1858t - 2.9588 \end{pmatrix}. \quad (F.32)$$

As can be seen from Figure F-2, the norm of e_{ss} is again reduced, but e_{ss1} has again been increased (see Figure F-1).

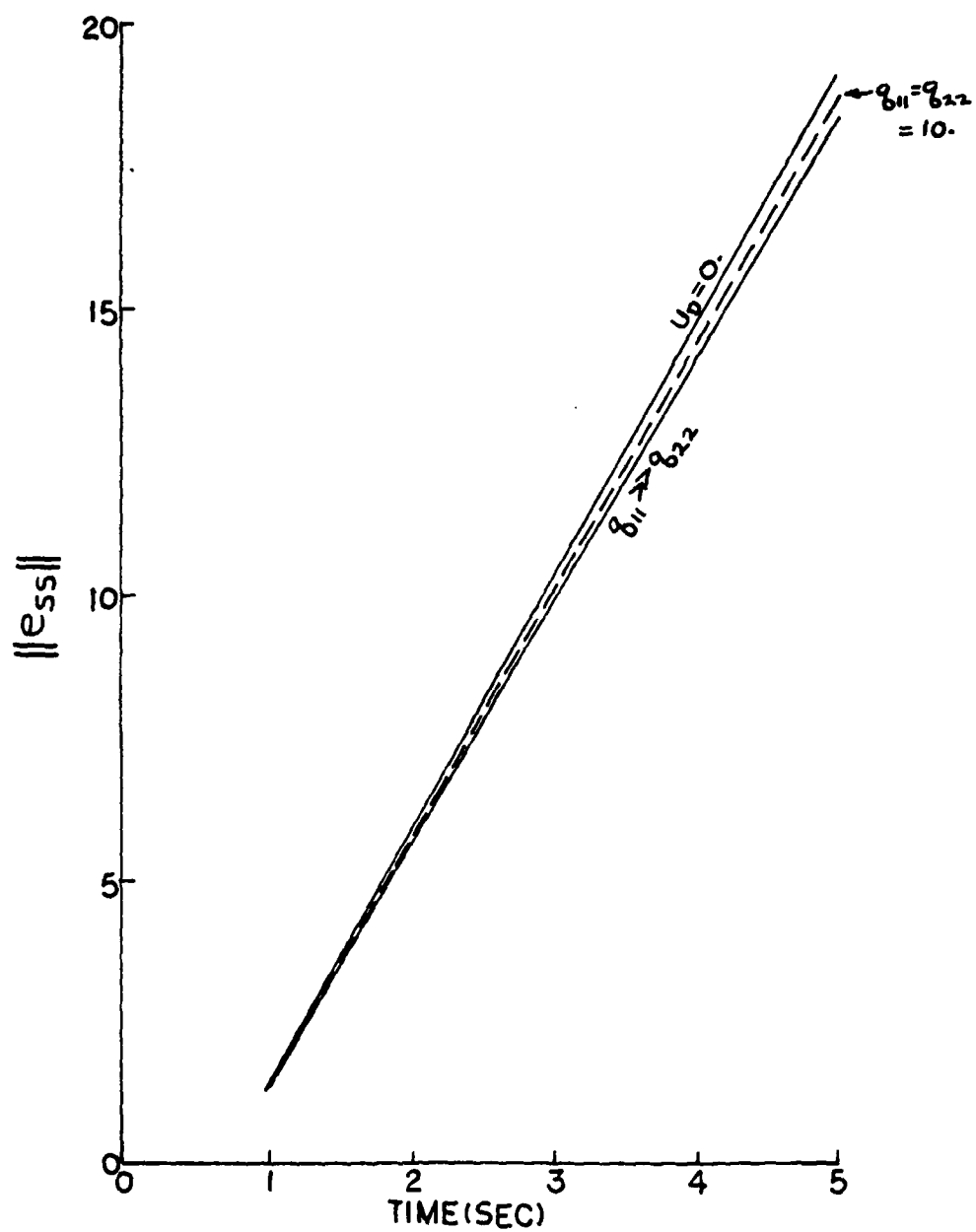


Figure F-2. Norm of e_{ss} versus time, norm minimization technique.

In this example C represents a contractive mapping (when $e_{ss2}(t) \neq 0$) and if one compares Figures F-1 and F-2 it can be seen that the norm of $\epsilon(t)$ is much smaller than the norm of $e_{ss}(t)$. However, the disturbance minimizing control vector actually produced an increase in the norm of $\epsilon(t)$ when it was designed to minimize $\|e_{ss}(t)\|$. Another minimization method might, therefore, prove to be of more benefit and Section F.4 will present another approach.

F.4 Solution II

In this section, a solution for the example problem from Section F.2 will be developed by assuming that the variable of interest, i.e., $e_{ss1}(t)$ is a critical-state variable. Using Equation (5.17), one can write $\dot{e}_{ss}(t)$ as

$$\begin{pmatrix} \dot{e}_{ss1} \\ \dot{e}_{ss2} \end{pmatrix} = \begin{bmatrix} -19. & 4.5 \\ -40. & 8. \end{bmatrix} \begin{pmatrix} e_{ss1} \\ e_{ss2} \end{pmatrix} - \begin{pmatrix} \beta t \\ \beta \end{pmatrix} - \begin{pmatrix} \alpha t \\ \alpha t \end{pmatrix} - \begin{pmatrix} 1. \\ 2. \end{pmatrix} u_d(t) \quad (F.33)$$

and from the equation for $\dot{e}_{ss1}(t)$ it can be seen that a choice of

$$u_d(t) = -(\alpha + \beta)t \quad (F.34)$$

will completely absorb the direct disturbance effects on $\dot{e}_{ss1}(t)$. If this value is used for $u_d(t)$, then Equation (F.33) becomes

$$\dot{e}_{ss}(t) = \begin{bmatrix} -19. & 4.5 \\ -40. & 8. \end{bmatrix} \begin{pmatrix} e_{ss1} \\ e_{ss2} \end{pmatrix} - \begin{pmatrix} 0 \\ \beta - (\alpha + 2\beta)t \end{pmatrix} \quad (F.35)$$

Equation (F.35) can be solved for $e_{ss}(t)$ by utilizing Equation (F.24) with $(c_1, c_2)^T$ replaced by $(0, -(\alpha + 2\beta))^T$ and $(-0.4\beta, 0.2\beta)^T$ replaced by $(0, \beta)^T$. The result is given by the following equations,

$$\begin{aligned}\epsilon(t) = e_{ss1}(t) = & (0.0935\alpha + 0.562\beta)e^{-4t} + (-0.0304\alpha - 0.275\beta)e^{-7t} \\ & + (0.1607\alpha + 0.3214\beta)t - (0.0631\alpha + 0.2869\beta)\end{aligned}\quad (F.36)$$

$$\begin{aligned}e_{ss2}(t) = & (0.312\alpha + 1.874\beta)e^{-4t} + (-0.082\alpha - 0.7355\beta)e^{-7t} \\ & + (-0.2309\alpha - 1.1404\beta) + (0.6786\alpha + 1.3572\beta)t.\end{aligned}\quad (F.37)$$

With the values assumed in Section F.3 for α and β , i.e., $\alpha=\beta=2$, the expression for $\epsilon(t)$ reduces to

$$\epsilon(t) = e_{ss1}(t) = 1.311e^{-4t} - 0.3054e^{-7t} + 0.9642t - 0.35. \quad (F.38)$$

If one compares Equation (F.38) with Equations (F.27) and (F.32), it can be seen that the result for Equation (F.38) is worse than that for the other two equations. Let us re-do the problem with e_{ss2} considered as the critical-state variable.

From Equation (F.33) one finds that for a choice of

$$u_d(t) = -\frac{1}{2}(\alpha t + \beta), \quad (F.39)$$

the direct disturbance effects on $\dot{e}_{ss2}(t)$ will be completely absorbed and $\dot{e}_{ss}(t)$ will be given by

$$\dot{e}_{ss}(t) = \begin{bmatrix} -19. & 4.5 \\ -40. & 8. \end{bmatrix} \begin{pmatrix} e_{ss1} \\ e_{ss2} \end{pmatrix} - \begin{pmatrix} -\frac{1}{2}\beta + (\beta + \frac{1}{2}\alpha)t \\ 0 \end{pmatrix}. \quad (F.40)$$

The vector $e_{ss}(t)$ can be found by using Equation (F.24) and the result for $e_{ss1}(t)$ is found to be

$$e_{ss1}(t) = 2.4774e^{-4t} - 1.018e^{-7t} - 0.7296 + 0.857t. \quad (F.41)$$

The values for $\epsilon(t) = e_{ss1}(t)$ from Equations (F.38), (F.41) and (F.27) are plotted on Figure F-3. As can be seen, when $e_{ss2}(t)$ is considered to be the critical-state variable, the resulting value of $\epsilon(t)$ is lower than those obtained in Section F.3 or for the case with $e_{ss1}(t)$ as the critical-state, but still higher than for the case with $u_d = 0$.

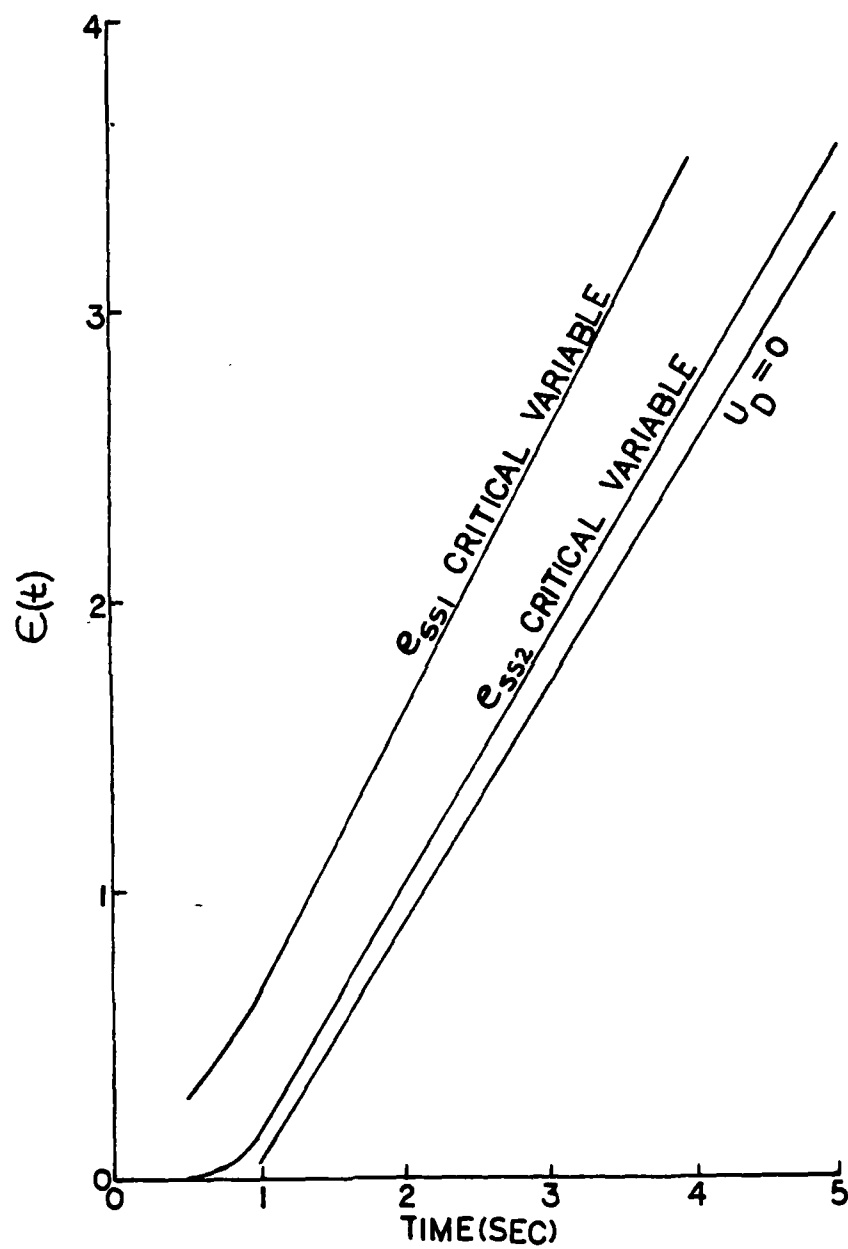


Figure F-3. $\epsilon(t)$ versus time, critical state-variable technique.

If the equation for $\dot{e}_{ss1}(t)$ is written out from Equation (F.33), one obtains the following,

$$\dot{e}_{ss1}(t) = -19e_{ss1}(t) + 4.5e_{ss2}(t) - (\beta + \alpha)t - u_d(t) . \quad (F.42)$$

Let the control vector $u_d(t)$ be designed to absorb the direct external and servo-command disturbance inputs to $\dot{e}_{ss1}(t)$ and also the coupling disturbance term (similar to $\tilde{A}_{12}\xi_2(t)$ in Equation (1.28)) given by $4.5e_{ss2}(t)$. The control $u_d(t)$ will thus be designed as

$$u_d(t) = -(\beta + \alpha)t + 4.5e_{ss2}(t) . \quad (F.43)$$

If one now substitutes this u_d into Equation (F.33), the resulting expression for $\dot{e}_{ss}$ is

$$\begin{aligned} \dot{e}_{ss}(t) &= \begin{bmatrix} -19. & 0. \\ -40. & -1. \end{bmatrix} \begin{pmatrix} e_{ss1} \\ e_{ss2} \end{pmatrix} - \begin{pmatrix} 0 \\ \beta - (\alpha + 2\beta)t \end{pmatrix} \\ &= \hat{A}_1 e_{ss}(t) - \hat{F} . \end{aligned} \quad (F.44)$$

In effect, the choice of u_d as in Equation (F.43) results in the eigenvalues of A_1 being placed at $\lambda_1 = -1$, $\lambda_2 = -19$.

For the matrix $\hat{A}_1$ one can calculate $e^{\hat{A}_1 t}$ to be as follows,

$$e^{\hat{A}_1 t} = \mathcal{L}^{-1}\{(sI - \hat{A}_1)^{-1}\} = \begin{bmatrix} e^{-19t} & 0 \\ 2.222(e^{-19t} - e^{-t}) & e^{-t} \end{bmatrix} . \quad (F.45)$$

Equation (F.24) can again be used to evaluate the integral of Equation (F.42) in order to obtain an expression for $e_{ss}(t)$. The result, with $\alpha = \beta = 2$, $e_{ss}(0) = 0$, $\hat{A}$ replaced by $\hat{A}_1$ and

$$\hat{A}^{-2} = \begin{bmatrix} 0.0028 & 0. \\ 1.9946 & 1. \end{bmatrix} ,$$

is calculated to be (the terms containing e^{-19t} are dropped since they approach zero rapidly)

$$e_{ss}(t) = (0., 8e^{-t-8+6t})^T . \quad (F.47)$$

The error vector $\epsilon(t) = e_{ss1}(t)$ is thus seen to theoretically be zero when the control vector is designed to include the "coupling disturbance" term due to $e_{ss2}(t)$. However, in order to implement the u_d shown in Equation (F.43), the external disturbance and servo-command states will have to be obtained from the external disturbance and servo-command state reconstructors, respectively. Since these reconstructors will have transients in their responses as the respective impulses arrive, the result obtained for $\epsilon(t)$ will not be zero. Instead, it will include the reconstructor transient errors.

F.5 Solution III

If one reviews the results obtained in Section F.3 one sees that the control vector $u_{dc}^*(t)$ (Equation F.14) is a weighted linear combination of $y_c(t)$ and $\dot{y}_c(t)$. From Equation (F.20) one also can see that the control vector $u_{dw}^*(t)$ is a weighted linear function of the external disturbance. In Section F.4, when $e_{ss1}(t)$ was chosen as a critical-state variable the control $u_d(t)$ (Equation (F.34)) was a combination of $y_c(t)$ and $w(t)$. When $e_{ss2}(t)$ was chosen as the critical-state the control (Equation (F.39)) was a combination of $w(t)$ and $\dot{y}_c(t)$. In this section (Section F.5) the isobasis design technique, whereby the disturbance minimizing control vectors are assumed a priori to be of the same form as $y_c(t)$ and $w(t)$, will be illustrated.

To begin, the assumption is made that $u_{dc}(t) = \bar{b}\beta t$, $u_{dw}(t) = \bar{a}\alpha t$ since the servo-command is given as being $y_c(t) = \beta t$ and the external disturbance as $w(t) = \alpha t$. If these assumed forms for $u_{dc}(t)$ and u_{dw} are substituted into Equation (5.17), and the resulting equation is solved for $e_{ss}(t)$, the result is given by

$$e_{ss}(t) = e^{\hat{A}t} e_{ss}(0) - \int_0^t e^{\hat{A}(t-\tau)} \left[\begin{pmatrix} (1+\bar{b})\beta \\ 2\bar{b}\beta \end{pmatrix} \tau + \begin{pmatrix} 0 \\ \beta \end{pmatrix} \right] d\tau$$

$$- \int_0^t e^{\hat{A}(t-\tau)} \begin{pmatrix} (1+\bar{a})\alpha \\ (1+2\bar{a})\alpha \end{pmatrix} \tau d\tau \quad (F.48)$$

The first integral term on the right-hand side of Equation (F.48) can be evaluated as

$$- \int_0^t e^{\hat{A}(t-\tau)} ((\cdot)\tau + (\cdot)) d\tau = -(\hat{A}^{-2}(e^{\hat{A}t} - I) - \hat{A}^{-1}t)((1+\bar{b})\beta, 2\bar{b}\beta)^T$$

$$- \hat{A}^{-1}(e^{\hat{A}t} - I)(0, \beta)^T$$

$$= \begin{pmatrix} (-0.3087+0.2857t)\beta + (-0.0218-0.0357t)\bar{b}\beta \\ (-1.2399+1.4286t)\beta + (-0.0995+0.0714t)\bar{b}\beta \end{pmatrix} \quad (F.49)$$

where the exponential terms have been dropped and we consider only the part of the response after the plant transients are approximately zero. The second integral on the right-hand side of Equation (F.48) can be evaluated as

$$- \int_0^t e^{\hat{A}(t-\tau)} (\cdot)\tau d\tau = -(\hat{A}^{-2}(e^{\hat{A}t} - I) - \hat{A}^{-1}t)((1+\bar{a})\alpha, (1+2\bar{a})\alpha)$$

$$= \begin{pmatrix} (-0.0849+0.125t)\alpha + (-0.0218-0.0357t)\bar{a}\alpha \\ (-0.3304+0.75t)\alpha + (-0.0995+0.0714t)\bar{a}\alpha \end{pmatrix} \quad (F.50)$$

If the right-hand side of Equation (F.49) is written in the form

$$\begin{pmatrix} -0.0218\dot{y}_c - 0.0357y_c \\ -0.0955\dot{y}_c + 0.0714y_c \end{pmatrix} \bar{b} + \begin{pmatrix} -0.3087\dot{y}_c + 0.2857y_c \\ -1.2399\dot{y}_c + 1.4286y_c \end{pmatrix} = 0 \quad (F.51)$$

or, in simpler form, $u_1 \bar{b} + u_2 = 0$, then the $\bar{b}^*$ of minimum norm which will minimize the norm of the residual error in Equation (F.51) is found to be

$$\bar{b}^* = -(u_1)^\dagger u_2 \quad (F.52)$$

with

$$u_1^* = (u_1^T u_1)^{-1} u_1^T \quad (F.53)$$

Substituting the appropriate vectors from Equation (F.51) into Equations (F.53) and (F.52) yields

$$\bar{b}^* = \frac{-0.1301\dot{y}_c^2 + 0.2254\dot{y}_c y_c - 0.0918y_c^2}{0.0104\dot{y}_c^2 - 0.0126\dot{y}_c y_c + 0.0064y_c^2} \quad (F.54)$$

In a similar fashion, if the right-hand side of Equation (F.50) is written in the form

$$\begin{pmatrix} -0.0218\dot{w} - 0.0357w \\ -0.0995\dot{w} + 0.0714w \end{pmatrix} \bar{a} + \begin{pmatrix} -0.0849\dot{w} + 0.125w \\ -0.3304\dot{w} + 0.75w \end{pmatrix} = 0 \quad (F.55)$$

then the $\bar{a}^*$ which will minimize the norm of the residual error in Equation (F.55) is as follows,

$$\bar{a}^* = \frac{-0.0348\dot{w}^2 + 0.979\dot{w}w - 0.0491w^2}{0.0104\dot{w}^2 - 0.0126\dot{w}w + 0.0064w^2} \quad (F.56)$$

If Equations (F.54) and (F.56) are substituted back into Equations (F.49) and (F.50), respectively, then $e_{ss}(t)$, as given by Equation (F.48), can be evaluated. For the same conditions used in the previous two sections, i.e., $e_{ss}(0) = 0$, $\alpha = \beta = 2$, Figure F-4 shows the results obtained for $e_{ss1}(t)$ and $||e_{ss}(t)||$ using the solution procedure of this section.

