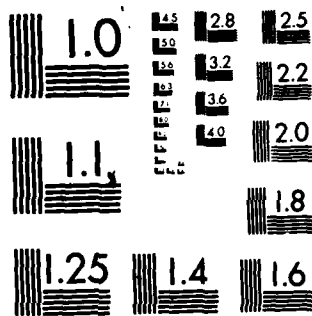MICROCOPY RESOLUTION TEST CHART

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REMARKS ON THE AMBROSETTI-PRODI PROBLEM

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MATHEMATICS RESEARCH CENTER

REMARKS ON THE AMBROSETTI-PRODI PROBLEM

D. G. Costa* and D. G. de Figueiredo**

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ABSTRACT

The Ambrosetti-Prodi boundary value problem with an asymptotically linear
nonlinearity is considered. Under general conditions on the nonlinearity it
is shown that there exist positive and negative solutions. In the case when
the domain is a ball in R_n and the nonlinearity "crosses" the first n
eigenvalues, corresponding to radial eigenfunctions, it is proved that there
are at least $n + 1$ radial solutions.

SIGNIFICANCE AND EXPLANATION

Let $\Omega \subset \mathbf{R}^N$ be a bounded smooth domain. Let λ_1 be the first eigenvalue of $-\Delta$ on $H_o^1(\Omega)$ and let $\phi_1 > 0$ be a corresponding eigenfunction. The Ambrosetti-Prodi problem

$(1)_t \qquad -\Delta u = f(x,u) + h(x) + t\phi_1(x)$ in Ω, $u = 0$ on $\partial\Omega$,

where $t \in \mathbf{R}$, $h \in C(\overline{\Omega})$, $\int h\phi_1 = 0$, is studied. First it is proved that if $\lim_{s \to -\infty} f(x,s)/s < \lambda_1 < \lim_{s \to +\infty} f(x,s)/s$ then problem $(1)_t$ has a large positive and a large negative solution for $t < 0$ large. The interest in this type of solution comes from the fact that they play a special role in the existence of other solutions. This has been observed before for special classes of problem $(1)_t$ by Lazer-McKenna, Solimini and Ambrosetti. If the limit of $f(x,s)/s$ at $+\infty$ is larger than a higher eigenvalue, it is expected that other solutions, besides the positive and negative ones, will appear. Previous results of Hofer, Solimini and Ambrosetti indicate that this is the case. They have been able to prove existence of up to six solutions. A connection between the number of solutions and the number of eigenvalues which are crossed has been proved to exist in the o.d.e. case by Lazer-McKenna. Here the authors consider the case of a ball in $\mathbf{R}^N$ and prove the existence of many radial solutions. These solutions are characterized by the number of their "nodal lines".

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REMARKS ON THE AMBROSETTI-PRODI PROBLEM

D. G. Costa* and D. G. de Figueiredo**

Let $\Omega \subset \mathbb{R}^N$ be a bounded smooth domain. Let $\lambda_1 < \lambda_2 < \cdots < \lambda_k < \cdots$ denote the eigenvalues of $-\Delta$ on Ω with Dirichlet boundary condition and $\phi_1 > 0$ an eigenfunction corresponding to λ_1. We consider the Ambrosetti-Prodi problem

$$(1)_t \qquad -\Delta u = f(x,u) + h(x) + t\phi_1(x) \quad \text{in } \Omega, \ u = 0 \ \text{ on } \partial\Omega,$$

where $h \in C(\overline{\Omega})$, $\int h\phi_1 = 0$, and $f:\overline{\Omega}\times\mathbb{R} \to \mathbb{R}$ is a continuous function such that there exist the limits

$$(2) \qquad \lim_{s\to\pm\infty} \frac{f(x,s)}{s} = f_{\pm} \quad \text{uniformly for } x \in \overline{\Omega}, \text{ with}$$

$$(3) \qquad f_- < \lambda_1 < f_+ \ , \quad f_+ \neq \lambda_j.$$

We assume without loss of generality that $f(x,0) \equiv 0$.

