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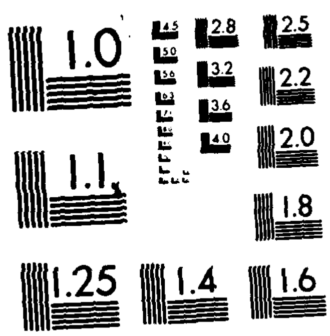

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LOCAL ASYMPTOTIC APPROXIMATION OF NONLINEAR CONTROL SYSTEMS

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ABSTRACT

This paper describes a canonical procedure to approximate an arbitrary family of C^∞ vector fields $\{g_1,\ldots,g_m\}$ on R^d with vector fields $\bar{g}_1,\ldots,\bar{g}_m$ on the same space R^d which generate a nilpotent Lie algebra. Each $\bar{g}_i$ can be either obtained from a Taylor expansion of the input-output map for the control system

$$(*) \qquad \dot{x} = \sum_{i=1}^{m} g_i(x)u_i, \qquad x(0) = 0 ,$$

or computed directly as an asymptotic limit of the corresponding vector field g_i. A useful consequence is that every control system of the form (*) can locally be regarded as an arbitrarily small C^∞ perturbation of a nilpotent system on the same state space, up to a suitable linear rescaling of coordinates.

AMS (MOS) Subject Classifications: 93B10, 93C10

Key Words: Dilation, nilpotent family of vector fields

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SIGNIFICANCE AND EXPLANATION

If a family of vector fields $\{\bar{g}_1,\ldots,\bar{g}_m\}$ generates a nilpotent Lie algebra, then the response of the control system

$$\dot{x}(t) = \sum_{i=1}^{m} \bar{g}_i(x)u_i(t), \qquad x(0) = 0 \in \mathbb{R}^d$$

can be written out explicitly in terms of integrals of the controls u_i. This and other nice consequences make nilpotency a highly desirable property from the point of view of mathematical analysis.

This paper

~~The present paper~~ describes a canonical method to locally approximate any

sub 1 sub m

family $\{g_1,\ldots,g_m\}$ of vector fields by one which generates a nilpotent Lie algebra. This is particularly useful in control theory, because it shows that an arbitrary control system can locally be obtained from a small perturbation of a nilpotent system, by a suitable rescaling of the coordinates.

The responsibility for the wording and views expressed in this descriptive summary lies with MRC, and not with the author of this report.

LOCAL ASYMPTOTIC APPROXIMATION OF NONLINEAR CONTROL SYSTEMS

Alberto Bressan*

1. INTRODUCTION

This paper is concerned with an autonomous nonlinear control system of the form

$$\dot{x}(t) = \sum_{i=1}^{m} g_i(x(t))u_i(t), \qquad x(0) = 0 \in \mathbb{R}^d, \tag{1.1}$$

where the g_i are C^∞, globally bounded vector fields on $\mathbb{R}^d$ and $u(\cdot) = (u_1(\cdot),\ldots,u_m) \in L^1([0,\infty);\mathbb{R}^m)$. The system (1.1) generates a smooth input-output map $\phi : u(\cdot) \to x(u,\cdot)$ from $L^1([0,\infty);\mathbb{R}^m)$ into $C^0([0,\infty);\mathbb{R}^d)$. Explicitly computable approximations of ϕ are of primary importance in the local study of (1.1). The p-th order Taylor expansion $T^p\phi$ of ϕ about the null control was studied in [1,3]. In [8] it is shown how the trajectories of (1.1) can be locally approximated by means of an additional (nilpotent) system, say

$$\dot{x}(t) = \sum_{i=1}^{m} f_i(x(t))u_i(t), \qquad x(0) = 0 \in \mathbb{R}^{d'}. \tag{1.2}$$

For certain applications, such as the computation of a local time-optimal feedback [2], both approaches seem unsatisfactory. In general, $T^p\phi$ is merely a sum of multilinear *integral mappings* from L^1 into C^0 and does not arise as the exact input-output map of any control system. The optimality of a given control $\bar{u}$ under $T^p\phi$ cannot therefore be tested by the Maximum Principle. On the other hand, the vector fields f_i in (1.2) represent a lifting of the g_i $(i = 1,\ldots,m)$ in a usually higher dimensional space $\mathbb{R}^{d'}$. It is not always possible to determine a local property of (1.1) by studying (1.2), since the two systems live on different spaces.

