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CHARACTERIZATION AND ESTIMATION*
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ABSTRACT

A class of finite order two dimensional autoregressive moving average (ARMA) is introduced having the ability to represent any process with rational spectral density. In this model, the driving noise is correlated and need not be Gaussian. Currently known classes of ARMA models or AR models are shown to be subsets of the above class. This document We discuss the three definitions of markov property and precisely states the class of ARMA models having the noncausal and semicausal markov property without imposing any specific boundary conditions. Next we considers it the estimation of parameters of a model to fit a given image. Two approaches are considered. The first method uses only the empirical correlations and involves the solution of linear equations. The second method is the likelihood approach. Since the exact likelihood function is difficult to compute, the author we resorts to approximations suggested by the toroidal models. The quality of the two estimation schemes are compared via numerical experiments. Finally, he we considers the problem of synthesizing a texture obeying an ARMA model.

Keywords:

Two dimensional ARMA models, noncausal markov, semi-causal markov, parameter estimation, synthesis of texture, nongaussian images.

I. Introduction

Parametric representations for two dimensional random fields are useful in many applications like image synthesis, classification, spectral estimation, etc. The aim of the paper is to develop a finite stochastic difference equation model for regular two dimensional random fields having rational spectral densities and discuss related topics like the various definitions of weak markov processes, parameter estimation and synthesis of textures resembling a given non Gaussian texture.

We will first give the background information regarding the structural representations. Rosanov [1], Woods [2] and Besag [3] have shown that any Gaussian markov field having an all pole spectral density (i.e., a reciprocal of a linear sinusoidal function) possesses a finite difference equation representation, the so-called conditional autoregressive (CAR) model in which the driving input noise is correlated, but does not have, in general, a moving average representation. The set of models suggested earlier such as the simultaneous AR models [3, 4], causal recursive models [5, 6] is a proper subset of the set of CAR models.

As discussed later, the various types of 2-D autoregressive moving average (ARMA) models discussed in [7, 8, 9, 11] have the restriction that the denominator of the corresponding spectral density, say $A(z)$, is factorable, i.e., $A(z) = D_1(z_1,z_2)D_1(z_1^{-1},z_2^{-1})$. Thus no general *finite* difference equation model is available for representing a discrete random field having a rational spectral density in which both the numerator and the denominator are not factorable. We emphasize the use of the word 'finite' since a simple spectral density such as $[1 + \phi(\cos\lambda_1 + \cos\lambda_2)]/[1 - \theta(\cos\lambda_1 + \cos\lambda_2)]$, $|\phi|, |\theta| < 0.5, \phi \neq \theta$, cannot be represented by any of the ARMA models in [7, 8, 9, 11] using a *finite* number of parameters, but can always be represented by these models using an *infinite* number of parameters. But the principle of parsimony precludes the use of a model having a large number of parameters especially in tasks such as fitting of models to the given data. In

contrast the class of recursive finite ARMA models in one dimension can represent any process with a rational spectral density of finite order.

The 2D case differs significantly from the 1D case in relation to the markov property. There are 3 types of weak markov property, namely, causal [10], semi-causal [8, 9] and non-causal [1-3]. In contrast with the 1D case where a process obeying an ARMA model is a projection of a vector markov process, the general ARMA model in the 2D case is neither markovian according to any of the three definitions nor a projection of another markov process. However, a particular subset of ARMA models is shown to possess the semi-causal markov property which was introduced in [8, 9]. We will clarify the precise structure of the ARMA models having the requisite semi-causal markov property without imposing any special boundary conditions.

The next topic to be covered is the estimation of parameters in a model to fit a given finite image. We present two approaches. In the first approach the parameter estimates are computed from the empirical correlations by solving linear equations. There are no iterations in contrast with the 1D ARMA model parameter estimation problems. The second approach utilizes the likelihood. The exact expression for the likelihood of the given observations in terms of the parameters is very complicated. We consider an approximation which is easy to handle. The approximation happens to be the exact likelihood when the observations obey a variant of the ARMA model, the so-called toroidal ARMA model. Finally we discuss a procedure for synthesizing a texture obeying a given ARMA model.

Section II deals with the general ARMA representation, the related markovian properties, and the relation to existing 2D difference equation models. Section III contains the parameter estimation using the estimated correlations. Section IV deals with the likelihood approach which includes the results of numerical experiments on the quality of estimates. The next section deals with the problems of synthesizing a texture to resemble a real texture.

II. The ARMA Model

Let $y(s)$, $s \in L$ be a two dimensional random field $L = \{(j,k): j,k \text{ are integers}\}$. Let $y(\cdot)$ be stationary and have the correlation function $R_y(s)$ and spectral density $S(z)$, $z = (z_1, z_2)$.

$$S(z) = \sum_{j=-\infty}^{\infty} \sum_{k=-\infty}^{\infty} R_y(j,k) z_1^j z_2^k, \tag{2.1}$$

$$= \nu B(z)/A(z), \ z = [\exp 2\pi i \lambda_1, \exp 2\pi i \lambda_2], \ i = \sqrt{-1}, \tag{2.2}$$

where $A(z) = 1 - \sum_{r \in N_1} \theta_r z^r$, $B(z) = 1 + \sum_{r \in N_2} \phi_r z^r$, $\theta_r = \theta_{-r}$, $\phi_r = \phi_{-r}$.

N_i are *finite* subsets of $(L-(0,0))$ and are symmetric, i.e., if $(j,k)\epsilon N_i$ then $(-j,-k)\epsilon N_i$, $i = 1,2$. A and B are finite order polynomials. $A(z_1,z_2) = A(z_1^{-1}, z_2^{-1})$. Similarly B.

$$A(z) \neq 0 \text{ and } B(z) \neq 0 \text{ for all } z \text{ such that } |z_i| = 1, \ i = 1,2. \tag{2.3}$$

$$A(z) \text{ and } B(z) \text{ have no common zero.} \tag{2.4}$$

Conditions (2.3) and (2.4) assure that $S(z)$ is finite, and positive for all real $\lambda = (\lambda_1, \lambda_2)$. Our aim is to develop a finite difference equation representation for $y(\cdot)$ valid for any spectral density $S(\cdot)$ obeying (2.2)-(2.4).

Theorem 1: The stationary random field $y(\cdot)$ with its spectral density in (2.2) obeying the conditions (2.3)-(2.4) obeys the bilateral autoregressive moving average model which can be represented as in (2.5) or (2.6).

$$y(s) = \sum_{r \in N_1} \theta_r y(s+r) + \sqrt{\nu} \, e(s). \tag{2.5}$$

$$A(z)y(s) = \sqrt{\nu}\, e(s). \tag{2.6}$$

In (2.5) or (2.6), e(s) is a zero mean stationary correlated sequence with the spectral density in (2.7).

$$S_e(z) = A(z)B(z) \tag{2.7}$$

Proof: If $y(\cdot)$ obeying (2.5) exists, then taking spectral density of both sides of the equation (2.6) and using (2.7) indicates that the spectral density of $y(\cdot)$ is as in (2.2). Thus all we have to show is the existence of a sequence $y(\cdot)$ obeying (2.5). This will be done by construction.

