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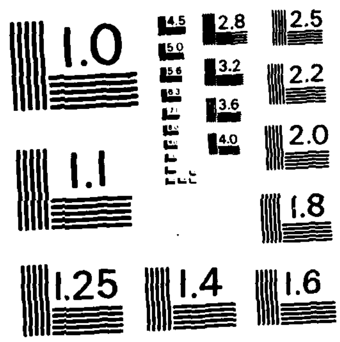

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MULTIVARIATE PERPENDICULAR INTERPOLATION

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UNIVERSITY OF WISCONSIN - MADISON
MATHEMATICS RESEARCH CENTER

MULTIVARIATE PERPENDICULAR INTERPOLATION

Peter Alfeld*

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ABSTRACT

An approach to multivariate interpolation is described. The algorithm is applicable in arbitrary dimensions, and can generate surfaces of arbitrary smoothness. This is accomplished by tesselating the (polyhedral) domain into simplices and using one dimensional algorithms to construct interpolants first on edges and then successively on higher order faces by blending methods. The result is a piecewise rational function of a high degree which has the prescribed global smoothness and interpolates to the original data. The interpolants are local, i.e. their evaluation at a point requires only data on the simplex that the point resides in. The schemes require data of the same degree as the degree of global smoothness. The degree of polynomial precision is greater than or equal to the degree of smoothness. The approach derives its power and simplicity from the fact that derivatives in directions perpendicularly across faces are incorporated directly as data.

AMS (MOS) Subject Classification: 65D05

Key Words: Scattered Data, Multivariate Interpolation, Multivariate Approximation

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SIGNIFICANCE AND EXPLANATION

An approach is described to the interpolation of data in arbitrarily many variables. The schemes are of arbitrary smoothness and require data of the same degree as the degree of smoothness. The domain is assumed to be tesselated into simplices (e.g. triangulated in the case of two variables). Evaluation at a point requires only data on the simplex that the point resides in. The methods described here constitute the only known local interpolation schemes in arbitrarily many variables and with an arbitrary degree of smoothness.

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MULTIVARIATE PERPENDICULAR INTERPOLATION

Peter Alfeld*

1. Introduction

The problem of interpolating to scattered multivariate data is becoming increasingly important. Applications include the modeling of physical phenomena involving space and time (e.g. combustion, temperature, pressure, etc.) and the design of geometric objects (e.g. the body of a car or an aircraft). For a recent survey of this area see Barnhill, 1983.

In this paper, we describe an approach to multivariate interpolation in an arbitrary number of variables, and with an arbitrary degree of smoothness. We assume that the domain of interest has been tesselated into simplices, and that at the data points we are given the values of all partial derivatives through the same order as the degree of smoothness. Any other directional derivatives can then be constructed from the partial derivatives. We do not address the question of how the tesselation may be accomplished, or how higher order data may be generated from lower order data. For some answers to these questions see Barnhill and Little, 1983, or Alfeld, 1984a.

Our interpolants are piecewise rational functions of the prescribed smoothness that interpolate to the given data. The basic idea consists of constructing the interpolant on an individual simplex as a blend of lower dimensional interpolants on faces of the simplex. Recursion of this procedure ultimately leads to one dimensional problems which are easy to solve.

The interpolants derive their power and simplicity from the fact that derivatives in directions perpendicular to faces of a simplex are incorporated directly as data. Such derivatives govern the smoothness of the interpolants between simplices. If derivatives in other directions, e.g. parallel to edges, are used, then an algebraic link to perpendicular

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derivatives has to be constructed. This leads to extremely complex algebraic manipulations that, moreover, at the present state of the art, cannot be carried out for a general degree of smoothness and a general number of variables. The central role of perpendicular derivatives gave rise to the title of this paper.

The schemes proposed here are local, i.e. their evaluation at a point requires only data on the simplex that the point resides in. In two variables there are many local C^1 and C^2 schemes (see Barnhill, 1983, and the references therein). There are very few local trivariate C^1 schemes (see Alfeld, 1984, and Barnhill and Little, 1983) and no alternative C^2 schemes. The methods proposed here are the only existing local schemes for scattered data in an arbitrary number of variables and with an arbitrary degree of smoothness.

The paper is organized as follows: In section 2, we develop some geometric concepts and notation. In section 3, we described two types of perpendicular interpolants. In section 4, some formulas are given that are useful for the implementation of bi- and trivariate schemes; and section 5 contains (trivariate) numerical examples and comparisons.

2. The Geometry of Perpendicular Interpolants

We shall be ultimately concerned with polyhedral regions in $\mathbb{R}^n$ that have been tesselated into n-dimensional simplices. Any such simplex is defined by its vertices which we denote by $V_1, V_2, \ldots, V_{n+1}$. A general point $x \in \mathbb{R}^n$ is described in terms of <u>barycentric coordinates</u> $b_1, \ldots, b_{n+1}$ as

$$x = \sum_{i=1}^{n+1} b_i V_i \quad \text{where} \quad \sum_{i=1}^{n+1} b_i = 1 \quad . \tag{2.1}$$

We will always assume that the general simplex is non-degenerate, i.e. that the linear system (2.1) possesses a unique solution.