If one now designs $\bar{b}$ and $\bar{a}$ in order to obtain $e_{ss1}(t) = 0$, the resulting expressions will be

$$\bar{b} = \frac{-0.3087\dot{y}_c + 0.2857y_c}{0.0218\dot{y}_c + 0.0357y_c} \quad (F.57)$$

$$\bar{a} = \frac{-0.0849\dot{w} + 0.125w}{0.0218\dot{w} + 0.0357w} \quad (F.58)$$

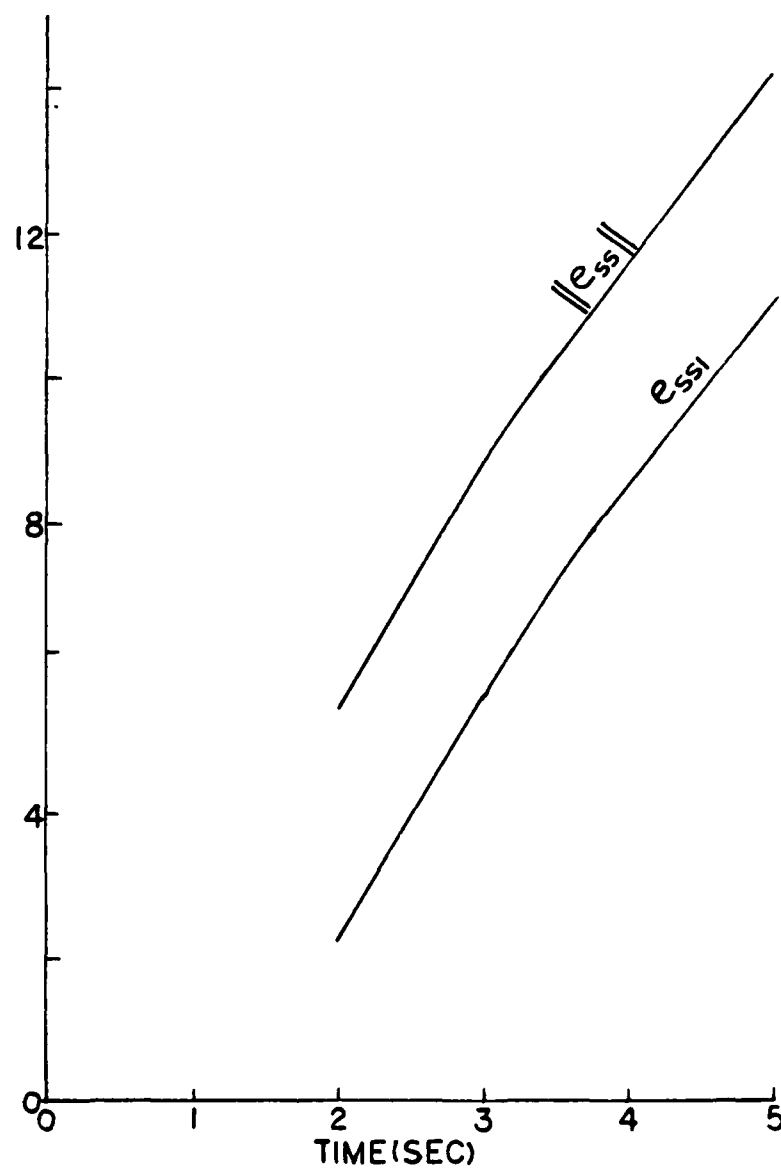


Figure F-4. e_{ss1} and $||e_{ss}||$ versus time, isobasis design technique.

The values which will result for $e_{ss2}(t)$ if Equations (F.57) and (F.58) are utilized in the control vectors are shown on Figure F-5. Also included on Figure F-5 are values for $e_{ss2}(t)$ corresponding to the data shown on Figure F-4 and to the case with $u_d = 0$.

The technique illustrated in this section results in larger values for $\epsilon(t)$ than did the techniques of Sections F.3 and F.4.

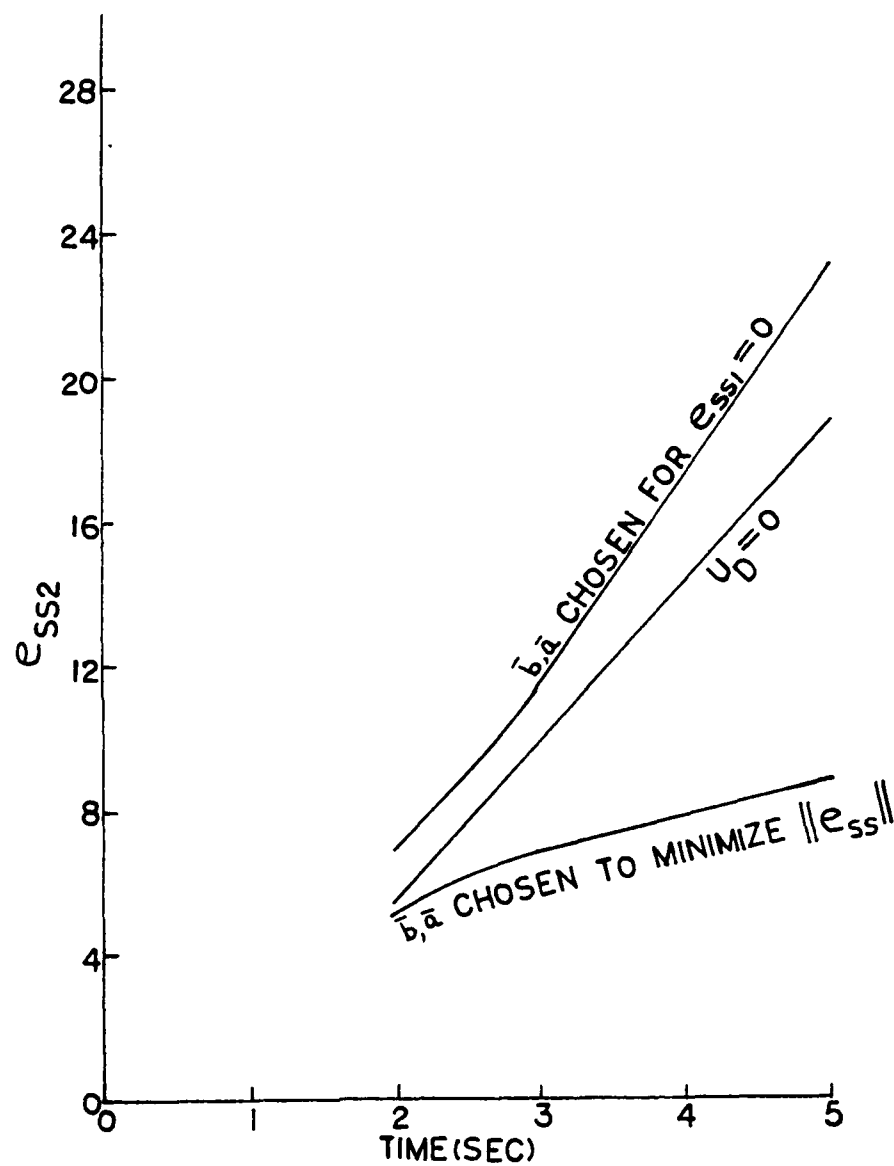


Figure F-5. e_{ss2} versus time, isobasis design technique.

APPENDIX G

APPENDIX G

A SECOND-ORDER STATE SERVO-COMMAND EXAMPLE

WITH DIRECT DISTURBANCE MINIMIZATION

G.1 Introduction

This Appendix describes the solution of an example which illustrates the methods developed in Chapter V to minimize disturbance effects in a time-invariant state servo-command problem. The problem will first be solved by minimizing the norms of the disturbance terms prior to solving for $\epsilon(t)$ (Section 5.5). Then, the same problem will be solved by using the isobasis design technique to minimize $||\epsilon(t)||$. Finally, the problem will be solved using the maximum partial absorption technique.

G.2 Plant, Disturbance and Servo-Command Models

The plant model for this example is given by

$$\begin{pmatrix} \dot{x}_1 \\ \dot{x}_2 \end{pmatrix} = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} + \begin{pmatrix} 1 \\ 2 \end{pmatrix} u + \begin{pmatrix} 1 \\ 1 \end{pmatrix} w$$

$$y = (1, 0)x. \quad (G.1)$$

The external disturbance will be considered to be an unknown ramp modelled as

$$w(t) = \alpha t, \quad (G.2)$$

where α is an arbitrary unknown constant. The state model for (G.2) is taken as

$$w = (1, 0)z; \quad \dot{z} = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} z + \sigma. \quad (G.3)$$

The input state servo-command will also be considered to be a ramp, given by

$$x_c(t) = \beta t = (\beta_1, \beta_2)^T t, \quad (G.4)$$

β_1, β_2 arbitrary constants, and with state model

$$\begin{pmatrix} \dot{x}_{c1} \\ \dot{x}_{c2} \end{pmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix} \begin{pmatrix} c_1 \\ c_2 \\ c_3 \\ c_4 \end{pmatrix} \quad (G.5)$$

$$\begin{pmatrix} \dot{c}_1 \\ \dot{c}_2 \\ \dot{c}_3 \\ \dot{c}_4 \end{pmatrix} = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix} \begin{pmatrix} c_1 \\ c_2 \\ c_3 \\ c_4 \end{pmatrix} + \begin{pmatrix} 0 \\ \mu_1 \\ 0 \\ \mu_2 \end{pmatrix}.$$

The objective is to design a control $u(t)$ such that the error given by

$$e(t) = x_c(t) - x(t) \quad (G.6)$$

is minimized in the face of all initial conditions, state servo-commands and external disturbances.

G.3 Solution I

In this section, the state servo-command disturbance minimization problem will be solved by utilizing the norm minimization process given in Section 5.5. The example will be developed in terms of the error vector defined by Equation (G.6). The error dynamics are given by Equation (5.72) as ($L=0$),

$$\dot{e}(t) = (A + BK)e(t) - (AR - RS)c(t) - Fw(t) - Bu_d(t). \quad (5.72)$$

The minimum norm $u_{dw}^*(t) = u_{dw}^*(t)$ which will give a minimum norm for the external disturbance residual term is given by Equation (5.22) as

$$u_{dw}^*(t) = -B^+ Fw(t), \quad (5.22)$$

with B^+ given by Equation (F.13) as

$$B^+ = (0.2, 0.4). \quad (G.7)$$

Substituting Equation (G.7) and the F matrix from Equation (G.1) into Equation (5.22), one obtains the external disturbance minimizing control as

$$u_{dw}^*(t) = -0.6\alpha t. \quad (G.8)$$

The minimum norm $u_{dc}^*(t)$ which will give a minimum norm for the servo-command disturbance residual term is given by Equation (5.23) (with θ replaced by R) as

$$u_{dc}^*(t) = -B^{\dagger}(AR - RS)c(t). \quad (5.23)$$

For the present example, $u_{dc}^*(t)$ is as follows (using the appropriate matrices defined in Equations (G.1) and (G.5) and with $c(t) = (\beta_1 t, \beta_1, \beta_2 t, \beta_2)^T$),

$$u_{dc}^*(t) = 0.2\beta_1(1-t) - 0.2\beta_2(2-3t). \quad (G.9)$$

For the control $u_p(t) = -K\varepsilon(t)$, the gain matrix K will be taken to be $K = (-20., 3.5)$ as in Appendix F.

A practical implementation of u_{dw}^* and u_{dc}^* would require use of a state reconstructor to obtain estimates $\hat{c}(t)$ and $\hat{z}(t)$ of the servo-command and external disturbance states $c(t)$ and $z(t)$, respectively. To proceed with this example, however, we shall assume perfect knowledge of $x_c(t)$ and $w(t)$.

If one now substitutes the data given by Equations (G.1), (G.2), (G.4), (G.8) and (G.12) into Equation (5.72), one arrives at the following expression for the error dynamics.

$$\dot{\varepsilon}(t) = \begin{bmatrix} -19. & 4.5 \\ -40. & 8. \end{bmatrix} \varepsilon(t) - \begin{pmatrix} 0.4 \\ -0.2 \end{pmatrix} w(t) - \begin{bmatrix} 0.8 & -0.8 & 0.4 & 0.4 \\ -0.4 & 0.4 & -0.2 & -0.2 \end{bmatrix} \begin{pmatrix} x_{c1} \\ \dot{x}_{c1} \\ x_{c2} \\ \dot{x}_{c2} \end{pmatrix} \quad (G.10)$$

This example was programmed on a digital computer. A block

diagram of the plant/control combination is shown in Figure G-1. The external disturbance and state servo-command ramp coefficients used for the first set of runs are as follows:

(a) External Disturbance

$$\alpha = \begin{cases} 0. & 0. \leq t < 6. \\ 1. & 6. \leq t < 12. \\ -1. & 12. \leq t < 20. \end{cases}$$

(b) State Servo-Command

(G.11)

$$\beta_1 = \begin{cases} 1. \\ 0. \\ 2. \end{cases} \quad \beta_2 = \begin{cases} -1. & 0. \leq t < 6. \\ 0. & 6. \leq t < 12. \\ -2. & 12. \leq t < 20. \end{cases}$$

As a check on the simulation output, the time response for the system of Figure G-1 was determined for $\epsilon_1(0) = \epsilon_2(0) = -1.$, $w(t) = 0.$, $x_c(t) = 0.$ For $\bar{A}$ as given by Equation (F.9), $e^{\bar{A}t}$ is given by Equation (F.23). The solution for $\epsilon(t)$ for the homogeneous part of Equation (5.72) is calculated to be

$$\epsilon(t) = \begin{pmatrix} 2.5e^{-4t} - 3.5e^{-7t} \\ 8.33e^{-4t} - 9.33e^{-7t} \end{pmatrix} \quad (G.12)$$

Figures G-2 and G-3 give the system response to an initial condition only, $x_1(0) = x_2(0) = 1.$ Several points calculated from Equations (G.12) are indicated on Figures G-2 and G-3. As can be seen, the calculated and simulated outputs match. Also, from Figure G-3 one notes that $\epsilon_2(t)$ exhibits a very large overshoot. A second run was made with $w(t)$ and $x_c(t)$ simulated as shown in expressions (G.11) and the resulting error components are given in Figures G-4 and G-5. Figure G-6 is a plot of $w(t)$. The first six seconds of data provide an indication of tracking capability with servo-command disturbances only. The next six

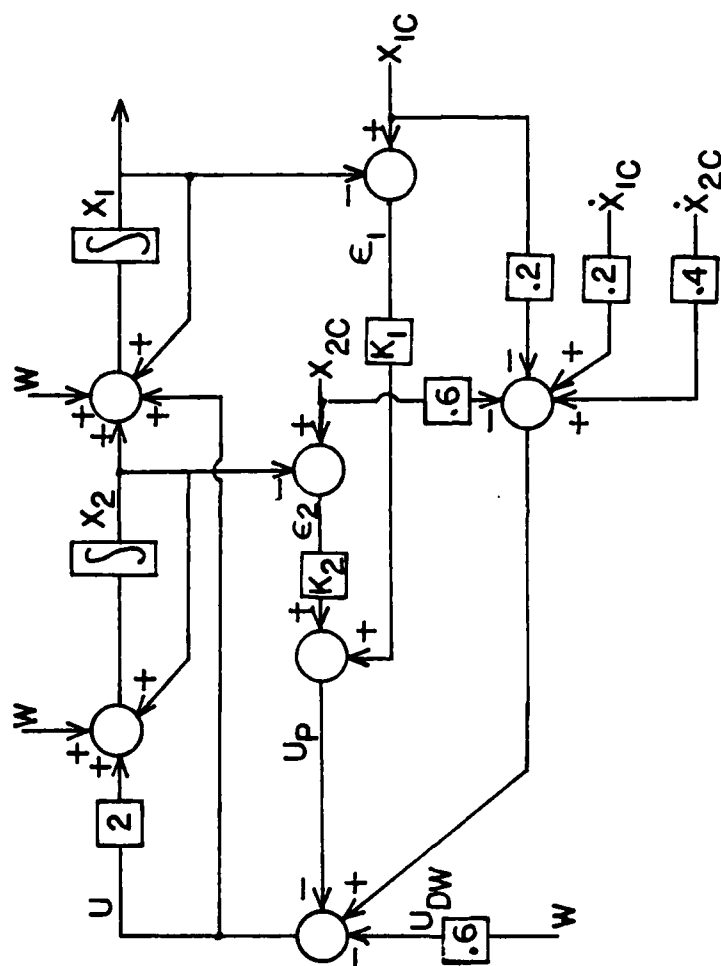


Figure G-1. Block diagram of plant/controller, norm minimization technique.

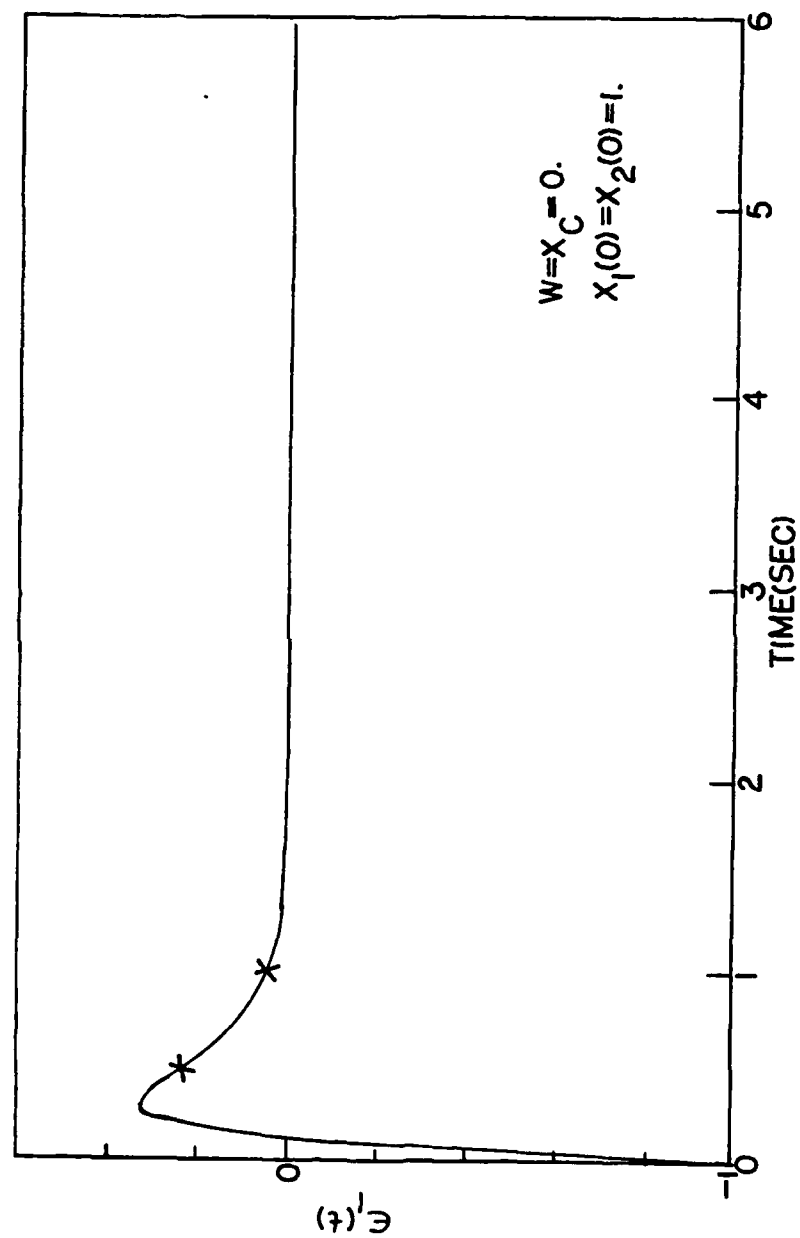


Figure G-2. ϵ_1 versus time, initial conditions only.

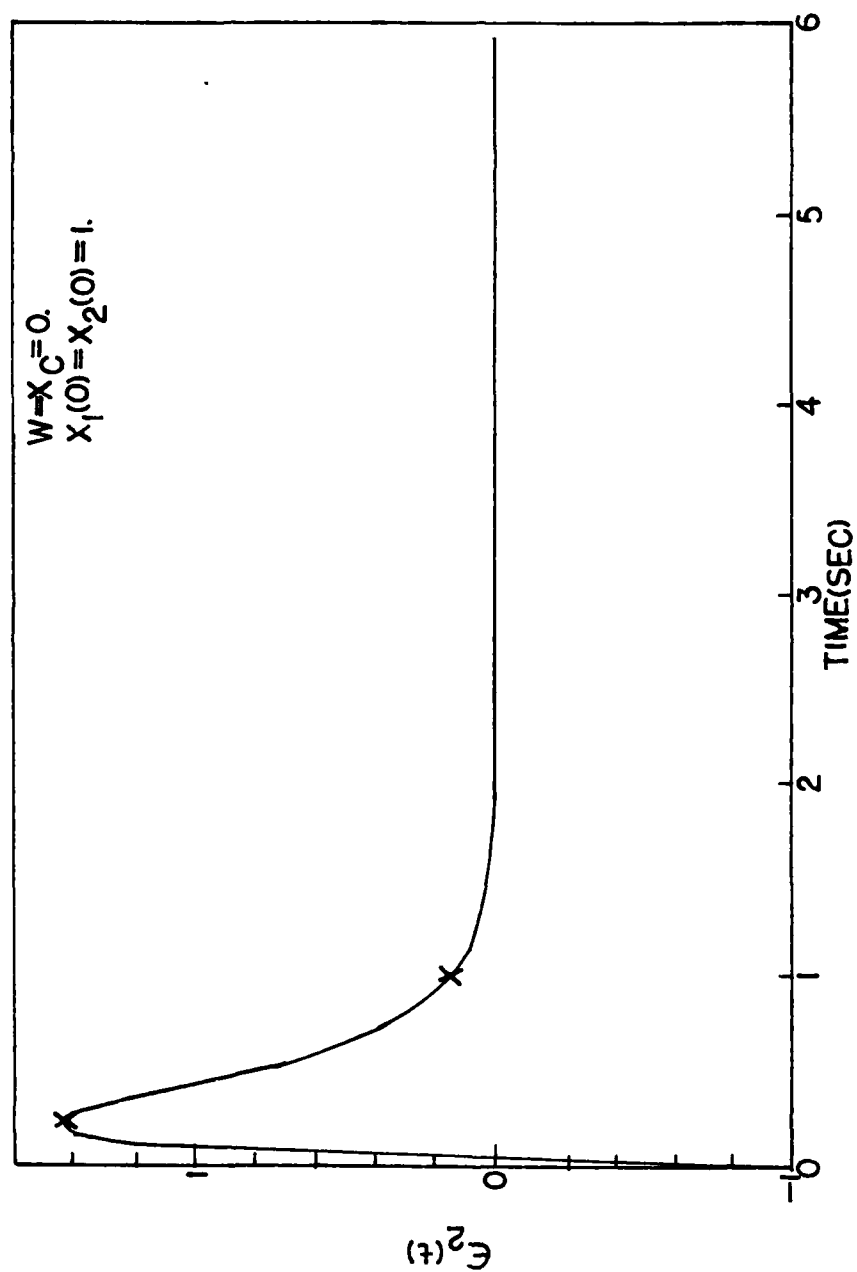


Figure G-3. ϵ_2 versus time, initial conditions only.