Let $\{w(s),\, s \in L\}$ be an infinite sequence of independent and identically distributed random variables having mean zero and variance unity. Let $\tilde{w}(\cdot)$, $\tilde{e}(\cdot)$, $\tilde{y}(\cdot)$ stand respectively for the fourier transform of the infinite sequence $w(\cdot)$, $e(\cdot)$, and $y(\cdot)$. Compute $\tilde{e}(z)$

$$\tilde{e}(z) = \sqrt{B(z)A(z)}\; \tilde{w}(z)$$

Then the fourier inverse of $\tilde{e}(z)$ yields the sequence $\{e(s),\, s \in L\}$ having zero mean and spectral density $B(z)A(z)$. Compute $\tilde{y}(z)$ as shown below, which is finite for all $|z_1| = 1$ and $|z_2| = 1$ in view of (2.3).

$$\tilde{y}(z) = \sqrt{\nu}\, \tilde{e}(z)/A(z)$$

Rearranging the above equation, we get

$$(1 - \sum \theta_r z^r)\tilde{y}(z) = \sqrt{\nu}\, \tilde{e}(z).$$

The fourier inverse of $\tilde{y}(\cdot)$ yields the desired sequence $y(\cdot)$ obeying (2.5).

Comment 1: In view of (2.7) the sequence e(s) has nonzero auto correlation only over a finite number of lags, as displayed in (2.8)

$$E[e(t)e(t+s)] = -\sum_{r \epsilon N_1} \phi_r \cdot \theta_{s-r} \text{ if } s \epsilon N'', \tag{2.8}$$

$$= 0, \qquad \text{otherwise}$$

where $N_i' = N_i \cup \{0,0\}$, $N'' = \{r+s: r \epsilon N_1', s \epsilon N_2'\}$,

$$\phi_{0,0} = 1, \quad \theta_{0,0} = -1,$$

$$\phi_s = 0, \text{ if } s \notin N_2', \quad \theta_r = 0 \text{ if } r \notin N_1'.$$

The sequence e(s) has non zero correlation with y(s+r) only for a *finite* number of values of r. To prove this statement, let us find the cross spectral density $S_{ey}(\cdot)$ from eq. (2.6).

$$S_{ey}(z) = (\sqrt{\nu}/A(z)) \, S_e(z)$$

$$= \sqrt{\nu} \, B(z), \text{ from (2.7),} \tag{2.9}$$

Equating the coefficients of z^r on either side, we get

$$E[e(s)y(s+r)] = \sqrt{\nu} \, \phi_r, \text{ if } r \epsilon N_2', \tag{2.10}$$

$$= 0, \text{ otherwise.}$$

Comment 2: An alternative representation for y($\cdot$) obeying (2.5) is given below:

$$\sqrt{A(z)} \, y(s) = \sqrt{\nu} \, \sqrt{B(z)} \, w(s), \tag{2.11}$$

where $\{w(s)\}$ is an *independent* and identically distributed (I.I.D.) sequence with zero mean and unit variance. $\sqrt{A(z)}$ and $\sqrt{B(z)}$ are infinite order symmetric polynomials

involving only a finite number of parameters θ_r and ϕ_r. One can verify that the spectral density of $y(\cdot)$ obeying (2.11) is as in (2.2). The representation in (2.11) is more suitable for synthesis of an image, as discussed in section V. Note that the probability density of $w(\cdot)$ can be chosen as desired.

Comment 3: Viewing (2.6) as an input-output system with $e(\cdot)$ as input and $y(\cdot)$ as the output, we can see, as in the proof of theorem 1, that a necessary and sufficient condition for the BIBO stability is that $A(z) \neq 0$ for $|z_1| = 1$ and $|z_2| = 1$. In section V a specific algorithm is given for synthesis using this condition. The condition $B(z) \neq 0$ for $|z_1| = 1$ and $|z_2| = 1$ is needed for constructing a whitened representation of $y(\cdot)$ as shown below, where $\tilde{w}(z)$ is the fourier transform of the whitened sequence and similarly $\tilde{y}(\cdot)$.

$$\tilde{w}(z) = \sqrt{A(z)/B(z)} \; \tilde{y}(z).$$

The condition (2.4) on A and B in addition to (2.3) is needed to ensure the identifiability of the parameters θ_r and ϕ_r, as indicated later.

Relation to currently known ARMA models

Case (1): The conditional auto regressive (CAR) model [1, 2, 3] is a special case of (2.6) and (2.7) with $B(z)=1$. The CAR models are called as minimum variance representations (MVR) in [9].

Case (2): The simultaneous AR model [3, 9, 10], also called as a white noise driven representation (WNDR) in [9] is a special case of (2.6) and (2.7) where $B(z)=1$ and $A(z)$ has a factorization as in (2.12).

$$A(z) = KD(z)D(z^{-1}), \quad D(z) = 1 - \sum_N \theta_r z^r, \quad \text{N need not be symmetric.} \tag{2.12}$$

A simultaneous ARMA model [9, 11, 24] is a special case of (2.6) and (2.7) in which both A and B have a factorization as in (2.12).

Case (3): Consider the 2-D ARMA models introduced in [7] in which $\Phi(z)$, a special 2-D transform of the correlation function defined in (2.13) is a rational function as in (2.14).

$$\Phi(z_1, z_2) = \sum_{i=0}^{\infty} \sum_{j=-\infty}^{\infty} R(i_1, j_2) \, z_1^i z_2^j, \tag{2.13}$$

$$= C(z)/D(z), \tag{2.14}$$

where

$$C(z) = \sum_{i=0}^{M_1} \sum_{j=0}^{M_2} c_{ij} \, z_1^i z_2^j, \qquad D(z) = \sum_{i=0}^{M_1} \sum_{j=0}^{M_2} d_{ij} \, z_1^i z_2^j$$

We emphasize that $\Phi(\cdot)$ is distinct from the spectral density S defined in (2.1) even though Φ is also called a spectral density in [7]. The ARMA models which possess a Φ function as in (2.14) is a *proper subset* of the processes having spectral density S as in (2.2) and hence a proper subset of the ARMA models defined in (2.6) and (2.7). This result is stated as theorem 2.