Most of the sequel will be concerned with the development of the interpolants as they are defined in the context of a single simplex. It will be convenient to think of the simplex as the affine space spanned by the vertices. We define

$$S := \{x \mid x = \sum_{i=1}^{n+1} b_i V_i, \sum_{i=1}^{n+1} b_i = 1\} \tag{2.2}$$

$$\text{bound}(S) := \{x \in S \mid \exists\, i \in \{1,2,\ldots,n+1\} : b_i = 0\} \quad . \tag{2.3}$$

When we need to refer to the simplex as the convex hull of its vertices we will use the notation

$$\text{conv}(S) = \{x = \sum_{i=1}^{n+1} b_i V_i \mid \sum_{i=1}^{n} b_i = 1,\ b_i > 0 \quad i = 1,\ldots,n+1\} \quad .$$

We will denote by F that $n-1$ dimensional face of S that is obtained by removing the vertex V_i. F is itself a simplex and the notations bound (F) and conv (F) apply. The set of all facets of S of dimension μ, say, is denoted by $\mathcal{F}_\mu$, and the set of all facets is $\mathcal{F}$.

The anchor of F is that face of S whose normal forms the smallest angle with the normal of F. With the point x and the face F we associate the straight line through x perpendicular to F. That line is the line of fixation of x with respect to F. F is also called the base face of the line of fixation. The point $B_F(x)$ where the line of fixation intersects its base face is the base point of x and the point $A_F(x)$ where it intersects the anchor is the top point of x (all with respect to the base face).

Before going into algebraic details in section 3 it is useful to visualize perpendicular interpolants geometrically. Consider a fixed point x and a fixed simplex S. In unanchored interpolants, we interpolate along all lines of fixation through x to all derivatives through given order at the base points. The final value of the interpolant is obtained by suitably blending the values of the univariate interpolants defined on the lines of fixation. In anchored interpolants, we interpolate additionally to function (but no derivative) values at the top points of x.

Note that perpendicular interpolants require data on the faces of S. These are in turn defined in terms of lower dimensional interpolants, giving rise to the recursive structure of perpendicular interpolants. It is possible that the top point of the line of fixation lies outside the convex hull of the points defining the anchor, necessitating extrapolation. This is why in definition (2.2) and (2.3) we do not restrict ourselves to the convex hull of $\{V_i\}$. In the presence of obtuse angles extrapolation may also be necessary in the base face.

The above geometric concepts are illustrated in Figures 1 and 2 for the case that $n = 2$. In section 4 on computational aspects algebraic expressions are given for $n = 2$ and $n = 3$.

3. The Interpolants

3.1. Unanchored Interpolants

We shall construct operators P_S^m which associate to each f a function $P_S^m f \in C^m$ which interpolates f and its derivatives of order $\leq m$ at the vertices of S. Actually $P_S^m f$ only depends on the values of f and its derivative values at the vertices and hence in actuality is defined for each set of discrete data. However, it is convenient to think of P_S^m as operating on functions. If P_S^m is applied piecewise on a tesselation of a polyhedral domain D into simplices, then the resulting piecewise defined function is in $C^m(D)$.

Now the operator P_S^m is built up from the lower dimensional operators P_F^k where F is any face of S of dimension $n - 1$ and $0 \leq k \leq m$. In this way, the operators P_S^m are defined recursively starting with simplices of dimension 0 (the vertices) and working up to higher dimensional simplices.

The operator P_S^m is gotten by blending the lower dimensional operators P_F^k. Suppose V_i is a vertex of S and F is the $n - 1$ dimensional face of S which does not contain V_i. For any point x as in (2.1) which is not in bound (S) we define

$$c_F^m(x) := \frac{\prod_{r \neq i} b_r^{\eta(m)}}{\sum_{j=1}^{m} \prod_{r \neq j} b_r^{\eta(m)}} \qquad (3.1)$$

where $\eta(m) = \begin{cases} m+2 & \text{if } m \text{ is even} \\ m+1 & \text{if } m \text{ is odd} \end{cases}$.

The exponents of the barycentric coordinates b_r are chosen to be even so that the blending functions c_F^m constitute a nonnegative partition of unity even if some bary-

