

BIFURCATION AND MULTIPLICITY RESULTS FOR NONLINEAR  
ELLIPTIC PROBLEMS INVOLVING  $\lambda$  WISCONSIN UNIV-MADISON  
MATHEMATICS RESEARCH CENTER G CERAMI ET AL. JAN 84

99

MRC-TSR-2630 DAAG29-80-C-0041

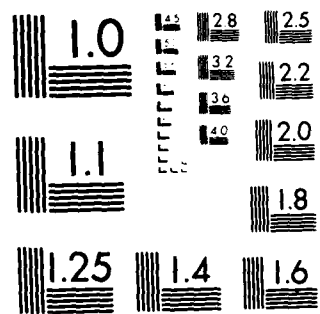
F/G 12/1

Ni

END

DATA

4 84  
DTIC



MICROCOPY RESOLUTION TEST CHART  
NATIONAL BUREAU OF STANDARDS-1963-A

AD A139332

MRC Technical Summary Report # 2630

BIFURCATION AND MULTIPLICITY RESULTS  
FOR NONLINEAR ELLIPTIC PROBLEMS  
INVOLVING CRITICAL SOBOLJEV EXPONENTS

Giovanna Cerami, Donato Fortunato and  
Michael Struwe

Mathematics Research Center  
University of Wisconsin—Madison  
610 Walnut Street  
Madison, Wisconsin 53705

January 1984

(Received September 15, 1983)

Approved for public release  
Distribution unlimited

Sponsored by

U. S. Army Research Office  
P. O. Box 12211  
Research Triangle Park  
North Carolina 27709

DTIC FILE COPY

DTIC  
ELECTE  
MAR 26 1984  
S D  
E

84 03 22 091

UNIVERSITY OF WISCONSIN-MADISON  
MATHEMATICS RESEARCH CENTER

BIFURCATION AND MULTIPLICITY RESULTS FOR NONLINEAR  
ELLIPTIC PROBLEMS INVOLVING CRITICAL SOBOLEV EXPONENTS

Giovanna Cerami<sup>1)</sup>, Donato Fortunato<sup>2)</sup>, and Michael Struwe<sup>3)</sup>

Technical Summary Report #2630  
January 1984

ABSTRACT

In this paper we study the existence of nontrivial solutions for the boundary value problem

$$\begin{cases} -\Delta u - \lambda u - u|u|^{2^*-2} = 0 & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega \end{cases}$$

when  $\Omega \subset \mathbb{R}^n$  is a bounded domain,  $n \geq 3$ ,  $2^* = \frac{2n}{n-2}$  is the critical exponent for the Sobolev embedding  $H_0^1(\Omega) \hookrightarrow L^P(\Omega)$ ,  $\lambda$  is a real parameter.

We prove that there is bifurcation from any eigenvalue  $\lambda_j$  of  $-\Delta$  and we give an estimate of the left neighborhoods  $[\lambda_j^*, \lambda_j]$  of  $\lambda_j$ ,  $j \in \mathbb{N}$ , in which the bifurcation branch can be extended. Moreover we prove that, if  $\lambda \in ]\lambda_j^*, \lambda_j[$ , the number of nontrivial solutions is at least twice the multiplicity of  $\lambda_j$ .

The same kind of results holds also when  $\Omega$  is a compact Riemannian manifold of dimension  $n \geq 3$ , without boundary and  $\Delta$  is the relative Laplace-Beltrami operator.

AMS (MOS) Subject Classifications: 35A15, 33932, 58E99

Key Words: Boundary value problem, critical Sobolev exponent, bifurcation, critical points, eigenvalue, variational problem, Riemannian manifold

Work Unit Number 1 (Applied Analysis)

---

1,2) Supported by Ministero P.I. (Italy) and by G.N.A.F.A. (CNR).

3) Supported by SFB72 of the Deutsche Forschungsgemeinschaft. The third author is indebted to the University of Bari for its kind hospitality.