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Using a singular perturbation technique, in this paper we derive a canonical procedure to approximate the vector fields g_i with a nilpotent family $\{\bar{g}_1,\dots,\bar{g}_m\}$ of vector fields defined on the same state space $\mathbb{R}^d$. The main construction is as follows. For small $\varepsilon > 0$, the restriction of the map ϕ to controls with support inside $[0,\varepsilon]$ can be rescaled to a map ϕ^ε from $L^1([0,1];\mathbb{R}^m)$ into $C^0([0,1];\mathbb{R}^d)$. This is achieved by means of the transformation $t \to \varepsilon^{-1}t$ of the time variable and by letting a suitable family of dilations δ_ε act on the space variables. For each $\varepsilon > 0$, ϕ^ε is thus the input-output map generated by the control system

$$\dot{x}(t) = \sum_{i=1}^{m} g_i^\varepsilon(x(t))u_i(t), \qquad x(0) = 0 \in \mathbb{R}^d , \tag{1.3}$$

where the vector fields g_i^ε are obtained from the g_i after a rescaling of coordinates. As $\varepsilon \to 0$, there exist a map $\bar{\phi}$ and vector fields $\bar{g}_i$ on $\mathbb{R}^d$ such that $\phi^\varepsilon \to \bar{\phi}$ and $g_i^\varepsilon \to \bar{g}_i$ together with all derivatives, uniformly on bounded sets. $\bar{\phi}$ is then the input-output map corresponding to the control system

$$\dot{x}(t) = \sum_{i=1}^{m} \bar{g}_i(x(t))u_i(t), \qquad x(0) = 0 . \tag{1.4}$$

The vector fields $\bar{g}_i$ have polynomial coefficients and are invariant under a 1-parameter group of transformations. Moreover, without any assumption on the Lie algebra generated by $g_1,\dots,g_m$, it turns out that Lie $\{\bar{g}_1,\dots,\bar{g}_m\}$ is always nilpotent, so that the solutions of (1.4) can be written in closed form [7]. This remarkable feature makes the system (1.4) an attractive object for a detailed mathematical analysis. Indeed, since (1.4) is obtained as a uniform limit of (1.3) as $\varepsilon \to 0$, any property of (1.4) which is retained under small C^∞ perturbations [4] yields a local property of (1.1).

2. PRELIMINARIES

In the following, we use $|\cdot|$ for the euclidean norm on $\mathbb{R}^d$ and $\|\cdot\|$ for the norm in Banach spaces. The closed ball centered at x with radius r is denoted $B(x,r)$, while $\otimes$ indicates tensor product. Given two Banach spaces E and F, $k > 0$, we denote by $L^k(E;F)$ the space of continuous k-linear mappings Λ from $\otimes_k E = E \otimes E \otimes \dots \otimes E$ (k times) into F with the operator norm

$$\|\Lambda\| = \sup\{\|\Lambda(v_1,\dots,v_k)\|_F;\ \|v_i\|_E \leq 1,\ i = 1,\dots,k\} .$$

If $\psi : E \to F$ is a smooth mapping, its k-th Fréchet derivative a a point $u \in E$ is $D^k\psi(u) \in L^k(E;F)$. The k-th order Taylor expansion of ψ at the origin is then

$$T_\psi^k(u) = \sum_{j=0}^{k} \frac{1}{j!} D^j\psi(0) \cdot u^{[j]} ,$$

where $u^{[j]} = (u,u,\dots,u) \in \otimes_j E$. The same notation $T^k g$ is used for the k-th order Taylor expansion of a vector field g on $\mathbb{R}^d$ at the origin. It was shown in [1] that the n-th order Taylor expansion $T^n\phi$ of the input-output map ϕ generated by (1.1) about the null control can be obtained by computing the n-th Picard iterate for the approximate system

$$\dot{x}(t) = \sum_{i=1}^{m} T^{n-1}g_i(x(t))u_i(t), \qquad x(0) = 0 , \tag{2.1}$$

discarding the terms of order $>n$. The multilinear integral mappings $\mu : L^1(0,\infty) \to C^0(0,\infty)$ arising from the above procedure were called in [1] <u>integral monomials</u>. If μ is k-linear, we say that μ has order k. The first Picard iterate for (2.1) is

$$P_1(u,t) = \sum_{i=1}^{m} \int_0^t g_i(0)u_i(s)ds .$$

In general, if the k-th Picard iterate for (2.1) is a sum of integral monomials, say

$$P_k(u,t) = \sum_{i=1}^{N(k)} \mu_i(u,t) ,$$

then P_{k+1} can again be written as a sum of terms of the type

$$\mu(u,t) = \int_0^t \frac{1}{j!} D^j g_i(0) \cdot (\mu_{i_1}(u,s),\ldots,\mu_{i_j}(u,s))u_i(s)ds \tag{2.2}$$