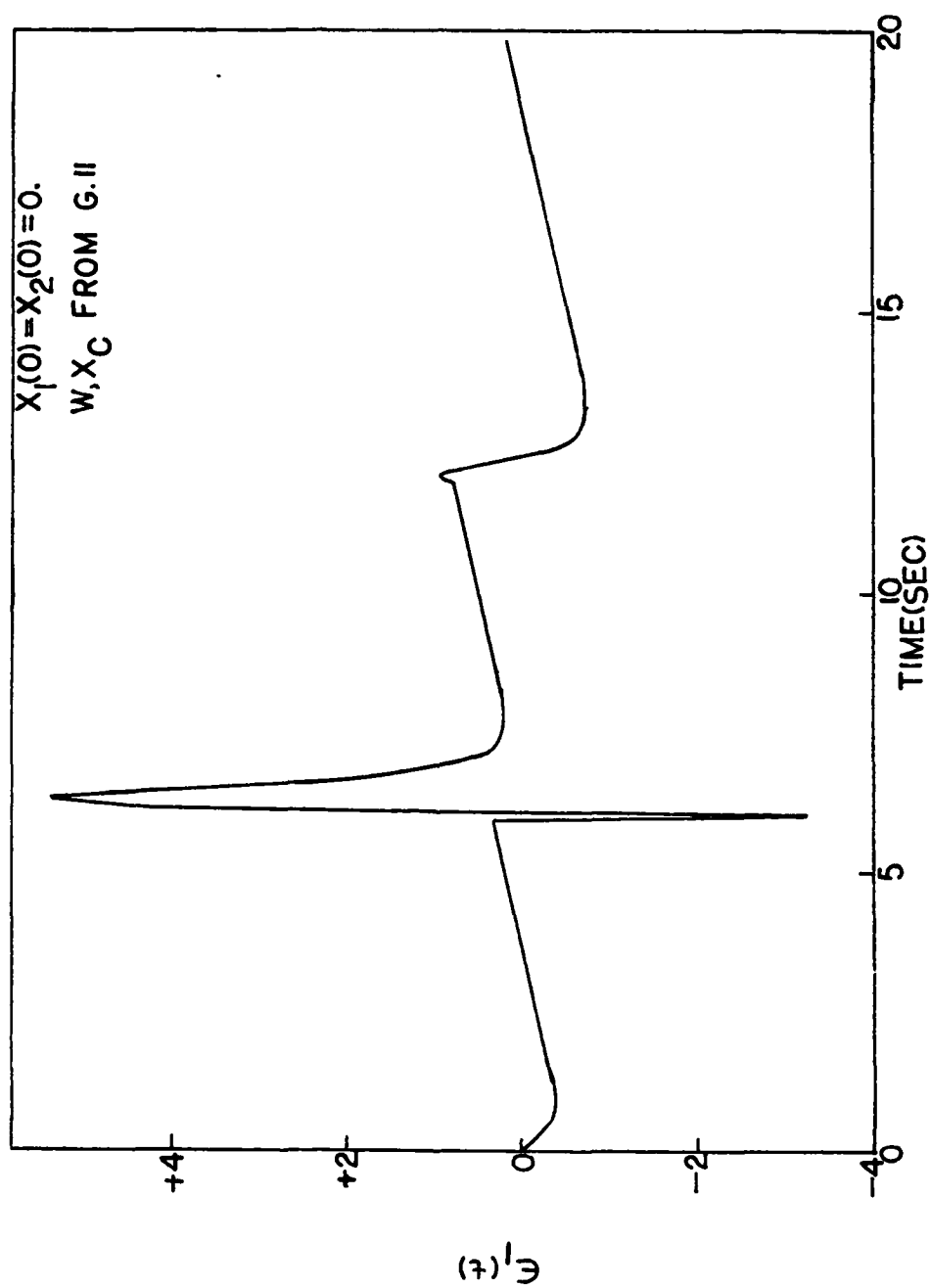


Figure C-4. ϵ_1 versus time, inputs from G.11

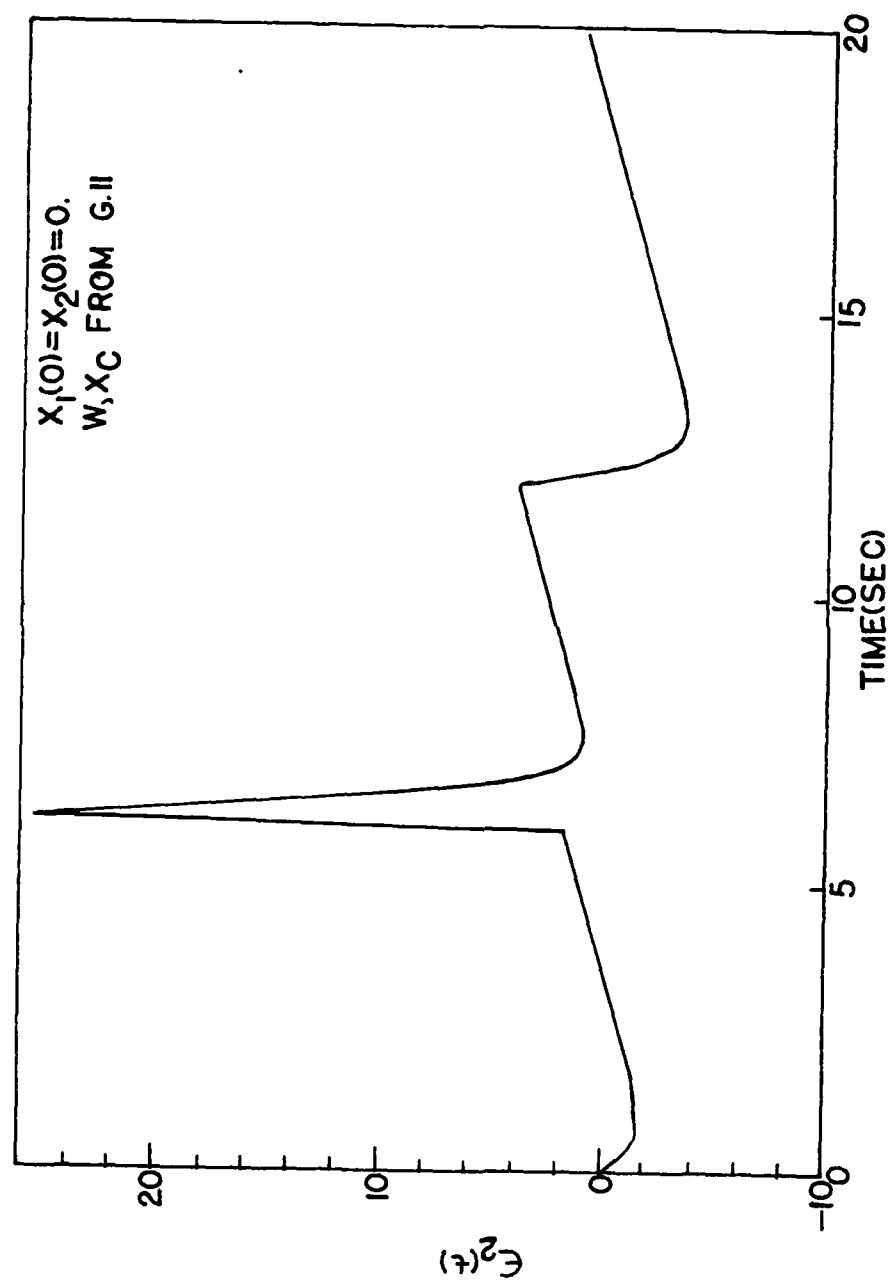


Figure G-5. ϵ_2 versus time, inputs from G.II

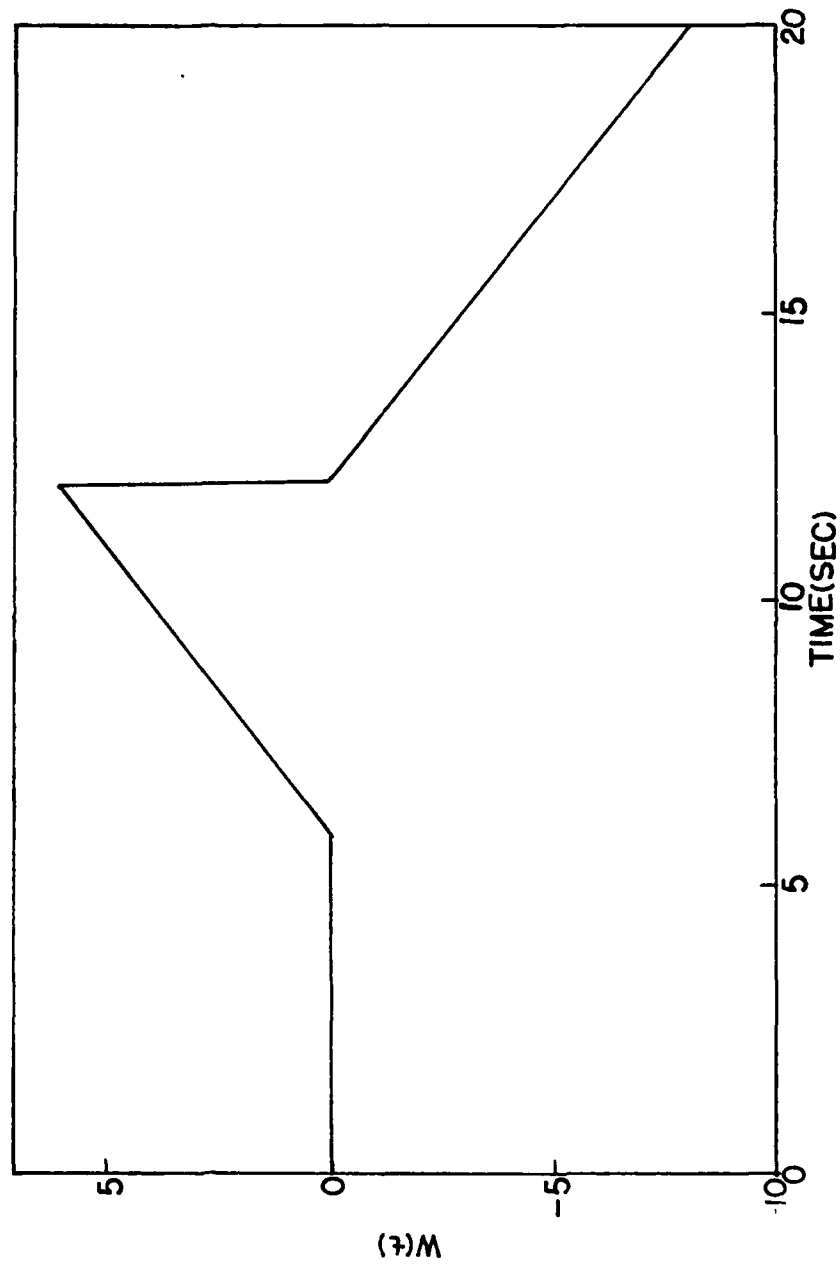


Figure G-6. External disturbance, G.11

seconds provide an indication of the response due to an external disturbance only and the last eight seconds show performance with both disturbance sources. The component $\epsilon_1(t)$ remains fairly small except at those times when an impulse arrives. The component $\epsilon_2(t)$ does not perform as well. Figures G-7 and G-8 give the data for $x_1(t)$ and $x_2(t)$ corresponding to Figures G-4 and G-5, respectively. The dashed lines on Figures G-7 and G-8 indicate the servo-command inputs.

Several other runs were made with $w(t)$ and $x_c(t)$ continuous on the time interval. Figures G-9 to G-11 give $\epsilon_1(t)$, $\epsilon_2(t)$ and $\epsilon_T(t)$ for a case with coefficients as shown in expressions (G.13),

(a) External Disturbance

$$\alpha = \begin{cases} 0. & 0. \leq t < 5. \\ 1. & 5. \leq t < 10. \\ -1. & 10. \leq t < 15. \end{cases}$$

(b) Servo-Command Input

(G.13)

$$\beta_1 = \begin{cases} 1. \\ 0. \\ 2. \end{cases} \quad \beta_2 = \begin{cases} -1. & 0. \leq t < 5. \\ 0. & 5. \leq t < 10. \\ -2. & 10. \leq t < 15. \end{cases}$$

for cases with and without the disturbance controls u_{dw}^* and u_{dc}^* . The vector $\epsilon_T(t)$ is the total error vector magnitude, i.e., $\epsilon_T = (\epsilon_1^2 + \epsilon_2^2)^{1/2}$. The overall effect of u_{dw}^* and u_{dc}^* was not large but the functioning of the control in playing one component against another can be seen. The same conditions were then rerun, but with the gain matrix K taken to be $K = (-3., -0.36)$. The results are shown in Figures G-12 to G-14. In this case the disturbance minimizing control had more effect.

In order to check the performance of the system to a situation in which the various ramp coefficients changed at different times instead of at the same times, as in expressions (G.13), two runs were made with the values given in expressions (G.14). One run was made with

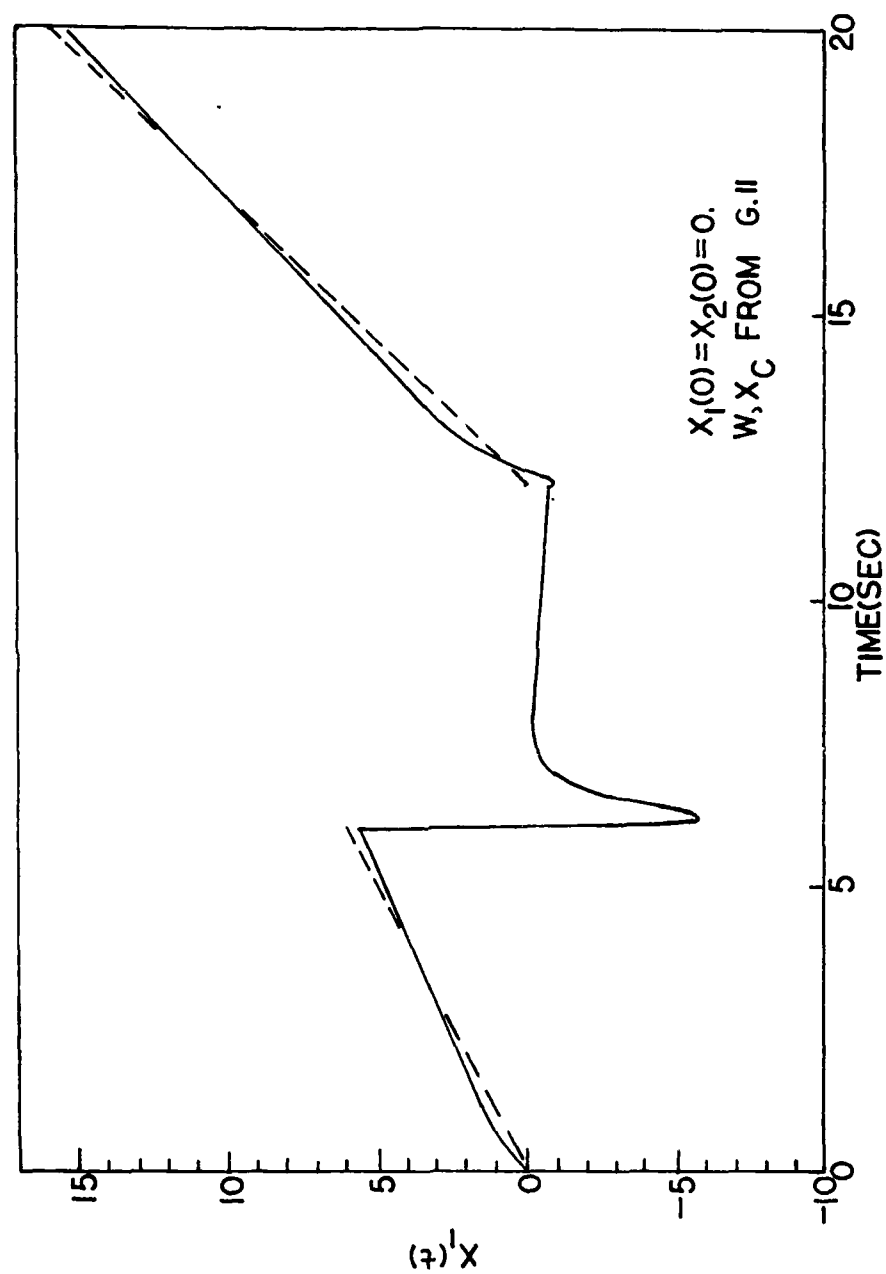


Figure G-7. X_1 versus time, inputs from G.11.

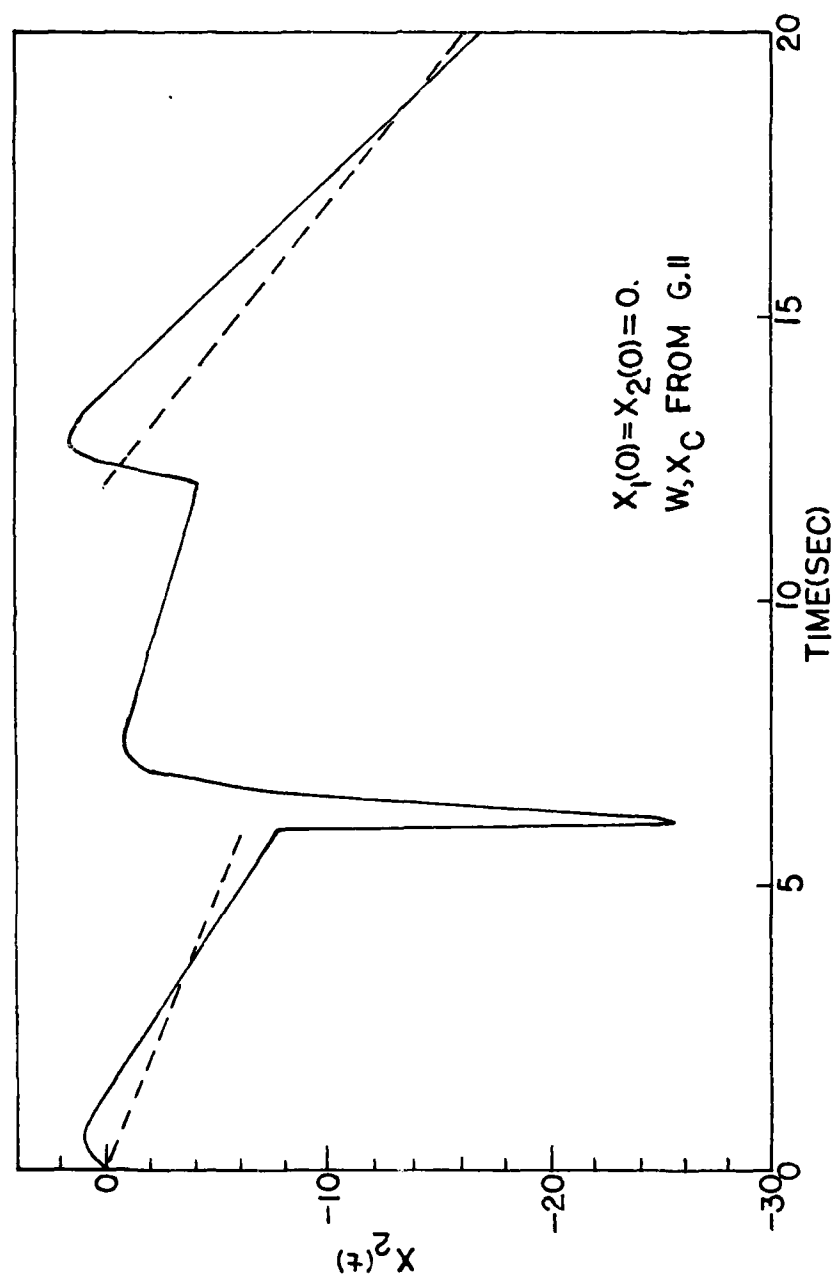


Figure G-8. X_2 versus time, inputs from G.11.

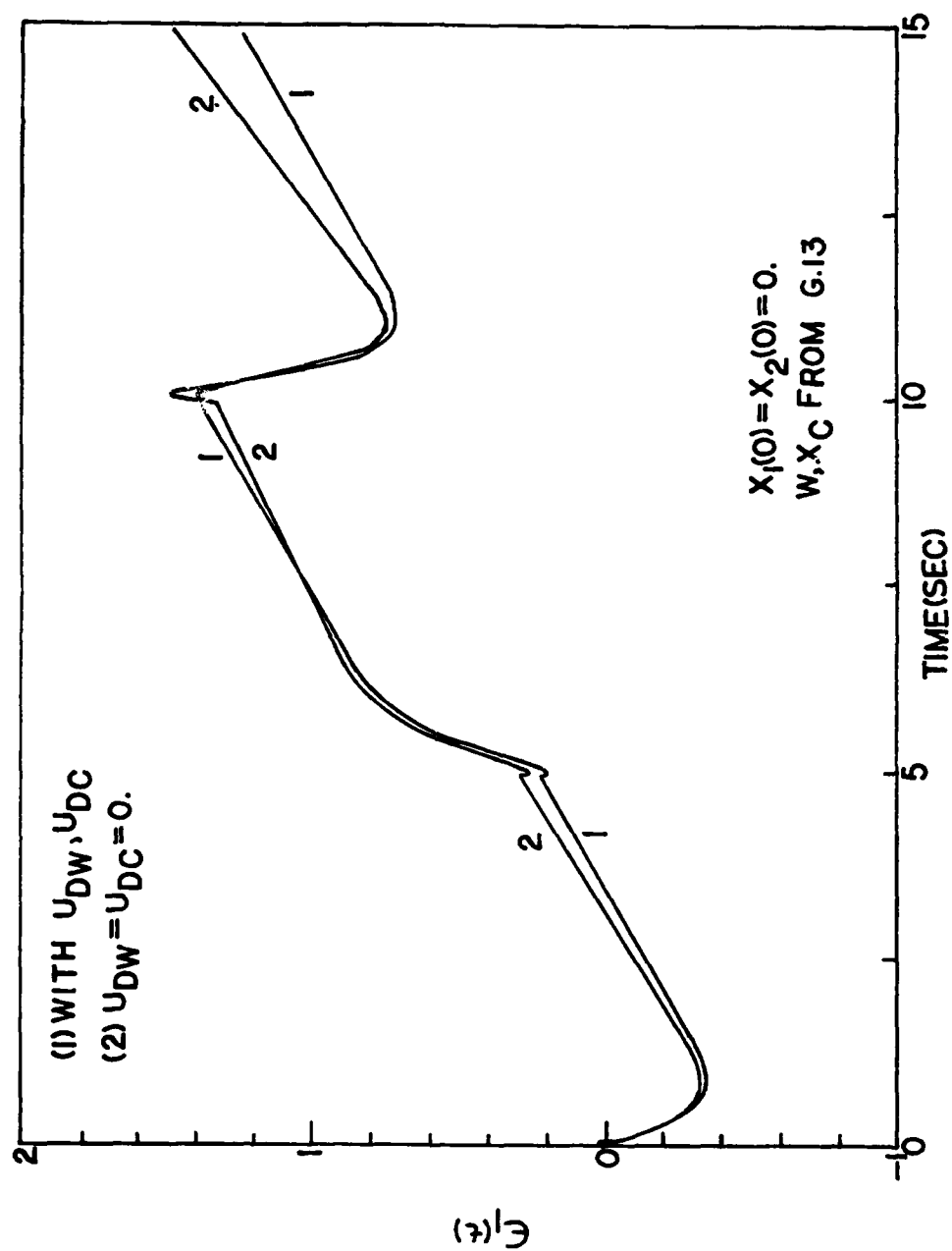


Figure G-9. e_1 versus time, inputs from G.13.