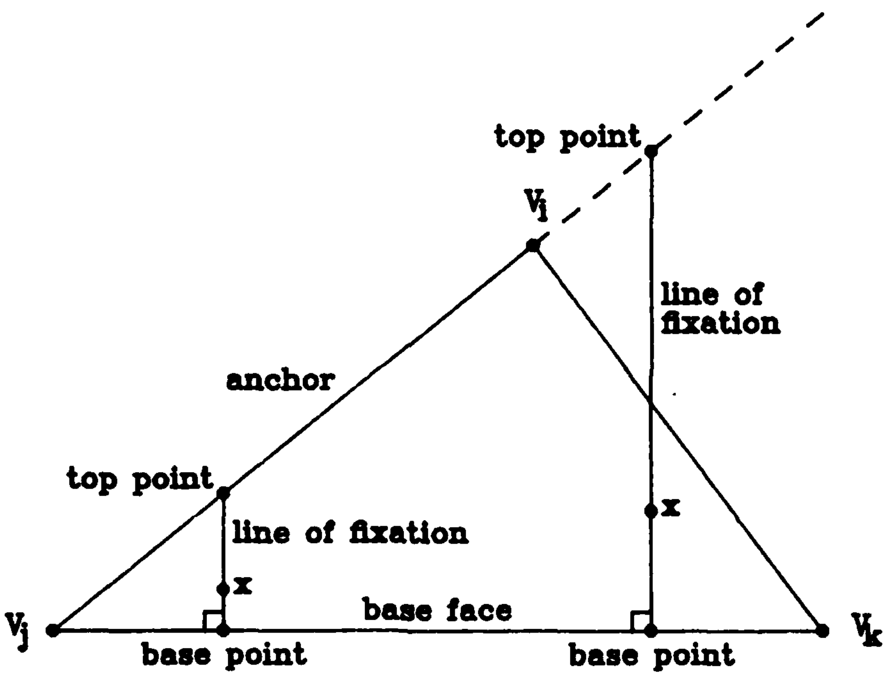

Figure 1: Extrapolation in the anchor

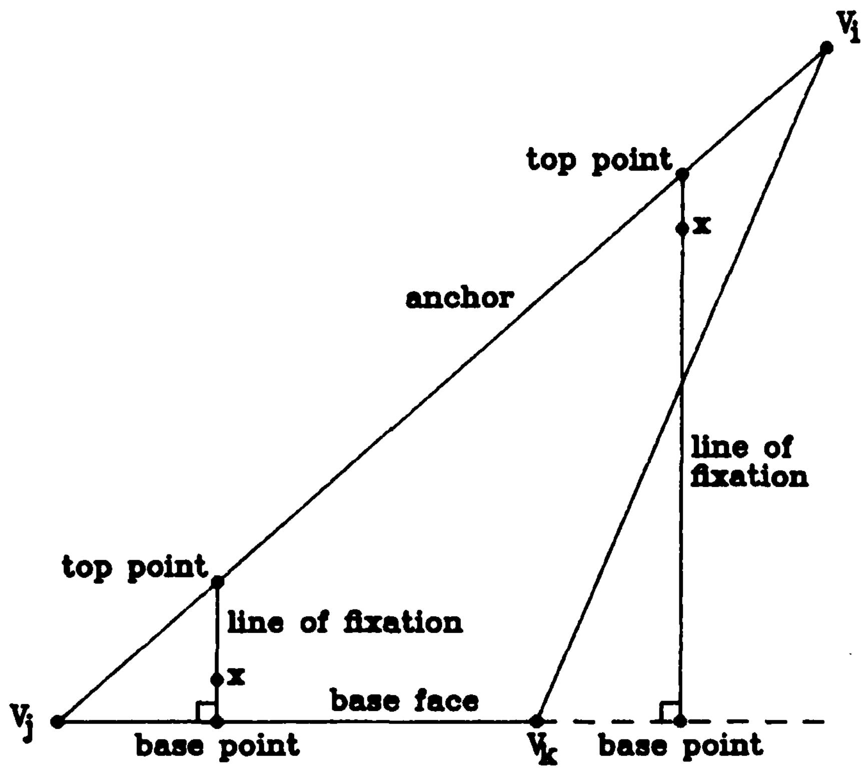

Figure 2: Extrapolation in the base face

centric coordinates are negative. This is necessary because as pointed out in section 2 some extrapolation on faces of S may be necessary.

We do not define c_F^m for points in bound(S); however, since $b_i = 0$ on F, c_F^m is to be thought of as a function which is 1 on F and 0 on the other faces of S. For such a face F we also introduce the operators G_F^m defined by

$$G_F^m(f)(x) := \sum_{\nu=0}^{m} \frac{b_i^\nu}{\nu!} P_F^{m-\nu}\left(\frac{\partial^\nu f}{\partial s_F^\nu}\right) (B_F(x)) \tag{3.2}$$

where the P_F^k are the lower dimensional operators, and, as introduced earlier, $B_F(x)$ is the base point of x and s_F is the normal to F, normalized to have length equal to the distance from V_i to F. Note that the operator G_F^m extends data on F by Taylor interpolation in the direction perpendicular to F, hence the name perpendicular interpolation.

Then, for $n = \dim(S) > 1$, and $m > 0$, the operators P_S^m are defined by

$$P_S^m(f)(x) := \begin{cases} \sum\limits_{F \in F_{n-1}} c_F^m(x) G_F^m(f)(x) & x \notin \text{bound}(S) \\ P_F^m(f)(x) & x \in F \in F_{n-1} \end{cases} . \tag{3.3}$$

We shall see shortly that $P_S^m(f)(x)$ is well defined on bound(S). Before doing this, however, we must define P_S^m if $\dim(S) \le 1$ or $m = 0$. This is accomplished by

$$P_S^0(f)\left(\sum_{i=1}^{n+1} b_i V_i\right) := \sum_{i=1}^{n+1} b_i \, f(V_i) \tag{3.4}$$

and

$$\dim(S) = 0 \implies P_S^m(f)(x) := f(V) \qquad (S = \{V\}) \tag{3.5}$$

$$\dim(S) = 1 \implies$$

$$P_S^m(f)(tV_2 + (1-t)V_1) := \sum_{j=0}^{m} [h_{0,j}^m(t) \frac{\partial^j f}{\partial(V_2-V_1)}(V_1) + h_{1,j}^m(t) \frac{\partial^j f}{\partial(V_2-V_1)}(V_2)] \tag{3.6}$$

where S is spanned by V_1 and V_2 and the $h^m_{i,j}$ are cardinal polynomials of degree $2m+1$ that satisfy and are defined uniquely by the properties

$$\left.\frac{\partial^{\mu} h^m_{i,s}(t)}{\partial x^{\mu}}\right|_{t=k} = \delta_{s\mu}\delta_{ik}$$

$s,\mu = 0,1,\ldots,m;\ i,k = 0,1,$ and δ the Kronecker delta.