Theorem 2: If there exists a stationary process $y(\cdot)$ with its Φ function as in (2.14), its spectral density S has the structure as in (2.15).

$$S = \nu B(z)/D_1(z) \, D_1(z^{-1}), \qquad D_1(z) = \sum_{i=0}^{M_1} \sum_{j=0}^{M_2'} d_{ij}' \, z_1^i \, z_2^j, \qquad (2.15)$$

$M_2' \geq M_2$, B need not be factorable.

Proof:

$$\sum_{i=-\infty}^{0} \sum_{j=-\infty}^{\infty} R(i,j) \, z_1^i \, z_2^j = \sum_{i=0}^{\infty} \sum_{j=-\infty}^{\infty} R(-i,j) z_1^{-i} \, z_2^j$$

$$= \sum_{i=0}^{\infty} \sum_{j=-\infty}^{\infty} R(-i,-j) z_1^{-i} \, z_2^{-j}, \text{ by replacing } j \text{ by } -j$$

$$= \sum_{i=0}^{\infty} \sum_{j=-\infty}^{\infty} R(i,j) z_1^{-i} \, z_2^{i}, \text{ since } R(i,j) = R(-i,-j)$$

$$= \Phi(z_1^{-1}, z_2^{-1}), \text{ in view of (2.13).}$$

$$\sum_{j=-\infty}^{\infty} R(0,j) z_2^j = \frac{G_1(z_2) G_1(z_2^{-1})}{G_2(z_2) G_2(z_2^{-1})},$$

where $G_i(z_2) = \sum_{k=0}^{M_{3i}} g_{ik} z_2^k$, in view of the factorability of 1-D polynomials.

From (2.1),

$$S(z_1, z_2) = \left[\sum_{i=0}^{\infty} \sum_{j=-\infty}^{\infty} + \sum_{i=-\infty}^{0} \sum_{j=-\infty}^{\infty} \right] R(i,j) z_1^i z_2^j - \sum_{j=-\infty}^{\infty} R(0,j) z_2^j$$

$$= \Phi(z_1, z_2) + \Phi(z_1^{-1}, z_2^{-1}) - G_1(z_2) G_1(z_2^{-1}) / G_2(z_2) G_2(z_2^{-1})$$

$$= \frac{C(z)}{D(z)} + \frac{C(z^{-1})}{D(z^{-1})} - \frac{\left| G_1(z_2) \right|^2}{\left| G_2(z_2) \right|^2}$$

$$= \frac{\nu B(z)}{[D(z)G_2(z_2)][D(z^{-1})G_2(z_2^{-1})]}$$

$$= \nu B(z)/D_1(z)D_1(z^{-1}),$$

where $B(z)$ is the numerator normalized so that its constant term is one and

$$D_1(z) = D(z)G_2(z_2)$$

$$= \left[\sum_{i=0}^{M_1} \sum_{j=0}^{M_2} d_{ij}z_1^i z_2^j \right] \left[\sum_{j=0}^{M_{32}} g_{2j}z_2^j \right]$$

$$= \sum_{i=0}^{M_1} \sum_{j=0}^{M_2'} d_{ij}'z_1^i z_2^j, \qquad M_2' > M_2.$$

Comment: A simple consequence of theorem 2 is that a process $y(\cdot)$ with spectral density as in (2.2) with a non factorable denominator A cannot have a Φ function as in (2.15) and thus cannot have the corresponding ARMA representation given in [7].

Case (4): Consider the spectral density in (2.2) in which both A and B have the following factorization.

$$A(z) = K_1 D_1(z)D_1(z^{-1}), \quad B(z) = K_2 D_2(z)D_2(z^{-1}),$$

$$D_1(z) = 1 - \sum_{r \in N_4} d_r z^r, \quad D_2(z) = 1 - \sum_{r \in N_5} d_r' z^r.$$

$N_4, N_5 \subset L^-$, $(0,0) \not\subset N_4$, $(0,0) \not\subset N_5$. Both N_4 and N_5 are subsets of the nonsymmetrical half plane (NSHP) L^- indicated in figure 1. Then the corresponding ARMA difference

equation can be written as:

$$y(s) = \sum_{r \in N_4} d_r y(s+r) + \sqrt{\nu''} \, (w(s) + \sum_{r \in N_5} d_r' w(s+r)),$$

where $w(\cdot)$ is I.I.D. $(0,1)$ sequence. The above equation is the analog of the traditional ARMA model in time series. If in the above equation, in addition, $B(z) = 1$ or equivalently, $d_r' = 0$, then the corresponding process $y(\cdot)$ is said to possess the weak linear *causal* markov property, i.e.

$$E[y(s) \mid \text{all } y(s+r), \, r \in L^-] = \sum_{r \in N_4} d_r y(s+r), \, N_4 \subset L^-, \tag{2.16}$$

The corresponding difference equation is called as a causal AR representation [9]. In this case, it is possible to divide the image at any point s into 3 parts, namely, s is the present, the set $\{s+r : r \in L^-\}$ is the past, and $\{s+r, \, r \neq (0,0), \, r \notin L^-\}$ is the future.

Case 5: Semicausal models [9] is shown to be a subclass of the general ARMA class in theorem 5 to be proved later.

Let us evaluate the conditional expectation of $y(s)$ given all other values for the general model in (2.5).

Theorem 3: The sequence $y(\cdot)$ in Th. 1 obeying (2.5) and having a Gaussian density has the following conditional expectation and variance

$$y_1(s) \triangleq E[y(s) \mid \text{all } y(s+r), \, r \neq (0,0)] = \sum_{r \neq (0,0)} g_r y(s+r), \tag{2.17}$$

$$E[(y(s) - y_1(s))^2] = \nu/K, \tag{2.18}$$

where K and g_r are defined as

$$K(1 - \sum_{r \in L'} g_r z^r) = A(z)/B(z), \tag{2.19}$$

$$L' = L - (0,0), \ g_r = g_{-r}.$$

Proof: Let $G(z) = \sum_{r \in L} g_r z^r$.

$$u(s) \triangleq y(s) - y_1(s) = (1 - G(z))y(s). \tag{2.20}$$

The cross spectral density between v and y is

$$S_{vy}(z) = (1 - G(z))S_{yy(z)},$$

$$= \nu/K, \text{ from } (2.2).$$

Hence

$$E[u(s)y(s+r)] = 0 \ \forall \ r \neq (0,0).$$

Hence (2.17) is true since $y(\cdot)$ is Gaussian. To prove (2.18),

$$S_{vv}(z) = (1 - G(z))^2 S_{yy}(z), \text{ from } (2.20)$$

$$= (\nu/K)(1 - \sum g_r z^r), \text{ from } (2.2) \text{ and } (2.19).$$

Hence $E[v^2(s)] = \nu/K$.

xxx

The conditional expectation in (2.17) has, in general, an infinite number of terms. The next question is the determination of conditions under which the conditional expectation in (2.17) has a finite number of terms. The answer is in Theorem 4.