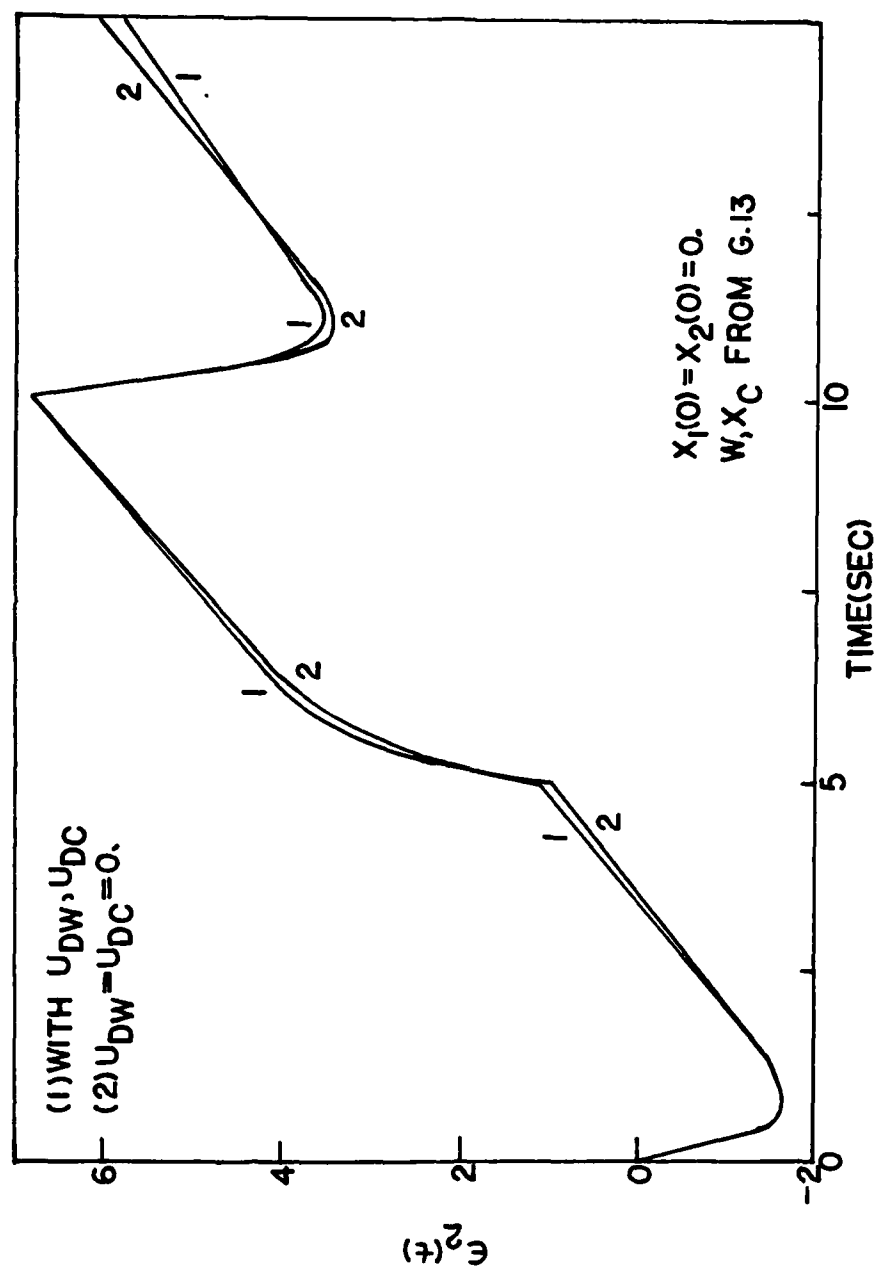


Figure G-10. ϵ_2 versus time, inputs from G.13

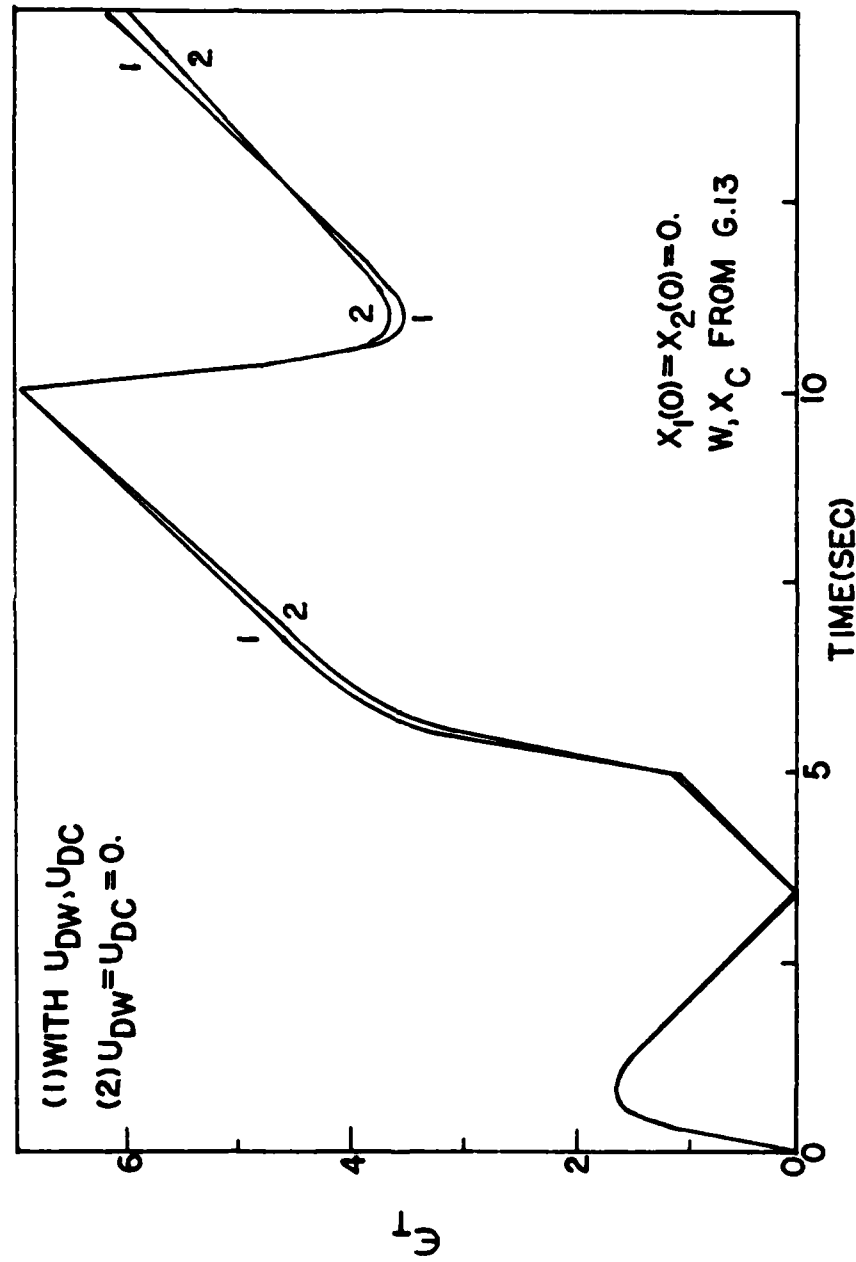


Figure G-11. ϵ_T versus time, inputs from G.13.

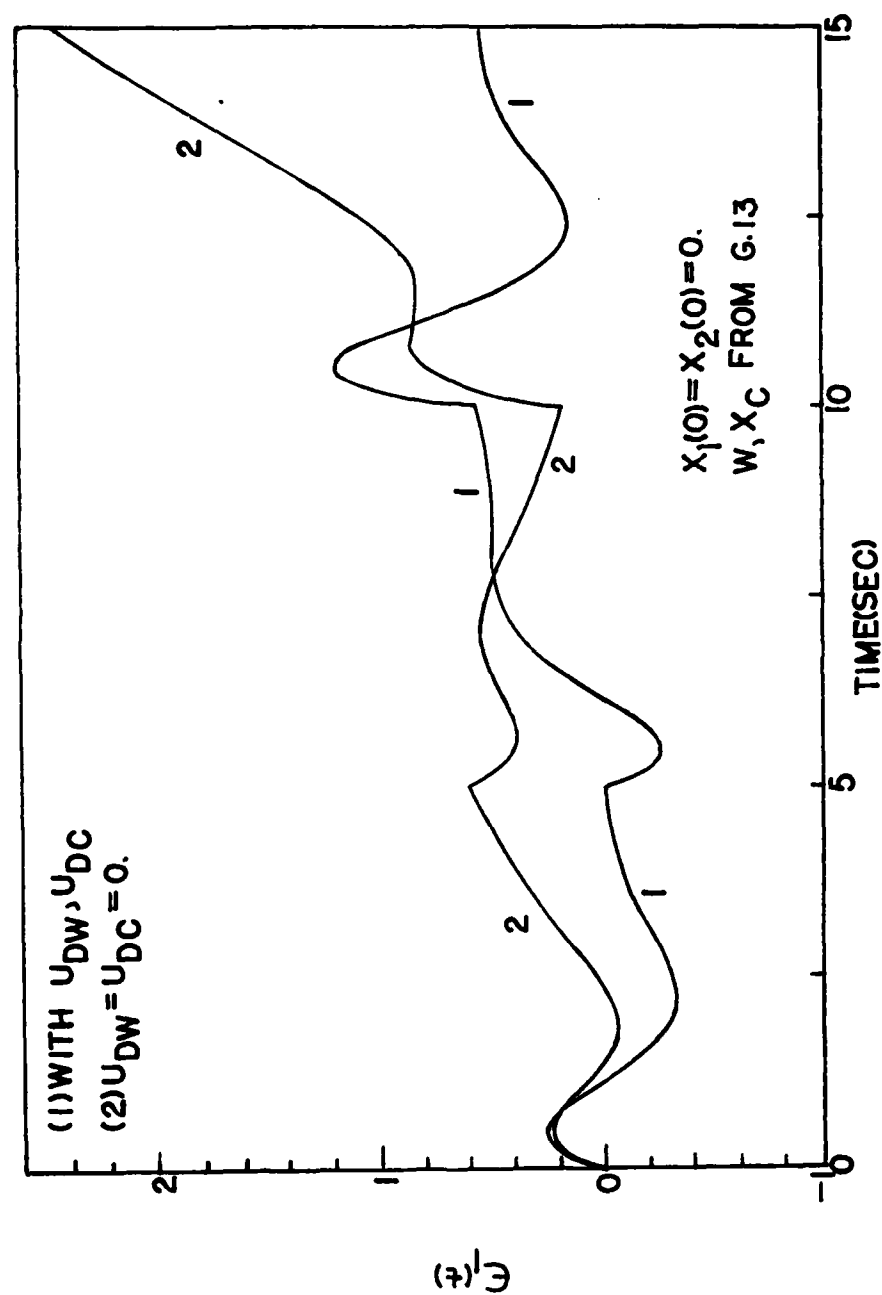


Figure G-12. ϵ_1 versus time, inputs from G.13, $K = (-3., -0.36)$.

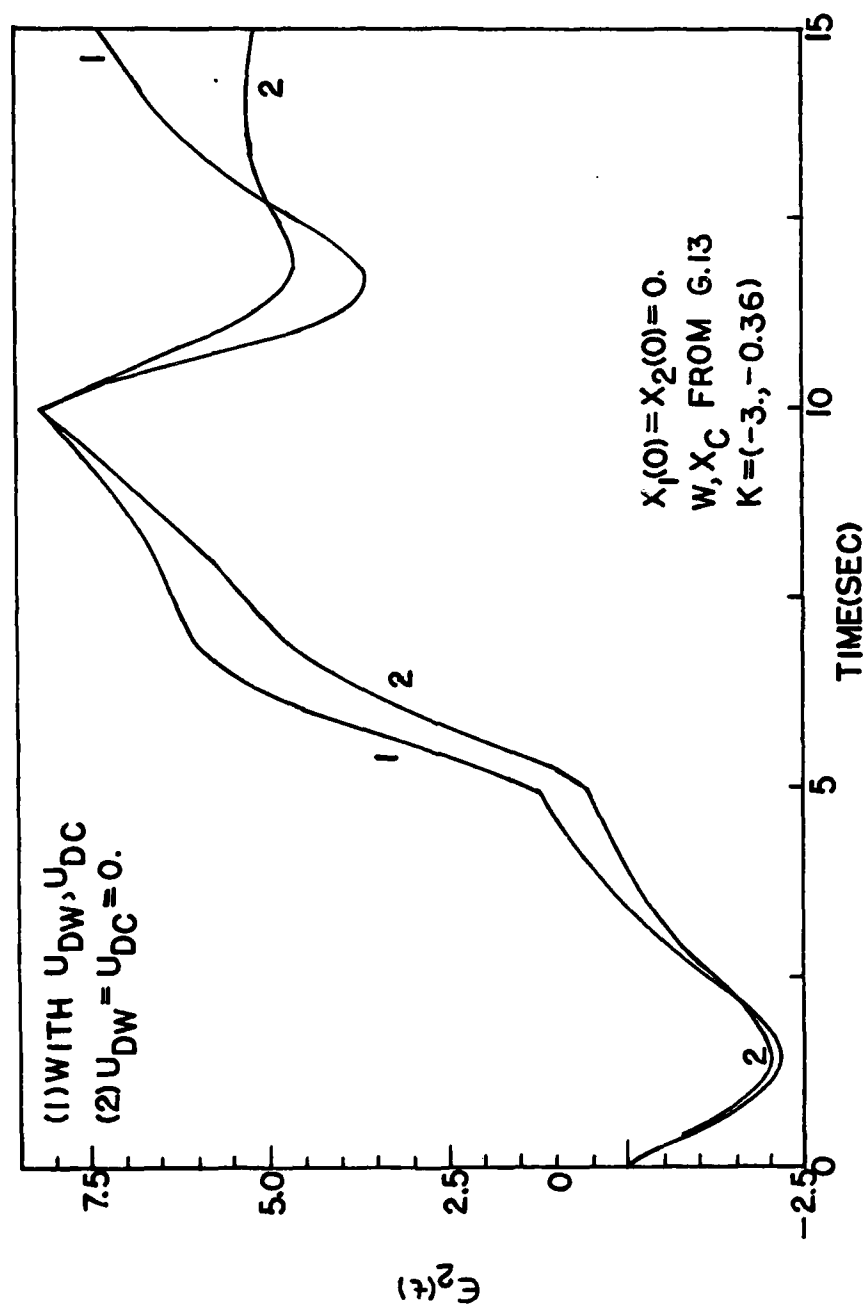


Figure G-13. ϵ_2 versus time, inputs from G.13, $K = (-3., -0.36)$.

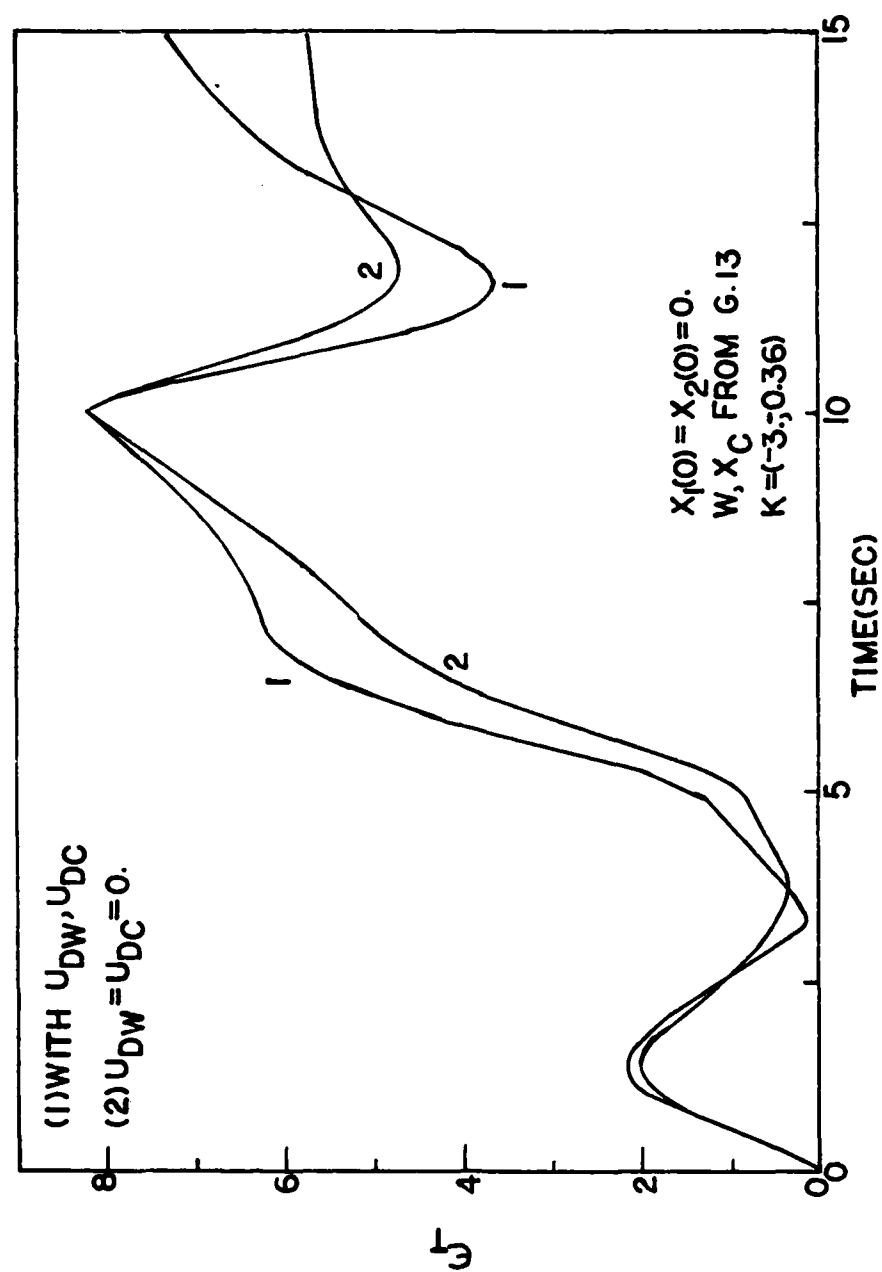


Figure G-14. ϵ_T versus time, inputs from G.13, $K = (-3., -0.36)$.

$u_{dw}^*(t)$ and $u_{dc}^*(t)$ and the other run with $u_{dw} = u_{dc} = 0$. The results obtained are shown in Figures G-15 to G-17. It can be seen that the minimizing control is allowing an increase in $\epsilon_2(t)$ in order to decrease $\epsilon_1(t)$.

(a) External Disturbance

$$\alpha = \begin{cases} 0.5 & 0. \leq t < 3. \\ -2. & 3. \leq t < 12. \\ 2. & 12. \leq t \end{cases}$$

(b) Servo-Command Input

(G.14)

$$\beta_1 = \begin{cases} 2. & 0. \leq t < 4. \\ -1. & 4. \leq t < 11. \\ 1. & 11. \leq t \end{cases} \quad \beta_2 = \begin{cases} -1. & 0. \leq t < 6. \\ -2. & 6. \leq t < 9. \\ 2. & 9. \leq t \end{cases}$$

G.4 Solution II

In this section the example will be solved by use of the isobasis design technique whereby the disturbance minimizing control vectors are assumed a priori to have the same form as the disturbances. The external disturbance minimizing control is assumed to be of the form $u_{dw}(t) = \bar{a}t$ and the servo-command disturbance minimizing control is assumed to be of the form $\bar{b}\beta t$. These assumed control parts can then be substituted into Equation (5.72) and the resulting expression for $\dot{\epsilon}(t)$ can be solved to obtain an expression for $\epsilon(t)$ which includes the parameters $\bar{a}$ and $\bar{b}$. By following the same procedure illustrated in Appendix F, Section F.5, one can calculate the $\bar{a}^*$ which will minimize the norm of the external disturbance residual in the solution for $\epsilon(t)$. The result is given by

$$\bar{a}^* = \frac{-0.0491w^2 + 0.0974\dot{w}w - 0.0345\dot{w}^2}{0.006374w^2 - 0.01256\dot{w}w + 0.01028\dot{w}^2} \quad (G.15)$$

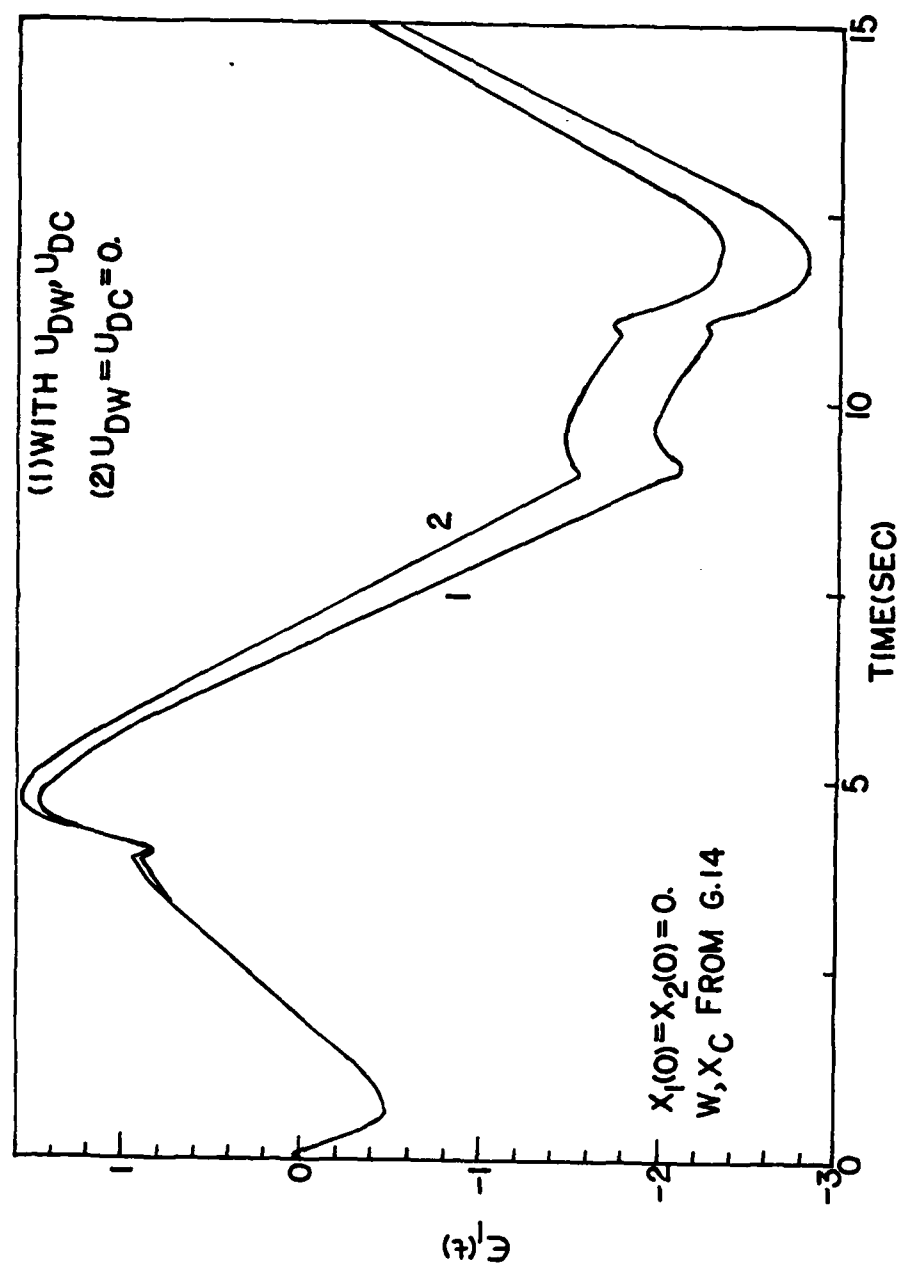


Figure G-15. ϵ_1 versus time, inputs from G.14.

