AD-A139 256 ON A THEOREM OF SZEGOE ON UNIVALENT CONVEX MAPS OF THE UNIT CIRCLE(U) WISCONSIN UNIV-MADISON MATHEMATICS RESEARCH CENTER I J SCHOENBERG FEB 84 MRC-TSR-2647 1/1

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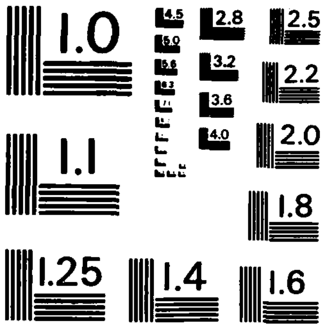
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MRC Technical Summary Report #2647

ON A THEOREM OF SZEGÖ ON UNIVALENT CONVEX MAPS OF THE UNIT CIRCLE

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February 1984

(Received January 30, 1984)

Sponsored by

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Research Triangle Park
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ON A THEOREM OF SZEGÖ ON UNIVALENT CONVEX MAPS OF THE UNIT CIRCLE

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ABSTRACT

For a positive constant λ we denote by $K(\lambda) = \{f(z)\}$ the class of function $f(z)$ which are regular and univalent in $|z| < \lambda$ and map this circle on a convex domain. In 1928 G. Szegö [3] proved the

<u>Theorem 1.</u> <u>If</u> $f(z) = \sum_0^\infty c_\nu z^\nu \in K(1)$, <u>then all its sections</u>

$$s_n(z) = \sum_0^n c_\nu z^\nu \in K\left(\frac{1}{4}\right) \quad \underline{\text{for}} \quad n = 1,2,\ldots .$$

An evident consequence is

<u>Corollary 1.</u> <u>Since the geometric series</u>

$$f_0(z) = \sum_0^\infty z^\nu \in K(1)$$

<u>we have</u>

$$\sum_0^n z^\nu \in K\left(\frac{1}{4}\right), \quad (n = 1,2,\ldots)$$

<u>and therefore, on replacing</u> z <u>by</u> $z/4$, <u>we have</u>

$$\sum_0^n \frac{1}{4^\nu} z^\nu \in K(1) \quad \underline{\text{for}} \quad n = 1,2,\ldots .$$

In the present paper we prove directly Corollary 1, and derive from it Szegö's Theorem 1. This is done by appealing to Theorem 2 which was conjectured by Polya and Schoenberg [1] in 1958, but only proved in 1973 by St. Ruscheweyh and T. Sheil-Small [2].

AMS (MOS) Subject Classifications: 30C20, 52A10

Key Words: Convexity, Geometric Function Theory

Work Unit Number 6 (Miscellaneous Topics)

Sponsored by the United States Army under Contract No. DAAG29-80-C-0041.

SIGNIFICANCE AND EXPLANATION

There is a fine interplay between two fundamental notions of geometry: Convexity and Conformal Mapping. The subject belongs to Geometric Function Theory. In 1928 Gabor Szegö showed that if a power series converges in the unit circle $|z| < 1$ and maps it onto a convex domain, then all its finite sections map the circle $|z| < \frac{1}{4}$ onto convex domains. The present paper shows that Szegö's theorem reduces to a study of the finite sections of the geometric series

1/16 z squared

$$1 + \frac{1}{4} z + \frac{1}{4^2} z^2 + \dots = \frac{1}{4^n} z^n + \dots .$$

The main tool is a result conjectured in 1958 by Polya and Schoenberg, but only established in 1973 by St. Ruscheweyh and T. Sheil-Small.

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ON A THEOREM OF SZEGŐ ON UNIVALENT CONVEX MAPS OF THE UNIT CIRCLE

I. J. Schoenberg

1. INTRODUCTION. There is an interesting interplay between two fundamental notions of geometry: Convexity and Conformal Mapping. The subject belongs to Geometric Function Theory. We use the following notation: If $r > 0$, we denote $K(r) = \{f(x)\}$ the class of functions $f(z)$ which are univalent in the circle $|z| < r$, and map it onto a convex domain.