Thus (3.4) describes multivariate linear interpolation and (3.6) univariate Hermite Interpolation of degree $2m+1$.

We turn now to the smoothness of $P^m_S f$. is clear that $P^m_S f$ is infinitely often differentiable at all points of S which a ot in bound(S). To study $P^m_S f$ on bound(S) we need the following lemma:

Lemma. The blending functions c^m_F have the following properties:

1. For any differentiation operator $\mathcal{D}$ of order ℓ, $1 \le \ell \le m$, and any $x_0 \in$ bound(S)

$$\lim_{\substack{x \to x_0 \\ x \notin \text{bound}(S)}} \mathcal{D}\, c^m_F(x) = 0 \ .$$

2. With $G = \{F \mid x_0 \in F\} \ne \emptyset$

$$\lim_{\substack{x \to x_0 \\ x \notin \text{bound}(S)}} \sum_{F \in G} c^m_F(x) = 1 \ .$$

Proof: The lemma is proved by inspection.

We now come to the fundamental theorem of this section:

Theorem 1:

The operator P^m_S has the properties:

(i) $P^m_S f$ is well defined for each f.

(ii) $P^m_S f \in C^m$ for each f.

(iii) For any $x_0 \in F \in \mathcal{F}_{n-1}$ and any differential operator $\mathcal{D}$ of order $\ell \le m$ involving only directions contained in S:

$$\mathcal{D} P_S^m(f)(x_0) = P_F^{m-k}(\mathcal{D} f)(x_0) \quad .$$

Proof: The theorem is proved by induction on $\dim(S)$. The statements are clear from the definition (3.6) of P_S^m if $\dim(s) = 1$. Now suppose we have established the theorem for all simplices of dimension $n - 1$ and let S be a simplex of dimension n.

Consider first statement (i). If $x_0 \in \mathrm{bound}(S)$ and $x_0 \in F \cap F'$ for two faces F and F' of S then $x_0 \in G$ with G a facet of dimension $n - 2$ contained in both F and F'. But then $P_F^m(f)(x_0) := P_G(f)(x_0)$ and similarly for F' by the induction hypothesis. This establishes (i).

Now consider (iii) and suppose $F \in \mathcal{F}_{n-1}$ a fixed face of S. Without loss of generality we may assume that $\mathcal{D}$ has been written as the sum of operators of the form $\mathcal{D}_F \overline{\mathcal{D}}$ where $\mathcal{D}_F$ is of order j involving only derivatives in directions contained in F and $\overline{\mathcal{D}} = \dfrac{\partial^k f}{\partial s_F^k}$ with s_F the normal to F and $j+k = \ell \le m$. We first observe that

$$\lim_{\substack{x \to x_0 \\ x \notin F}} \mathcal{D}(G_F f)(x) = P_F^{m-\ell}(\mathcal{D} f)(x_0) \quad . \tag{3.7}$$

Indeed, $\mathcal{D}_F b_i = 0$, and $\overline{\mathcal{D}}(b_i^\nu) = 0$ at x_0 unless $\nu = k$ (in which case $\overline{\mathcal{D}}(b_i^k) = k!$). Hence, from our induction hypothesis

$$\begin{aligned}
\lim_{\substack{x \to x_0 \\ x \notin F}} \mathcal{D}(G_F f)(x) &= \lim_{\substack{x \to x_0 \\ x \notin F}} \sum_{\nu=1}^{m} \Big(\overline{\mathcal{D}}\Big(\frac{b_i^\nu}{\nu!}\Big)\Big)\Big(\mathcal{D}_F P_F^{m-\nu}\Big(\frac{\partial^\nu f}{\partial s_F^\nu}\Big)(B_F(x))\Big) \\
&= \lim_{\substack{x \to x_0 \\ x \notin F}} \mathcal{D}_F\Big(P_F^{m-k}\Big(\frac{\partial^k f}{\partial s_F^k}\Big)\Big)(B_F(x)) \\
&= P_F^{m-\ell}\Big(\mathcal{D}_F \frac{\partial^k f}{\partial s_F^k}\Big)(B_F(x_0)) \\
&= P_F^{m-\ell}(\mathcal{D} f)(x_0) \quad .
\end{aligned}$$

The letter equality holds because $\mathcal{D}_F \dfrac{\partial^k f}{\partial s_F^k} = \mathcal{D}f$ and $B_F(x_0) = x_0$. This shows (3.7).