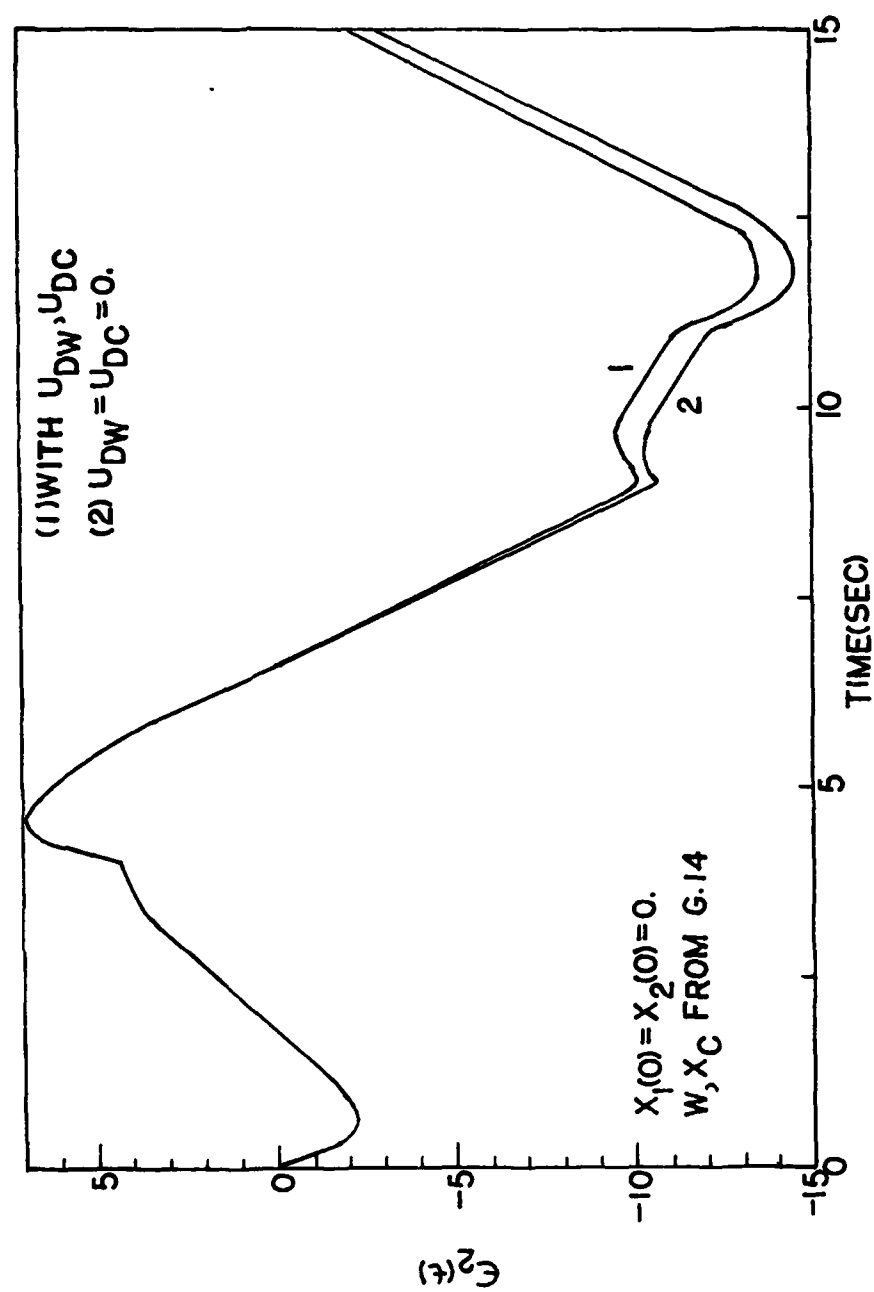
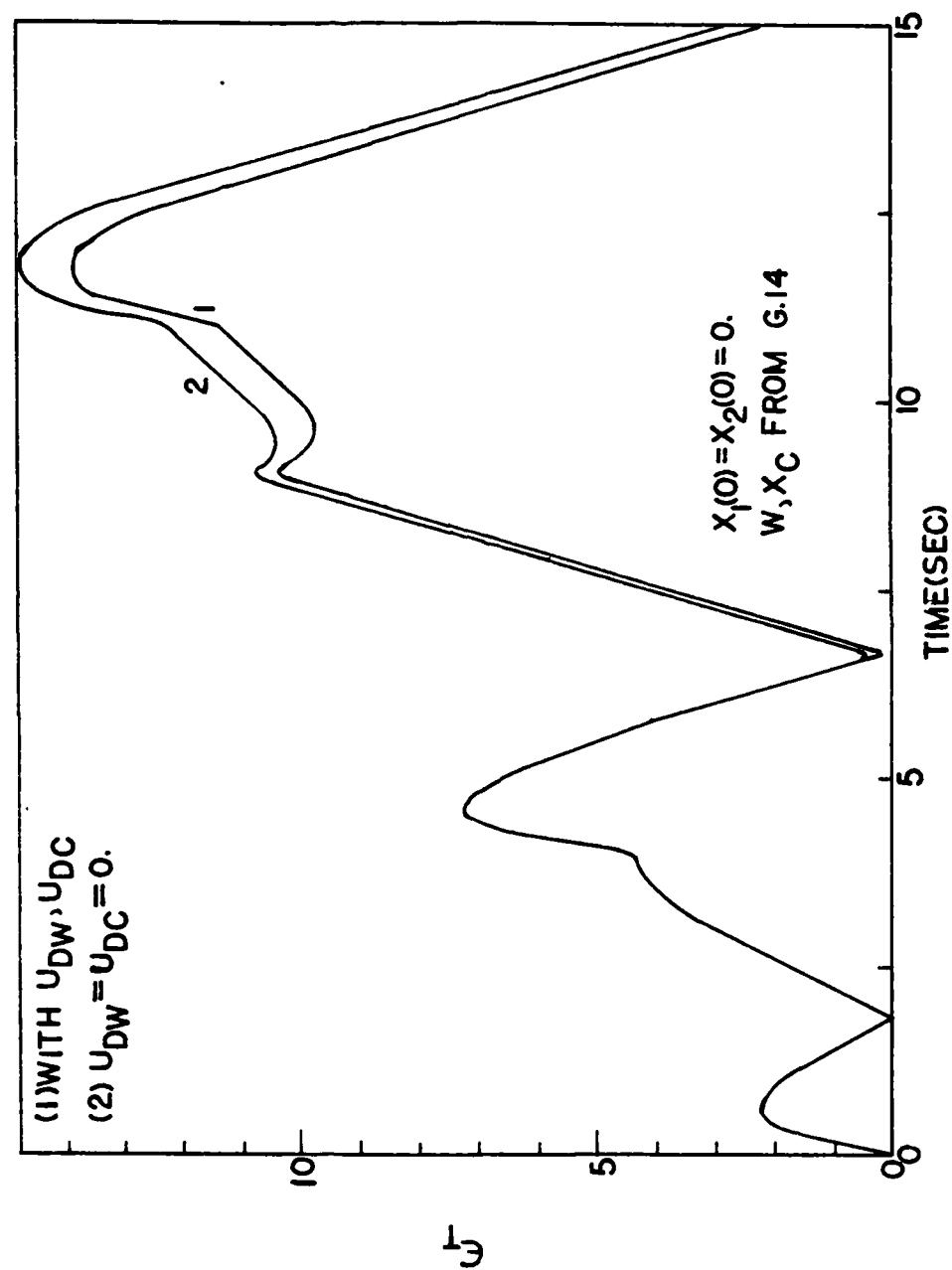


Figure G-16. ϵ_2 versus time, inputs from G.14.

Figure G-17. e_T versus time, inputs from G.14.

and u_{dw}^* is then given as

$$u_{dw}^*(t) = \bar{a}^* w(t) = k_{dw} w(t) . \quad (G.16)$$

In a similar fashion, one can calculate the $\bar{b}^*$ which will minimize the norm of the servo-command residual in $\epsilon(t)$. The result is given by

$$\bar{b}^* = \left(\frac{-0.0918\dot{y}_c^2 + 0.2743\dot{y}_c y_c - 0.198\dot{y}_c^2}{0.006374\dot{y}_c^2 - 0.01256\dot{y}_c y_c + 0.01028\dot{y}_c^2} , \right. \\ \left. \frac{-0.0581\dot{y}_c^2 + 0.0548\dot{y}_c y_c + 0.03617\dot{y}_c^2}{0.006374\dot{y}_c^2 - 0.01256\dot{y}_c y_c + 0.01028\dot{y}_c^2} \right) \quad (G.17)$$

and $u_{dc}^*(t)$ is expressible as

$$u_{dc}^*(t) = \bar{b}^* \beta t = (k_{dc1}, k_{dc2}) x_c(t) . \quad (G.18)$$

This example was also programmed on a digital computer and a block diagram of the system as programmed is shown in Figure G-18. A run was first made with the external disturbance and servo-command disturbance coefficients as given in expressions (G.11). Figures G-19 and G-20 show the results obtained (compare with Figures G-4 and G-5). Two more runs were made using the ramp coefficients in expressions (G.13) and the same conditions as for Figures G-9 and G-10. The results are given in Figures G-21 to G-23 and indicate the manner in which the controller developed in this section "played off" the two error components in order to provide a smaller total error vector magnitude (after about time $t=6.5$ seconds) than the controller from Section G.3.

Two more runs were also made for comparison with results in Section G.3 and used the coefficients in expressions (G.14). The results which were obtained are shown in Figures G-24 to G-26. The controller from this section makes a large difference in the magnitude

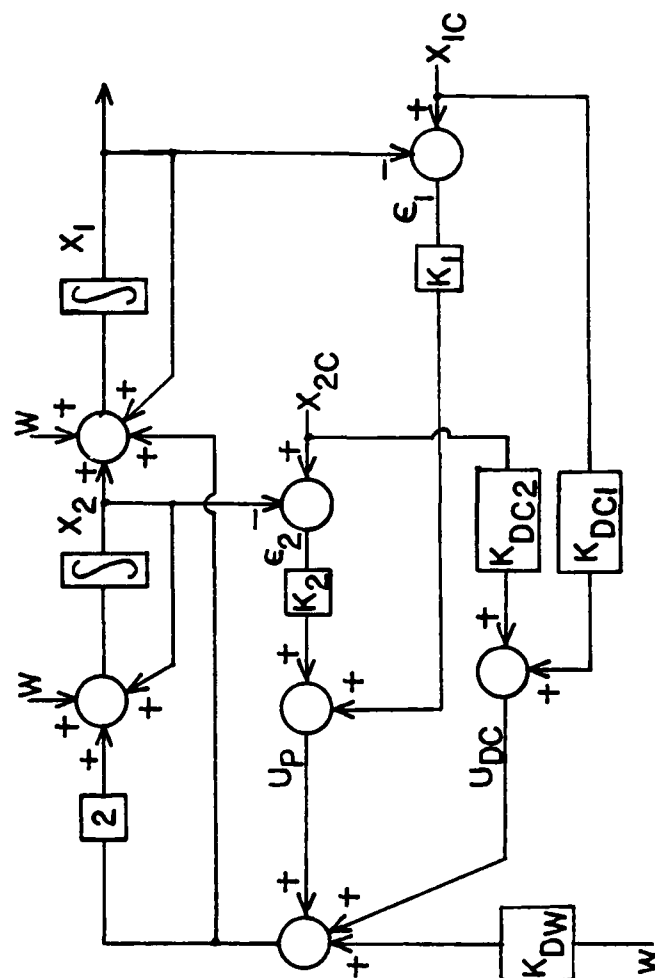


Figure G-18. Block diagram of plant/controller, isobasis design technique.

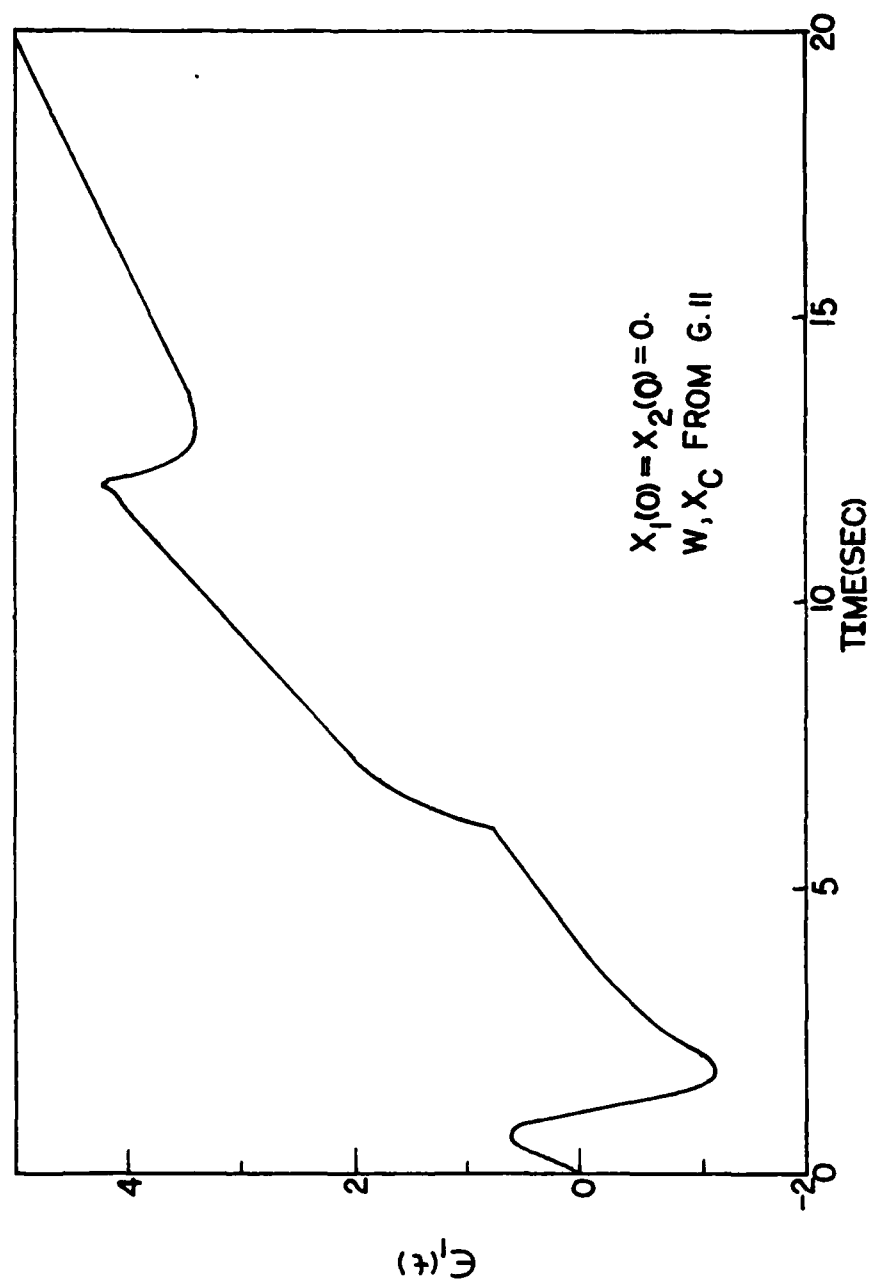


Figure G-19. ϵ_1 versus time, inputs from G.11

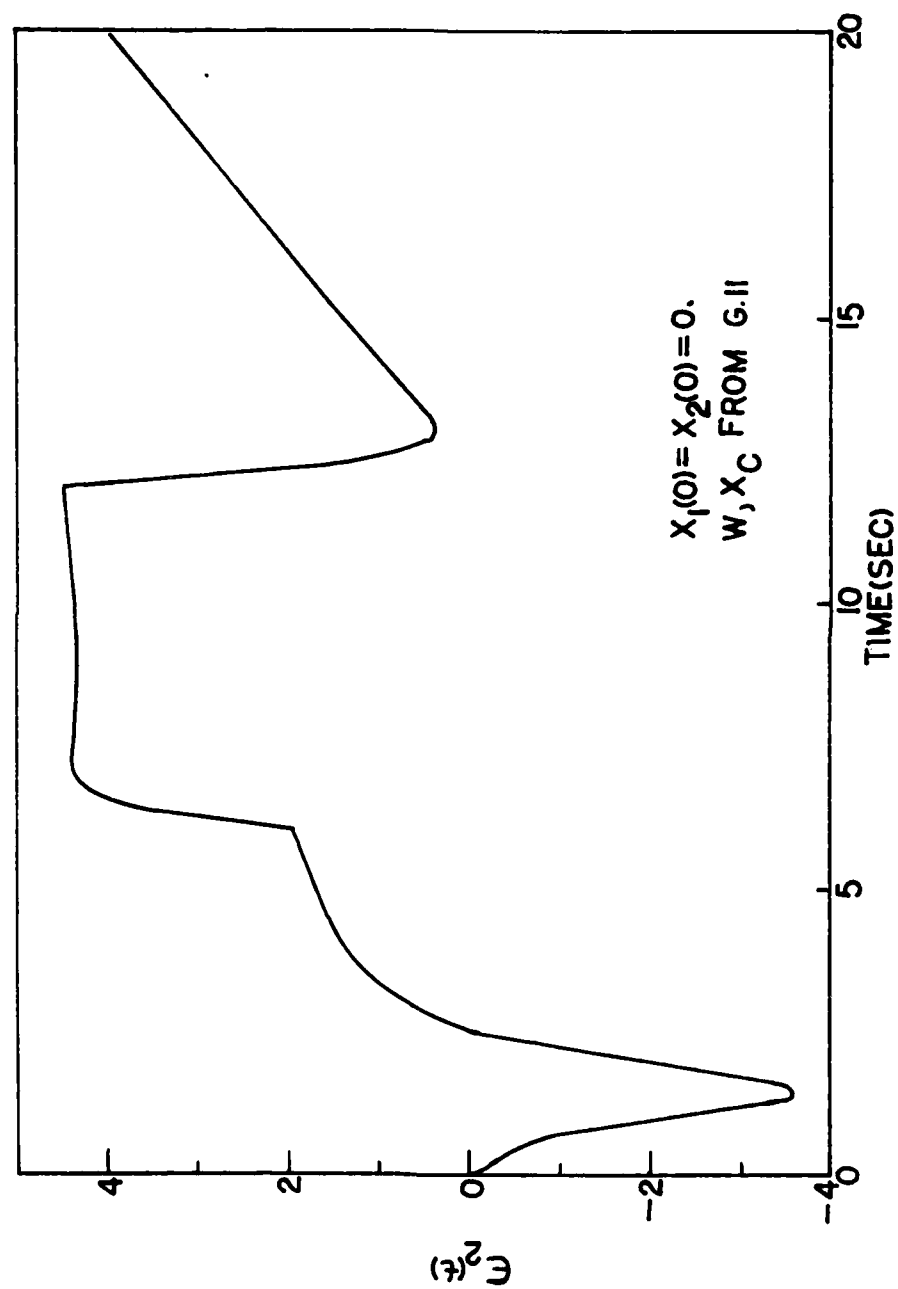


Figure G-20. ϵ_2 versus time, inputs from G.11.