In Section 1 of the present paper we shall show that for $t < 0$ sufficiently large problem $(1)_t$ possesses a large positive solution and a large negative solution. We remark that for a special class of problems $(1)_t$ existence of a negative solution was first observed by Lazer-McKenna [3]. Subsequently, Ambrosetti [1] and Solimini [5], again for a special class of problems $(1)_t$ proved the existence of both a negative and a positive solution for $t < 0$ sufficiently large. In [2], one of the authors proved existence of a negative solution for a general class of Ambrosetti-Prodi problems, including the superlinear case. At present we do not know whether a positive solution exists in the case of nonlinearities f which grow more rapidly than linear.

In Section 2 we consider the case when Ω is a ball and f is a C^1 function which does not depend explicitly on n. Let us denote by $\mu_1 < \mu_2 < \cdots$ the eigenvalues

of $-\Delta$ acting on the radial functions of $H_o^1(\Omega)$ (It is well-known that $\mu_1 = \lambda_1$). We assume there that the limits below exist and satisfy

(4) $\quad f_- \equiv \lim\limits_{s \to -\infty} f'(s) < \mu_1$, $\qquad \mu_n < f_+ \equiv \lim\limits_{s \to +\infty} f'(s) < \mu_{n+1}$, $\qquad n > 1$.

It is then proved that for $t < 0$ sufficiently large $(1)_t$ possesses a radial positive solution U_t and a radial negative solution u_t. It is also proved that given any $c \neq 0$ there exists a radial solution u of the equation in $(1)_t$ such that $u - c \in H_o^1(\Omega)$ and $u - U_t$ has at least n "nodal lines". More precisely there are at least n concentric spheres where the function $u - U_t$ vanishes.

In Section 3 we prove under the same assumptions of the previous section that problem $(1)_t$ has at least $n + 1$ radial solutions for $t < 0$ sufficiently large. Here we use a shooting argument in a manner similar to the work of Lazer-McKenna [4]. In their paper they treated a two-point boundary value problem, which is the o.d.e. analogue of $(1)_t$, with $-\Delta u$ replaced by $-u''$. Our analysis would correspond in their case to looking for solutions which are symmetric with respect to the middle point of the interval. Without this assumption of symmetry they can actually obtain 2n solutions. We believe that this is also true in the p.d.e. case because we are in fact crossing also a number of eigenvalues that correspond to non-radial eigenfunctions. As a matter of fact we prove in Section 4 that this is the case when $N = 3$. Finally we observe that in the case when Ω is a region between two concentric spheres there are at least 2n solutions for problem $(1)_t$, $t < 0$ large.

1. Existence of large positive and negative solutions

In this section we let $\Omega \subset \mathbb{R}^N$ be an arbitrary bounded smooth domain. Let $\lambda_1 < \lambda_2 < \cdots < \lambda_k < \cdots$ denote the eigenvalues of $-\Delta$ on Ω with Dirichlet boundary condition and $\phi_1 > 0$ an eigenfunction corresponding to λ_1. We consider the Ambrosetti-Prodi problem

$(1)_t \quad -\Delta u = f(x,u) + h(x) + t\phi_1(x) \text{ in } \Omega, \qquad u = 0 \text{ on } \partial\Omega,$

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where $h \in C(\overline{\Omega})$, $\int h\phi_1 = 0$, and $f: \overline{\Omega} \times \mathbb{R} \to \mathbb{R}$ is a continuous function such that

$f(x,0) \equiv 0$ and there exist the limits

(2) $\qquad \lim_{s \to \pm\infty} \dfrac{f(x,s)}{s} = f_{\pm}$, uniformly for $x \in \overline{\Omega}$, with

(3) $\qquad f_- < \lambda_1 < f_+$, $\quad f_+ \neq \lambda_j$.

We shall show that for $t < 0$ sufficiently large problem $(1)_t$ possesses a large

positive solution and a large negative solution. To that end consider the related

problems

$(1)_\alpha^\pm \qquad -\Delta v = f_\pm v + \xi(x, v+\alpha\phi_1) + h$ in Ω, $\qquad v = 0$ on $\partial\Omega$, $\qquad (\alpha \in \mathbb{R})$,

where $\xi(x,s) = f(x,s) - (f_+ s^+ - f_- s^-)$, so that

(4) $\qquad \lim_{|s| \to \infty} \dfrac{\xi(x,s)}{s} = 0$ uniformly for $x \in \overline{\Omega}$.