From (3.7) we can conclude that under the same hypotheses

$$\lim_{\substack{x \to x_0 \\ x \notin F}} \mathcal{D}(P_S^m f)(x) = P_F^{m-\ell}(\mathcal{D}f)(x_0) \quad . \tag{3.8}$$

To see this, we first observe that as in the proof of (i), the right hand side in (3.8) is well defined if x_0 lies in two different $n - 1$ dimensional faces of S. Let us denote by L the common value of $P_{F'}^{m-1}(\mathcal{D}f)(x_0)$ for all faces F' that contain x_0. Using the Lemma and (3.7) we have

$$\lim_{\substack{x \to x_0 \\ x \notin \text{bound}(S)}} \mathcal{D}(P_S^m f)(x) = \lim_{x \to x_0} \Big(\sum_{\substack{F \\ x_0 \in F}} c_F^m(x) \Big) L = L \quad . \tag{3.9}$$

The last equality holds because in the limit all derivatives of all c_F^m, and all c_F^m with $x_0 \notin F$ vanish, rendering the sum in (3.9) equal to 1, and because the $G_F^m f(x)$ are continuous and hence bounded in a neighborhood of x_0. This establishes (iii).

The statement (ii) now follows by induction on the degree $\ell < m$ of a differentiation operator. Let $x_0 \in \text{bound}(S)$. In the proof of (iii) we have seen that $\lim_{x \to x_0} P_S^m f(x)$ is well-defined and assumes the value required in (3.3). It is hence continuous. Now assume $P_S^m \in C^{\ell-1}(S)$, $\ell < m$, and $\mathcal{D}$ is a differentiation operator of degree ℓ. Then $\lim_{x \to x_0} \mathcal{D}P_S^m f(x)$ is again well defined. Since $P_S^m \in C^{\ell-1}(S)$, $\mathcal{D}P_S^m$ is continuous. This establishes (ii) and completes the proof of the theorem.

The properties of interpolation and global smoothness of piecewise applied unanchored interpolants follow immediately from Theorem 1.

Corollary: Assume a polyhedral region D has been tesselated into simplices and let $P^m(f)(x) = P_S^m(f)(x)$ whenever $x \in \text{conv}(S)$. Then $P^m f$ is well defined, $P^{-m} f \in C^m(D)$, and for all vertices V and differential operators $\mathcal{D}$ of order $k < m$

$$\mathcal{D}(P^m f)(V) = \mathcal{D} f(V) \quad .$$

Proof: Nothing needs to be shown in the interior of a simplex. So let F be a common facet of two simplices S and S'. Then by a possibly repeated application of Theorem 1 (iii),

$$\mathcal{D} P_S^m f \Big|_F = P_F^{m-k} \mathcal{D} f = \mathcal{D} P_{S'}^m f \quad . \tag{3.10}$$

Hence $P^m f$ and its derivatives are well defined. Its smoothness follows as in the proof of theorem 1 (ii). Interpolation follows from (3.10) and the definition (3.5).

We now turn to the question of polynomial precision.

Theorem 2: Let f be any polynomial in n variables of degree $\leq m$, and let $P^m f$ be defined as in the corollary of Theorem 1. Then

$$P^m f = f \quad .$$

Proof: It is sufficient to consider only an individual simplex S. If $m = 0$, the operator P_S^m even reproduces linear functions which are of degree $m + 1$. For $m > 0$, the proof is again by induction on $\dim (S)$. The statement is clear from the definition (3.6) if $\dim (S) = 1$ since univariate Hermite Interpolation reproduces polynomials of degree up to $2m + 1$. This follows from the existence and uniqueness of the interpolant which is shown e.g. in Davis, 1975, p. 29.

Now suppose we have established the theorem for all simplices of dimension $n - 1$ and let S be a simplex of dimension n. Let f be a polynomial of degree $\leq m$. Then f reduces to a univariate polynomial of degree $\leq m$ along any line of fixation. It follows that for any face F of S the operator G_F^m reproduces f exactly because by (3.2) G_F^m just applies Taylor interpolation to data on F which are exact by the induction hypothesis. The theorem now follows from (3.3) since the c_F^m constitute a partition of unity.

3.2 Anchored Interpolants

In this subsection, we increase the degree of precision of unanchored interpolants by also interpolating to function (but no derivative) values at the top point of the line of fixation. It is infeasible to interpolate to derivatives at the top point since the direction of interpolation is not perpendicular to the anchor. The necessary changes in the interpolant consist of the addition of another term in the definition of the face operators and a modification of the cardinal functions. The notation is essentially the same as that for unanchored interpolants, except that differing objects have been distinguished by bars. We define

$$\overline{G}_F^m(x) := \sum_{\nu=0}^{m} \overline{p}_\nu^m(b_F,t_F)\ \overline{P}_F^{m-\nu}\left(\frac{\partial^\nu f}{\partial s_F^\nu}\right)(B_F(x))$$

$$+\ \overline{p}_{m+1}^m(b_F,t_F)\ \overline{P}_A^m(f)(T_F(x))$$

where A is the anchor of F, $T_F(x)$ is the top point of the line of fixation with respect to F through x, and $B_F(x)$ and s_F are as before. The quantities b_F and t_F are defined by

$$b_F := \|x - B_F(x)\|_2$$

and

$$t_F := \|x - T_F(x)\|_2$$

and the $\overline{p}_\nu^m$ replace the Taylor polynomials. They are polynomials of degree $m + 1$ defined by the cardinal properties

$$\left.\frac{\partial^r\ \overline{p}_\nu^m(b,t)}{\partial b^r}\right|_{b=0} = \delta_{r\nu} \qquad \begin{array}{l} r = 0,1,\ldots,m \\ \nu = 0,1,\ldots,m+1 \end{array}$$

$$\text{and} \quad p_r^m(t,t) = \delta_{r,m+1} \qquad r = 0,1,\ldots,m+1 \quad .$$

Explicitly they are given by

$$p_\nu^m(b,t) = \frac{b^\nu}{\nu!}\left(1 - \left(\frac{b}{\nu}\right)^{m-\nu+1}\right)$$

and

$$p^m_{m+1}(b,t) = \left(\frac{b}{t}\right)^{m+1} .$$

Thus the operator $\overline{G}^m_F$ extends data on F and its anchor by univariate interpolation along the line of fixation.