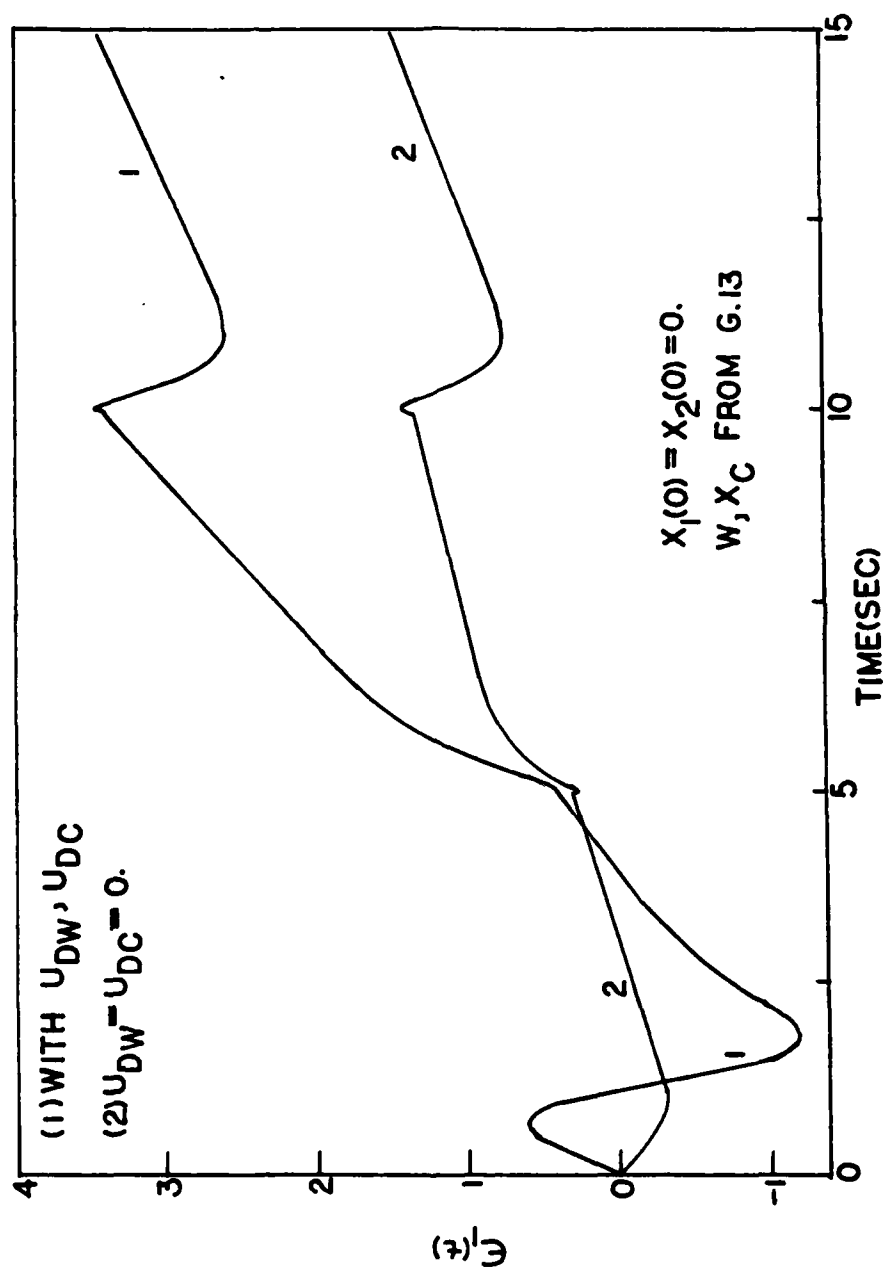
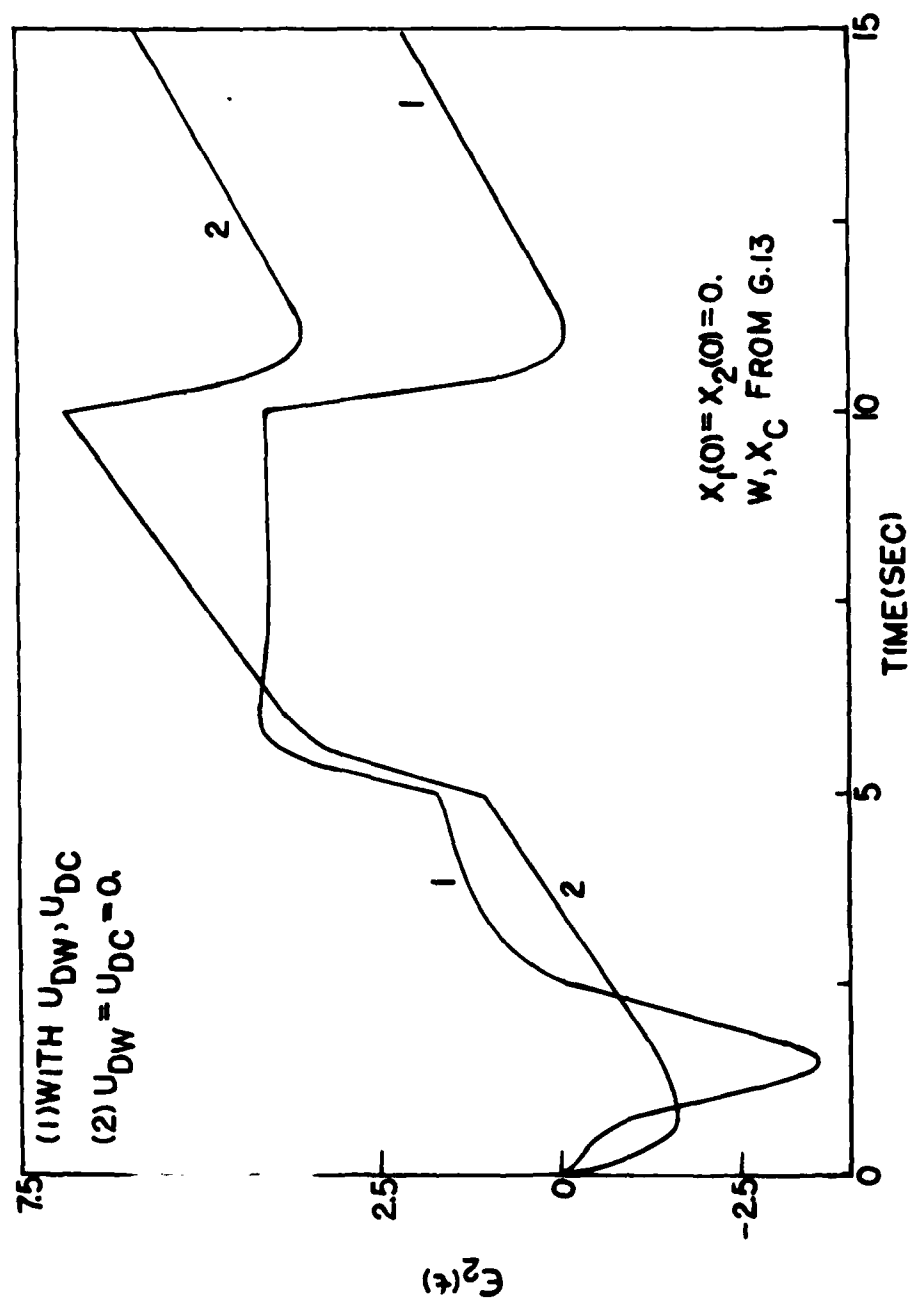


Figure G-21. ϵ_1 versus time, inputs from G.13.

Figure G-22. ϵ_2 versus time, inputs from G.13.

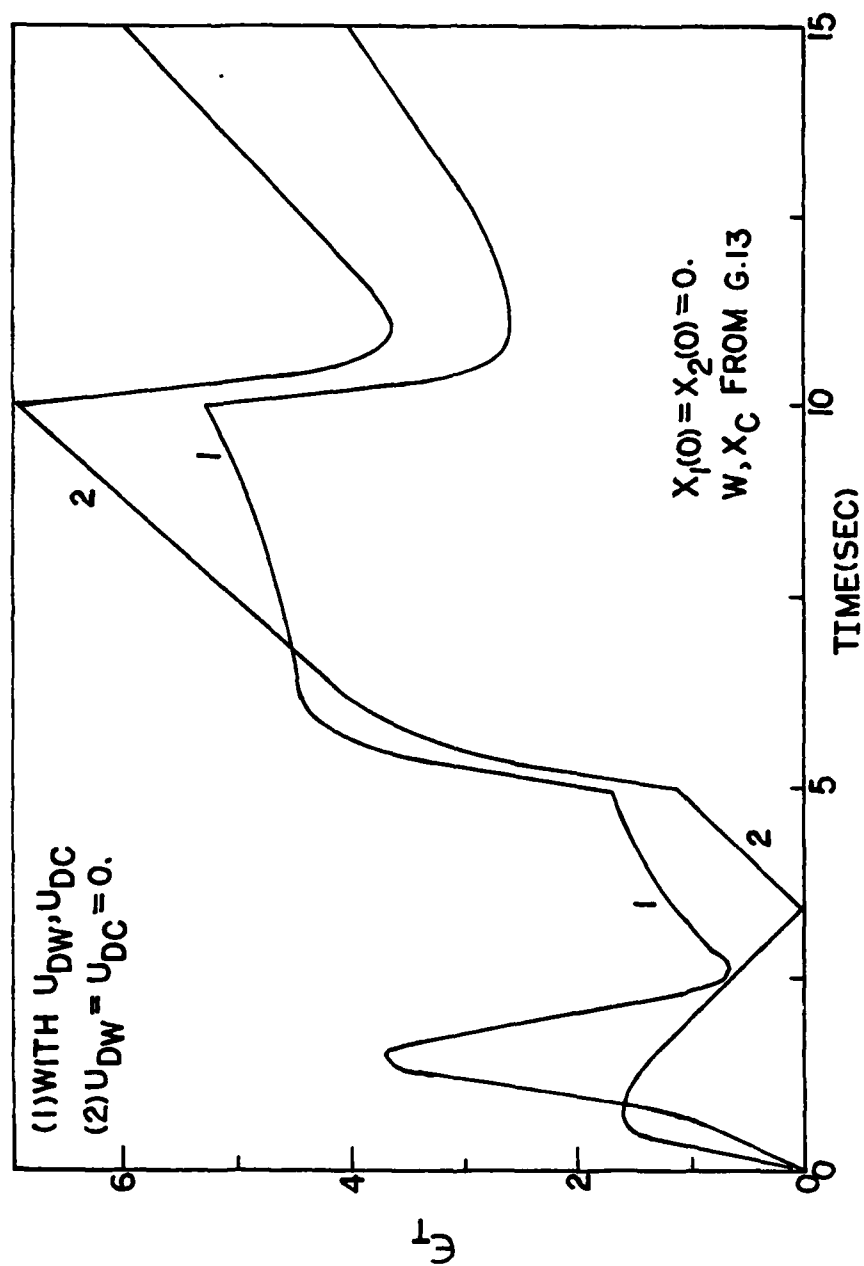


Figure G-23. ϵ_T versus time, inputs from G.13.

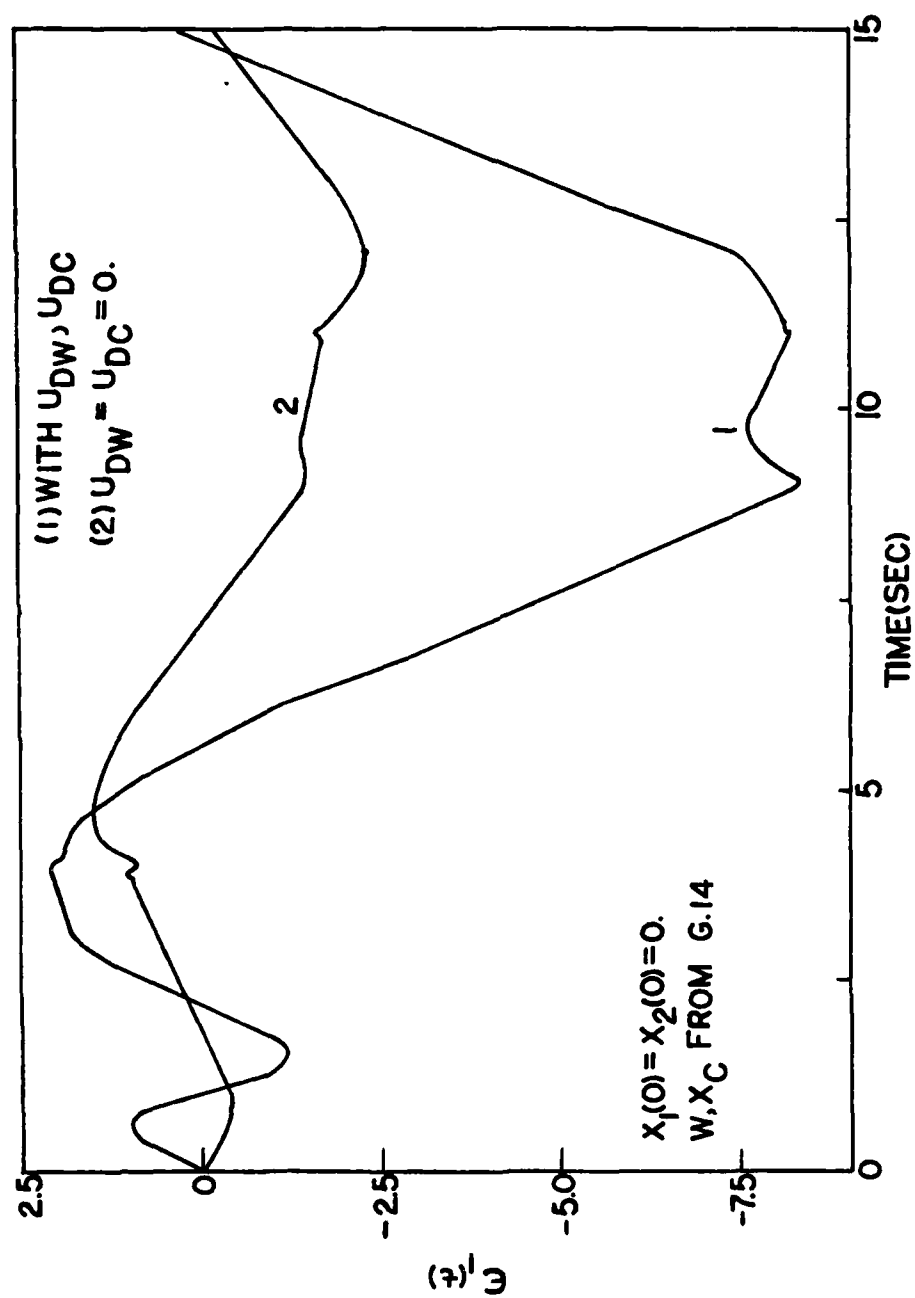


Figure G-24. ϵ_1 versus time, inputs from G.14.

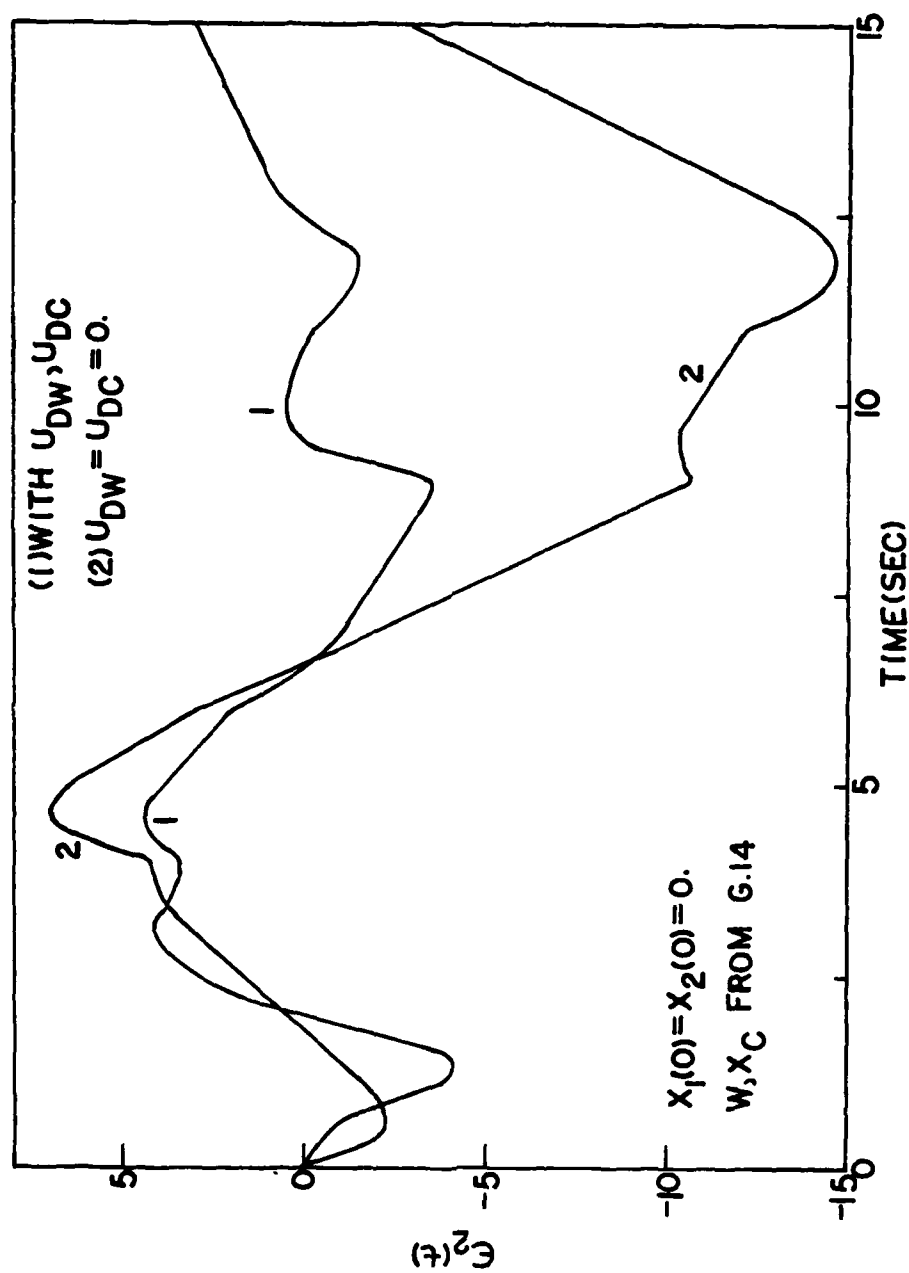


Figure G-25. ϵ_2 versus time, inputs from G.14.

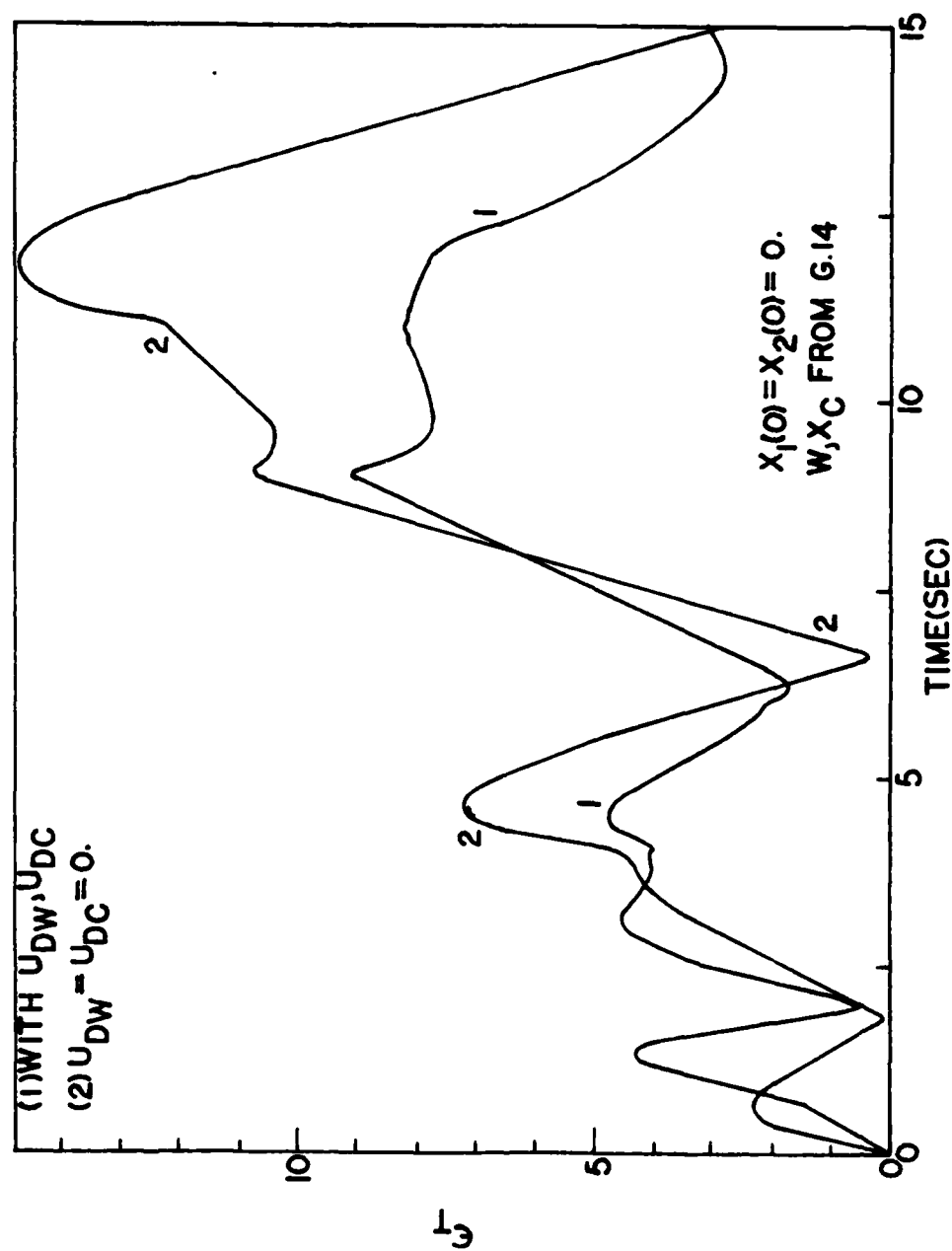


Figure G-26. ϵ_T versus time, inputs from G.14.

of $\epsilon_2(t)$ (as can be seen by comparing Figures G-16 and G-25) in order to decrease the total error magnitude. To see what effect the controller of this section has with respect to the external disturbance only, two runs were made using the following coefficients for $w(t)$, with $x_c(t) \equiv 0$.

$$\alpha = \begin{cases} 0. & 0. \leq t < 5. \\ 2. & 5. \leq t < 10. \\ -1. & 10. \leq t \end{cases} \quad (G.19)$$

The results are shown in Figures G-27 to G-29. During the first five seconds of data the system response to an initial condition of $x(0)=1$ is shown. From $t=5$ seconds to $t=15$ seconds, the effects of the controller on an external disturbance is shown. The controller allows an increase in $\epsilon_1(t)$ in order to decrease $\epsilon_2(t)$ and provide an overall decrease in $\epsilon_T(t)$.