# SIGNIFICANCE AND EXPLANATION

✓ This paper deals with the problem of existence of nontrivial solutions for a nonlinear elliptic boundary value problem in which the nonlinear term involves the "critical Sobolev exponent", which is associated with a loss of compactness. The motivation for investigating this type of problem comes from the fact that mathematical models of some interesting problems in geometry (Yamabe's problem) and in physics (existence of nonminimal solutions for Yang-Mills functionals) have this character and involve a lack of compactness. Variational arguments are used here to prove some bifurcation and multiplicity results for these problems.

|                    |                                     |
|--------------------|-------------------------------------|
| Accession For      |                                     |
| NTIS GRA&I         | <input checked="" type="checkbox"/> |
| DTIC TAB           | <input type="checkbox"/>            |
| Unannounced        | <input type="checkbox"/>            |
| Justification      |                                     |
| By                 |                                     |
| Distribution/      |                                     |
| Availability Codes |                                     |
| Dist               | and/or                              |
| A-1                |                                     |



The responsibility for the wording and views expressed in this descriptive summary lies with MRC, and not with the authors of this report.

BIFURCATION AND MULTIPLICITY RESULTS FOR NONLINEAR  
ELLIPTIC PROBLEMS INVOLVING CRITICAL SOBOLEV EXPONENTS

Giovanna Cerami<sup>1)</sup>, Donato Fortunato<sup>2)</sup>, and Michael Struwe<sup>3)</sup>

Introduction

Let  $\Omega$  be a bounded domain in  $\mathbb{R}^n$ ,  $n \geq 3$ ,  $2^* = \frac{2n}{n-2}$  the critical exponent for the Sobolev embedding  $H_0^1(\Omega) \rightarrow L^{2^*}(\Omega)$ . For a real parameter  $\lambda \in \mathbb{R}$  consider the boundary value problem

$$(0.1) \quad \begin{cases} -\Delta u - \lambda u - |u|^{2^*-2} u = 0 & \text{in } \Omega \\ u|_{\partial\Omega} = 0 \end{cases}$$

corresponding to the functional  $f_\lambda : H_0^1(\Omega) \rightarrow \mathbb{R}$  given by

$$(0.2) \quad f_\lambda(u) = \frac{1}{2} \int_\Omega (|\nabla u|^2 - \lambda |u|^2) dx - \frac{1}{2^*} \int_\Omega |u|^{2^*} dx$$

Since the embedding  $H_0^1(\Omega) \rightarrow L^{2^*}(\Omega)$  is not compact the functional  $f_\lambda$  in general will not satisfy the Palais-Smale condition.

However, recently Brezis and Nirenberg [5] were able to establish the existence of positive solutions of (0.1) for any  $\lambda$  in a certain range  $]\lambda^*, \lambda_1[$ , where

$\lambda_j$ ,  $j \in \mathbb{N}$  ( $\lambda_1 < \lambda_2 < \dots < \lambda_k$ ), denote the eigenvalues of the operator  $-\Delta : H_0^1(\Omega) \rightarrow H^{-1}(\Omega) = (H_0^1(\Omega))^*$ , and  $\lambda^* > 0$  is some constant depending on  $n$  and  $\Omega$ .

In this paper we study the existence of nontrivial solutions for (0.1) also for  $\lambda > \lambda_1$  to obtain bifurcation from any eigenvalue  $\lambda_j$ . We give an estimate of the left

1,2) Supported by Ministero P.I. (Italy) and by G.N.A.F.A. (CNR).

3) Supported by SFB72 of the Deutsche Forschungsgemeinschaft. The third author is indebted to the University of Bari for its kind hospitality

moreover we prove that, if  $\lambda \in ]\lambda_j^*, \lambda_j[$ , the number of nontrivial solutions of (0.1) is at least twice the multiplicity of  $\lambda_j$  (cp. Theorem 1.1).

Our results are based on the observation that although the Palais-Smale condition does not hold globally for  $f_\lambda$  (cp Remark 2.3) it is satisfied locally in a certain energy range (cp. Lemma 2.1 or [5, Remark 2.2]).

We observe that the tools used in proving the above results do not depend on the shape of  $\Omega$  and on the dimension  $n$ .

With suitable modifications the existence and bifurcation results also apply to problem (0.1) posed on a compact Riemannian manifold without boundary of dimension  $n \geq 3$  (cp. Theorem 1.3).

We thank Prof. H. Brezis for his useful comments.

# 1. Results.

Let  $\|u\| = (\int_{\Omega} |\nabla u|^2 dx)^{1/2}$ ,  $\|u\|_p = (\int_{\Omega} |u|^p dx)^{1/p}$  denote the norms in  $H_0^1(\Omega)$ ,  $L^p(\Omega)$ , respectively, and let

$$S = \inf \{ \|u\|^2 / \|u\|_{2^*}^2 : u \in H_0^1(\Omega) \setminus \{0\} \}$$

denote the best constant for the embedding  $H_0^1(\Omega) \rightarrow L^{2^*}(\Omega)$ .