The operators $\overline{P}^m_S$ are now defined analogously to P^m_S by

$$\overline{P}^m_S(f)(x) := \begin{cases} \sum_{F \in F_{n-1}} c^m_F(x)\overline{G}^m_F(f)(x) & x \notin \text{bound}(S) \\ \overline{P}^m_F(f)(x) & x \in F \in F_{n-1} \end{cases}$$

if $\dim(S) > 1$ and $m > 0$, and

$$\overline{P}^m_S(x) = P^m_S(x),\ \overline{P}^0_S(x) = P^0_S(x)$$

otherwise.

The following theorem is analogous to Theorem 1:

Theorem 3:

The operator $\overline{P}^m_S$ has the properties

(i) $\overline{P}^m_S f$ is well defined for each f

(ii) $\overline{P}^m_S f \in C^m$ for each F

(iii) For any $x_0 \in F \in F_{n-1}$ and any differential operator $\mathcal{D}$ of order $\ell \le m$ involving only directions contained in S:

$$\mathcal{D}\overline{P}^m_S(f)(x_0) = \overline{P}^{m-k}_F(\mathcal{D}f)(x_0) .$$

Proof: The proof of Theorem is like that of Theorem 1. The only difference is that the $\overline{G}^m_F(f)$ differ from the $G^m_F(t)$ on faces other than F. However, all that is required is the boundedness of $\overline{G}^m_F(f)(x)$ as x approaches a point in a face other than F. This, however, follows from the continuity of $\overline{G}^m_F(f)$.

Global Smoothness and interpolation follows as for Theorem 1:

Corollary: Assume a polyhedral region D has been tesselated into simplices and let $\overline{P}^m(f)(x) = \overline{P}_S^m(f)(x)$ whenever $x \in \text{conv}(S)$. Then $\overline{P}^m f$ is well defined, $\overline{P}^m f \in C^m(D)$, and for all vertices V and differential operators $\mathcal{D}$ of order $k \leq m$

$$\mathcal{D}(\overline{P}^m f)(V) = \mathcal{D}f(V) \quad .$$

The advantage of anchored interpolants is that the degree of polynomial precision is increased without raising the degree of the required data:

Theorem 4: Let f be any polynomial in n variables of degree $\leq m+1$, and let $\overline{P}^m f$ be defined as in the corollary of Theorem 3. Then

$$P^m f = f \quad .$$

The proof of theorem 4 is analogous to that of theorem 2.

4. Implementation of Interpolants in Two or Three Variables

In this section, we develop algebraic representations of the geometric concepts introduced in the preceding sections. We restrict our attention to two or three variables. An edge of a simplex will be denoted by $e_{ij} = V_i - V_j$.

4.1 Two Variables

In this case, the generic simplex S is a triangle, and its faces are edges. The anchor A of the base face e is that edge other than e that maximizes the expression

$$|\cos \sigma| = \frac{|e^T A|}{\|e\|_2 \|A\|_2}$$

where σ is the angle formed by e and A. In the case of an isosceles triangle, the resulting tie may be broken arbitrarily, but this must be done consistently. Some care may be necessary to avoid different assignments of the anchor at different stages of the computation due to round-off errors.

To maintain generality we consider a general triangle S with vertices V_i, V_j, V_k, where e_{jk} is the base face and e_{ij} is its anchor, as depicted in Figures 1 and 2. Any actually occuring situation can be described by assigning suitable values to i, j, and k.

The normal Φ to the base face is expressed in the form

$$\Phi = e_{ij} + \gamma e_{jk} \quad \text{where}$$

$$\Phi^T e_{jk} = 0 \quad \text{i.e.} \quad \gamma = -\frac{e_{ij}^T e_{jk}}{e_{jk}^T e_{jk}} \quad . \tag{4.1}$$

A general point x is written in terms of barycentric coordinates as

$$x = b_i V_i + b_j V_j + b_k V_k, \; b_i + b_j + b_k = 1 \quad .$$

Let B denote the base point of the line of fixation through x, and let T denote its top point. With the normalization implied by (4.1) we have that b_i is the natural parameter along Φ, i.e. $\frac{\partial b_i}{\partial \Phi} = 1$. We obtain

$$x = B + b_i \Phi, \; B = a_j V_j + a_k V_k \quad ,$$

where $a_j = b_j + (1-\gamma) b_i$, $a_k = 1 - a_j$, and

$$T = c_i V_i + c_j V_j \quad \text{where}$$

$$c_i = \frac{a_k}{\gamma} \quad \text{and} \quad c_j = 1 - c_i \quad .$$

Note that always $\gamma \neq 0$ because the choice of the anchor avoids right angles σ.