G.5 Solution III

Since the example in this Appendix is a second-order system, one can easily determine a disturbance minimizing control component which will remove the direct disturbance effects from one or the other of the two states. The complete expression for the error dynamics is given by

$$\begin{pmatrix} \dot{\epsilon}_1 \\ \dot{\epsilon}_2 \end{pmatrix} = \begin{bmatrix} -19. & 4.5 \\ -40. & 8. \end{bmatrix} \begin{pmatrix} \epsilon_1 \\ \epsilon_2 \end{pmatrix} - \begin{pmatrix} 1 \\ 2 \end{pmatrix} u_d - \begin{pmatrix} 1 \\ 1 \end{pmatrix} w - \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} x_c + \dot{x}_c. \quad (G.20)$$

If one were to choose $\epsilon_1(t)$ as the critical state-variable then $u_d(t)$ would be given by

$$u_d(t) = -w(t) - x_{c1}(t) - x_{c2}(t) + \dot{x}_{c1}(t) \quad (G.21)$$

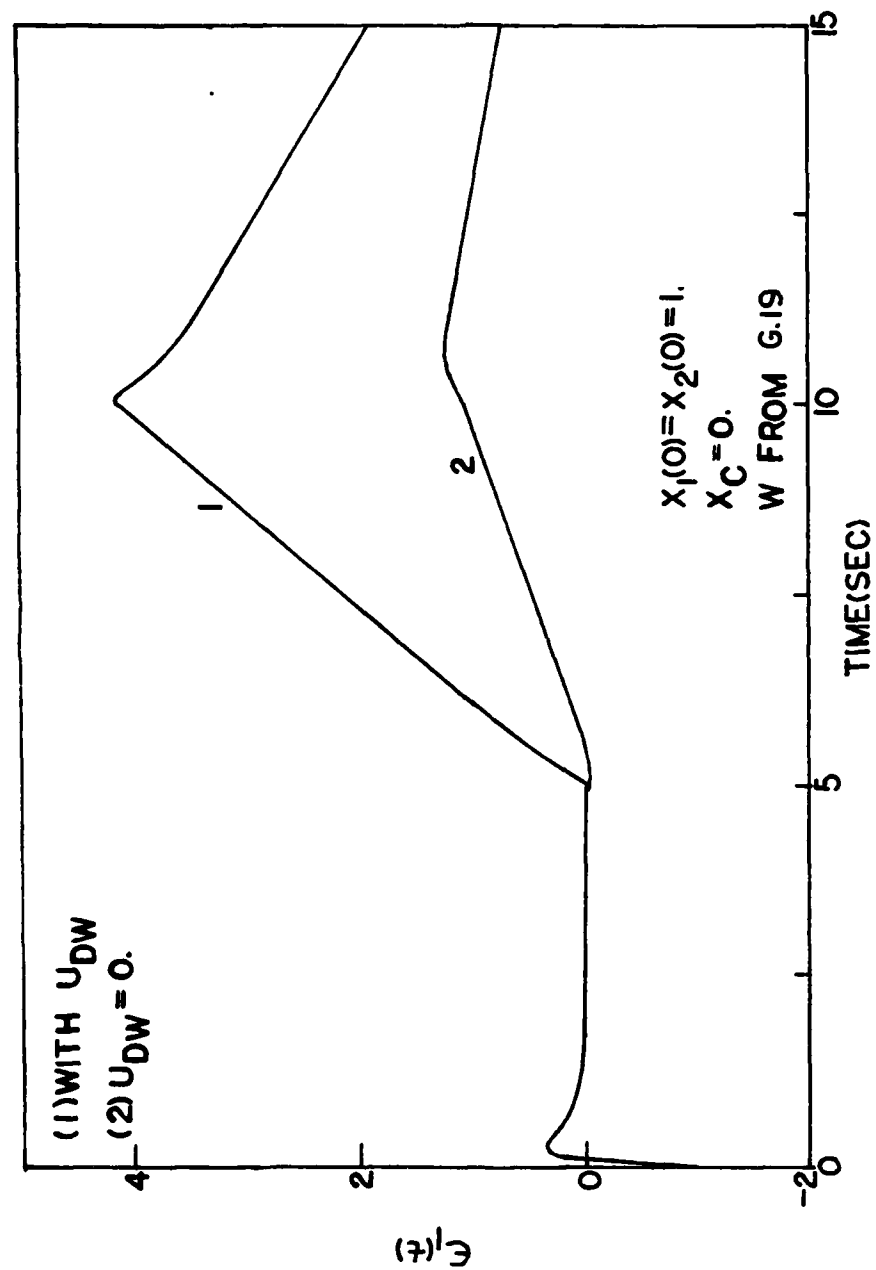


Figure G-27. ϵ_1 versus time, inputs from G.19.

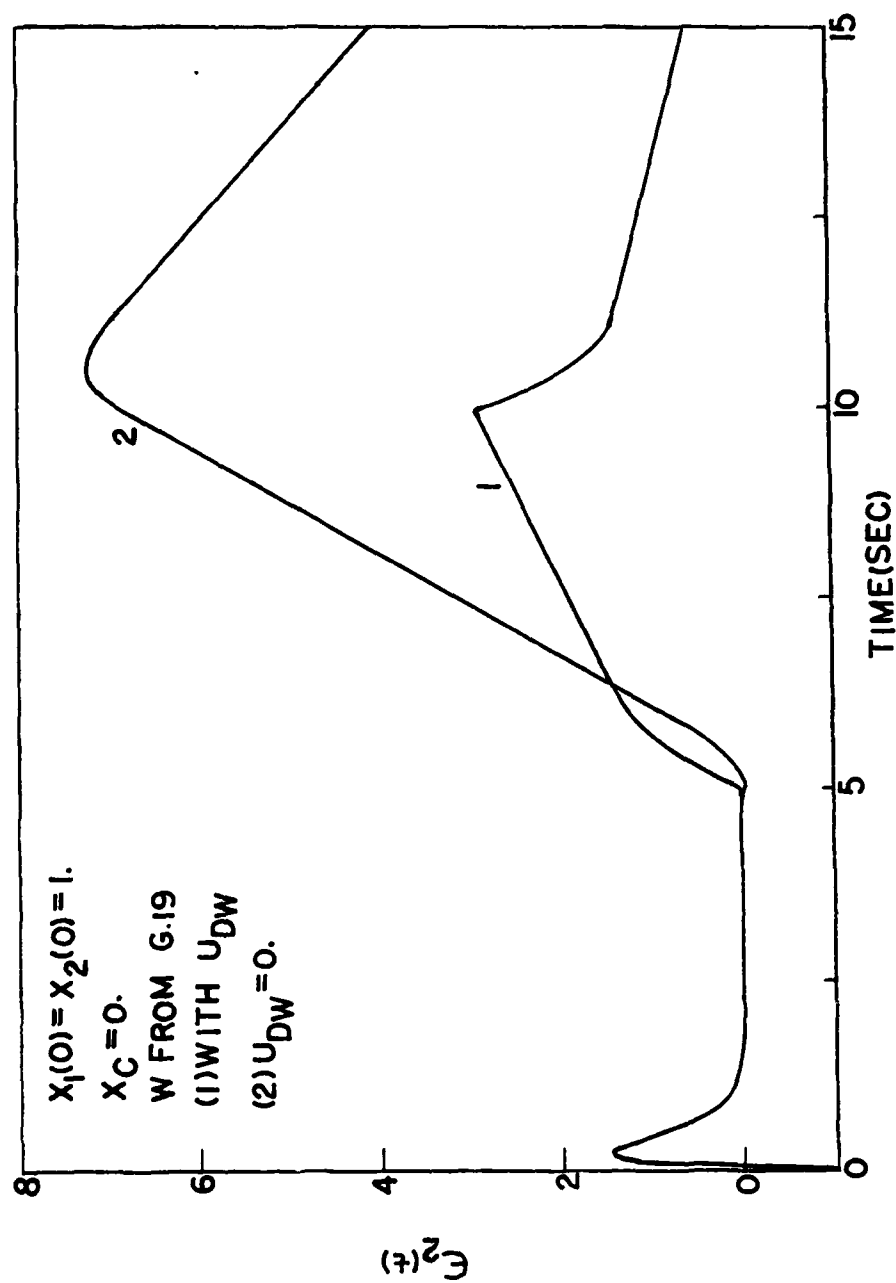
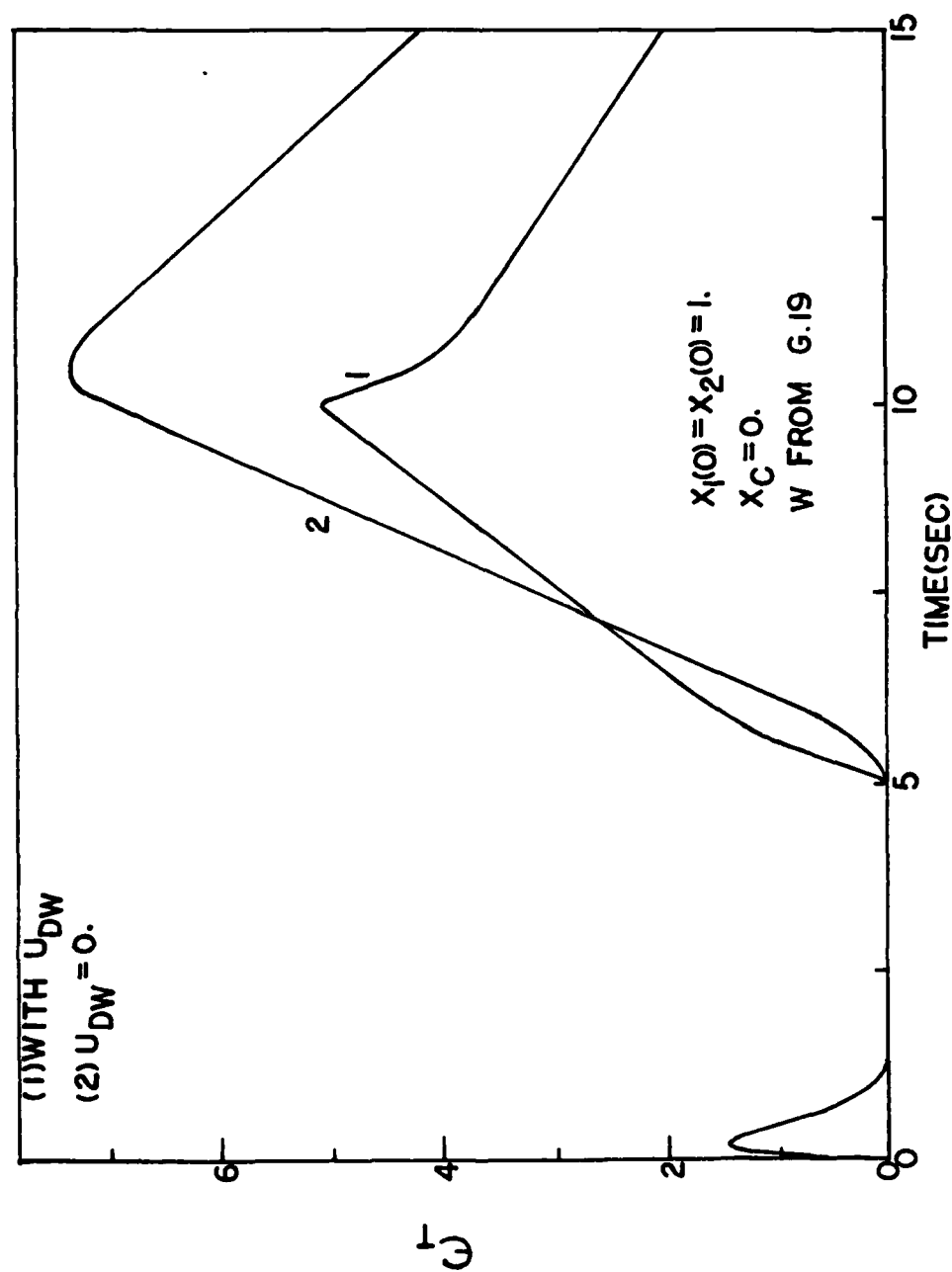


Figure G-28. ϵ_2 versus time, inputs from G.19.

Figure G-29. ϵ_T versus time, inputs from G.19.

and the resulting expression for $\dot{\epsilon}_1(t)$ would be

$$\dot{\epsilon}_1(t) = -19\epsilon_1(t) + 4.5\epsilon_2(t) . \quad (G.22)$$

If $\epsilon_2(t)$ were chosen as the critical state-variable then $u_d(t)$ would be

$$u_d(t) = 0.5(-w(t) - x_{c2}(t) + \dot{x}_{c2}(t)) \quad (G.23)$$

and the resulting expression for $\dot{\epsilon}_2(t)$ would be

$$\dot{\epsilon}_2(t) = -40\epsilon_1(t) + 8\epsilon_2(t) . \quad (G.24)$$

Figures G-30 to G-32 show the results obtained, using coefficients in G.19, when $\epsilon_1(t)$ is chosen as the critical state variable and Equation (G.21) is used as the disturbance minimizing control. Comparing Figures G-30 and G-27, one can see that $\epsilon_1(t)$ is reduced at the expense of an increase in $\epsilon_2(t)$. Figures G-33 to G-35 show the results, using the coefficients in G.19, when $\epsilon_2(t)$ is chosen as the critical-state variable and Equation (G.23) is used as the disturbance minimizing control. Comparing Figures G-33 to G-35 with Figures G-27 to G-29, it can be seen that the results obtained with $\epsilon_2(t)$ as a critical state-variable are about the same as for the case when the control objective was to minimize the total error, which is logical since $\epsilon_2(t)$ is the largest contributor to total error.

As a matter of interest, the run conditions of Figures G-30 to G-33 were repeated with the gain matrix K taken to be $K = (-3., -0.36)$ and using Equation (G.23) for $u_d(t)$. The results, shown in Figures G-36 to G-38 indicate that for a lower gain, the disturbance minimizing control makes a larger difference between the u_{dw} -on and u_{dw} -off cases than it does for the higher gain case.

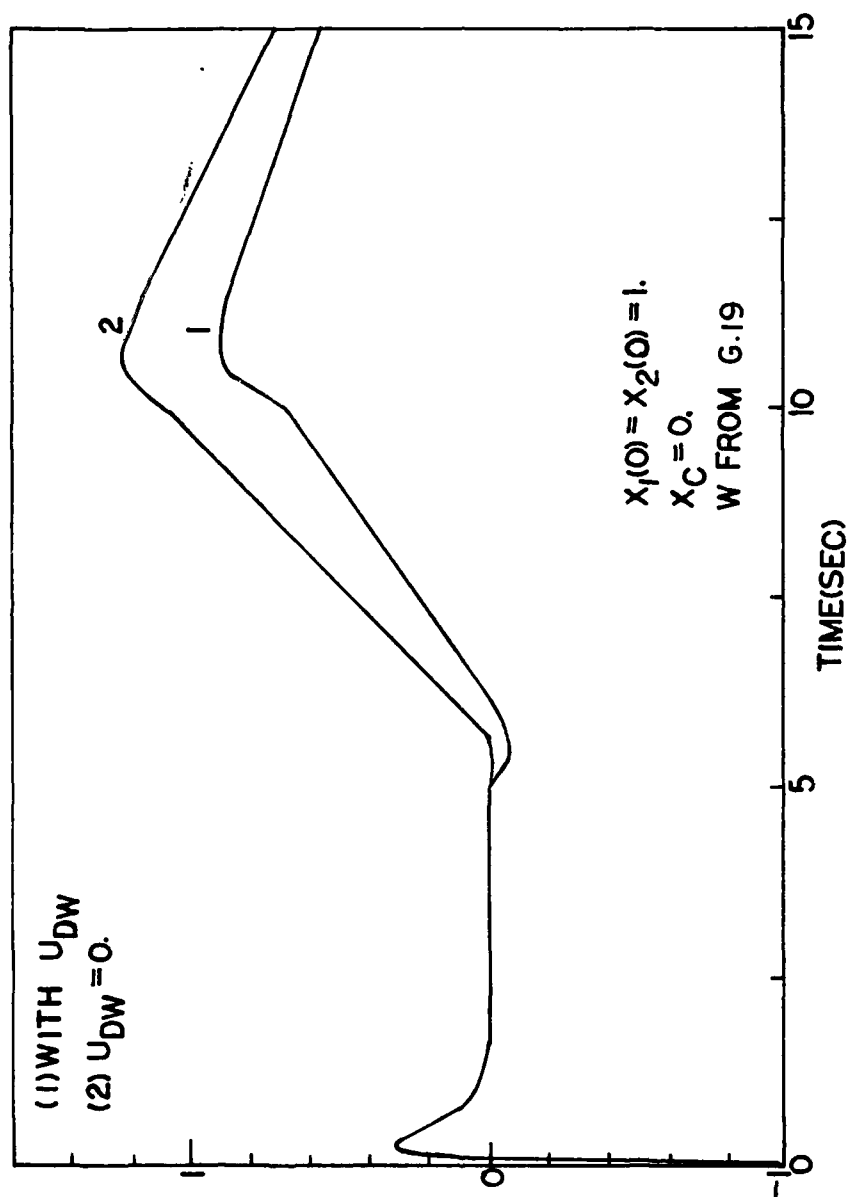


Figure G-30. ϵ_1 versus time, ϵ_1 as critical state-variable.

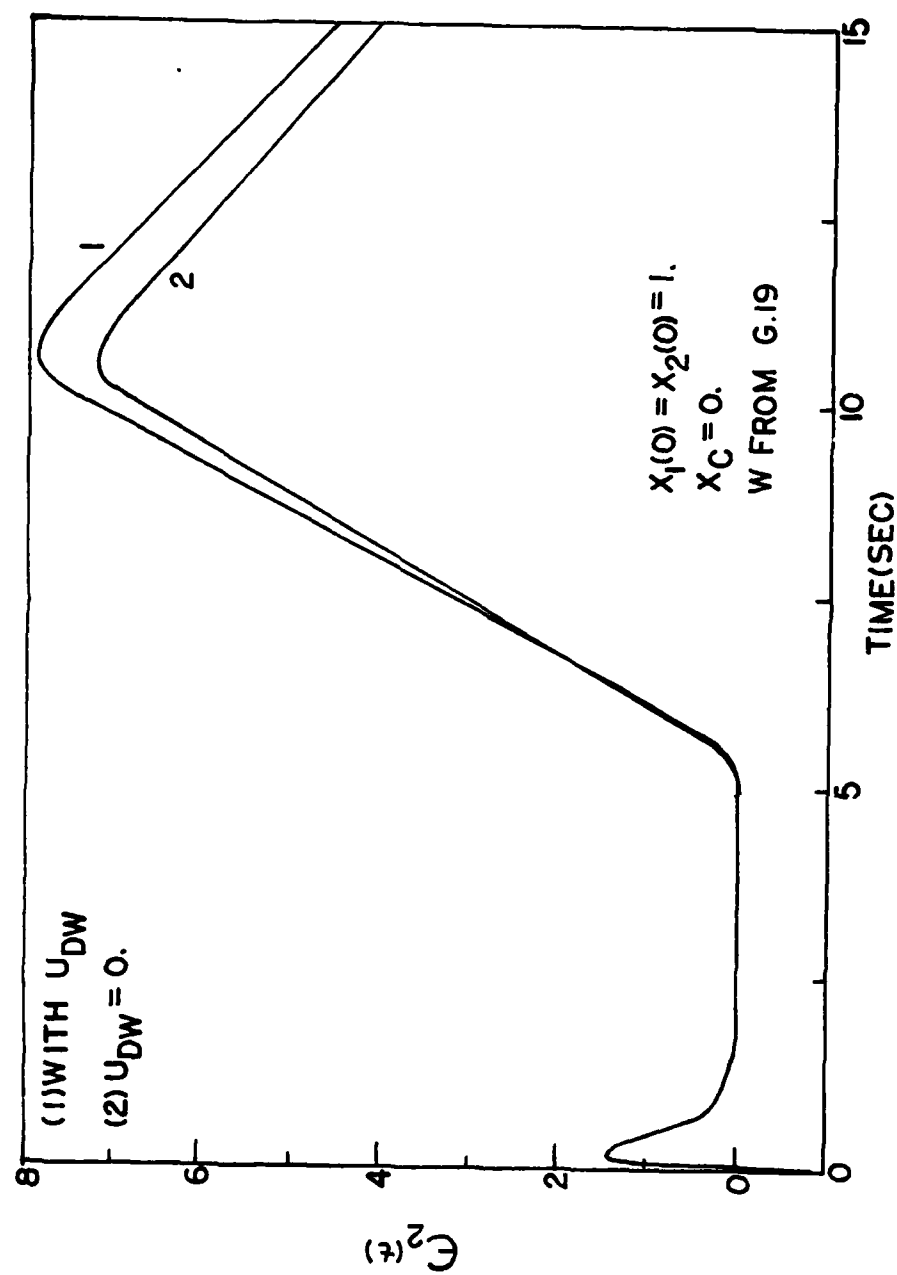


Figure G-31. ϵ_2 versus time, ϵ_1 as critical state-variable.

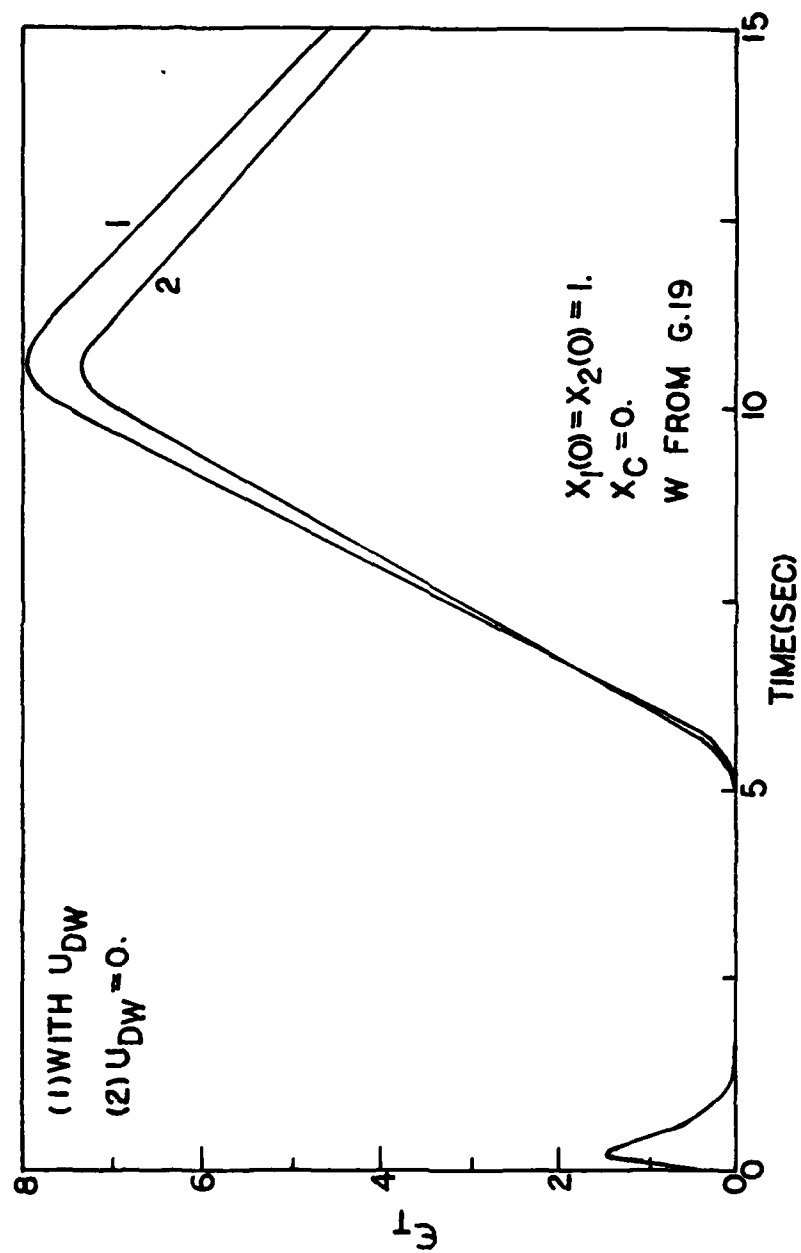


Figure G-32. ϵ_T versus time, ϵ_1 as critical state-variable.