Denote by $S^\pm(\alpha)$ the set of solutions of $(1)_\alpha^\pm$ and let $M^\pm(\alpha) = \sup\{|v|_{C^1} : v \in S^\pm(\alpha)\}$.

<u>Lemma 1.</u> $\quad \lim_{|\alpha| \to \infty} \dfrac{M^\pm(\alpha)}{\alpha} = 0.$

<u>Proof.</u> We shall consider only the *case of* $M^+(\alpha)$, the other case being entirely

similar. Any solution $v \in S^+(\alpha)$ is a solution of the equation $v = K_+[\xi(\cdot, v+\alpha\phi_1) + h]$,

where $K_+: L^2(\Omega) \to L^2(\Omega)$ is the resolvent operator $K_+ = (-\Delta - f_+)^{-1}$ (which exists in view

of (3)). As it is well-known, $K_+ : C^0(\overline{\Omega}) \to C^1(\overline{\Omega})$ is a bounded operator and, therefore,

any $v \in S^+(\alpha)$ satisfies

$$|v|_{C^1} \leq c_+ |\xi(\cdot, v+\alpha\phi_1) + h|_{C^0},$$

or yet, since one has by (4) that $|\xi(x,s)| \leq \varepsilon|s| + c_\varepsilon$ for all $x \in \overline{\Omega}$, $s \in \mathbb{R}$, where

$\varepsilon > 0$ can be chosen arbitrarily (and c_ε depends only on ε),

$$|v|_{C^1} \leq c_+ [\varepsilon|v|_{C^0} + \varepsilon|\alpha| |\phi_1|_{C^0} + c_\varepsilon + |h|_{C^0}].$$

Choosing $\varepsilon > 0$ so that $1 - c_+ \varepsilon > 1/2$ one obtains

$$|v|_{C^1} \leq 2 c_+ \varepsilon|\alpha| |\phi_1|_{C^0} + 2 c_+(c_\varepsilon + |h|_{C^0}),$$

from which the claim follows readily. ∎

Theorem 1. _Suppose that_ (2) _and_ (3) _hold. Then there exists_ $t_o < 0$ _such that for all_ $t < t_o$ _problem_ $(1)_t$ _has a positive solution_ U_t _and a negative solution_ u_t _satisfying_

$$(5) \qquad \lim_{t \to -\infty} \Big| \frac{U_t}{t} - \frac{\phi_1}{\lambda_1 - f_+} \Big|_{C^1} = \lim_{t \to -\infty} \Big| \frac{u_t}{t} - \frac{\phi_1}{\lambda_1 - f_-} \Big|_{C^1} = 0.$$

Proof. Let $\eta > 0$ be such that $w + \phi_1 > 0$ for all w satisfying $|w|_{C^1} < \eta$, $w|\partial\Omega = 0$. By Lemma 1 there exists $\alpha_o > 0$ such that $M^+(\alpha) < \eta \, \alpha$ for all $\alpha > \alpha_o$, that is,

$$(6) \qquad |v|_{C^1} < \eta \, \alpha$$

for any solution v of $(1)^+_\alpha$, $\alpha > \alpha_o$. For any such v, in view of the choice of η and the fact that $|v/\alpha|_{C^1} < \eta$, we have that

$$(7) \qquad U \equiv v + \alpha \, \phi_1 = \alpha\Big(\frac{v}{\alpha} + \phi_1\Big) > 0.$$