In 1958 Polya and Schoenberg conjectured that for $r = 1$ the class $K(1)$ form a semi-group with respect to Hadamard multiplication of power series. This was established in 1973 by St. Ruscheweyh and T. Sheil-Small in [2] by the following

Theorem 1. (Ruscheweyh and Sheil-Small). If

$$\sum_0^\infty a_\nu z^\nu \in K(1) \quad \text{and} \quad \sum_0^\infty b_\nu z^\nu \in K(1)$$

then

$$\sum_0^\infty a_\nu b_\nu z^\nu \in K(1) .$$

Before we pass to the work of Szegö, let us use Theorem 1 to establish a simple and well known

Proposition. If

$$f(z) = \sum_0^\infty a_\nu z^\nu \in K(1) \tag{1.1}$$

and $0 < \lambda < 1$, then

$$f(z) = \sum_0^\infty a_\nu z^\nu \in K(\lambda) . \tag{1.2}$$

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Proof: We start from the geometric series

$$(1.3) \qquad f_0(z) = \sum_0^{\infty} z^{\nu} = \frac{1}{1 - z} \in K(1) ,$$

because it maps $|z| < 1$ onto the half-plane $\mathrm{Re}\, z > \frac{1}{2}$. However, $w = f_0(z) = 1/(1 - z)$ clearly maps $|z| < \lambda$ onto a circle, and so

$$\sum_0^{\infty} z^{\nu} \in K(\lambda) .$$

Replacing z by λz, we obtain that

$$(1.4) \qquad \sum_0^{\infty} \lambda^{\nu} z^{\nu} \in K(1) .$$

However, from (1.1) and (1.4), by Theorem 1 we obtain

$$\sum_0^{\infty} \lambda^{\nu} a_{\nu} z^{\nu} \in K(1) ,$$

or, replacing λz by z, we have

$$\sum_0^{\infty} a_{\nu} z^{\nu} \in K(\lambda) ,$$

which is the desired conclusion (1.2). ∎

Remark. Our derivation of the conclusion (1.2) from the special case of the geometric series (1.3) justifies Polya's statement that within the class $K(1)$ the geometric series (1.3) "sets the fashion" (in German: "tonangebend").

Szegö's question: If

$$(1.5) \qquad f(z) = \sum_0^{\infty} c_{\nu} z^{\nu} \in K(1) ,$$

what can we say about its sections

$$S_n(z) = \sum_{\nu=0}^{n} c_{\nu} z^{\nu} ?$$

In 1928 G. Szegö [3, Satz II', page 204] established

Theorem 2 (Szegö). If (1.5) holds, then

(1.6) $$S_n(z) = \sum_0^n c_\nu z^\nu \in K(\tfrac{1}{4}) \quad \text{for} \quad n = 1,2,\ldots .$$

Using Theorem 1 we will show in §2 that we can reduce Theorem 2 to the question concerning the section of the geometric series

$$f_0(z) = 1 + \frac{1}{4} z + \ldots + \frac{1}{4^n} z^n + \ldots .$$

Theorem 3. Let λ be a constant satisfying

$$0 < \lambda < 1 .$$

If (1.5) holds, then

(1.7) $$S_n(z) = \sum_0^n c_\nu z^\nu \in K(\lambda) \quad \text{for all} \quad n \geqq m(\lambda) ,$$

where

(1.8) $m(\lambda)$ is the least integer such that (1.7) holds.

Evidently

(1.9) $$m(\lambda) = 1 \quad \text{if} \quad \lambda \leqq \frac{1}{4} ,$$

by Theorem 2. It is equally evident that

(1.10) $$\lambda_1 < \lambda_2 \quad \text{implies that} \quad m(\lambda_1) \leqq m(\lambda_2) ,$$

for if $S_n(z) \in K(\lambda_2)$, then also $S_n(z) \in K(\lambda_1)$.

The main difficulty in Theorem 3 is an explicit determination of $m(\lambda)$, if $\lambda > \frac{1}{4}$, and we will do that for the value

(1.11) $$\lambda = \frac{1}{3}$$

only, and find that

(1.12) $$m(\tfrac{1}{3}) = 4 .$$

We state this result as

Theorem 4. If

$$f(z) = \sum_{0}^{\infty} c_\nu z^\nu \in K(1) \tag{1.13}$$

then

$$S_n(z) = \sum_{0}^{n} c_\nu z^\nu \in K\left(\tfrac{1}{3}\right) \quad \text{for} \quad n \geq 4 \,, \tag{1.14}$$

but not necessarily for $n = 2$ or 3.