Theorem 1.1: For  $\lambda > 0$  let  $\lambda_+ = \min \{ \lambda_j | \lambda < \lambda_j \}$ , and suppose

$$\lambda_+ - \lambda < S [\text{meas } (\Omega)]^{-2/n}$$

Let  $m$  be the multiplicity of  $\lambda_+$ . Then problem (0.1) admits at least  $m$  pairs of nontrivial solutions

$$\{u_k(\lambda), -u_k(\lambda)\} \quad k = 1, \dots, m$$

such that

$$\|u_k(\lambda)\| \rightarrow 0 \quad \text{as} \quad \lambda \rightarrow \lambda_+.$$

Remark 1.2: If  $\Omega$  is starshaped, it is well known that (0.1) admits only the trivial solution for  $\lambda < 0$  (cp [5], [8]).

A result analogous to Theorem 1.1 holds for the problem

$$(1.1) \quad -\Delta_M u - \lambda u - u|u|^{2^*-2} = 0$$

on a compact Riemannian manifold  $M$  of dimension  $> 3$  and without boundary. Here  $\Delta_M$

is the Laplace-Beltrami operator on  $M$ ,  $\lambda > 0$  a parameter and  $2^* = \frac{2n}{n-2}$  as before.

Denote by  $H^1(M)$  the closure of  $C^\infty(M)$  with respect to the norm

$$\|u\|_M = (\int_M (|\nabla u|^2 + |u|^2) dM)^{1/2}$$

which in local coordinates on a covering  $\{T_h\}$  of  $M$  is given by

$$\|u\|_M^2 = \left( \sum_h \int_{T_h} \left( \sum_{i,j=1}^n g^{ij} \frac{\partial u}{\partial x_i} \frac{\partial u}{\partial x_j} + |u|^2 \right) \sqrt{g} dx \right)^{1/2}$$

$g^{ij}$  denoting the metric tensor, and  $g = \det(g^{ij})$ . Note that the quadratic form

$\int_M |\nabla u|^2 dM$  is only positive semidefinite in  $H^1(M)$ , then the operator

$$- \Delta_M : H^1(M) \rightarrow H^{-1}(M) := (H^1(M))'$$

possesses eigenvalues  $\mu_1 < \mu_2 < \dots < \mu_k < \dots$  which are  $> 0$  (cp. Appendix 1 of [4]).

Theorem 1.3. For  $\lambda > 0$  let  $\mu_+ = \min \{\mu_j | \lambda < \mu_j\}$  and suppose

$$\mu_+ - \lambda < S \left( \int_M dM \right)^{-2/n}$$

Let  $m$  be the multiplicity of  $\mu_+$ . Then problem (1.1) admits at least  $m$  pairs of nonconstant solutions

$$\{u_k(\lambda), -u_k(\lambda)\} \quad k = 1, \dots, m$$

such that

$$\|u_k(\lambda)\|_M \rightarrow 0 \quad \text{as} \quad \lambda \rightarrow \mu_+.$$

## 2. Proof of Theorems 1.1, 1.3

The proof of Theorem 1.1 requires some lemmata

Lemma 2.1: For any  $\lambda \in \mathbb{R}$  the functional  $f_\lambda$  (see (0.2)) satisfies the Palais-Smale condition in  $]-\infty, \frac{1}{n} s^{n/2}[$  in the following sense:

(P.S.) If  $c < \frac{1}{n} s^{n/2}$  and  $\{u_m\}$  is a sequence in  $H_0^1(\Omega)$  such that as  $m \rightarrow \infty$   $f_\lambda(u_m) \rightarrow c$ ,  $df_\lambda(u_m) \rightarrow 0$  strongly in  $H^{-1}(\Omega)$ , then  $\{u_m\}$  contains a subsequence converging strongly in  $H_0^1(\Omega)$ .

Remark 2.2. An analogous result has been proved in [5]. Nevertheless for completeness we give here a proof of lemma 2.1 which is slightly different from that contained in [5].