Definition: A sequence $y(\cdot)$ is weak *noncausal* markov if the following is true:

$$E[y(s)| \text{ all } y(s+r), \ r \neq 0,0] = E[y(s)| \text{ all } y(s+r), \ r \epsilon N, \ N \text{ is finite, symmetric, } (0,0) \notin N]$$

Theorem 4: A stationary sequence is weak noncausal markov and possesses a finite linear conditional expectation indicated in (2.21) if and only if the process $y(\cdot)$ has an all pole spectral density, i.e., $B(z)$ in (2.2) is a constant

$$E[y(s)| \text{ all } y(s+r), \ r \neq (0,0)] = \sum_{r \epsilon N_1} g_r y(s+r), \tag{2.21}$$

where $(0,0) \notin N_1$, N_1 is symmetric and finite, $g_r = g_{-r}$.

Proof: 'If' part: Let the spectral density of y be $\nu/A(z)$. By theorem 3, the conditional expectation is defined in terms of g_r in (2.19),

$$g_r = \theta_r, \text{ if } r \epsilon N_1,$$

$$= 0, \quad r \neq (0,0), \ r \notin N_1.$$

Hence the conditional expectation has a finite number of terms.

'only if' part:

Let $u(s) = y(s) - y_1(s)$

$$= y(s) - (\sum_{r \epsilon N_1} g_r z^r) y(s). \tag{2.22}$$

Since $y_1(\cdot)$ is the conditional expectation,

$$E[u(s)y(s+t)] = 0, \ \forall t \neq (0,0). \tag{2.23}$$

Let $E[u(s)y(s)] = K_1$. Multiply (2.22) on both sides by $y(s+t)$ and take expectation. Let $R(t) = E[y(s)y(s+t)]$.

$$R(t) - \sum_{r \epsilon N} g_r R(t-r) = 0, \text{ if } t \neq (0,0),$$

$$= K_1, \text{ if } t = (0,0), \text{by}(2.23).$$

Take fourier transform on both sides of the above equation.

$$(1 - \sum_{r \epsilon N_1} g_r z) S_y(z) = K_1$$

i.e., $S_y(z)$ has an all pole spectral density.

Comment 1: Parts of the theorem 4 have been known in the literature [1-3]. The aim of giving the theorem is to show the equivalence of the following three statements.

(i) $y(\cdot)$ has the conditional expectation in (2.21).

(ii) $y(\cdot)$ has an all pole spectral density $\nu/A(z)$

(iii) $y(\cdot)$ obeys the *conditional AR* model in (2.5) where the driving input $e(\cdot)$ has the spectral density $A(z)$.

This equivalence is never explicitly stated in the literature. For instance in [2] both (i) and (iii) are together used in defining the CAR model.

Comment 2: Every sequence $y(\cdot)$ which is causal markov and has the linear expectation in (2.16) defined by a neighbor set N_4 also possesses the noncausal markov property in (2.21) with neighbor set N_1, $N_1 \supset N_4$. The reverse is not true [3].

We mentioned earlier that $y(\cdot)$ obeying a general ARMA model in (2.5) does not possess the noncausal markov property. However a small subset of ARMA models possesses another markov property called as semi-causal or half-plane markov.

Definition: (semi-causal or half-plane markov): $y(\cdot)$ is said to be linear *half-plane* markov with respect to the neighbor set N if

$$E[y(s)/ \text{ all } y(s+r), \ r \ \epsilon \ L^+] = \sum_{r \epsilon N} \theta_r y(s+r), \qquad (2.24)$$

where $L^+ = \{(j,k): k \leq 0, (j,k) \neq (0,0)\}$. L^+ is displayed in figure 1. N is any subset of L^+ defined below.

$$N = N_1 \cup N_2,$$

$$N_1 = \{(i,0), (-i,0); i = 1,...,m_1\} \qquad (2.25)$$

$$N_2 = \{(i,j), j = -1,...,-m_2, i = \pm 1, \pm 2,...,\pm m\}$$

We will presently display a subset of ARMA models, the so-called semi-causal models which have the semi-causal markov property.

$$A(z)y(s) = \sqrt{\nu} \ e(s) \qquad (2.26)$$

where

$$A(z) = 1 - \sum_{r \epsilon N} \theta_r z^r, \ N \text{ in } (2.25)$$

$$= 1 - \sum_{k=1}^{m_1} \theta_{k,0}(z_1^k + z_1^{-k})$$

$$- \sum_{k=-m_1}^{m_1} \sum_{\ell=1}^{m_2} \theta_{k,\ell} z_1^k z_2^{-\ell},$$

The correlated sequence $e(\cdot)$ has zero mean, Gaussian probability density and spectral density in (2.28) or correlation function in (2.29)

$$S_e(z) \triangleq B(z) \triangleq 1 - \sum_{p=0}^{m_1} \theta_{p,0}(z_1^p + z_1^{-p}), \tag{2.28}$$

$$R_{ee}(k,\ell) = 0 \quad , \quad \text{if } \ell \neq 0$$

$$R_{ee}(k,0) = -\theta_{k,0} \ , \quad \text{if } k = \pm 1,...,\pm m_1, \ \theta_{k,0} = \theta_{-k,0} \tag{2.29}$$

$$= 1 \quad , \quad \text{if } k = 0$$

$$= 0 \quad , \quad \text{otherwise}$$

Eq. (2.26) can be written as the difference equation in (2.30)

$$y(i,j) = \sum_{k=1}^{m_1} \theta_{k,0}[y(i+k,j) + y(i-k,j)]$$

$$+ \sum_{k=-m_1}^{m_1} \sum_{\ell=1}^{m_2} \theta_{k,\ell} \ y(i+k,j-\ell) + \sqrt{\nu}e(i,j). \tag{2.30}$$

A necessary and sufficient condition for the stability of (2.30) is given below [Thm. 5 of [9]].

$$A(z_1,z_2) \neq 0 \text{ for } |z_1| = 1, |z_2| \geq 1. \tag{2.31}$$

The model in (2.26) is called semi-causal because it is causal in the index j, i.e., in the RHS of (2.30), $j+k$, $k \geq 1$ does not appear.

Theorem 5:

The stationary sequence y($\cdot$) defined in (2.26) and (2.27) possesses the weak half-plane markov property in (2.24) *if and only if* the input sequence e($\cdot$) in it has the

correlation function in (2.29) or equivalently $y(\cdot)$ has the spectral density $\nu B(z)/||A(z)||^2$.