In establishing Theorems 3 and 4 I greatfully acknowledge the help of Fred W. Sauer, of the Computing Staff of the Mathematics Research Center.

2. THE NEW APPROACH TO SZEGÖ'S THEOREM 2.

Since evidently

$$f_0(z) = \sum_{0}^{\infty} z^\nu \in K(1) \tag{2.1}$$

we conclude by Szegö's Theorem 2 that

$$\sum_{0}^{n} z^\nu \in K\left(\tfrac{1}{4}\right) \quad \text{for all} \quad n = 1,2,\ldots \,. \tag{2.2}$$

Replacing z by $z/4$ we may restate (2.2) as

Corollary 1. We have

$$S_n^{(0)}(z) = \sum_{0}^{n} \frac{1}{4^\nu} z^\nu \in K(1) \quad \underline{\text{for}} \quad n = 1,2,\ldots \,, \tag{2.3}$$

The new approach to Theorem 1 is to establish Corollary 1 *directly*. If now

$$f(z) = \sum_{0}^{\infty} c_\nu z^\nu \in K(1) \tag{2.4}$$

is an arbitrary element of $K(1)$ we argue as follows: Applying Theorem 1 to (2.3) and

(2.4) we conclude that

$$\sum_{\nu=0}^{n} \frac{1}{4^\nu} c_\nu z^\nu \in K(1)$$

and therefore

$$\sum_{\nu=0}^{n} c_\nu z^\nu \in K\left(\frac{1}{4}\right) \quad \text{for} \quad n = 1,2,\ldots , \tag{2.5}$$

which is the conclusion (1.6) of Theorem 2.

<u>Szegö's theorem is therefore made to depend on the proof that all sections of the geometric series</u>

$$f(z) = \sum_{0}^{\infty} \frac{1}{4^\nu} z^\nu , \quad (|z| < 4) \tag{2.6}$$

<u>are in</u> $K(1)$.

3. <u>A DIRECT PROOF OF COROLLARY 1</u>. We begin with

<u>Lemma 1</u>. <u>All sections</u>

$$f_n(z) = \sum_{0}^{n} \frac{1}{4^\nu} z^\nu \quad (n = 1,2,\ldots) \tag{3.1}$$

<u>are univalent in the closed unit circle</u> $\bar{U} = \{|z| \leq 1\}$

<u>Proof</u>: We are to show that

$$|z| \leq 1, \quad |z_2| \leq 1, \quad z_1 \neq z_2 \quad \underline{\text{imply that}} \quad f_n(z_1 \neq f(z_2) . \tag{3.2}$$

Observe that

$$|f_n(z_1) - f_n(z_2)| = |\sum_{1}^{n} \frac{1}{4^\nu} (z_1^\nu - z_2^\nu)| \tag{3.3}$$

$$= \frac{|z_1 - z_2|}{4} \cdot \left| 1 + \frac{z_1 + z_2}{4} + \frac{z_1^2 + z_1 z_2 + z_2^2}{4^2} + \ldots + \frac{z_1^{n-1} + z_1^{n-2} z_2 + \ldots + z_2^{n-1}}{4^{n-1}} \right|$$

However, by (3.2),

$$\left|\frac{z_1 + z_2}{4} + \dots + \frac{z_1^{n-1} + \dots + z_2^{n-1}}{4^{n-1}}\right| \leq \frac{2}{4} + \frac{3}{4^2} + \dots + \frac{n}{4^{n-1}} = \frac{7}{9} - \frac{3n + 4}{9 \cdot 4^{n-1}} ,$$

the last equality being easily obtained by summation, or by complete induction. The last member being < 1. We obtain from (3.3) that

$$|f_n(z_1) - f_n(z_2)| > \frac{|z_1 - z_2|}{4} \left|1 - \left(\frac{7}{9} - \frac{3n + 4}{9 \cdot 4^{n-1}}\right)\right| = \frac{|z_1 - z_2|}{4} \left|\frac{2}{9} + \frac{3n + 4}{9 \cdot 4^{n-1}}\right| > 0 ,$$

and (3.2) is established.