Proof. Let  $\lambda \in \mathbb{R}$ , and suppose  $\{u_m\}$  is a sequence in  $H_0^1(\Omega)$  such that as  $m \rightarrow \infty$

$$(2.1) \quad f_\lambda(u_m) \rightarrow c_1 < \frac{1}{n} s^{n/2}$$

$$(2.2) \quad df_\lambda(u_m) \rightarrow 0 \text{ strongly in } H^{-1}(\Omega)$$

As in [5, estimates (2.18)] from (2.1), (2.2) we obtain that

$$(2.3) \quad \{\|u_m\|\} \text{ is bounded}$$

Hence we may extract a subsequence  $\{u_m\}$  (reabeled) such that

$$(2.4) \quad u_m \rightharpoonup u \text{ weakly in } H_0^1(\Omega)$$

$$(2.5) \quad u_m \rightarrow u \text{ strongly in } L^p(\Omega) \text{ for any } p \in [1, 2^*[$$

Moreover  $u$  is a solution of (0.1). Indeed, letting  $\phi \in C_0^\infty(\Omega)$ , by (2.4), (2.5) and (2.2) we deduce that

$$\langle df_\lambda(u), \phi \rangle = \langle df_\lambda(u_m), \phi \rangle + o(1) = o(1)$$

Hence  $u$  weakly solves (0.1). But by regularity results (cp. [5], [6], [7], and [10]) it follows that

$$(2.6) \quad u \in L^\infty(\Omega)$$

and hence that  $u$  is regular and is a solution of (0.1) in the classical sense.

To show that  $u_m \rightarrow u$  strongly in  $H_0^1(\Omega)$  as  $m \rightarrow \infty$ , let  $v_m = u_m - u$ . Testing (2.2) with  $v_m$  we obtain

$$(2.7) \quad \begin{aligned} o(1) &= \langle df_\lambda(u_m), v_m \rangle = \\ &= \int_{\Omega} (\nabla u \nabla v_m + |\nabla v_m|^2 - \lambda(u+v_m)v_m - |u+v_m|^{2^*-2}(u+v_m)v_m) dx \end{aligned}$$

By (2.4) and (2.5) we have

$$(2.8) \quad \int_{\Omega} (\nabla u \nabla v_m - \lambda(u+v_m)v_m) dx = o(1)$$

Whence from (2.7), (2.8) we deduce that

$$(2.9) \quad \|v_m\|^2 = \int_{\Omega} |u+v_m|^{2^*-2}(u+v_m)v_m dx + o(1)$$

Now we claim that

$$(2.10) \quad \|v_m\|^2 = \|v_m\|_{2^*}^{2^*} + o(1)$$

In fact, by using (2.5) and (2.6), we have

$$(2.11) \quad \begin{aligned} & \left| \int_{\Omega} (u+v_m)|u+v_m|^{2^*-2}v_m dx - \int_{\Omega} |v_m|^{2^*} dx \right| = \\ &= \left| \int_{\Omega} \int_0^{u(x)} \frac{\partial}{\partial \xi} [(v_m + \xi)|v_m + \xi|^{2^*-2}] v_m d\xi dx \right| = \end{aligned}$$

$$= |(2^*-1) \int_{\Omega} \int_0^1 |v_m + tu|^{2^*-2} v_m u \, dt \, dx|$$

$$< \text{const.} \left[ \int_{\Omega} (|u| |v_m|^{2^*-1} |v_m| |u|^{2^*-1}) \, dx \right] = o(1)$$

and (2.10) easily follows from (2.9) and (2.11).

Since

$$\langle df_{\lambda}(u_m), u_m \rangle = o(1)$$

we have

$$|u_m|_{2^*}^{2^*} = \int_{\Omega} (|\nabla u_m|^2 - \lambda |u_m|^2) \, dx + o(1)$$

Inserting into the expression for  $f_{\lambda}(u_m)$  we obtain

$$(2.12) \quad f_{\lambda}(u_m) = \frac{1}{n} \int_{\Omega} (|\nabla u_m|^2 - \lambda |u_m|^2) \, dx + o(1) =$$

$$= \frac{1}{n} \int_{\Omega} (|\nabla u|^2 - \lambda |u|^2) \, dx + \frac{1}{n} \int_{\Omega} |\nabla v_m|^2 \, dx + o(1)$$

Moreover, since  $u$  is a solution of (0.1)

$$\int_{\Omega} (|\nabla u|^2 - \lambda |u|^2) \, dx - \int_{\Omega} |u|^{2^*} \, dx = \langle df_{\lambda}(u), u \rangle = 0$$