Proof: *'If' part:*

$S_{ey}(z) = $ Cross spectral density of $e(\cdot)$ and $y(\cdot)$,

$$= \frac{\nu S_{ee}(z)}{A(z^{-1})} = \nu B(z)/A(z^{-1}).$$

Expanding $S_{ey}(z)$ in power series we see that the coefficient of any term involving $z_2^{-\ell}$, $\ell \geq 1$ is zero. Hence,

$$R_{ey}(k,\ell) \triangleq E[e(i,j)y(i-k,j-\ell)] = 0, \text{ if } \ell > 0. \tag{2.32}$$

Let

$$R(k) \triangleq R_{ey}(k,0). \tag{2.33}$$

Multiply (2.30) by $e(i+k,j)$, take expectation on both sides and use (2.32) and (2.33).

$$R(k) - \sum_{p=1}^{m_1} \theta_{p,0}[R(k+p) - R(k-p)] = R_{ee}(k,0), \tag{2.34}$$

Let $S(z_1)$ be the one dimensional discrete fourier transform of $R(k)$. Multiply (2.34) by z^{-k}, sum from $k = -\infty$ to ∞ and use $R_{ee}(k,0)$ in (2.29).

$$B(z_1)S(z_1) = \nu B(z_1)$$

Hence $S(z_1) = \nu$

$$\text{or } R_{ey}(k,0) = 0, \text{ if } k \neq 0 \tag{2.35}$$

Hence (2.32) and (2.35) yield

$$E[e(s)y(s+r)] = 0 \quad \forall\, r \in L^+,$$

(2.36a)

Since $e(\cdot)$ and $y(\cdot)$ are Gauss, the above equation yields

$$E[e(s)|\ \text{all } y(s+r),\ r \in L^+] = 0.$$

(2.36b)

Taking conditional expectation of $y(s)$ given all $y(s+r)$, $r \in L^+$ on both sides of (2.26) and using (2.36b) yields (2.24).

'Only if' part:

Since (2.24) is true, (2.36) is true. Multiplying (2.30) by $e(k,\ell)$, $\ell \neq j$, and taking expectations on both sides by using (2.36) we get

$$E[e(i,j)e(k,\ell)] = 0 \quad \forall j \neq \ell$$

(2.37)

Multiplying (2.30) by $e(i,j)$ and taking expectation on both sides using (2.36) we get

$$E[e(i,j)y(i,j)] = E[e^2(i,j)] = \nu$$

(2.38)

Multiplying (2.30) by $e(i+k,j)$ and take expectations on both sides

$$R_{ey}(k,0) = \sum_{p=1}^{m_1} \theta_{p,0}[R_{ey}(k+p,0) + R_{ey}(k-p,0)] + R_{ee}(k,0)$$

(2.39)

By (2.36), $R_{ey}(k,0) = 0$ if $k \neq 0$

(2.40)

Substituting for $R_{ey}(k,0)$ in (2.39) from (2.40) and (2.38), we get the desired expression for $R_{ee}(k,0)$ in (2.29).

Comment 1: For images with *specific boundary* conditions Jain [8, 9] has shown that the models in (2.26) have the weak semi-causal property. In theorem 5, we have established the converse also without imposing any specific boundary conditions.

Comment 2: A semi-causal markov sequence is also causal markov *only* in the degenerate case $B(z) = 1$, i.e., $\theta_{k,0} = 0$ if $k \neq 0$. In this degenerate case, it also possesses the noncausal markov property w.r.t. to a suitable symmetric set N_4. Apart from this case, a semi-causal markov sequence is never noncausal markov or vice versa.

Comment 3: In this entire section, we have discussed only the weak markov property. In the Gaussian case the weak markov property is the same as the strong markov property involving the factorization of probability density. Some additional results connected with the strong (noncausal) markov property can be found in [3, 13].

III. Parameter Estimation from Correlations

Given a finite image over the M×M grid Ω, we want to develop a procedure for fitting a ARMA model in (2.5) to it, i.e., estimating the unknown parameters in it after fixing the neighbor sets N_1 and N_2. We will give 2 procedures. In this section the parameters are estimated using the estimated correlations and it is independent of the density of the image. The method is computationally easy, and does not involve any iteration. This state of affairs is in contrast to the parameter estimation of ARMA models in the time series case. We will point out the reason for the difference. Note that here $\theta_r = \theta(r)$ and $R_r = R(r)$. N_i, $i=1,2,...$ are finite symmetric subsets of $\{L-(0,0)\}$. N_{Si} and $\overline{N}_{Si}$ are mutually exclusive *antisymmetric* subsets defined by the following relations:

$$N_i = N_{Si} \cup \bar{N}_{Si}, \quad N_{Si} \cap \bar{N}_{Si} = 0, \text{ If } (i,j) \in N_{Sk},(-i,-j) \nsubseteq N_{Sk}.$$

Without loss of generality we can make N_{Si} a subset of $(L-L^- -(0,0))$ where L^- is defined in Figure 1. Let the polynomials A and B in the spectral density of the process $y(\cdot)$ in (2.2) be:

$$A(z) = 1 - \sum_{r \in N_{S1}} \theta_r(z^r + z^{-r}) \quad , \#N_{S1} = m_1 \cdot$$

$$B(z) = 1 + \sum_{r \in N_{S2}} \phi_r(z^r + z^{-r}) \quad \#N_{S2} = m_2.$$

Let $N_{S1} = \{r_1,...,r_{m_1}\}$, $N_{S2} = \{s_1,...,s_{m_2}\}$

The corresponding ARMA model equation is

$$y(s) = \sum_{j=1}^{m_1} \theta(r_j)(y(s+r_j)+y(s-r_j)) + \sqrt{\nu}\, e(s), \tag{3.1}$$

where $e(\cdot)$ has the following cross spectral density

$$S_{ey}(z) = \sqrt{\nu}\, B(z),$$

i.e., $E[e(s)y(s+r)] = \sqrt{\nu}$, if $r = \underset{\sim}{0}$, $\tag{3.2a}$

$$= \sqrt{\nu}\, \phi_r, \text{ if } r \epsilon N_2, \tag{3.2b}$$

$$= 0, \text{ otherwise.} \tag{3.2c}$$

Choose a symmetric set N_3 having $2m_1$ *nearest* neighbors of $(0,0)$ so that $N_3 \cap N_2 = 0$. Note N_3 is not unique.

$$N_{S3} = \{t_1,t_2,...,t_{m_2},t_{m_2+1},...,t_{m_1}\}.$$

We will obtain an explicit expression for the coefficients θ_r, ϕ_r, ν in terms of the correlations $R(s)$.