Lemma 1 shows that the polynomial (3.1) maps the circle $|z| = 1$ onto a closed Jordan curve C_n. Separating real and imaginary parts by

(3.4) $$f_n(e^{it}) = x_n(t) + iy_n(t)$$

we obtain for C_n the parametric representation

(3.5) $$C_n : x = x_n(t),\ y = y_n(t), \quad (0 \leq t \leq 2\pi) .$$

Wishing to study its curvature

(3.6) $$\frac{1}{R} = \frac{x'y'' - y'x''}{(x'^2 + y'^2)^{3/2}}$$

we first establish

<u>Lemma 2</u>. <u>Defining</u>

(3.7) $$T_n(t) = x_n'(t)y_n''(t) - y_n'(t)x_n''(t) ,$$

<u>we have</u>

(3.8) $$T_n(t) \geq 0 \ \underline{\text{for}} \ -\pi \leq t \leq \pi, \ \underline{\text{and}} \ n = 1,2,\dots .$$

<u>Proof</u>: From (3.1) and (3.4) we have

(3.9) $$x_n(t) = 1 + \sum_{\alpha=1}^{n} \frac{\cos \alpha t}{4^\alpha}, \quad y_n(t) = \sum_{\alpha=1}^{n} \frac{\sin \alpha t}{4^\alpha}$$

and (3.7) becomes

$$T_n(t) = \begin{vmatrix} x'_n & x''_n \\ y'_n & x''_n \end{vmatrix} = \begin{vmatrix} -\sum_1^n 4^{-\alpha}\alpha \sin \alpha t & -\sum_1^n 4^{-\beta}\beta^2 \cos \beta t \\ \sum_1^n 4^{-\alpha}\alpha \cos \alpha t & -\sum_1^n 4^{-\beta}\beta^2 \sin \beta t \end{vmatrix} .$$

By splitting the columns we rewrite the determinant as a sum of n^2 determinants obtaining

$$T_n(t) = \sum_{\alpha=1}^{n} \sum_{\beta=1}^{n} 4^{-\alpha-\beta}\alpha\beta^2(\sin \alpha t \sin \beta t + \cos \alpha t \cos \beta t) ,$$

and finally

$$(3.10) \qquad T_n(t) = \sum_{\alpha,\beta=1}^{n} \frac{\alpha\beta^2}{4^{\alpha+\beta}} \cos(\alpha - \beta)t ,$$

which is a cosine polynomial of order $n - 1$.

Establishing the non-negativity of a cosine polynomial is in general a difficult problem. Fortunately, in our case it is easy due to the structure of the infinite matrix

$$(3.11) \qquad \left\| \frac{\alpha\beta^2}{4^{\alpha+\beta}} \right\|_{\alpha,\beta = 1,2,\ldots}$$

of the coefficients of all (3.10). For instance, we obtain $T_3(t)$ as the sum of the elements of the 3×3 matrix (3.11) provided with appropriate cosine factors:

$$T_3(t) = \begin{cases} \frac{1.1^2}{4^2} + \frac{1.2^2}{4^3}\cos t + \frac{1.3^2}{4^4}\cos 2t \\ + \frac{2.1^2}{4^3}\cos t + \frac{2.2^2}{4^4} + \frac{2.3^2}{4^5}\cos t \\ + \frac{3.1^2}{4^4}\cos 2t + \frac{3.2^2}{4^5}\cos t + \frac{3.3^2}{4^6} \end{cases} .$$

Since $T_n(t)$ is a cosine polynomial, we may restrict t to $0 \leq t \leq \pi$.

For $n = 2$ (3.8) presents no difficulty since

$$T_2(t) = \frac{3}{32}(1 + \cos t), \tag{3.13}$$

which is non-negative. I owe to Fred Sauer the positivity of $T_3(t)$, $T_4(t)$ and $T_5(t)$ who provided the following (positive) minima

$$\min_t T_3(t) = .01309$$

$$\min_t T_4(t) = .01221$$

$$\min_t T_5(t) = .01600\ . \tag{3.14}$$

I claim that <u>the last result</u> (3.14) <u>allows us to show that</u>

$$\min_t T_n(t) > 0 \quad \underline{\text{for all}} \quad n \geq 6\ , \tag{3.15}$$

but this requires some elementary Algebraic Analysis.