where $0 \le j \le n$, $1 \le i \le m$. If μ_{i_ℓ} has order ν_ℓ $(\ell = 1,\ldots,j)$, then μ in (2.2) has order $\nu_1 + \ldots + \nu_j + 1$. The above construction canonically determines an increasing sequence of subspaces $Z_p \subseteq \mathbb{R}^d$, namely $Z_0 = \{0\}$, $Z_1 = \text{span}\{g_i(0);\ i = 1,\ldots,m\}$ and inductively

$$Z_p = \text{span}\{D^j g_i(0) \cdot (v_1,\ldots,v_j);\ 1 \le i \le m,\ 0 \le j < p,$$
$$v_j \in Z_{p_j},\ p_1 + \ldots + p_j < p\} . \tag{2.3}$$

Comparing (2.3) with (2.2), it is clear that

$$T^p\phi(u)(t) \in Z_p \tag{2.4}$$

for all $p \ge 0$, $u \in L^1$, $t \ge 0$. Indeed, Z_p is precisely the subspace of $\mathbb{R}^d$ spanned by the coefficients of the integral monomials of order $\le p$ in the Taylor expansion of ϕ about the null control.

3. A CLASS OF NILPOTENT LIE ALGEBRAS

All of the vector fields arising from the asymptotic limit procedure considered in this paper generate a special type of Lie algebras which we now briefly describe. For a different approach, based on the dual action of vector fields as derivations on C^∞ functions, see [6]. Given an orthogonal decomposition $\mathbb{R}^d = W_1 \oplus \dots \oplus W_{\bar{p}}$, let π_p be the canonical projection of $\mathbb{R}^d$ and $\frac{\partial}{\partial x_p}$ the differentiation w.r.t. the p-th component of x. Define L (L_r, for $r \geq 0$) to be the set of all smooth vector fields $f = (f_1,\dots,f_{\bar{p}})$ on $\mathbb{R}^d$ such that

$$\frac{\partial^k f_q}{\partial x_{p_1} \dots \partial x_{p_k}} \equiv 0$$

whenever $k > 0$, $1 \leq q \leq \bar{p}$ and $p_1 + \dots + p_k \geq q$ ($p_1 + \dots + p_k \geq q - r$). For $r \geq \bar{p}$, L_r contains only the null vector field. From the above definitions it follows

LEMMA 1. L is a finite dimensional nilpotent Lie algebra of vector fields with polynomial components, with the usual bracket operation $[f,g] = (Dg) \cdot f - (Df) \cdot g$, and $L = L_0 \supseteq L_1 \supseteq \dots \supseteq L_{\bar{p}} = \{0\}$ is a decreasing sequence of ideals, indeed

$$[L_r,L_s] \subseteq L_{r+s+1} \tag{3.1}$$

PROOF. If $f \in L$, any $\bar{p}$-th derivative of f vanishes identically. Hence L contains only vector fields whose components are polynomials of degree less than $\bar{p}$. In particular, L is finite dimensional. To prove (3.1), let $f \in L_r$, $g \in L_s$, $k > 0$, $q,p_1,\dots,p_k \in \{1,\dots,\bar{p}\}$ with $p_1 + \dots + p_k \geq q-r-s-1$. Denoting $f_q = \pi_q f$, $g_q = \pi_q g$ the q-th components of f and g, one has

$$\frac{\partial^k}{\partial x_{p_1} \dots \partial x_{p_k}} \pi_q[(Df) \cdot g] = \frac{\partial^k}{\partial x_{p_1} \dots \partial x_{p_k}} \Big[\sum_{i=1}^{\bar{p}} \big(\frac{\partial}{\partial x_i} f_q\big) \cdot g_i \Big] . \tag{3.2}$$

Notice that the right-hand side of (3.2) can be written as a sum of terms having the form

$$\Big[\frac{\partial^{h+1}}{\partial x_{p_{\sigma(1)}} \dots \partial x_{p_{\sigma(h)}} \partial x_i} f_q\Big] \bullet \Big[\frac{\partial^{k-h}}{\partial x_{p_{\sigma(h+1)}} \dots \partial x_{p_{\sigma(k)}}} g_i\Big] \tag{3.3}$$

where $0 \leq h \leq k$, $1 \leq i \leq \bar{p}$ and σ is a permutation of the set $\{1,\ldots,k\}$. If $p_{\sigma(1)} + \cdots + p_{\sigma(h)} + i \geq q - r$, the first term in the tensor product (3.3) vanishes identically. Otherwise $p_{\sigma(1)} + \cdots + p_{\sigma(h)} \leq q-r-i-1$, hence

$$p_{\sigma(h+1)} + \cdots + p_{\sigma(k)} \geq (q-r-s-1) - (q-r-i-1) = i - s$$

and the second factor in (3.3) vanishes. Therefore $(Df) \circ g \in L_{r+s+1}$. Similarly $(Dg) \circ f \in L_{r+s+1}$, hence (3.1) holds, proving the nilpotency of L.