4.2. Three Variables

We now repeat this development for the case of three variables and consider a general tetrahedron S with vertices V_i, V_j, V_k, V_ℓ where the base face is spanned by V_j, V_k, and V_ℓ. The normal to the base face is written as

$$\Phi = e_{ij} + \alpha e_{jk} + \beta e_{k\ell}$$

where α and β are defined by the linear system

$$\Phi^T e_{jk} = \Phi^T e_{k\ell} = 0 \quad .$$

Explicit expressions for α and β are given by:

$$\alpha = \frac{e_{jk}^T e_{k\ell} e_{ij}^T e_{k\ell} - e_{k\ell}^T e_{k\ell} e_{ij}^T e_{jk}}{e_{jk}^T e_{jk} e_{k\ell}^T e_{k\ell} - (e_{jk}^T e_{k\ell})^2}$$

and

$$\beta = \frac{e_{jk}^T e_{k\ell} e_{ij}^T e_{jk} - e_{jk}^T e_{jk} e_{ij}^T e_{k\ell}}{e_{jk}^T e_{jk} e_{k\ell}^T e_{k\ell} - (e_{jk}^T e_{k\ell})^2} .$$

The anchor of the base face is that face other than the base face whose normal q, say, maximizes the expression

$$\frac{|\phi^T q|}{\|\phi\|_2 \|q\|_2} .$$

A general point x is now written as

$$x = b_i V_i + b_j V_j + b_k V_k + b_\ell V_\ell$$

where

$$b_i + b_j + b_k + b_\ell = 1 .$$

Again we denote the base point by B and the top point by T. We obtain

$$B = a_j V_j + a_k V_k + a_\ell V_\ell$$

where

$$a_j = b_j + (1-\alpha) b_i$$

$$a_k = b_k + (\alpha-\beta) b_i$$

$$a_\ell = 1 - a_j - a_k$$

and, assuming the anchor is spanned by V_i, V_j, V_k

$$T = c_i V_i + c_j V_j + c_k V_k$$

where

$$c_i = \frac{a_\ell}{\beta} ,$$

$$c_j = a_j + (\alpha-1) c_i$$

$$c_k = 1 - c_i - c_j .$$

5. Computational Aspects and Numerical Results

5.1. Computational Aspects

We have implemented all the schemes described in this paper into **FORTRAN** research codes. A major tool in constructing the codes was the symbol manipulation language **REDUCE** (Hearn, 1983), which can be instructed to write mathematical formulas directly in **FORTRAN** notation. We have examined all interpolants using the smoothness tester **MICROSCOPE** (Alfeld and Harris, 1984). This investigation confirmed that the codes do indeed possess the smoothness and precision properties implied by the mathematical construction.

5.2. Numerical Results

In this section, we compare several trivariate schemes on a simple problem. The domain is a distorted cube that has been tesselated into 12 tetrahedra. The tesselation is described in detail in Alfeld, 1984b. The guiding principle behind the construction of the domain was to avoid artifacts due to e.g. edges or faces being parallel to coordinate axes, etc.

For ease of comparison an underlying function was used. This was given by

$$F(x,y,z) = \sqrt{r^2-x^2-y^2-z^2} \tag{5.1}$$

for several values of r. The derivative data required for the schemes were exact. The distance of the vertices of the domain from the origin varied between 18.71 and 187.6.

For comparison, we also include transfinite variants of our interpolants, i.e., schemes where the data on faces are given exactly rather than being defined by lower dimensional interpolants. Thus W^m and $\overline{W}^m$ denote the transfinite interpolants corresponding to P^m and $\overline{P}^m$, respectively.

Table 1 contains the numerical results. The interpolation schemes are ordered by increasing complexity as defined by the amount of CPU time (on the DEC System 20 of the College of Science at the University of Utah) required for one evaluation of the interpolant. The individual schemes were not programmed in the most efficient fashion possible, but their implementations are sufficiently similar such that the comparison is valid. The following is a more detailed description of the table by columns:

1. (Scheme) Short notation for the scheme. For most of the scheme the notation is defined in this paper, but the schemes defined in Alfeld, 1984b are included for comparison.

2. (Discrete versus Transfinite) Indicates whether the scheme is a transfinite or a discrete one.

3. (Anchor) Indicates whether the scheme is anchored or not. N. A. means that the scheme is not based on perpendicular interpolation so that this distinction does not apply.

4. (Precision) All (trivariate) polynomial of degree up to the given number are reproduced exactly by the scheme.

5. (Smoothness) All derivatives through the given order are continuous.

6. (r) The value of r in (5.1).

7. (Error) The maximum relative error (computed over the entire distorted cube). This quantity was computed by searching in each tetrahedron with Winfields Method (Winfield, 1973) starting at the centroid of the tetrahedrom. This algorithm identifies local maxima. Hence there is a slim possibility that the overall global maximum was overlooked by the method. However, this is an unlikely occurrence because on each tetrahedron the primitive function (although not the relative error) is uniformly convex.

8. (CPU) The CPU time in milliseconds for one evaluation of the interpolant (not including the setting up of the relevant data structure).

9. (COMM.) Comments:

(1) This is the standard linear interpolant that is included here for comparison.