So U is a positive solution of the problem

$$-\Delta u = f_+(u - \alpha\phi_1) + \xi(x,u) + h + \alpha\lambda_1\phi_1 \quad \text{in } \Omega, \qquad u = 0 \quad \text{on } \partial\Omega,$$

for $\alpha > \alpha_o$, that is, U is a positive solution of $(1)_t$ for $t = \alpha(\lambda_1 - f_+) < \alpha_o(\lambda_1 - f_+) \equiv t^+_o$. And, in view of (6) and (7), U/t satisfies the inequality

$$\Big| \frac{U}{t} - \frac{\phi_1}{\lambda_1 - f_+} \Big|_{C^1} = \Big| \frac{v}{t} \Big|_{C^1} < \frac{\eta}{f_+ - \lambda_1} \, .$$

Similarly, we prove the existence of a negative solution u_t of $(1)_t$ for $t < -\alpha_o(\lambda_1 - f_-) \equiv t^-_o$ satisfying

$$\Big| \frac{u_t}{t} - \frac{\phi_1}{\lambda_1 - f_-} \Big|_{C^1} < \frac{\eta}{\lambda_1 - f_-} \, .$$

The proof is complete by letting $t_o = \min\{t^+_o, t^-_o\}$ and noticing that the chosen $\eta > 0$ can be taken arbitrarily small. ∎

2. The case when Ω is a ball

We now let Ω be the unit ball $B = B_1(0) \subset \mathbb{R}^N$ and consider the Ambrosetti-Prodi problem

$(8)_t$ $\qquad -\Delta u = f(u) + h + t\phi_1$ in B, $\qquad u = 0$ on ∂B,

where $h \in C(\overline{B})$ is a given radial function with $\int h\phi_1 = 0$ and $f: R \to R$ is a C^1 function satisfying $f(0) = 0$,

(9) $\quad \lim\limits_{s \to -\infty} f'(s) \equiv f_- < \mu_1$, $\qquad \mu_n < \lim\limits_{s \to +\infty} f'(s) \equiv f_+ < \mu_{n+1}$, $n \geq 1$.

Here we are denoting by $\mu_1 = \lambda_1 < \mu_2 < \cdots$ the eigenvalues of $-\Delta$ acting on the radial functions of $H_o^1(B)$, that is, the eigenvalues of the problem

(10) $\qquad -u'' - \dfrac{N-1}{r} u' \equiv \mu u$, $0 < r < 1$, $u'(0) = u(1) = 0$.

As it is well-known, these eigenvalues are given by $\mu_j = \nu_j^2$ where the ν_j's are the positive zeros of the Bessel function $J_{(N-2)/2}$, and corresponding eigenfunctions are given by $\phi_j(r) = r^{-(N-2)/2} J_{(N-2)/2}(\nu_j r)$.

Theorem 2. Under assumption (9), there exists $t_1 < 0$ such that for all $t < t_1$ problem $(8)_t$ has a radial positive solution U_t and a radial negative solution u_t satisfying (5).

Proof. In view of Theorem 1 there exists $t_o < 0$ such that for all $t < t_o$ problem $(8)_t$ has a positive solution U_t and a negative solution u_t satisfying (5). It remains to show that both U_t and u_t are radial functions for $t < t_o$ sufficiently large. Indeed, letting $z_t = \partial U_t / \partial \omega$ denote any angular derivative of U_t, z_t satisfies $-\Delta z_t = f'(U_t)z_t$ in B, $z_t = 0$ on ∂B. Therefore if we assume that $z_t \not\equiv 0$, z_t is an eigenfunction of the above problem corresponding to the eigenvalue $1 = \lambda_j (f'(U_t))$. But then the fact that $U_t(x) \to +\infty$ for all $x \in B$ and the Lebesgue dominated convergence theorem imply that $f'(U_t) \to f_+$ in L^p , $p > 1$, and hence that $1 = \lambda_j(f'(U_t)) \to \lambda_j(f_+) = \lambda_j/f_+$ as $t \to -\infty$, which is a contradiction. Similarly we show that u_t is a radial function for $t < t_o$ sufficiently large. ∎