4. STATEMENT OF THE MAIN RESULT

With the increasing sequence of subspaces $Z_p \subseteq \mathbb{R}^d$ defined at (2.3) one can associate an orthogonal decomposition

$$\mathbb{R}^d = W_1 \oplus \dots \oplus W_{\bar{p}} \tag{4.1}$$

as follows. Fix any $\bar{p} > 1$. For $1 < p < \bar{p}$ let W_p be the orthogonal complement of Z_{p-1} in Z_p, and let $W_{\bar{p}}$ be the orthogonal complement of $Z_{\bar{p}-1}$ in $\mathbb{R}^d$. This clearly yields (4.1). The canonical projection of $\mathbb{R}^d$ onto W_p is denoted π_p. In addition to the input-output map $\phi : u(\cdot) \to x(u,\cdot)$ generated by (1.1), for $0 < \varepsilon \leq 1$ we can now define the rescaled maps $\phi^\varepsilon : u(\cdot) \to x^\varepsilon(u,\cdot)$ by setting

$$x^\varepsilon(u,t) = \sum_{p=1}^{\bar{p}} \varepsilon^{-p}\pi_p(x(\varepsilon u,t)) \ . \tag{4.2}$$

By direct computation one checks that

$$\dot{x}^\varepsilon(u,t) = \sum_{i=1}^{m} g_i^\varepsilon(x^\varepsilon(u,t))u_i(t), \qquad x^\varepsilon(u,0) = 0 \tag{4.3}$$

with

$$g_i^\varepsilon(x) = \sum_{p=1}^{\bar{p}} \varepsilon^{1-p}\pi_p\Big[g_i\Big(\sum_{j=1}^{\bar{p}} \varepsilon^j\pi_j(x)\Big)\Big] \ . \tag{4.4}$$

Our major interest is in the behavior of the rescaled system (4.3) as $\varepsilon \to 0$.

THEOREM. Let g_i $(i = 1,\dots,m)$ be C^∞, globally bounded vector fields on $\mathbb{R}^d$, and let $\phi^\varepsilon, x^\varepsilon, g_i^\varepsilon$ be defined by (4.2), (4.4), corresponding to the decomposition (4.1) obtained from (2.3). Then as $\varepsilon \to 0$

i) ϕ^ε converges to $\bar{\phi} : u(\cdot) \to \bar{x}(u,\cdot)$, defined by

$$\bar{\phi}(u)(t) = \sum_{p=1}^{\bar{p}} \pi_p[T^p\phi(u)(t)] \ . \tag{4.5}$$

More precisely, for all $k > 0$, $D^k\phi^\varepsilon$ tends to $D^k\bar{\phi}$ uniformly on bounded subsets of $L^1([0,\infty);\mathbb{R}^m)$.

ii) For all $i = 1,\ldots,m$, g_i^ε converges to $\bar{g}_i$, defined by

$$\bar{g}_i(x) = \sum_{p=1}^{\bar{p}} \pi_p\Big[\sum_{j=0}^{p-1} \sum_{\sigma\in\Gamma(p,j)} \frac{1}{j!} D^j g_i(0)(\pi_{\sigma(1)}(x),\ldots,\pi_{\sigma(j)}(x))\Big] , \tag{4.6}$$

$\Gamma(p,j)$ being the set of all maps $\sigma : \{1,\ldots,j\} \to \{1,\ldots,p\}$ for which $\sigma(1) + \ldots + \sigma(j) = p - 1$. For all $k > 0$, $D^k g_i^\varepsilon$ tends to $D^k \bar{g}_i$ uniformly on bounded subsets of $\mathbb{R}^d$.

iii) For all $u(\cdot) \in L^1$, the trajectory $t \to \bar{x}(u,t) = \bar{\phi}(u)(t)$ is the solution of

$$\dot{x}(t) = \sum_{i=1}^{m} \bar{g}_i(x(t))u_i(t), \qquad x(0) = 0 . \tag{4.7}$$

iv) Lie $\{\bar{g}_1,\ldots,\bar{g}_m\}$ is nilpotent.