(2) This is the transfinite scheme described in Alfeld, 1984b. It is also included for comparison.

(3) Here the necessary extrapolation on the anchor caused an evaluation of the primitive function outside of its domain where $x^2 + y^2 + z^2 > r^2$. The computing system proceeded by replacing negative radicands by their absolute values.

(4) This is the discrete interpolant described in Alfeld, 1984b. It is also included for comparison.

The following points emerge from the table:

1. Most schemes give reasonable results, considering that a large part of the domain of the primitive function is covered by only 12 tetrahedra.

2. The discrete schemes are significantly more complex than the transfinite schemes. The only exception to this is the very simple linear interpolation scheme, which yields a surface that is only continuous.

3. The discrete anchored C^1 interpolant $\bar{P}^1$ is much more efficient than its competitor described in Alfeld, 1984b which has very similar properties. Indeed, an attempt was made to construct a C^2 scheme along the lines described in that paper. This attempt failed for purely practical reasons, the C^2 scheme was simply too complex for all practical purposes. Since the smoothness and precision of the C^1 schemes are identical, and the maximum relative errors are very similar, this demonstrates the superiority of the perpendicular interpolation approach described in this paper.

4. As one would expect, the discretization of a transfinite scheme increases the error substantially. This is a price that simply has to be paid in a practical application since transfinite data will usually not be available.

5. The accuracy also deteriorates as the radius of the domain decreases and the boundary of the domain approaches the outmost point of the tesselation.

6. The gain in accuracy due to employing a C^2 rather than a C^1 scheme is surprisingly small.

1	2	3	4	5	6	7	8	9
SCHEME	DISCRETE versus TRANSFINITE	ANCHOR	PRECISION	SMOOTHNESS	r	ERROR	CPU	COMM.
					200	2.1E-1		(1)
P^0	Discr.	both	1	0	300	4.0E-2	4.9	
					400	1.9E-2		
					200	7.9E-3		
W^1	Trnsf.	no	1	1	300	2.4E-3	8.7	
					400	1.2E-3		
					200	3.3E-3		
$\bar{W}^1$	Trnsf.	yes	2	1	300	1.1E-4	10.1	
					400	2.6E-5		
					200	6.7E-4		
BBG	Trnsf.	N.A.	2	1	300	3.2E-5	10.3	(2)
					400	6.8E-6		
					200	2.5E-4		
W^2	Trnsf.	no	2	2	300	2.8E-5	12.3	
					400	7.5E-6		
					200	6.4E-4		(3)
$\bar{W}^2$	Trnsf.	yes	3	2	300	1.0E-5	16.1	
					400	2.1E-6		
					200	1.2E-1		
P^1	Discr.	no	1	1	300	9.7E-3	18.4	
					400	4.4E-3		
					200	8.7E-2		
$\bar{P}^1$	Discr.	yes	2	1	300	1.4E-3	44.2	
					400	2.8E-4		
					200	1.3E-1		(3)
P^2	Discr.	no	2	2	300	6.4E-4	65.8	
					400	1.2E-4		
					200	9.9E-2		(3)
$\bar{P}^2$	Discr.	yes	3	2	300	3.9E-4	142.5	
					400	1.1E-4		
					200	8.4E-2		
BBG	Discr.	N.A.	2	1	300	1.4E-3	280.3	(4)
					400	2.6E-4		

Table 1: Numerical Results

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REFERENCES

Peter Alfeld, 1984a, Scattered Data Derivative Generation by Functional Minimization. Submitted for publication.

Peter Alfeld, 1984b, A Discrete C^1 Interpolant for Tetrahedral Data, special issue of the Rocky Mountain Journal of Mathematics, to appear January 1984.

Peter Alfeld and Bill Harris, 1984, MICROSCOPE: A Software Tool for the Analysis of Multivariate Functions, in preparation.

R. E. Barnhill, 1983, A Survey of the Representation and Design of Surfaces, IEEE Computer Graphics and Applications, October 1983, pp. 9-16.

R. E. Barnhill and F. F. Little, 1983, Three- and Four-Dimensional Surfaces, special issue of the Rocky Mountain Journal of Mathematics, to appear January 1984.

Philip J. Davis, 1975, Interpolation and Approximation, Dover Publication.

A. C. Hearn, 1983, REDUCE User's Manual, Version 3.0, The Rand Corporation, Santa Monica, CA 90406.

D. Winfield, 1973, Function Minimization by Interpolation in a Data Table. JIMA 12, pp. 339-347.

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An approach to multivariate interpolation is described. The algorithm is applicable in arbitrary dimensions, and can generate surfaces of arbitrary smoothness. This is accomplished by tesselating the (polyhedral) domain into simplices and using one dimensional algorithms to construct interpolants first on edges and then successively on higher order faces by blending methods. The result is a piecewise rational function of a high degree which has the prescribed global smoothness and interpolates to the original data. The interpolants are local, i.e. their evaluation at a point requires only data on the simplex that the

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ABSTRACT (continued)

point resides in. The schemes require data of the same degree as the degree of global smoothness. The degree of polynomial precision is greater than or equal to the degree of smoothness. The approach derives its power and simplicity from the fact that derivatives in directions perpendicularly across faces are incorporated directly as data.

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