Whence in particular

$$(2.13) \quad \int_{\Omega} (|\nabla u|^2 - \lambda |u|^2) \, dx > 0$$

From (2.12) and (2.13) we now infer

$$|\nabla v_m|^2 \leq n f_{\lambda}(u_m) + o(1)$$

Then, by (2.1), for  $n$  sufficiently large we obtain

$$(2.14) \quad \|v_m\|^2 \leq c_2 < s^{n/2}$$

Now, by (2.10)

$$\|v_m\|^2 \leq s^{-2^*/2} \|v_m\|^{2^*} + o(1)$$

Or equivalently

$$\|v_m\|^2 (s^{2^*/2} - \|v_m\|^{2^*-2}) \leq o(1)$$

Taking account of (2.14) this implies that  $v_m \rightarrow 0$  strongly in  $H_0^1(\Omega)$ , concluding the proof.  $\square$

Remark 2.3. Complementing the preceding lemma we have a non-compactness result for energies  $> \frac{1}{n} s^{n/2}$ . In fact we now show that for any  $\lambda \in \mathbb{R}$  there exists a sequence  $\{u_m\} \subset H_0^1(\Omega)$  satisfying the P-S assumptions in  $c = \frac{1}{n} s^{n/2}$ , which is not relatively compact in  $H_0^1(\Omega)$ .

Let  $x_0 \in \Omega$  and choose a function  $\phi \in C_0^\infty(\Omega)$  such that  $\phi \equiv 1$  in a neighborhood  $N$  of  $x_0$ . The functions  $u_\mu : \mathbb{R}^n \rightarrow \mathbb{R}$

$$u_\mu(x) = \frac{[n(n-2)\mu^2]^{\frac{n-2}{4}}}{[\mu^2 + |x-x_0|^2]^{\frac{n-2}{2}}}$$

solve the equation

$$(2.15) \quad -\Delta u_\mu = u_\mu |u_\mu|^{2^*-2} \quad \text{in } \mathbb{R}^n$$

Let

$$u_m = \phi u_{\mu_m} \quad \mu_m = \frac{1}{m}$$

Note that  $u_m \in H_0^1(\Omega)$  and moreover

$$(2.16) \quad \{u_m\} \text{ is uniformly bounded in } H_0^1(\Omega)$$

Also we easily derive that as  $m \rightarrow \infty$

$$(2.17) \quad \nabla u_m \rightarrow 0 \text{ in } L^2(\mathbb{R}^n \setminus N)$$

$$(2.18) \quad u_m \rightarrow 0 \text{ in } L_{loc}^\infty(\Omega \setminus \{x_0\})$$

Hence also

$$(2.19) \quad u_m \rightarrow 0 \text{ weakly in } H_0^1(\Omega) \text{ (} m \rightarrow \infty \text{)}$$

Using (2.17) and (2.18) we deduce that

$$(2.20) \quad \begin{aligned} f_\lambda(u_m) &= \frac{1}{2} \int_{\mathbb{R}^n} |\nabla u_m|^2 dx - \frac{1}{2^*} \int_{\mathbb{R}^n} |u_m|^{2^*} dx + o(1) = \\ &= \frac{1}{n} S^{n/2} + o(1) \end{aligned} \quad (\text{cp}[1,9]).$$

Also using (2.15) - (2.18) we obtain

$$\begin{aligned} \|df_\lambda(u_m)\|_{H^{-1}(\Omega)} &= \sup_{\substack{v \in H_0^1(\Omega) \\ \|v\|_{H_0^1} = 1}} \int_{\mathbb{R}^n} (\nabla u_m \nabla v - u_m |u_m|^{2^*-2} v) dx + o(1) = o(1) \\ \|v\|_{H_0^1} &= 1 \end{aligned}$$

Hence  $\{u_m\}$  satisfies the (P-S) assumptions with  $c = \frac{1}{n} S^{n/2}$ , however, by (2.19) and (2.20),  $\{u_m\}$  cannot be relatively compact in  $H_0^1(\Omega)$ .