5. PROOF OF THE THEOREM

Fix any $p \in \{1,\ldots,\bar{p}\}$ and write

$$\pi_p(\phi^\varepsilon(u)(t)) = \varepsilon^{-p}\pi_p(\phi(\varepsilon u)(t))$$
$$= \varepsilon^{-p}\pi_p(T^p\phi(\varepsilon u)(t)) + \varepsilon^{-p}\pi_p(\eta(\varepsilon u)(t))$$

where $\eta = \phi - T^p\phi$, the remainder of the p-th order Taylor expansion of ϕ about the null control, is a C^∞ map from $L^1([0,\infty);R^m)$ into $C([0,\infty);R^d)$ with $D^j\eta(0) = 0$ for $0 < j < p$. By (2.4), $\pi_p T^{p-1}\phi \equiv 0$. Therefore, $\pi_p T^p\phi$ is a homogeneous p-linear mapping, and

$$\varepsilon^{-p}\pi_p T^p\phi(\varepsilon u) = \pi_p T^p\phi(u) \qquad \forall\, \varepsilon > 0 \,. \tag{5.1}$$

To prove i) it now suffices to show that, for all $k > 0$, $\varepsilon^{-p}D^k(\pi_p \circ \eta^\varepsilon)(u)$ converges to zero uniformly as u ranges on bounded subsets of L^1, η^ε being the map $u \to \eta(\varepsilon u)$. The assumptions on η imply the existence of a constant $C > 0$ such that

$$\|D^k\eta(u)\| < C\min\{\|u\|^{p-k+1},1\}$$

whenever $\|u\| < C^{-1}$. If U is a bounded subset of L^1, choose $\varepsilon_0 > 0$ so small that $\varepsilon_0 U \subset B(0,C^{-1})$. If $u \in U$ and $0 < \varepsilon < \varepsilon_0$, then

$$\|\varepsilon^{-p}D^k(\pi_p \circ \eta^\varepsilon)(u)\| < \varepsilon^{k-p}\|D^k\eta(\varepsilon u)\| < C\varepsilon^{k-p}\cdot\min\{\varepsilon^{p-k+1}\|u\|,1\} \,. \tag{5.2}$$

In both cases $k > p$ or $k < p$, as $\varepsilon \to 0$ the right-hand side of (5.2) converges to zero uniformly on U. This establishes i). The proof of ii) is similar: fix $i \in \{1,\ldots,m\}$ and $p \in \{1,\ldots,\bar{p}\}$. Using a Taylor expansion of g_i of order $\bar{p} - 1$ with remainder $h(\cdot)$, from (4.6) one obtains

$$\pi_p g_i^\varepsilon(x) = \pi_p\Big[\varepsilon^{1-p}\sum_{j=0}^{\bar{p}-1}\frac{1}{j!}\sum_{\sigma\in\Gamma_j} D^j g_i(0)(\varepsilon^{\sigma(1)}\pi_{\sigma(1)}(x),\ldots,\varepsilon^{\sigma(j)}\pi_{\sigma(j)}(x))\Big]$$
$$+ \pi_p\varepsilon^{1-p}\, h\Big(\sum_{\ell=1}^{\bar{p}} \varepsilon^\ell\pi_\ell(x)\Big) \tag{5.3}$$

where Γ_j is the set of all maps σ from $\{1,\ldots,j\}$ into $\{1,\ldots,\bar{p}\}$ and h is a C^∞ vector field on R^d with $D^jh(0) = 0$ for $j = 1,\ldots,\bar{p} - 1$. The generic term in the first summation in (5.3) is a homogeneous j-linear map from R^d into W_p of the form

$$\pi_p\Psi(\varepsilon,x) = \pi_p\left[\frac{1}{j!}\,\varepsilon^{1-p}D^j g_i(0)(\varepsilon^{\sigma(1)}\pi_{\sigma(1)}(x),\ldots,\varepsilon^{\sigma(j)}\pi_{\sigma(j)}(x))\right]$$

$$= \pi_p\left[\frac{1}{j!}\,\varepsilon^{1-p-S_\sigma}D^j g_i(0)(\pi_{\sigma(1)}(x),\ldots,\pi_{\sigma(j)}(x))\right]$$