Now we make the change of (dependent) variables $u = w + U_t$ to rewrite problem $(8)_t$ as

$$-\Delta w = f(w+U_t) - f(U_t) \quad \text{in } B, \qquad w = 0 \text{ on } \partial B,$$

or yet as

$(11)_t \qquad -\Delta w = f_+ w + g(w+U_t) - g(U_t) \quad \text{in } B, \qquad w = 0 \text{ on } \partial B,$

where $g(s) \equiv f(s) - f_+ s$. Notice that, in view of (9), $g : R \to R$ is a C^1 function satisfying

$(12) \qquad \lim\limits_{s \to -\infty} g'(s) = f_- - f_+ \ , \ \lim\limits_{s \to +\infty} g'(s) = 0.$

On the other hand, given $0 \neq \bar{c} \in R$, let us define the function

$$\bar{v}(r) = \frac{\bar{c}}{r^{(N-2)/2}} \frac{J_{(N-2)/2}(\sqrt{f_+}\, r)}{J_{(N-2)/2}(\sqrt{f_+})} \ , \ r = |x|.$$

Then it is easy to see that $\bar{v}$ is a solution of the problem

$(13) \qquad -\Delta\bar{v} = f_+ \bar{v} \quad \text{in } B, \qquad \bar{v} = \bar{c} \text{ on } \partial B,$

and, since $\nu_n < \sqrt{f_+} < \nu_{n+1}$ by (9), $\bar{v}$ vanishes on the n concentric spheres $r = r_j = \nu_j / \sqrt{f_+}$ $(j=1,\ldots,n)$ in B.

Theorem 3. Suppose (9) holds and let $0 \neq \bar{c} \in R$ be given. Then there exists $\bar{t} = \bar{t}(\bar{c}) < 0$ such that for all $t < \bar{t}$ problem

$(14)_{t,\bar{c}} \qquad -\Delta u = f(u) + h + t\phi_1 \quad \text{in } B, \qquad u = \bar{c} \text{ on } \partial B,$

has a radial solution $\bar{u}$ with the property that $\bar{u} - U_t$ vanishes on n concentric spheres in B.

Proof. We want to find a radial solution w of $(11)_t$, with the boundary condition replaced by $w = \bar{c}$ on ∂B, and such that w vanishes on n concentric spheres in B. Since the function $\bar{v}$ has this latter property and its n zeros in $(0,1)$ (as a function of r) are all simple, it suffices to find a solution z of the problem

$(15)_t \qquad -\Delta z = f_+ z + g(z+\bar{v}+U_t) - g(U_t) \quad \text{in } B, \ z = 0 \text{ on } \partial B,$

with $|z|_{C^0} < \bar{\varepsilon}, \ \bar{\varepsilon} = \bar{\varepsilon}(\bar{c})$ sufficiently small, for then $\bar{u} = z + \bar{v} + U_t$ is a solution of

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(14)$_{t,\bar{c}}$ with the desired property that $\bar{u} - U_t = z + \bar{v}$ vanishes on n concentric

spheres in B. (We point out that the same argument used in the proof of Theorem 2 shows

that the solutions of (15)$_t$ are radial for $t < 0$ sufficiently large). So Lemma 2 below

concludes the proof.

__Lemma 2__. __Given__ $\bar{\epsilon} > 0$ __there exists__ $\bar{t} = \bar{t}(\bar{\epsilon})$ __such that for all__ $t < \bar{t}$ __problem__ (15)$_t$

__has a unique solution__ z __satisfying__ $|z|_{C^0} < \bar{\epsilon}$.

__Proof__. We observe that solving (15)$_t$ is equivalent to solving the operator equation

$$z = KG_t(z) \equiv T_t(z),$$

where $K : L^2(B) \to H_0^1(B)$ is the resolvent operator $K = (-\Delta - f_+)^{-1}$ and G_t is the