The sum of all elements of the infinite matrix (3.11) is

$$\sum_{\alpha;\beta=1}^{\infty} \frac{\alpha\beta^2}{4^{\alpha+\beta}} = \Big(\sum_{\alpha=1}^{\infty} \frac{\alpha}{4^\alpha}\Big)\Big(\sum_{\beta=1}^{\infty} \frac{\beta^2}{4^\beta}\Big)\ . \tag{3.16}$$

From this sum we subtract the sum of the elements of the principal $n \times n$ minor of (3.11), and define the new sequence

$$N_n = \Big(\sum_1^\infty \frac{\alpha}{4^\alpha}\Big)\Big(\sum_1^\infty \frac{\beta^2}{4^\beta}\Big) - \Big(\sum_1^n \frac{\alpha}{4^\alpha}\Big)\Big(\sum_1^n \frac{\beta^2}{4^\beta}\Big)\ . \tag{3.17}$$

By iteration of $z(d/dz)$ applied to $\sum_1^n (z/4)^\nu$ we obtain the identities

$$\sum_1^n \frac{\nu}{4^\nu} = \frac{4.4^n - 3n - 4}{9.4^n}, \quad \sum_1^n \frac{\nu^2}{4^\nu} = \frac{20.4^n - 9n^2 - 24n - 20}{27.4^n}\ , \tag{3.18}$$

and letting $n \to \infty$ we obtain

(3.19)
$$\sum_1^\infty \frac{\alpha}{4^\alpha} = \frac{4}{9}, \quad \sum_1^\infty \frac{\beta^2}{4^\beta} = \frac{20}{27} .$$

Now the sequence (3.17) may be explicitly written as

(3.20)
$$N_n = \frac{4}{9} \cdot \frac{20}{27} - \frac{4.4^n - 3n - 4}{9.4^n} \cdot \frac{20.4^n - 9n^2 - 24n - 20}{27.4^n} ,$$

which yield the numerical values

$$N_4 = .0216009195$$

and

(3.21)
$$N_5 = .0073673303 < .0074 .$$

I claim that this last inequality completes our proof of Lemma 2: Indeed, from (3.21) and the definition (3.17) of N_5 we conclude that all $T_n(t)$ are positive for all $n > 5$, for in view of the relations (3.14) and (3.21) w have for $n > 5$ and all real t

$$T_n(t) > T_5(t) - N_5 \geqq \min_t T_5(t) - N_5 > .0160 - .0074 = .0086 > 0 .$$

4. PROOF OF THEOREM 3. Our previous discussion makes it clear that it suffices to consider the geometric series

(4.1)
$$f(z) = \sum_0^\infty \lambda^\nu z^\nu = \frac{1}{1 - \lambda z} \qquad (|z| < \lambda^{-1}) ,$$

and prove

Lemma 3. For its partial sums we have

(4.2)
$$S_n(z) = \sum_0^n \lambda^\nu z^\nu \in K(1) \quad \underline{\text{for all}} \quad n \geqq m(\lambda) ,$$

where

(4.3) $m(\lambda)$ *is the least integer such that (4.2) holds.*

__Proof of Lemma 3__. Notice that

$$f(e^{it}) = \frac{1}{1 - \lambda e^{it}} = x(t) + iy(t) \tag{4.4}$$

traces out a circle C having the interval of reals $[1/(1 + \lambda),\ 1/(1 - \lambda)]$ as diameter and therefore the radius

$$R = \frac{\lambda}{1 - \lambda^2}\,. \tag{4.5}$$

Setting

$$s_n(e^{it}) = x_n(t) + iy_n(t)\ , \tag{4.6}$$

and observing that $|z| = 1$ is well within the circle of convergence of (4.1), it should be clear that the real periodic functions

$$x_n(t),\ y_n(t),\ x(t),\ y(t)$$

are regular in a neighborhood of the real t-axis. Also that $y_n(t)$ and $y_n(t)$, as well as their derivatives, converge uniformly to the corresponding derivatives of $x(t)$ and $y(t)$, respectively. It follows that the closed curve $C_n = (x_n(t),\ y_n(t))$ converges to the circle C and that its curvature

$$\frac{x_n'(t)y_n''(t) - y_n'(t)x_n''(t)}{(x_n'(t)^2 + y_n'(t)^2)^{3/2}}$$

converges uniformly in t, as $n \to \infty$, to the curvature $1/R$ of C. This establishes Lamma 3.