Multiply (3.1) by y(0) on both sides and take expectation on both sides using (3.2a).

$$R(0) = 2 \sum_{j=1}^{m_1} \theta(r_j)R(r_j) + \nu. \tag{3.3}$$

Multiply (3.1) by $y(s+s_i)$, $s_i \epsilon N_{S2}$, on both sides and take expectation using (3.2b).

$$R(s_i) = \sum_{j=1}^{m_1} \theta(r_j)[R(r_i+s_j) + R(s_i-r_j)] + \nu\phi(s_i), \ i=1,...,m_2. \tag{3.4}$$

Multiply (3.1) by $y(s+t_i)$, $t_i \epsilon N_{S3}$ on both sides and take expectation using (3.2c).

$$R(t_i) = \sum_{j=1}^{m_1} \theta(r_j) \{R(t_i+r_j)+R(t_i-r_j)\}, \ i=1,...,m_1, \tag{3.5}$$

In (3.3), (3.4) and (3.5), the true correlations R( ) can be replaced by their estimates and the resulting equations can be solved for $\theta(\cdot)$, $\phi(\cdot)$ and $\nu(\cdot)$ as indicated below. The steps are:

(i) Choose the set N_{S3}.

(ii) Estimate the various correlation needed in (3.3), (3.4) and (3.5).

$$\hat{R}(r) = \frac{1}{M_1-1} \sum_{s} y(s)y(s+r),$$

where summation extends over all valid s in Ω and M_1 is the number of admissible values of s.

(iii) Solve the m_1 linear equations in (3.5) for $\theta(r_1), \ldots, \theta(r_{m_1})$ to yield the corresponding estimates $\hat{\theta}$

(iv) Solve the linear equation (3.3) for ν after replacing $\theta(\cdot)$ by $\hat{\theta}$, yielding the estimate $\hat{\nu}$.

(v)   Solve the linear equation (3.4) for $\phi(\cdot)$ after replacing θ by $\hat{\theta}$ and ν by $\hat{\nu}$.

It is important to note that the computation does not involve any iteration. One can show that the estimates are consistent, i.e., as the size of the image M goes to infinity, the estimates tend to their true values provided $A(z)$ and $B(z)$ do not have any common factors.

Comment 1: The procedure needs the choice of the set N_3. The set could be arbitrary as long as it is exclusive of N_2. Empirical evidence indicates that the one suggested here, namely having $2m_1$ *nearest* neighbors, leads to the estimate with higher accuracy than other choices.

Comment 2: As noted earlier, no iteration is needed in the computation. In contrast, the estimation of parameters by the covariance method in the one dimensional ARMA model is much more complicated and involves iteration. The reason for the different behavior is the difference in the model equations. In the 2D case the input sequence $e(\cdot)$ is correlated. In the 1-D ARMA case the input sequence, $w(\cdot)$, is independent. If we convert the 1-D ARMA equation into a form similar to eq. (3.1), then the computation procedure indicated in this section can be used for 1-D case also. Note that in this section, we aim at estimating directly the coefficients occurring in the spectral density, whereas in traditional 1-D ARMA case, we estimate the coefficients of $C(z_1)$ and $D(z_1)$ where $S(z_1) = ||C(z_1)||^2/||D(z_1)||^2$.

The sequence of computation is illustrated by an example.

Example 1: Let the ARMA model be as in (3.6) and (3.7).

$$y(s) = \theta \sum_{r \epsilon N_1} y(s+r) + \sqrt{\nu}\, e(s) \ , \tag{3.6}$$

$$N_1 = \{(0,1), (0-1), (1,0), (-1,0)\}$$

$$B(z) = 1 + \phi \sum_{r \epsilon N_2} z^r, \, N_2 = N_1, \tag{3.7}$$

The required choice of $N_{S3} = (1,1)$. Note $R_{i,j} = R_{i,-j} = R_{j,i}$. From (3.4), we get

$$R_{0,0} = 4\theta R_{1,0} + \nu. \tag{3.8}$$

From (3.5), we get

$$R_{1,0} = \theta[R_{0,0} + R_{2,0} + 2R_{1,1}] + \nu\phi, \tag{3.9}$$

From (3.6), we get

$$R_{1,1} = 2\theta[R_{1,0} + R_{1,2}]. \tag{3.10}$$

Solving (3.8)-(3.10) for θ, ϕ and ν is straightforward. The numerical results are given in the next section.

IV. Likelihood

When the number of parameters to be estimated is not very small compared to the image size M^2, then the estimates given earlier may not be accurate. Hence, we introduce the more accurate method of estimation, the so called likelihood method. As before let the given set of observations be $\underline{y}$.

$$\underline{y} = \text{Col.}[y(s), s \epsilon \Omega]$$

$$\Omega = \{(i,j), 0 \leq i, j \leq M-1\}$$

Let the spectral density of y be $\nu B(z)/A(z)$. Let us assume that y is Gauss. Then we can find the correlation function R(s) of y(·) as a function of θ_i, ϕ_i and ν from the spectral density. Then we can find the covariance of the vector $\underset{\sim}{y}$, say C. Thus $\underset{\sim}{y}$ is Gauss $[0, C(\underset{\sim}{\theta}, \underset{\sim}{\phi}, \nu)]$. But the matrix C is of dimension $M^2 \times M^2$ and R(s) is not a simple function of s, θ, ν and ϕ. Hence the above density expression for $\underset{\sim}{y}$ is not useful for tasks like maximizing it to find the parameter estimates. We have to be content with an approximation to the probability density function of $\underset{\sim}{y}$ so that it is amenable for optimization.

Let $\{Y_r, r\epsilon\Omega\}$ be the finite fourier transform of the finite sequence $\{y(s), s\epsilon\Omega\}$. Then as M tends to infinity, the sequence $\{Y_r\}$ is independent and Gauss with mean zero and variance $M^2 S_r(\underset{\sim}{\theta}, \underset{\sim}{\phi}, \nu)$ [14] in (4.2).

$$S_r(\underset{\sim}{\theta}, \underset{\sim}{\phi}, \nu) = \frac{\nu B(z = \exp(i(2\pi/M)r))}{A(z = \exp(i(2\pi/M)r))}, \quad i = \sqrt{-1}, \tag{4.2}$$

$$= \frac{\nu(1 + \underset{\sim}{\phi}^T \underset{\sim}{\psi}_r)}{1 - \underset{\sim}{\theta}^T \underset{\sim}{\alpha}_r}$$

$\underset{\sim}{\psi}_r = \text{Col.}(2\cos(2\pi/M)r^T s, s\epsilon N_{S2})$, $\underset{\sim}{\alpha}_r = \text{Col.}(2\cos(2\pi/M)r^T s, s\epsilon N_{S1})$

$\underset{\sim}{\phi} = \text{Col.}(\phi_r, r\epsilon N_{S2})$, $\underset{\sim}{\theta} = \text{Col.}(\theta_r, r\epsilon N_{S1})$