Nemytskii operator associated with the function $g(z + \bar{v}(r) + U_t(r)) - g(U_t(r))$. We also

observe that the regularity theory for elliptic equations implies that K maps $L^p(B)$

continuously into $W^{2,p}(B)$ and (12) implies that G_t maps $L^p(B)$ continuously into

itself, so that T_t maps $L^p(B)$ into $W^{2,p}(B)$. Given the ball $B_{\bar{\epsilon}}(0)$ in $C^0(\bar{B})$, we

shall show that for $t < 0$ sufficiently large we have $T_t(B_{\bar{\epsilon}}(0)) \subset B_{\bar{\epsilon}}(0)$ and T_t is a

contraction. Indeed, fixing $p > N/2$ and using the Sobolev imbedding theorem, we have

$$|T_t(t)|_{C^0} \leq c|T_t(z)|_{W^{2,p}} = c|KG_t(t)|_{W^{2,p}} \leq \text{const.} \, |G_t(z)|_{L^p} \, ,$$

and $|G_t(z)|_{L^p} \to 0$ as $t \to -\infty$ uniformly for $z \in B_{\bar{\epsilon}}(0)$, since we can estimate

$|G_t(z)|_{L^p} \leq \delta(t)|z + \bar{v}|_{C^0}$ where $\delta(t) \to 0$ as $t \to -\infty$ (In computing $|G_t(z)|_{L^p}$ apply

the mean value theorem for the function g and use the Lebesgue dominated convergence

theorem, keeping in mind that $z + \bar{v} + U_t \to +\infty$ pointwisely in B and that

$g'(s) \to 0$ as $s \to +\infty$). Therefore, $T_t(B_{\bar{\epsilon}}(0)) \subset B_{\bar{\epsilon}}(0)$ for $t < 0$ sufficiently large.

Similarly, for arbitrary $z_1, z_2 \in B_{\bar{\epsilon}}(0)$, we have the estimate

$$|T_t(z_1) - T_t(z_1)|_{C^0} \leq \text{const.} \, |G_t(z_1) - G_t(z_2)|_{L^p} \leq \text{const.} \, \delta(t)|z_1 - z_2|_{C^0},$$

where $\delta(t) \to 0$ as $t \to -\infty$, so that $T_t : B_{\bar{\epsilon}}(0) \to B_{\bar{\epsilon}}(0)$ is a contraction for $t < 0$

sufficiently large. The proof is complete.   ∎

Remark. It should be pointed out that we stated and proved Theorem 3 as a p.d.e. result, ignoring for the time being its natural 1-dimensional character (i.e., search of radial solutions in a ball). The reason for that is the fact that indeed, for an arbitrary bounded smooth domain Ω, Theorem 3 has the following analogue which we now describe.

For given $0 \neq c \in \mathbb{R}$, let v_c denote the solution of the problem

$$-\Delta v = f_+ v \text{ in } \Omega, \quad v = c \text{ on } \partial\Omega \quad (\text{i.e., } v_c = (-\Delta)(-\Delta - f_+)^{-1}c).$$

Suppose that for some $\bar{c} \neq 0$, $\bar{v} = v_{\bar{c}}$ satisfies the condition that $\nabla\bar{v}(x) \neq 0$ whenever $\bar{v}(x) = 0$, that is, the graph of $\bar{v}$ is transversal to $\Omega \times \{0\}$ at the points of $M \times \{0\}$, where

$M = \{x \in \Omega | \bar{v}(x) = 0\}$ is the "null manifold" of $\bar{v}$. Let m be the r~~er~~ of components of M. Then, for $t < 0$ sufficiently large (and under hypotheses (9)' ~~~~here exists a solution u of the problem

$$-\Delta u = f(u) + h + t\phi_1 \text{ in } \Omega, \quad u = c \text{ on } \partial\Omega,$$

such that $u - U_t$ has at least m "nodal lines", that is, the null manifold of $u - U_t$ has at least m components.

3. Existence of many solutions

In this section we consider again the case when Ω is the unit ball $B = B_1(0) \subset \mathbb{R}^N$ and prove under the same assumption (9) of the previous section that the Ambrosetti-Prodi problem $(8)_t$ has at least $n + 1$ radial solutions for $t < 0$ sufficiently large. For that matter we shall use a shooting argument as in [5].

It should be remarked that only from now on is that the one-dimensional character of problem $(8)_t$ will play an important role in proving the existence of other solutions besides $U_t > 0$ and $u_t < 0$, when $n \geq 2$ in (9).