Lemma 2.4: For  $\lambda > 0$  let  $\lambda_+ = \inf \{\lambda_j | \lambda < \lambda_j\}$  and set

$$M_+ = \bigcup_{\lambda_j > \lambda_+} M(\lambda_j) \quad (\text{the closure is taken in } H_0^1(\Omega)),$$

$$M_- = \bigcup_{\lambda_j \leq \lambda_+} M(\lambda_j)$$

where  $M(\lambda_j)$  denotes the eigenspace of  $-\Delta$  corresponding to  $\lambda_j$ . Then

$$\beta_\lambda := \sup_{u \in M_-} f_\lambda(u) < (\lambda_+ - \lambda)^{n/2} \frac{\text{meas}(\Omega)}{n},$$

moreover, there exist constants  $\rho_\lambda > 0$ ,  $\delta_\lambda \in ]0, \beta_\lambda[$  such that

$$f_\lambda(u) > \delta_\lambda \text{ for any } u \in M_+, |u| = \rho_\lambda.$$

Proof: For any  $u \in M_-$  we have

$$\begin{aligned} f_\lambda(u) &= \frac{1}{2} \int_{\Omega} (|\nabla u|^2 - \lambda |u|^2) dx - \frac{1}{2^*} \int_{\Omega} |u|^{2^*} dx < \\ &< \frac{1}{2} (\lambda_+ - \lambda) \int_{\Omega} |u|^2 dx - \frac{1}{2^*} \int_{\Omega} |u|^{2^*} dx < \\ &< \frac{1}{2} (\lambda_+ - \lambda) \text{meas}(\Omega)^{2/n} \left( \int_{\Omega} |u|^{2^*} dx \right)^{2/2^*} - \frac{1}{2^*} \int_{\Omega} |u|^{2^*} dx \end{aligned}$$

Let

$$g(\rho) = \frac{1}{2} (\lambda_+ - \lambda) \text{meas}(\Omega)^{2/n} \rho^2 - \frac{1}{2^*} \rho^{2^*}$$

Then

$$\sup_{u \in M_-} f_\lambda(u) < \sup_{\rho > 0} g(\rho) = \frac{1}{n} (\lambda_+ - \lambda)^{n/2} \text{meas}(\Omega)$$

proving the first part of the lemma.

Since for  $u \in M_+$  we obtain

$$\int_{\Omega} (|\nabla u|^2 - \lambda |u|^2) dx > \left(1 - \frac{\lambda}{\lambda_+}\right) |u|^2$$

while

$$|u|_{2^*}^{2^*} < \text{const } |u|^{2^*}$$

The second part of the claim is immediate. ■

By lemmata 2.1, 2.4, Theorem 1.1 can be deduced by the following result - Bartolo, Benci, Fortunato (cp Theorem 2.4 of [3]).

Theorem 2.5: Let  $H$  be a real Hilbert space with norm  $\|\cdot\|$  and suppose  $I \in C^1(H, \mathbb{R})$  is a functional on  $H$  satisfying the following conditions:

$I_1)$   $I(u) = I(-u)$ ,  $I(0) = 0$ ;

$I_2)$  There exists a constant  $\beta > 0$  such that the Palais-Smale condition (P-S) holds in  $]0, \beta[$ ;

$I_3)$  There exist two closed subspaces  $V, W \subset H$  and positive constants  $\rho, \delta, \beta'$ , with  $\delta < \beta' < \beta$  such that

(i)  $I(u) < \beta'$  for any  $u \in W$

(ii)  $I(u) > \delta$  for any  $u \in V$ ,  $\|u\| = \rho$

(iii)  $\text{codim } V < +\infty$  and  $\dim W > \text{codim } V$

Then there exists at least

$$\dim W - \text{codim } V$$

pairs of critical points of  $I$  with critical values belonging to the interval

$$[\delta, \beta'] . \quad \blacksquare$$

We are now ready to prove Theorem 1.1.

Proof of Theorem 1.1: Let  $H = H_0^1(\Omega)$ ,  $I = f_\lambda$ ,  $V = M_+$ ,  $W = M_-$ ,  $\beta = \frac{1}{n} S^{n/2}$ ,  $\beta' = \beta_\lambda$ ,  $\delta = \delta_\lambda$ ,  $\rho = \rho_\lambda$  and apply Theorem 2.5 together with lemmata 2.1, 2.4.  $\blacksquare$

For the proof of Theorem 1.3 the following result from [2] is needed.