with $S_\sigma = \sigma(1) + \ldots + \sigma(j)$. If $S_\sigma > p$, then $\lim_{\varepsilon\to 0}\Psi(\varepsilon,x) = \lim_{\varepsilon\to 0}\varepsilon^{1-p+S_\sigma}\Psi(1,x) = 0$ and the same holds for all derivatives of Ψ w.r.t. x, uniformly on bounded subsets of $\mathbb{R}^d$. If $S_\sigma = p-1$, then $\Psi(\varepsilon,x) = \Psi(1,x)$ does not depend on ε. If $S_\sigma < p-1$, since $\pi_{\sigma(\ell)}(x) \in Z_{\sigma(\ell)}$ $(\ell = 1,\ldots,j)$, the definition of Z_{p-1} implies $\Psi(\varepsilon,x) \in Z_{p-1}$, hence $\pi_p \circ \Psi(\varepsilon,x) = 0$. To prove ii) it now suffices to show that the second summand on the right-hand side of (5.3) converges to zero as $\varepsilon \to 0$. For every $k > 0$, the assumptions on $h(\cdot)$ imply the existence of a constant $C > 0$ for which

$$|D^k h(x)| < C\cdot\min\{|x|^{\bar p-k},1\} \tag{5.4}$$

whenever $|x| < C^1$. Let V be a bounded subset of $\mathbb{R}^d$, and choose $\varepsilon_0 \in (0,1]$ so small that $\varepsilon_0 V \subseteq B(0,C^{-1})$. For $0 < \varepsilon < \varepsilon_0$, set $h^\varepsilon(x) = h\left(\sum_{p=1}^{\bar p}\varepsilon^p\pi_p(x)\right)$. From (5.4) it follows the estimate

$$\begin{aligned}
\varepsilon^{1-\bar p}|D^k h^\varepsilon(x)| &= \varepsilon^{1-\bar p}\sup\{|D^k h^\varepsilon(x)\cdot y^{[k]}|\cdot|y|^{-k},\ y \in \mathbb{R}^d\backslash\{0\}\} \\
&< \varepsilon^{1-\bar p}\sup\Big\{\Big|D^k h\Big(\sum_{p=1}^{\bar p}\varepsilon^p\pi_p(x)\Big)\cdot\Big(\sum_{p=1}^{\bar p}\varepsilon^p\pi_p(y)\Big)^{[k]}\Big|\cdot|y|^{-k},\ y \neq 0\Big\} \\
&< \varepsilon^{1-\bar p}\sup\{C\cdot\min\{|\varepsilon x|^{\bar p-k},1\}\cdot|\varepsilon y|^k\cdot|y|^{-k},\ y \neq 0\} \\
&< C\varepsilon^{1-\bar p+k}\min\{\varepsilon^{\bar p-k}|x|^{\bar p-k},1\}\ .
\end{aligned} \tag{5.5}$$

In both cases $k > \bar p$ or $k < \bar p$, the right-hand side of (5.5) converges to zero uniformly on V, as $\varepsilon \to 0$. Since the above holds for every i and p, ii) is proved. Using i) and ii), iii) follows from (4.3), letting $\varepsilon \to 0$ in the equality

$$x^\varepsilon(u,t) = \int_0^t \sum_{i=1}^m g_i^\varepsilon(x^\varepsilon(u,s))u(s)ds\ .$$

Finally, we check that each $\bar g_i$ belongs to the Lie algebra L defined in §3, corresponding to the decomposition (4.1). Let $k > 0$, $q,p_1,\ldots,p_k \in \{1,\ldots,\bar p\}$, $p_1 + \ldots + p_k > q$. By (4.6), the q-th component of $\bar g_i$ is a sum of terms of the form

$$\phi(x) = \frac{1}{j!}\,\pi_q[D^j g_i(0)(\pi_{q_1}(x),\ldots,\pi_{q_j}(x))]$$

with $q_1 + \ldots + q_j = q - 1$. We therefore have

$$\frac{\partial^k \phi}{\partial x_{p_1} \ldots \partial x_{p_k}}(x) = 0$$

for all $x \in \mathbb{R}^d$ unless there exists an injection σ of the set $\{1,\ldots,k\}$ into the set $\{1,\ldots,j\}$ such that $q_{\sigma(\ell)} = p_\ell$ for all $\ell = 1,\ldots,k$. But the existence of such an injection would contradict the assumption $p_1 + \ldots + p_k > q$. Therefore $\bar{g}_i \in L$ for all $i \in \{1,\ldots,m\}$. By Lemma 1, Lie $\{\bar{g}_1,\ldots,\bar{g}_m\}$ is a *subalgebra of a* nilpotent Lie algebra. This completes the proof.