Asymptotically, the probability density of $\{Y_r\}$ is:

$$p(Y_r, r\epsilon\Omega; \underset{\sim}{\theta}, \underset{\sim}{\phi}, \nu) \simeq [\prod_{r\epsilon\Omega} 1/(2\pi S_r(\underset{\sim}{\theta}, \underset{\sim}{\phi}, \nu)]^{1/2} \exp[-(1/2) \sum_{r\epsilon\Omega} ||Y_r||^2/M^2 S_r(\underset{\sim}{\theta}, \underset{\sim}{\phi}, \nu)]. \tag{4.3}$$

We can show [12] that, if we transform $\{Y_r, r\epsilon\Omega\}$ in (4.3) into $\{y(s), s\epsilon\Omega\}$, then the RHS of (4.3) is the *exact* probability density of a set of observations obeying the toroidal variant of the ARMA model described below which is valid for the region Ω only.

$$y(s) = \sum_{r \epsilon N_{S1}} \theta_r(y(s \oplus r) + y(s \ominus r) + \sqrt{\nu} \, e(s), \; s\epsilon\Omega, \qquad (4.4)$$

where $\oplus$ denotes summation modulo M,

$$y(s) = y(s \bmod M), \; e(s) = e(s \bmod M).$$

$\{e(s), s\epsilon\Omega\}$ have the correlation function described earlier.

The expression on the RHS of (4.3) has to be maximized w.r.t. $\underset{\sim}{\theta}, \underset{\sim}{\phi}$ and ν. It is more convenient to work with the log likelihood.

$$J_1(\underset{\sim}{\theta},\underset{\sim}{\phi},\nu) = -(M^2/2)\ln(2\pi\nu)-1/2 \sum_r \log[1+\phi^T\psi_r)/1-\theta^T\alpha_r)]$$

$$-(1/2) \sum_{r\epsilon\Omega} ||Y_r||^2(1-\theta^T\alpha_r)/\nu(1+\phi^T\psi_r)M^2, \qquad (4.5)$$

Maximizing J_1 w.r.t. ν yields

$$\nu = (\sum_r ||Y_r||^2(1-\underset{\sim}{\theta}^T\underset{\sim}{\alpha}_r)/(1+\underset{\sim}{\phi}^T\underset{\sim}{\psi}_r))/M^4 \qquad (4.6)$$

Substituting it back, and simplifying, we see that the ML estimates of $\underset{\sim}{\theta}$ and $\underset{\sim}{\phi}$ are obtained by minimizing $J(\underset{\sim}{\theta},\underset{\sim}{\phi})$ w.r.t. $\underset{\sim}{\theta}$ and $\underset{\sim}{\phi}$.

$$J(\underset{\sim}{\theta},\underset{\sim}{\phi}) = \sum_{r\epsilon\Omega} \log[(1+\underset{\sim}{\phi}^T\underset{\sim}{\psi}_r)/(1-\underset{\sim}{\theta}^T\underset{\sim}{\alpha}_r)]$$

$$+ M^2\log \sum_{r\epsilon\Omega} ||Y_r||^2 \frac{(1-\underset{\sim}{\theta}^T\underset{\sim}{\alpha}_r)}{1+\underset{\sim}{\phi}^T\underset{\sim}{\psi}_r}$$

Since the minimizing value of $(\underset{\sim}{\theta},\underset{\sim}{\phi})$ has to yield a finite value for $J(\cdot)$, the ML estimates of θ and ϕ automatically satisfy the conditions and $1+\underset{\sim}{\phi}^T\underset{\sim}{\psi}_r \neq 0$ $1-\underset{\sim}{\theta}^T\underset{\sim}{\alpha}_r \neq 0$ for all r. Thus the ML estimates of $\underset{\sim}{\theta}$ and $\underset{\sim}{\phi}$ satisfy (2.3). We cannot make such a claim for the estimates obtained by the correlation methods especially for small M. The numerical aspects of maximization have been discussed in [4,15] for the case of AR models.

The likelihood approach can be adapted to the particular situation on hand. If we know that the observation is the sum of a signal obeying a CAR model and an additive noise, then we can directly write the likelihood and estimate the parameters of the CAR model and the corresponding spectrum. If the signal plus noise assumption is true, any spectral estimation method which ignores the noise will not give good results. This feature has been documented in [16].

Numerical Experiments

The correlation and maximum likelihood estimates are compared via numerical experiments. The image model in example 1 of section IV is considered, with numerical values $\theta = .22$, $\phi = .2$ and $\nu = 1$. Ten different images of size 64×64 obeying this model were synthesized using different random sequences, as discussed in section V. In each case, the parameter estimates were computed by both the methods. For ϕ, θ and ν, the mean of the 10 estimates, the standard duration (SD) of the 10 estimates and the root mean square value of the deviation of the estimate from the true value (RMSE) are given in table 1 for both correlation and ML estimates. Similar experiments were performed with images of size 32×32 and 16×16 and the results are also given in table 1.

For 64×64 images, the SD and the RMSE are close to one another. Further the correlation estimates and the ML estimates of ϕ and θ have similar accuracy. The correlation estimate of ν appears to be slightly biased. But as the size of the image decreases, the RMSE values of the ML estimates are less than the corresponding values of the correlation estimate. This feature is to be expected. But the quality of correlation estimate is not unduly low. For instance for 32×32 image, the RMSE values for ML and correlation estimates of θ are .0124 and .0182, not very drastic. In many image processing problems the correlation estimates appear to be adequate, especially in view of their low computational demand.

V. Synthesis

An interesting problem in image processing is the synthesis of an image which resembles a real texture. There are many methods of synthesizing images, each one based on a different type of model. Synthesis has been done using various types of 2D AR models [5, 13, 17-20], and mosaic models [21-22]. Our aim is to explore the use of ARMA models. In an ARMA model in (2.11), there are three sets of parameters:

(i)   the order of ARMA model, i.e., the sets N_{S1} and N_{S2}

(ii)   the values of the coefficients

(iii) the histogram or density of the input process $w(\cdot)$ in (2.11).

The choice of the appropriate order of the 2D AR models, suitable for a given image is given in [12, 23]. A similar procedure is suitable for ARMA model. The estimation of parameters has already been discussed. We will discuss a convenient method of synthesizing an image obeying a ARMA model. Techniques for synthesizing images via CAR and SAR models to resemble a given texture which takes into account all the aspects mentioned above is given in [13, 19].