Lemma 2.6 : If  $\{v_m\}$  is a sequence in  $H^1(M)$  such that  $v_m \rightarrow 0$  weakly in  $H^1(M)$  as  $m \rightarrow \infty$ , then

$$\|v_m\|_{2^*}^2 \leq S^{-1} \|v_m\|_M^2 + o(1)$$

Proof : By [2, Theorem 2.21] for all  $\phi \in H^1(M)$ ,  $\epsilon > 0$

$$\|\phi\|_{2^*}^2 \leq (S^{-1} + \epsilon) \|\nabla \phi\|_2^2 + \Lambda(\epsilon) \|\phi\|_2^2$$

with a constant  $\Lambda(\epsilon)$  independent of  $\phi$ . Applying this inequality with  $\phi = v_m$ , and noting that by weak convergence  $v_m \rightarrow 0 (m \rightarrow +\infty)$  we have

$$\|v_m\|_2 \rightarrow 0 \quad m \rightarrow +\infty$$

we deduce that for any  $\epsilon > 0$

$$\|v_m\|_{2^*}^2 \leq (S^{-1} + \epsilon) \|v_m\|_M^2 + o(1)$$

The lemma follows on letting  $\epsilon \rightarrow 0$ .  $\square$

### Proof of Theorem 1.3

Going through the proof of Lemma 2.1. - keeping in mind Lemma 2.6 and the fact that, for any sequence  $\{v_m\}$  in  $H^1(M)$  tending to 0 weakly in this space,  $\|v_m\|_2 = o(1)$  - it is now immediate that also for the functional on  $H^1(M)$

$$\tilde{I}_\lambda(u) = \frac{1}{2} \int_M (|\nabla u|^2 - \lambda |u|^2) \, dM - \frac{1}{2^*} \int_M |u|^{2^*} \, dM$$

corresponding to problem (1.1) the Palais-Smale condition is satisfied in the interval

$$\left] -\infty, \frac{1}{n} S^{n/2} \right[.$$

Moreover it is easy to see that the same estimates of lemma 2.4 continue to hold

(obviously  $\lambda_j, \lambda_+, H_0^1(\Omega)$ , meas  $\Omega$  are replaced respectively by  $\mu_j, \mu_+, H^1(M), \int_M dM$ ).

Then Theorem 1.3 can be proved by using again the abstract critical point Theorem 2.5.  $\square$

# References

- [1] Th. Aubin, Problemes isoperimetriques et espaces de Sobolev, J. Diff. Geom. 11(1976) 573-598.
- [2] Th. Aubin, Nonlinear analysis on manifolds, Monge-Ampere equations: Springer Grundlehren 252, 1982.
- [3] P. Bartolo, V. Benci, D. Fortunato: Abstract critical point theorems and applications to some nonlinear problems with "strong resonance" at infinity, to appear in Journal of nonlinear Anal. T.M.A.
- [4] V. Benci, D. Fortunato: The dual method in critical point Theory. Multiplicity results for indefinite functionals, Ann. Mat. Pura Appl. 32 ;(1982), 215-242.
- [5] H. Brezis, L. Nirenberg: Positive solutions of nonlinear elliptic equations involving critical Sobolev exponents, Comm. Pure Appl. Math. XXXVI(1983).
- [6] H. Brezis, T. Kato: Remarks on the Schrodinger operator with singular complex potential, J. Math. Pures et Appl. 58 (1979), 137-151.
- [7] S. Luckhaus: Existence and regularity of weak solutions to the Dirichlet problem for semilinear elliptic systems of high order, J. Reine und Angew. Math. 306 (1979), 192-207.
- [8] S. J. Pohozaev: Eigenfunctions of the equation  $\Delta u + \lambda f(u) = 0$ , Soviet Math. Doklady 6 (1965), 1408-1411 (Translated from the Russian Dokl. Akad. Nauk SSSR 165 (1965) 33-36.
- [9] G. Talenti: Best constants in Sobolev inequality, Ann. Mat. Pure Appl. 110 (1976) 353-372.
- [10] M. Trudinger: Remarks concerning the conformal deformation of Riemannian structure on compact manifolds, Ann. Sc. Norm. Sup. Pisa 22 (1968), 265-274.

Author's Addresses:

G. Cerami - Istituto Matematico - Universita' - via Archirafi 34,  
90123 Palermo, (Italy) and Mathematics Research Center,  
610 Walnut Street, Madison, WI 53705 (U.S.A.)