6. CONCLUDING REMARKS

The above results also apply to control systems

$$\dot{x}(t) = \sum_{i=1}^{m} X_i(x(t))u_i(t), \qquad x(0) = \bar{\xi} \tag{6.1}$$

on a d-dimensional manifold M. Indeed, if the Lie algebra generated by $X_1,\ldots,X_m$ has full rank at $\bar{\xi}$, then among .. iterated brackets of the X_i's one can choose d vector fields, say $Y_1,\ldots,Y_d$, which are linearly independent at $\bar{\xi}$. The map

$$\theta : (s_1,\ldots,s_d) \to (\exp s_1Y_1) \circ \ldots \circ (\exp s_dY_d)(\bar{\xi}) \tag{6.2}$$

thus defines a local chart of a neighborhood of $\bar{\xi}$ in M [8]. The asymptotic expansion considered in §4 can now be performed for the system on $\mathbb{R}^d$ corresponding to (6.1) in this canonical chart.

The present approximation technique enables one to study a local problem concerning (1.1) by first solving the corresponding problem for the system (4.7) and by then proving that the structure of the solutions is "stable" under suitably small C^m perturbations of the vector fields $\bar{g}_i$. This approach is adopted in [2] to study the local time-optimal stabilizing feedback for a generic three-dimensional nonlinear system with scalar control.

The analysis of the control system (4.7) takes advantage from the explicit representation (4.5) of trajectories in terms of integrals of the controls. One can also make use of the special properties of the vector fields $\bar{g}_i$ in connection with the family of dilations $\delta_\xi : \mathbb{R}^d \to \mathbb{R}^d$,

$$\delta_\xi(x) = \sum_{p=1}^{\bar{p}} \xi^p \pi_p(x)$$

(see [6]). Some useful consequences deserve mention. Fix a compact convex set $\Omega \subseteq \mathbb{R}^m$ and define the admissible set of controls

$$U = \{u(\cdot) \in L^1([0,\infty);\mathbb{R}^m);\ u(t) \in \Omega \quad \forall\ t > 0\} .$$

Given a control $u \in U$, for $\xi > 0$ define $u_\xi(t) = u(\xi t)$. Call $R(\tau)$ the set reachable at time τ by trajectories $t \to \bar{x}(u,t)$ of (4.7) with controls in U. With these conventions we have

LEMMA 2. Let $\xi > 0$, $t > 0$, $i \in \{1,\ldots,m\}$, $u \in U$. Then

$$\bar{g}_i(\delta_\xi(x)) = \sum_{p=1}^{\bar{p}} \xi^{p-1}\pi_p(\bar{g}_i(x)) , \tag{7.1}$$

$$\xi\bar{g}_i(\delta_\xi(x)) = \delta_\xi(\bar{g}_i(x)) , \tag{7.2}$$

$$\bar{x}(\xi u_\xi,t) = \bar{x}(u,\xi t) = \delta_\xi(\bar{x}(u_\xi,t)) , \tag{7.3}$$

$$R(\xi) = \delta_\xi R(1) . \tag{7.4}$$

PROOF. By (4.6), $\pi_p\bar{g}_i(\delta_\xi(x))$ is a sum of terms having the form

$$\Psi(\delta_\xi(x)) = \pi_p\Big[\frac{1}{j!}\, D^j g_i(0) \cdot \big(\pi_{\sigma(1)}(\delta_\xi(x)),\ldots,\pi_{\sigma(j)}(\delta_\xi(x))\big)\Big]$$

with $\sigma(1) + \ldots + \sigma(j) = p - 1$. Therefore $\Psi(\delta_\xi(x)) = \xi^{p-1}\Psi(x)$. Thus yields (7.1). Multiplying both sides of (7.1) by ξ, one gets (7.2). The first equality in (7.3) holds simply because (4.7) is an autonomous system, linear in u. Moreover

$$\bar{x}(u,\xi t) - \delta_\xi(\bar{x}(u_\xi,t)) = \sum_{i=1}^{m} \int_0^t [\xi\bar{g}_i(\bar{x}(u,\xi t)) - \delta_\xi(\bar{g}_i(\bar{x}(u_\xi,t)))]dt .$$

Gronwall's lemma and (7.2) now yield the second equality in (7.3), from which (7.4) follows by setting $t = 1$.