So, we start by rewriting $(8)_t$ as

$(16)_t \qquad -u'' - \dfrac{N-1}{r} u' = f(u) + h + t\phi_1, \quad 0 < r < 1, \quad u'(0) = u(1) = 0,$

which in turn, again through the change of dependent variables $u(r) = w(r) + U_t(r)$, can be rewritten as

$(17a)_t \qquad -w'' - \dfrac{N-1}{r} w' = f_+ w + g(w+U_t) - g(U_t), \quad 0 < r < 1,$

(17b) $w'(0) = w(1) = 0$,

where we recall that $g(s) \equiv f(s) - f_+ s$.

Lemma 3. **The initial value problem**

$$-w'' - \frac{N-1}{r} w' = f_+ w + g(w + U_t(r)) - g(U_t(r)), \quad 0 < r < 1, \qquad w(0) = a, \; w'(0) = 0,$$

has a unique solution $w(r) = w(r;a)$ **which is defined for** $0 < r < 1$ **and depends**

continuously on $a \in R$.

Proof. Since $- w'' - \frac{N-1}{r} w' = - \frac{1}{r^{N-1}} (r^{N-1} w')'$, it can be seen that the initial value

problem in question is equivalent to the following Volterra integral equation

$$w(r) = a + \int_0^r K(r,\sigma) \, F(\sigma, w(\sigma)) d\sigma,$$

where $F(\sigma, s) \equiv f_+ s + g(s + U_t(\sigma)) - g(U_t(\sigma))$ and $K(r,\sigma) = \frac{\sigma}{N-2} [(\frac{\sigma}{r})^{N-2} - 1]$.   Since the

kernel $K(r,\sigma)$ is nice, the result follows from the standard theory of Volterra

equations.   ∎

Theorem 4. **Under assumption (9), there exists** $t_2 < 0$ **such that for all** $t < t_2$ **problem**

$(17)_t$ **has at least** n **distinct non-trivial solutions** $w_0, w_1, \ldots, w_{n-1}$ **with the**

property that $w_j(r)$ **has exactly** j **simple zeros in the open interval** $(0,1)$ **and**

$w_j(0) < 0$, $j = 0, \ldots, n-1$. (**Therefore, for** $t < t_2$ **problem** $(8)_t$ **has at least the**

$n + 1$ **distinct radial solutions** $U_t, \; w_j + U_t, \; j = 0, \ldots, n-1$.)

Proof. From Theorem 2 we already know 2 solutions for problem $(17)_t$, namely $w \equiv 0$

(corresponding to the positive solution U_t of $(8)_t$) and $w_0 = u_t - U_t < 0$ (corresponding

to the negative solution u_t of $(8)_t$), provided $t < t_1$. On the other hand, from Theorem

3 we have a solution $\bar{w} = \bar{u} - U_t$ of equation $(17a)_t$, for $t < \bar{t}$, such that $\bar{w}(r)$ has

n simple zeros in the open interval $(0,1)$ and $\bar{w}(0) < 0$ (take $\bar{c}$ in $(14)_{t,\bar{c}}$ so that the

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function $\bar{v}$ in (13) satisfies $\bar{v}(0) < 0$). Therefore, in view of Lemma 3, we can use the idea of Lemma 2.3 in [4] to show that, for each $0 < j < n$, problem $(17)_t$ has a solution $w_j(r)$ with exactly j simple zeros in $(0,1)$ and such that $w_j(0) < 0$. ∎

4. The case $N = 3$

In the case that Ω is the unit ball $B = B_1(0)$ in $\mathbb{R}^3$ we are able to improve Theorem 4 and obtain $2n$ radial solutions for the Ambrosetti-Prodi problem $(8)_t$, $t < 0$ sufficiently large. Namely, we have

Theorem 5. Let $N = 3$ and suppose (9) holds. Then, for all $t < 0$ sufficiently large, problem $(17)_t$ has at least $2n-1$ distinct non-trivial solutions w_j, $j = 0,\ldots,n-1$, and $\tilde{w}_j$, $j = 1,\ldots,n-1$, with the property that w_j and $\tilde{w}_j$ have exactly j simple zeros in $(0,1)$ and $w_j(0) < 0$, $\tilde{w}_j(0) > 0$. (Therefore, for $t < 0$ sufficiently large, problem $(8)_t$ has at least the following $2n$ distinct radial solutions: U_t, $w_j + U_t$, $j = 0, \ldots, n-1$, $\tilde{w}_j + U_t$, $j = 1, \ldots, n-1$.)