The determination of $m(\lambda)$, satisfying (4.3), is difficult and will be solved for $\lambda = 1/3$ only.

5. __THE CASE__ $\lambda = \frac{1}{3}$: __PROOF OF THEOREM 4__. As an analogue of Lemma 1 we should prove that the image of $|z| = 1$ by

$$f_n(z) = \sum_{0}^{n} \frac{1}{3^\nu} z^\nu \qquad (n = 1,2,\ldots) \tag{5.1}$$

are all simple closed curves. However, our simple proof of Lemma 1 does not generalize. Rather we consider

$$w = f_n(z) - 1 = \sum_1^n \frac{1}{3^\nu} z^\nu = \frac{z}{3}\,\frac{1-(z/3)^n}{1-(z/3)} \tag{5.2}$$

and show that it maps $z = e^{it}$ into a curve which is star-shaped with respect to the origin. This requires two facts: 1°. That as t varies from $-\pi$ to $+\pi$, the argument of the function (5.2) increases by 2π. 2°. That we have

$$\operatorname{Im}\left(\frac{\partial w}{dt} \,/\, w\right) > 0 \quad \text{for all } t\ . \tag{5.3}$$

However, we omit the tedious calculations.

We rather pass to considering the curvature of the curve; here matters are very close to those of §3 and obtained from them by replacing $\frac{1}{4}$ by $\frac{1}{3}$. We shall also use the same notations.

Setting

$$f_n(e^{it}) = x_n(t) + iy_n(t)\ , \tag{5.4}$$

we have as an analogue of Lemma 2 the

<u>Lemma 3</u>. <u>Defining</u>

$$T_n(t) = x_n'(t)y_n''(t) - y_n'(t)x_n''(t)\ , \tag{5.5}$$

<u>we have</u>

$$T_n(t) > 0 \ \ \underline{\text{for}}\ \ -\pi \leqq t \leqq \pi \ \ \underline{\text{and}}\ \ n \geqq 4, \tag{5.6}$$

<u>but not for</u> $n = 2$ <u>and</u> $n = 3$.

<u>Proof</u>: For the analogues of (3.10) and (3.17) we find

$$T_n(t) = \sum_{\alpha,\beta=1}^{n} \frac{\alpha\beta^2}{3^{\alpha+\beta}} \cos(\alpha-\beta)t \tag{5.7}$$

and

$$N_n = \left(\sum_1^\infty \frac{\alpha}{3^\alpha}\right)\left(\sum_1^\infty \frac{\beta^2}{3^\beta}\right) - \left(\sum_1^n \frac{\alpha}{3^\alpha}\right)\left(\sum_1^n \frac{\beta^2}{3^\beta}\right) \tag{5.8}$$

and explicitly

$$N_n = \frac{9}{8} - \left(\frac{3^{n+1} - 2n - 3}{3^n \cdot 4}\right)\left(\frac{3^{n+1} - n^2 - 3n - 3}{3^n \cdot 2}\right) . \tag{5.9}$$

Rounding off Fred Sauer's values to six decimal places, we have

$$\min_t T_2(t) = -.012346$$

$$\min_t T_3(t) = -.002057$$

$$\min_t T_4(t) = .004268$$

$$\min_t T_5(t) = .013733$$

$$\min_t T_6(t) = .014480, \qquad N_6 = .036836$$

$$\min_t T_7(t) = .017663, \qquad N_7 = .015400 . \tag{5.10}$$

The first two minima being negative shows that the curves $C_2 = (x_2(t),y_2(t))$ and $C_3 = (x_3(t),y_3(t))$ are not convex. However, <u>from (5.10) we can conclude that (5.6) holds</u>: From the above data we see that $T_4(z)$, $T_5(t)$, $T_6(t)$, and $T_7(t)$ are everywhere positive. Now (5.10) show that if $n > 7$ then

$$T_n(t) \geq T_7(t) - N_7 \geq \min_t T_7(t) - N_7 > .0176 - .0155 = .0021 > 0 .$$

Thus $T_n(t) \geq 0$ for all t and $n = 4,5,\ldots$, proving Theorem 4.