D. Fortunato - Dipartimento di Matematica - Universita' Via Nicolai 2,  
70100 Bari (Italy)

M. Struwe - Mathematisches Institut der Universität, Beringstrasse 4-6  
D5300, Bonn 1, West Germany

GC/DF/MS;jgb

| REPORT DOCUMENTATION PAGE                                                                                                                                                                                                                                                                                                |                       | READ INSTRUCTIONS<br>BEFORE COMPLETING FORM                                                                |
|--------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------|-----------------------|------------------------------------------------------------------------------------------------------------|
| 1. REPORT NUMBER<br>2630                                                                                                                                                                                                                                                                                                 | 2. GOVT ACCESSION NO. | 3. RECIPIENT'S CATALOG NUMBER                                                                              |
| 4. TITLE (and Subtitle)<br>BIFURCATION AND MULTIPLICITY RESULTS FOR<br>NONLINEAR ELLIPTIC PROBLEMS INVOLVING CRITICAL<br>SOBOLEV EXPONENTS                                                                                                                                                                               |                       | 5. TYPE OF REPORT & PERIOD COVERED<br>Summary Report - no specific<br>reporting period                     |
|                                                                                                                                                                                                                                                                                                                          |                       | 6. PERFORMING ORG. REPORT NUMBER                                                                           |
| 7. AUTHOR(s)<br><br>Giovanna Cerami, Donato Fortunato and<br>Michael Struwe                                                                                                                                                                                                                                              |                       | 8. CONTRACT OR GRANT NUMBER(s)<br><br>DAAG29-80-C-0041                                                     |
| 9. PERFORMING ORGANIZATION NAME AND ADDRESS<br>Mathematics Research Center, University of<br>610 Walnut Street Wisconsin<br>Madison, Wisconsin 53706                                                                                                                                                                     |                       | 10. PROGRAM ELEMENT, PROJECT, TASK<br>AREA & WORK UNIT NUMBERS<br>Work Unit Number 1 -<br>Applied Analysis |
| 11. CONTROLLING OFFICE NAME AND ADDRESS<br>U. S. Army Research Office<br>P. O. Box 12211<br>Research Triangle Park, North Carolina 27709                                                                                                                                                                                 |                       | 12. REPORT DATE<br>January 1984                                                                            |
|                                                                                                                                                                                                                                                                                                                          |                       | 13. NUMBER OF PAGES<br>14                                                                                  |
| 14. MONITORING AGENCY NAME & ADDRESS (if different from Controlling Office)                                                                                                                                                                                                                                              |                       | 15. SECURITY CLASS. (of this report)<br><br>UNCLASSIFIED                                                   |
|                                                                                                                                                                                                                                                                                                                          |                       | 15a. DECLASSIFICATION/DOWNGRADING<br>SCHEDULE                                                              |
| 16. DISTRIBUTION STATEMENT (of this Report)<br><br>Approved for public release; distribution unlimited.                                                                                                                                                                                                                  |                       |                                                                                                            |
| 17. DISTRIBUTION STATEMENT (of the abstract entered in Block 20, if different from Report)                                                                                                                                                                                                                               |                       |                                                                                                            |
| 18. SUPPLEMENTARY NOTES                                                                                                                                                                                                                                                                                                  |                       |                                                                                                            |
| 19. KEY WORDS (Continue on reverse side if necessary and identify by block number)<br><br>Boundary value problem, critical Sobolev exponent, bifurcation, critical<br>points, eigenvalue, variational problem, Riemannian manifold                                                                                       |                       |                                                                                                            |
| 20. ABSTRACT (Continue on reverse side if necessary and identify by block number)<br><br>In this paper we study the existence of nontrivial solutions for the<br>boundary value problem<br>$\begin{cases} -\Delta u - \lambda u - u u ^{2^*-2} = 0 & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega \end{cases}$ |                       |                                                                                                            |

ABSTRACT (Continued)

when  $\Omega \subset \mathbb{R}^n$  is a bounded domain,  $n > 3$ ,  $2^* = \frac{2n}{n-2}$  is the critical exponent for the Sobolev embedding  $H_0^1(\Omega) \hookrightarrow L^{2^*}(\Omega)$ ,  $\lambda$  is a real parameter.

We prove that there is bifurcation from any eigenvalue  $\lambda_j$  of  $-\Delta$  and we give an estimate of the left neighborhoods  $[\lambda_j^*, \lambda_j]$  of  $\lambda_j$ ,  $j \in \mathbb{N}$ , in which the bifurcation branch can be extended. Moreover we prove that, if  $\lambda \in ]\lambda_j^*, \lambda_j[$ , the number of nontrivial solutions is at least twice the multiplicity of  $\lambda_j$ .

The same kind of results holds also when  $\Omega$  is a compact Riemannian manifold of dimension  $n > 3$ , without boundary and  $\Delta$  is the relative Laplace-Beltrami operator.