The synthesis of a finite M×M image to obey *exactly* the difference equation in (2.5) or (2.11) is very difficult. It involves the factorization of a $M^2 \times M^2$ matrix whose elements are the correlation of various lags. It is still more difficult to ensure that the density of the synthesized $y(\cdot)$ has the prespecified form. Instead consider the toroidal variant of the ARMA model in (2.11) which can be compactly written in terms of $\{Y_r, r\epsilon\Omega\}$, the FFT of $(y(s), s\epsilon\Omega)$

$$\sqrt{A_r}Y_r = \sqrt{\nu}\,\sqrt{B_r}W_r, \qquad (6.1)$$

$$A_r = A(z = \exp(\sqrt{-1}(2\pi/M)r), \; B_r = B(z = \exp\sqrt{-1}2\pi r/M).$$

$\{W_r, r\epsilon\Omega\}$ is the FFT of $\{w(s), s\epsilon\Omega\}$, $\{w(s), s\epsilon\Omega\}$ being drawn from an I.I.D. sequence with zero mean, unit variance and the histogram P mentioned later on. As M tends to

infinity, the second order properties of the sequence obeying the toroidal ARMA model in (6.1) tend to that of the general ARMA model in (2.5) or (2.11). Eq. (4.4) is also an equivalent toroidal representation which is the direct analog of (2.5).

To generate the histogram P of $w(\cdot)$, for the given image we proceed as follows. Using the *given image* say $\{y'(s), s\epsilon\Omega\}$ and the (estimated) parameters θ, ϕ, ν etc., generate residuals $\{w'(s), s\epsilon\Omega\}$. The corresponding FFT can be computed as

$$W_r' = Y_r' \sqrt{A_r/B_r\nu}$$

where $\{Y_r', r\epsilon\Omega\}$ is FFT of $\{y'(s), s\epsilon\Omega\}$. The inverse FFT of $\{W_r, r\epsilon\Omega\}$ yields the $\{w'(s), s\epsilon\Omega\}$. The histogram of w' is the required histogram P.

A.   The Synthesis Procedure is as follows:

(i)   Generate a sequence $\{w(s), s\epsilon\Omega\}$ drawn from an I.I.D. population with zero mean, unit variance and histogram P.

(ii)   Compute W_r, $r\epsilon\Omega$

$$W_r = \sum_{s\epsilon\Omega} f_{r,s}w(s)$$

where $f_{r,s}$ is the fourier array, $r\epsilon\Omega$, $s\epsilon\Omega$

$$f_{r,s} = \exp[\sqrt{-1}\ \frac{2\pi}{M}\ s^T r],\ s,r\epsilon\Omega$$

(iii)   Compute $y(s)$, $s\epsilon\Omega$ the required image matrix

$$y(s) = \sum_{r\epsilon\Omega} f_{r,s}^* W_r \sqrt{\frac{B_r}{A_r}}\ \frac{\sqrt{\nu}}{M^2}$$

VI. Conclusions

We have introduced the general class of two dimensional ARMA models which can represent any discrete rational spectral density and shown that the various classes of two dimensional difference equation models discussed in the literature are subclasses of this general class. We have also given various definitions of weak markov processes and precisely characterized subclasses of ARMA models having the various types of markov properties. Two methods are given for estimating the parameters in the model. Finally, a technique is given for synthesizing an image obeying a given ARMA model.

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Grid Size		Θ			φ			ν		
		Mean	S.D.	RMSE	Mean	S.D.	RMSE	Mean	S.D.	RMSE
64*64	Corr.	.2216	.0050	.0053	.1988	.0110	.0110	.8987	.0206	.1033
	ML.	.2240	.0044	.0060	.1973	.0044	.0051	.9542	.1092	.1142
32*32	Corr.	.2019	.0146	.0182	.2125	.0300	.0325	.8060	.0618	.2036
	ML.	.2238	.0119	.0124	.1930	.0130	.0148	.8930	.1257	.1651
16*16	Corr.	.2148	.0379	.0383	.1974	.0528	.0529	.5845	.0971	.4269
	ML.	.2447	.0070	.0257	.1759	.0179	.0300	1.0861	.3535	.3639

Table 1: The summary of the parameter estimates of ten runs by the correlation method and ML method in the model of example 1 of section IV. 'Mean' is the mean of the 10 estimate values of the 10 images and S.D. is the corresponding standard duration. RMSE is the square root of the mean of the squared error between the estimates and the true value. The true values of the parameters are $\Theta = .22$, $\phi = 0.2$ and $\nu = 1.0$.
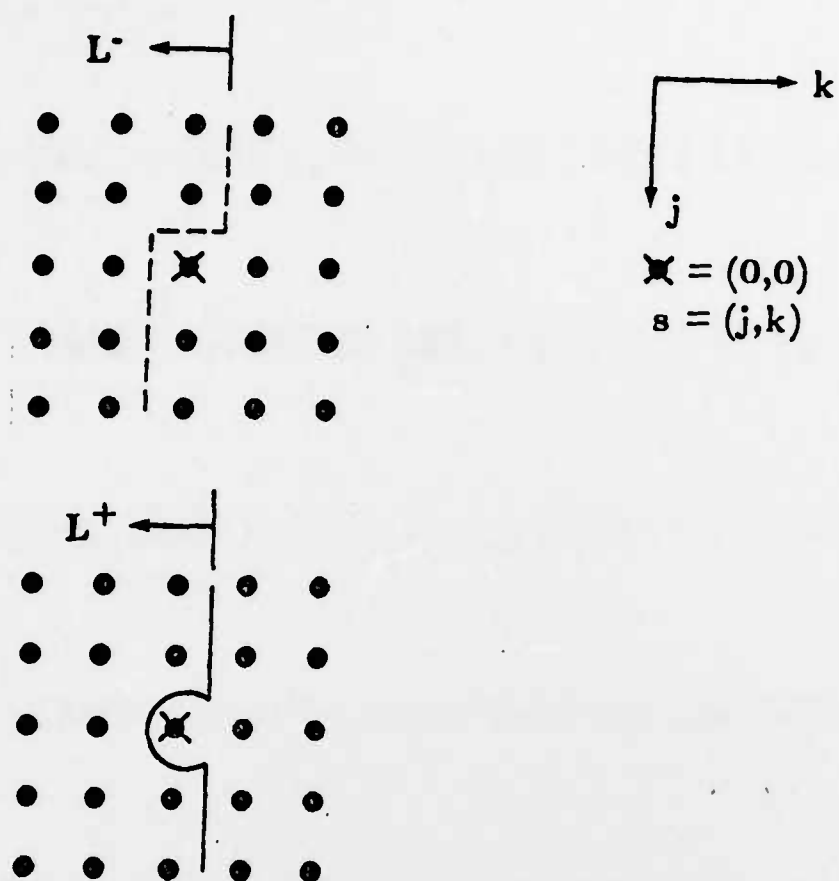
Figure 1: The nonsymmetrical half plane L⁻ and the half plane L⁺

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