REFERENCES

1. G. Polya and I. J. Schoenberg, Remarks on de la Vallée Poussin means and convex conformal maps of the circle, Pacific J. of Mathematics 8 (1958), 295-334.
2. St. Ruscheweyh and T. Sheil-Small, Hadamard products of schlicht functions and the Polya-Schoenberg conjecture, Comment. Math. Helvet. 48 (1973), 119-135.
3. Gabor Szegö, Zur Theorie der schlichten Abbildungen, Math. Annalen, 100 (1928), 188-211.

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SECURITY CLASSIFICATION OF THIS PAGE (When Data Entered)

REPORT DOCUMENTATION PAGE		READ INSTRUCTIONS BEFORE COMPLETING FORM
1. REPORT NUMBER 2647	2. GOVT ACCESSION NO. AD-A139256	3. RECIPIENT'S CATALOG NUMBER
4. TITLE (and Subtitle) ON A THEOREM OF SZEGÖ ON UNIVALENT CONVEX MAPS OF THE UNIT CIRCLE		5. TYPE OF REPORT & PERIOD COVERED Summary Report - no specific reporting period
		6. PERFORMING ORG. REPORT NUMBER
7. AUTHOR(s) I. J. Schoenberg		8. CONTRACT OR GRANT NUMBER(s) DAAG29-80-C-0041
9. PERFORMING ORGANIZATION NAME AND ADDRESS Mathematics Research Center, University of Wisconsin, 610 Walnut Street, Madison, Wisconsin 53706		10. PROGRAM ELEMENT, PROJECT, TASK AREA & WORK UNIT NUMBERS Work Unit Number 6 - Miscellaneous Topics
11. CONTROLLING OFFICE NAME AND ADDRESS U. S. Army Research Office, P.O. Box 12211, Research Triangle Park, North Carolina 27709		12. REPORT DATE February 1984
		13. NUMBER OF PAGES 13
14. MONITORING AGENCY NAME & ADDRESS(if different from Controlling Office)		15. SECURITY CLASS. (of this report) UNCLASSIFIED
		15a. DECLASSIFICATION/DOWNGRADING SCHEDULE

16. DISTRIBUTION STATEMENT (of this Report)

Approved for public release; distribution unlimited.

17. DISTRIBUTION STATEMENT (of the abstract entered in Block 20, if different from Report)

18. SUPPLEMENTARY NOTES

19. KEY WORDS (Continue on reverse side if necessary and identify by block number)

Convexity, Geometric Function Theory

20. ABSTRACT (Continue on reverse side if necessary and identify by block number)

For a positive constant λ we denote by $K(\lambda) = \{f(z)\}$ the class of function $f(z)$ which are regular and univalent in $|z| < \lambda$ and map this circle on a convex domain. In 1928 G. Szegö [3] proved the

(cont.)

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ABSTRACT (cont.)

Theorem 1. If $f(z) = \sum_0^\infty c_\nu z^\nu \in K(1)$, *then all its sections*

$$s_n(z) = \sum_0^n c_\nu z^\nu \in K\left(\frac{1}{4}\right) \quad \textit{for} \quad n = 1,2,\ldots .$$

An evident consequence is

Corollary 1. Since the geometric series

$$f_0(z) = \sum_0^\infty z^\nu \in K(1)$$

we have

$$\sum_0^n z^\nu \in K\left(\frac{1}{4}\right), \quad (n = 1,2,\ldots)$$

and therefore, on replacing z *by* $z/4$, *we have*

$$\sum_0^n \frac{1}{4^\nu} z^\nu \in K(1) \quad \textit{for} \quad n = 1,2,\ldots .$$

In the present paper we prove directly Corollary 1, and derive from it Szegö's Theorem 1. This is done by appealing to Theorem 2 which was conjectured by Polya and Schoenberg [1] in 1958, but only proved in 1973 by St. Ruscheweyh and T. Sheil-Small [2].

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