Proof. By making the change of variables $v(r) = rw(r)$ and letting $V_t(r) \equiv rU_t(r)$, we transform $(17)_t$ into the problem

$(18)_t$ $\qquad -v'' = f_+ v + \tilde{g}(r, v+V_t) - \tilde{g}(V_t)$, $0 < r < 1$, $v(0) = v(1) = 0$,

where $\tilde{g}(r,v) \equiv rg(v/r)$. Notice that (9) (hence (12)) implies that $\tilde{g}$ is continuous on $[0,1] \times \mathbb{R}$ (by defining $\tilde{g}(0,v) = (f_+ - f_-)v^-$), $\tilde{g}$ is of class C^1 on $(0,1) \times \mathbb{R}$, $\tilde{g}_v$ is continuous on $[0,1] \times (\mathbb{R}\backslash 0)$ and satisfies $\lim_{v \to -\infty} \tilde{g}_v(r,v) = f_- - f_+$, $\lim_{v \to +\infty} \tilde{g}_v(r,v) = 0$ uniformly for $r \in [0,1]$. Therefore, problem $(18)_t$ can be treated in a similar manner as the problem in [4]. It then follows that $(18)_t$ has solutions v_j, $j = 0, \ldots, n-1$, and $\tilde{v}_j$, $j = 1, \ldots, n-1$, such that v_j and $\tilde{v}_j$ have exactly j simple zeros in $(0,1)$ and $v_j'(0) < 0$, $\tilde{v}_j'(0) > 0$. Going back to our original (dependent) variable we first claim that if $v(r)$ is a solution of $(18)_t$ then $w(r) = v(r)/r$ is a solution of $(17)_t$. Indeed, the only non-obvious property to check is that

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$w(r)$ satisfies the boundary condition $w'(0) = 0$. For that we observe that any

solution $v(r)$ of $(18)_t$ satisfies $v''(0) = 0$, so that using ℓ'Hospital's rule we obtain

$$\lim_{r \downarrow 0} w'(r) = \lim_{r \downarrow 0} \frac{rw'(r)-v(r)}{r^2} = \lim_{r \downarrow 0} \frac{r\, v''(r)}{2r} = \frac{v''(0)}{2} = 0.$$

Consequently $w_j(r) = v_j(r)/r$ and $\tilde{w}_j(r) = \tilde{v}_j(r)/r$ are solutions of $(17)_t$ with the

stated properties.   ∎

Remark. We observe that in the case Ω is the region between two concentric spheres in

$\mathbb{R}^N$, say $\varepsilon < r < 1$, we again obtain $2n$ radial solutions for the corresponding

Ambrosetti-Prodi problem. Indeed, we are led to the o.d.e. problem $(17a)_t$,

$w(\varepsilon) = w(1) = 0$, or yet, making the change of variable $v(r) = r^{(N-1)/2} w(r)$, to the

self-adjoint problem

$$-v'' + \frac{C_N}{r^2} v = f_+ v + \tilde{g}(r,v+V_t) - \tilde{g}(r,V_t), \quad \varepsilon < r < 1, \qquad v(\varepsilon) = v(1) = 0,$$

where $C_N = (N-1)(N-3)/4$, and $V_t(r)$ and $\tilde{g}(r,v)$ are as before.

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The Ambrosetti-Prodi boundary value problem with an asymptotically linear nonlinearity is considered. Under general conditions on the nonlinearity it is shown that there exist positive and negative solutions. In the case when the domain is a ball in $\mathbb{R}^n$ and the nonlinearity "crosses" the first n eigenvalues, corresponding to radial eigenfunctions, it is proved that there are at least n + 1 radial solutions.

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