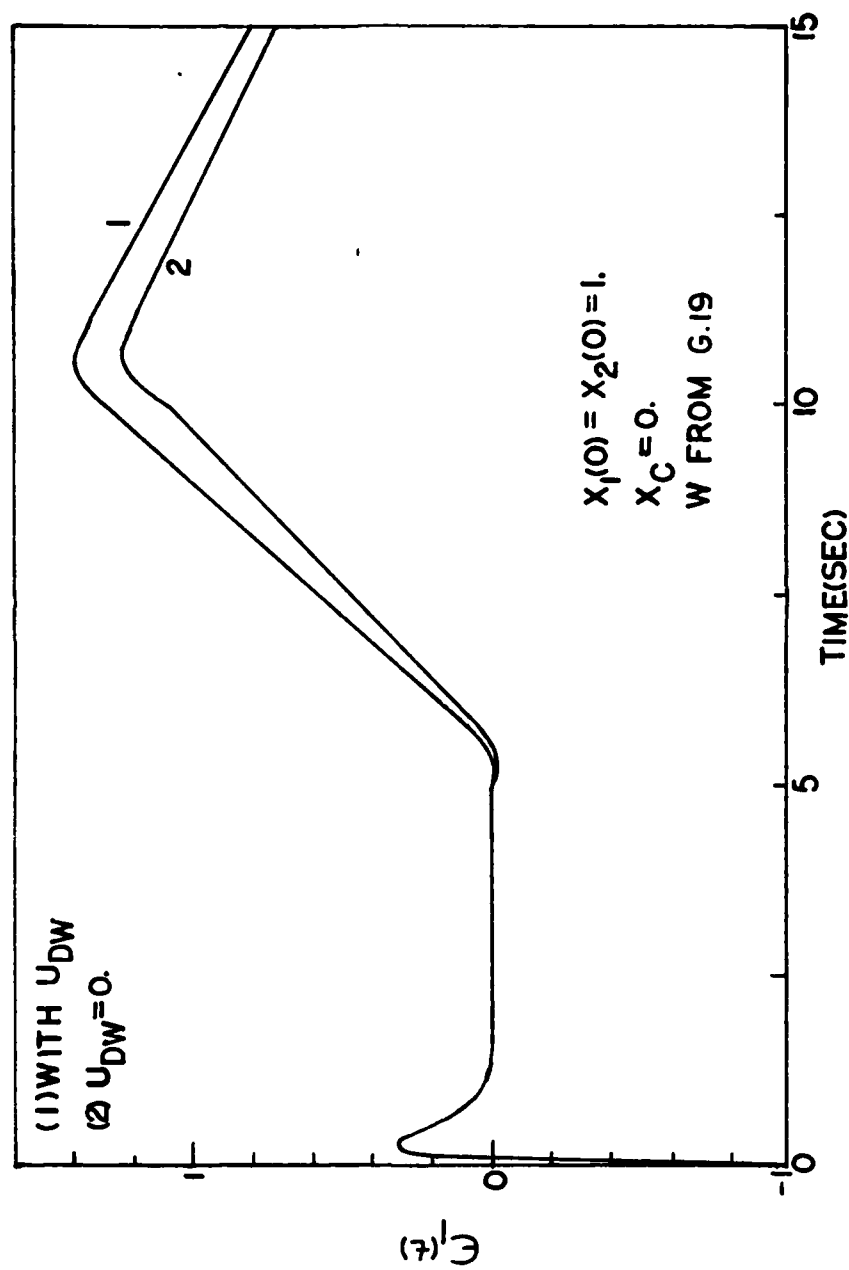


Figure G-33. ϵ_1 versus time, ϵ_2 as critical state-variable.

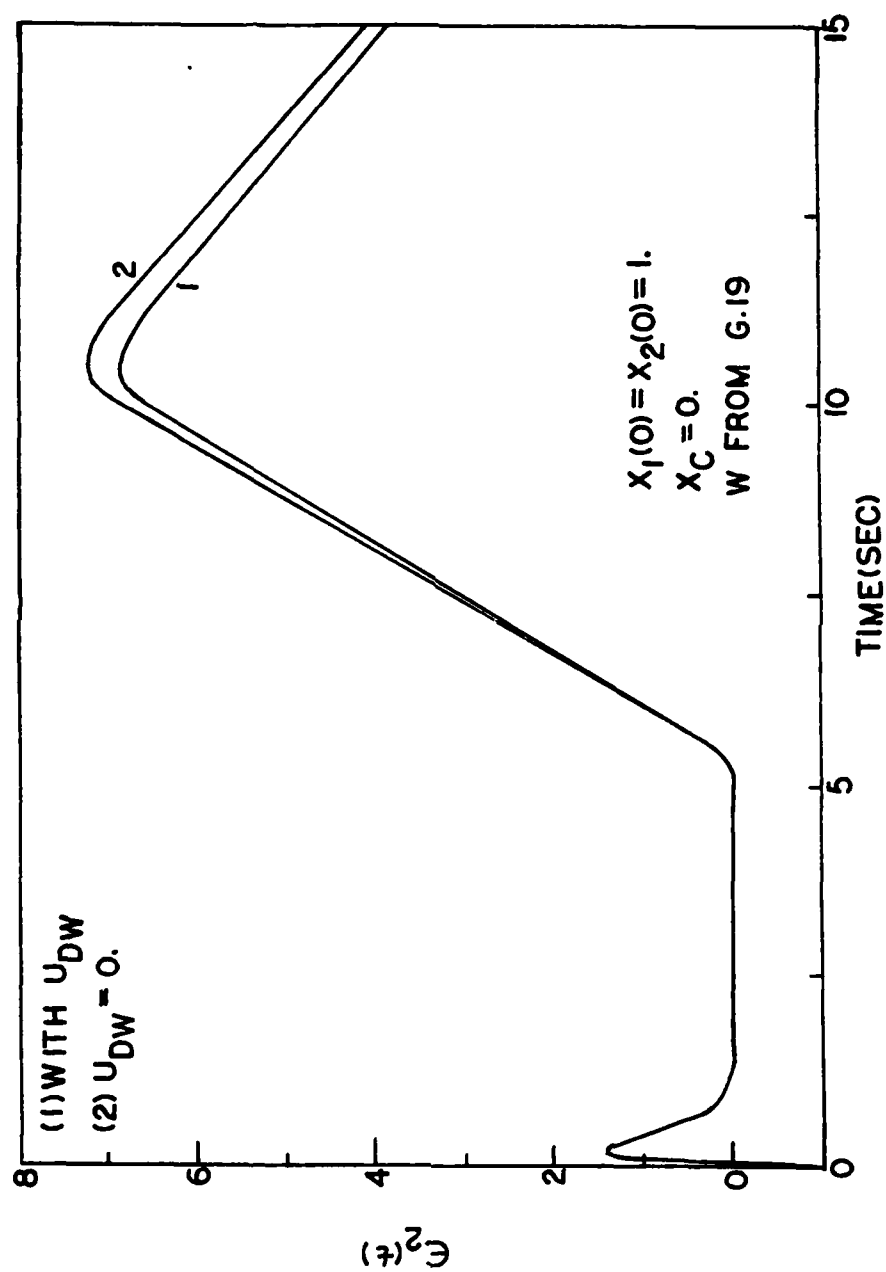


Figure G-34. ϵ_2 versus time, ϵ_2 as critical state-variable.

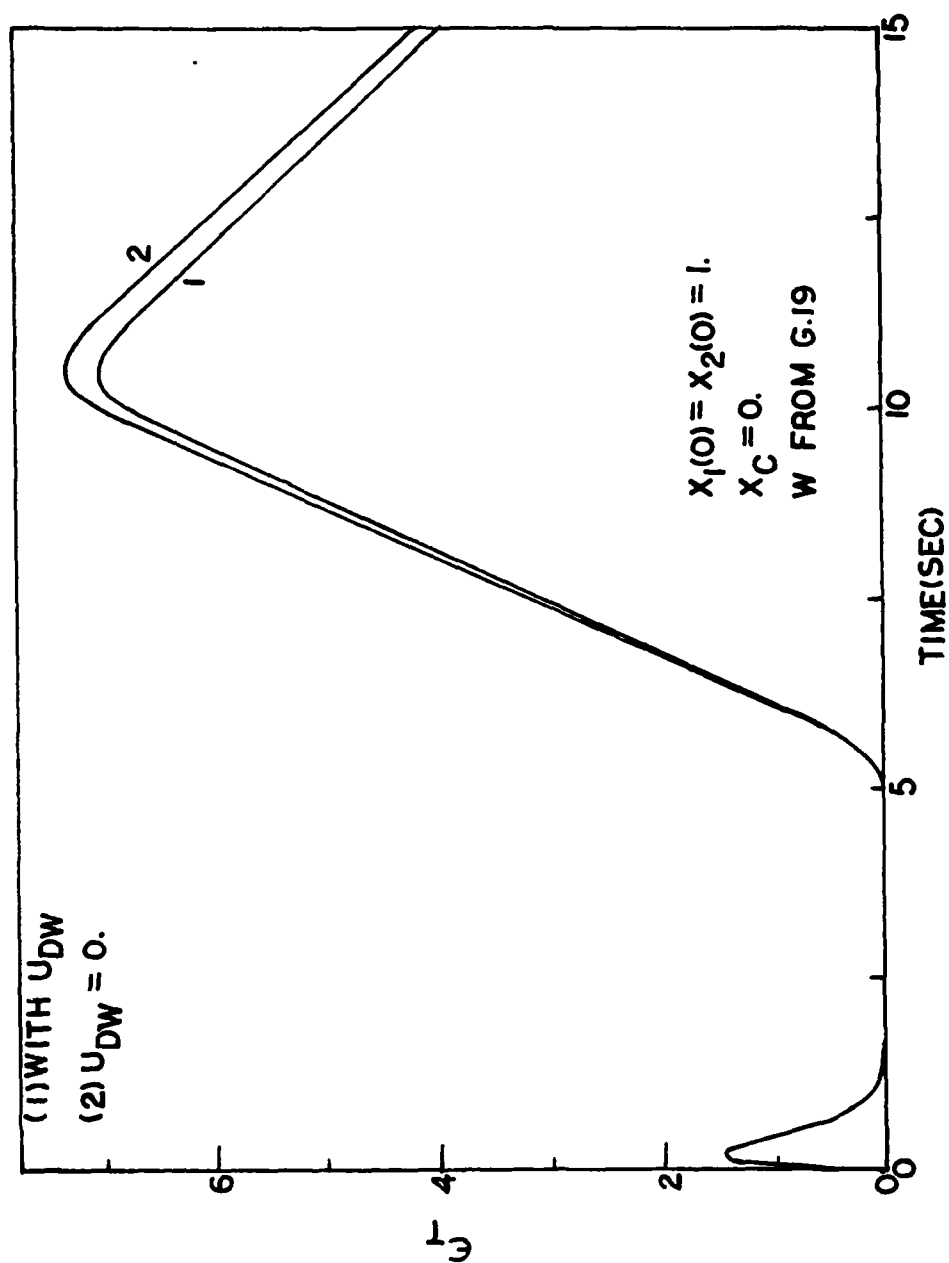


Figure G-35. ϵ_T versus time, ϵ_2 as critical state-variable.

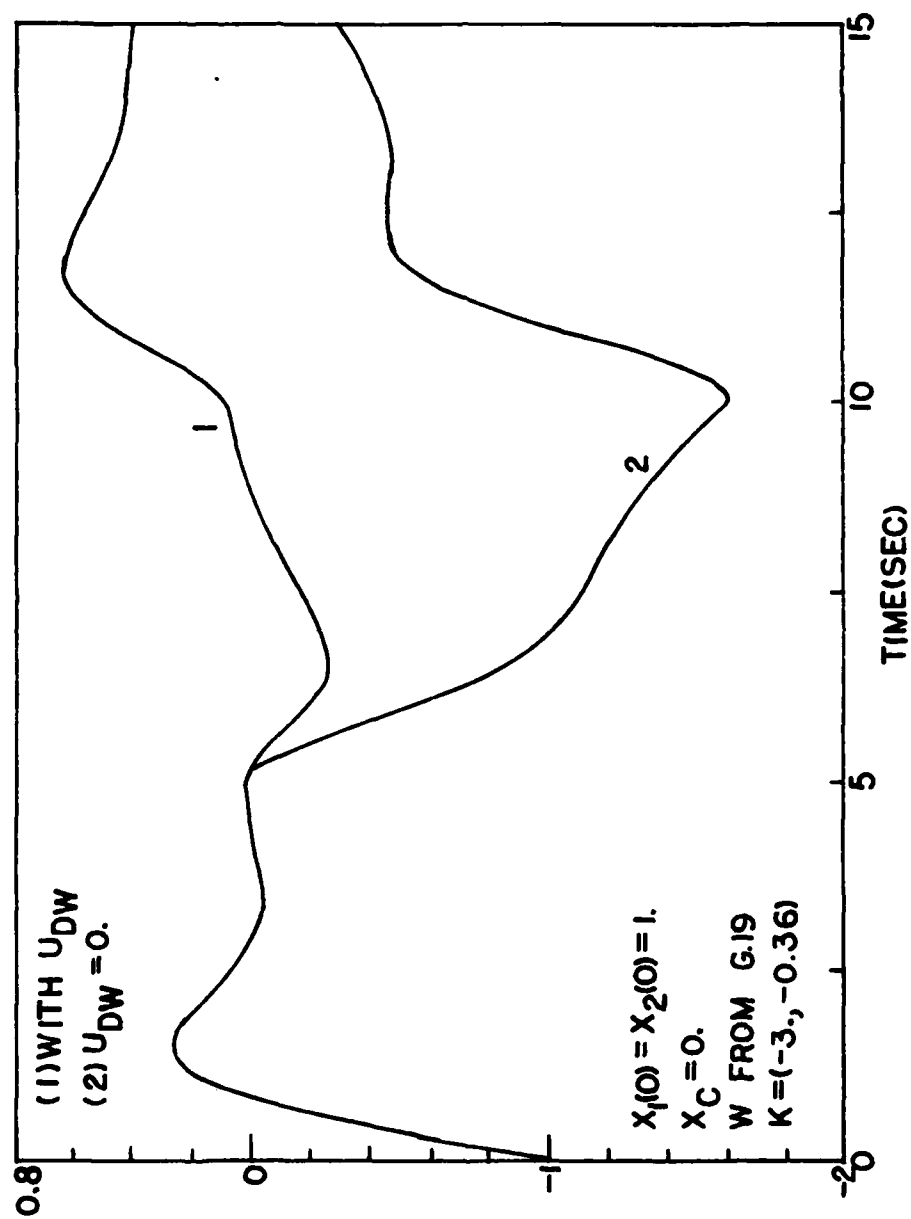


Figure G-36. ϵ_1 versus time, ϵ_1 as critical state-variable, $K = (-3, -0.36)$.

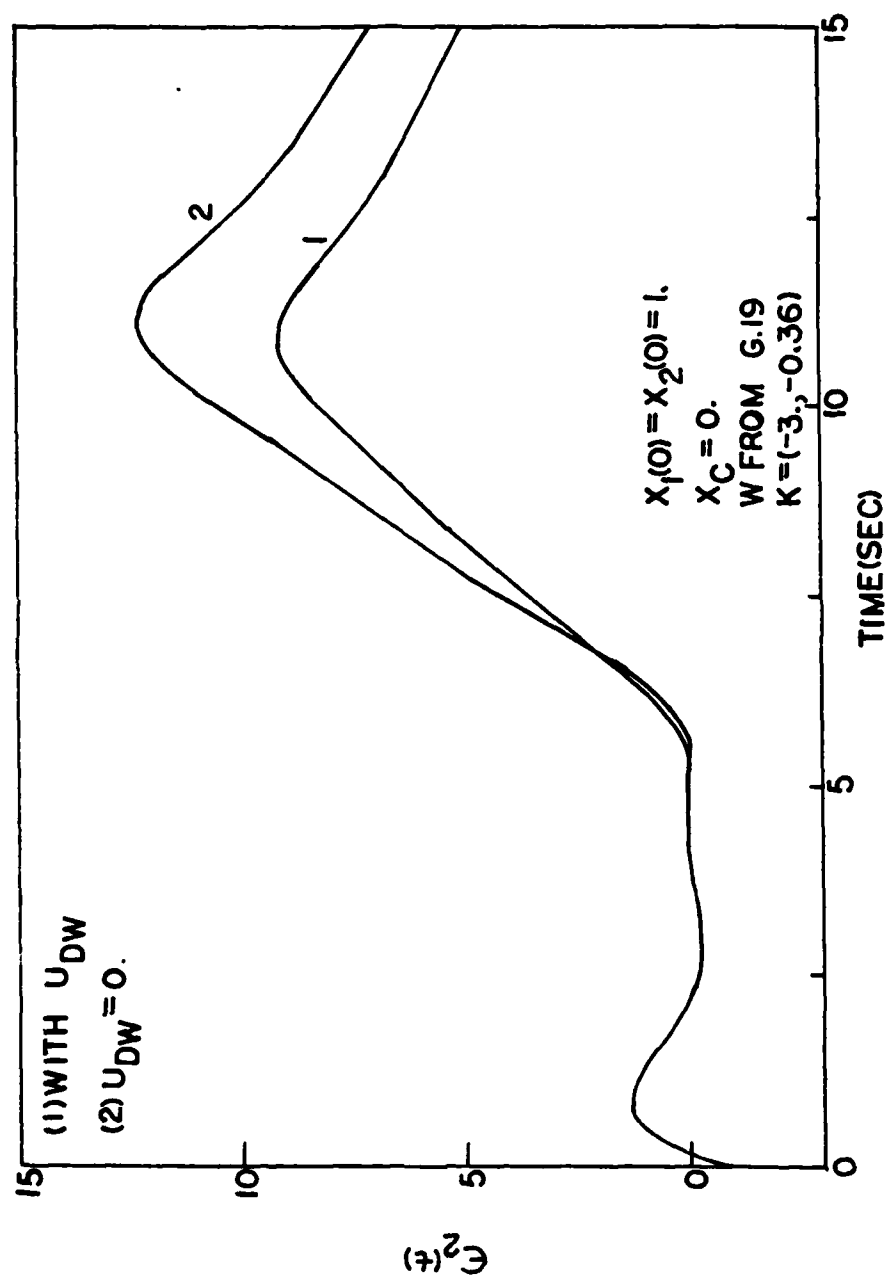


Figure G-37. ϵ_2 versus time, ϵ_1 as critical state-variable, $K = (-3, -0.36)$.

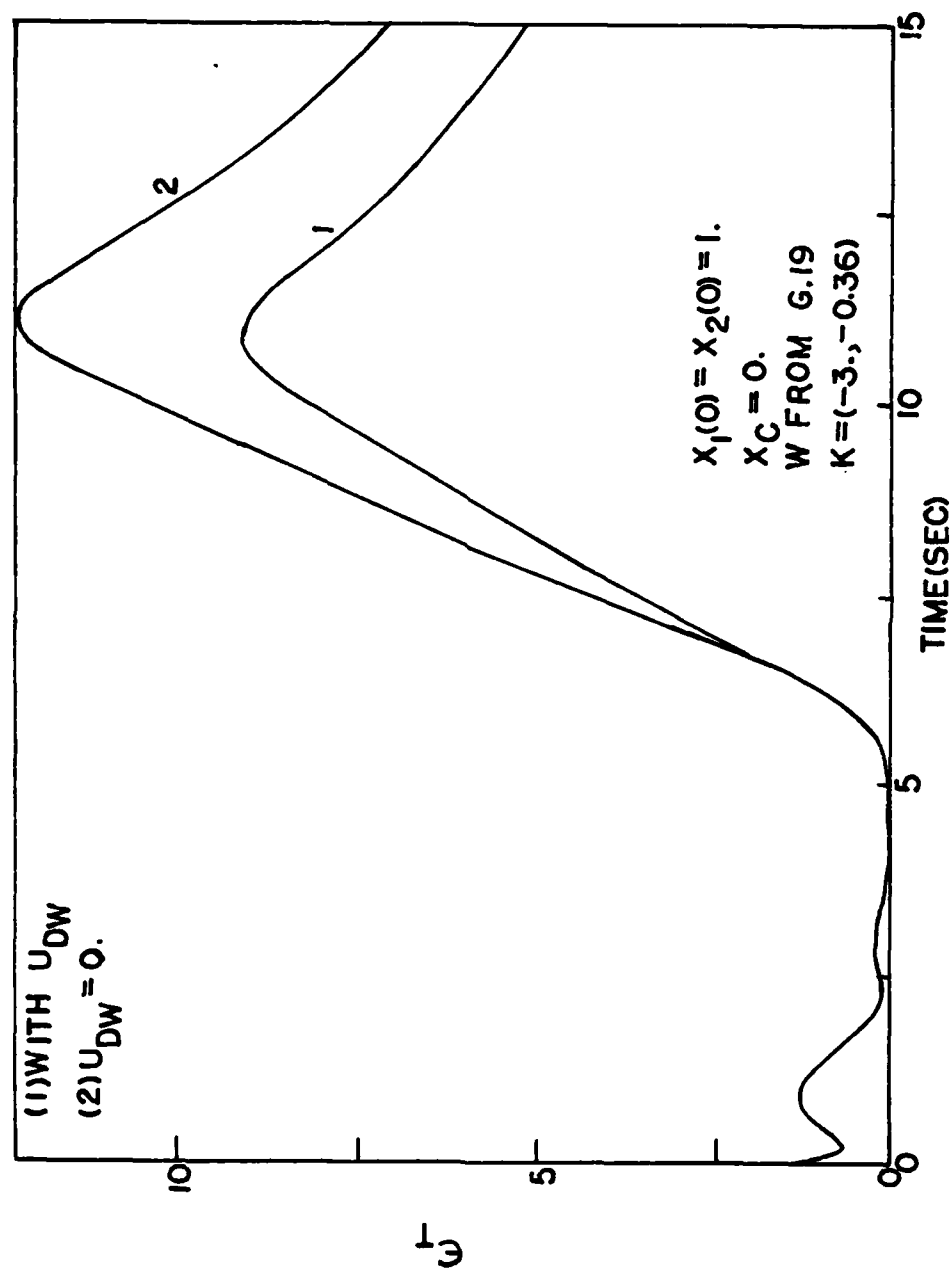


Figure G-38. ϵ_T versus time, ϵ_1 as critical state-variable, $K = (-3., -0.36)$.

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