Notice that a control $u \in U$ is time-optimal on $[0,\xi]$ iff u_ξ is time-optimal on $[0,1]$. If (4.7) admits a regular time-optimal feedback u_F and if $t \to u(t)$ is optimal on $[0,T]$, then for $t \in [0,T]$

$$u_F(\bar{x}(u,\xi t)) = u(\xi t) = u_\xi(t) = u_F(\bar{x}(u_\xi,t)) = u_F(\delta_{\xi^{-1}}(\bar{x}(u,\xi t))) .$$

Therefore we expect u_F to be invariant under dilations.

7. AN EXAMPLE

Consider the two-dimensional system

$$\frac{d}{dt}(x,y) = (u_1 + \sin(x+y)u_2,\ \ (1-\cos(x+y))u_2)\ , \qquad (7.1)$$
$$(x(0),y(0)) = (0,0)\ .$$

Computing the third Picard iterate for the reduced system

$$\frac{d}{dt}\ (x,y) = (u_1 + (x+y)u_2,\ \ ½(x+y)^2u_2),\ \ (x(0),y(0)) = (0,0)$$

and discarding terms of order > 3, one obtains the third order Taylor expansion of the input-output map ϕ generated by (7.1):

$$T^3\phi(u)(t) = (\int_0^t u_1(s)ds + \int_0^t (\int_0^s u_1(\sigma)d\sigma)u_2(s)ds$$
$$+ \int_0^t (\int_0^{\sigma_1} (\int_0^{\sigma_2} u_1(\sigma_3)d\sigma_3)u_2(\sigma_2)d\sigma_2)u_2(\sigma_1)d\sigma_1,$$
$$\frac{1}{2}\int_0^t (\int_0^s u_1(\sigma)d\sigma)^2u_2(s)ds)\ .$$

In this case $Z_1 = Z_2 = W_1 = \{(x,0);x \in R\}$, $Z_3 = R^2$,

$$W_2 = \{(0,0)\},\ \ W_3 = \{(0,y);y \in R\}\ .$$

The "homogenized" expansion (4.5) is

$$\bar{\phi}(u)(t) = (\int_0^t u_1(s)ds,\ \frac{1}{2}\int_0^t (\int_0^s u_1(\sigma)d\sigma)^2u_2(s)ds)$$

which exactly represents the response of

$$\frac{d}{dt}\ (x,y) = (u_1, ½x^2u_2),\ \ (x(0),y(0)) = (0,0)\ . \qquad (7.2)$$

If the set of admissible controls is

$$U = \{u = (u_1,u_2) \in L^1([0,\infty);R^2);\ |u_1(t)| \le 1,\ u_2(t) = 1,\ \forall\ t \ge 0\}\ ,$$

the reachable sets for (7.2) are

$$R(t) = \{(x,y) \subseteq R^2;\ \frac{1}{6}\ |x|^3 \le y \le \frac{1}{8}\ (\frac{t^3}{3} + t^2|x| + t|x|^2 - |x|^3)$$

and the relations

$$(x,y) \in R(t) \quad \text{iff} \quad (\xi x, \xi^3 y) \in R(\xi t)$$

hold for every $t, \xi > 0$. A time-optimal feedback for (7.2) is

$$u_F(x,y) = (1,1) \quad \text{on} \quad A^+, \quad u_F(x,y) = (-1,1) \quad \text{on} \quad A^- ,$$

with

$$A^+ = \{(x,y);\ x > 0,\ y = \frac{1}{6} x^3\} \cup \{(x,y);\ x < 0,\ y > \frac{1}{6} |x|^3\}$$

$$A^- = \{(x,y);\ (-x,y) \in A^+\} .$$

Notice that u_F is invariant w.r.t. all dilations $\delta_\xi : (x,y) \to (\xi x, \xi^3 y)$, $\xi > 0$.

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This paper describes a canonical procedure to approximate an arbitrary family of C^∞ vector fields $\{g_1,\dots,g_m\}$ on $\mathbb{R}^d$ with vector fields $\bar{g}_1,\dots,\bar{g}_m$ on the same space $\mathbb{R}^d$ which generate a nilpotent Lie algebra. Each $\bar{g}_i$ can be either obtained from a Taylor expansion of the input-output

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ABSTRACT (cont.)

map for the control system

$$(*) \qquad \dot{x} = \sum_{i=1}^{m} g_i(x)u_i, \qquad x(0) = 0 ,$$

or computed directly as an asymptotic limit of the corresponding vector field g_i. A useful consequence is that every control system of the form (*) can locally be regarded as an arbitrarily small C^∞ perturbation of a nilpotent system on the same state space, up to a suitable linear rescaling of coordinates.

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