

AD-A140 047

SWITCHING NET MODELS: RUDIMENTARY BEHAVIOR
REPRESENTATIONS AND APPLICATIONS(U) STANFORD UNIV CA
DEPT OF STATISTICS A E GELFAND 04 APR 84 TR-343

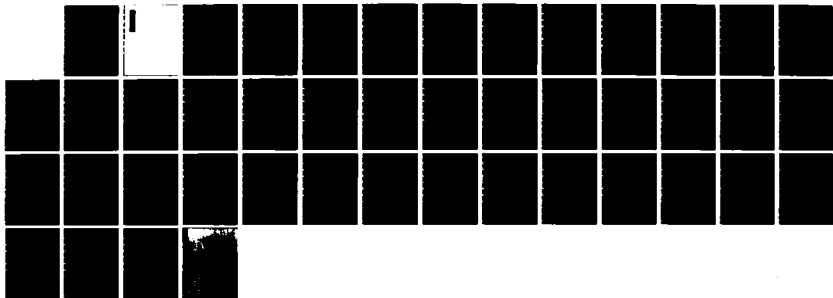
1/1

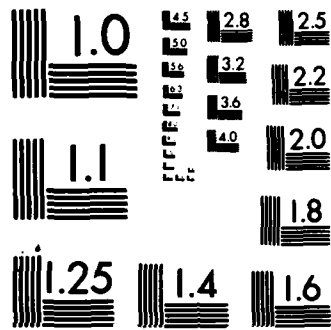
UNCLASSIFIED

N00014-76-C-0475

F/G 12/1

NL





MICROCOPY RESOLUTION TEST CHART
NATIONAL BUREAU OF STANDARDS-1963-A

SWITCHING NET MODELS: RUDIMENTARY BEHAVIOR,
REPRESENTATIONS AND APPLICATIONS

By

Alan E. Gelfand

1. Introduction.

It is the purpose of this report to introduce the reader to a class of abstract models called switching net models. Such models are quite simple in conception but nonetheless offer considerable behavioral diversity and complexity. In this sense they become quite attractive models for examination. The former aspect allows them significant receptivity to mathematical analysis and to computer simulation which the latter enables their applicability in the general systems sense to a variety of problems. We shall endeavor to describe the fundamentals of switching net behavior, to offer several different representations of switching nets and to describe three appealing applications of these models. We shall do this in sections 3, 4, and 5 respectively. Beforehand, in section 2 we present basic definitions and a brief historical perspective.

2. Basic Definitions.

We begin by briefly introducing some terminology and notation. For definitional purposes we shall take a binary switching net to be a network with an associated set of Boolean transformations. In our work we shall assume no external inputs to the net. Strictly speaking, this

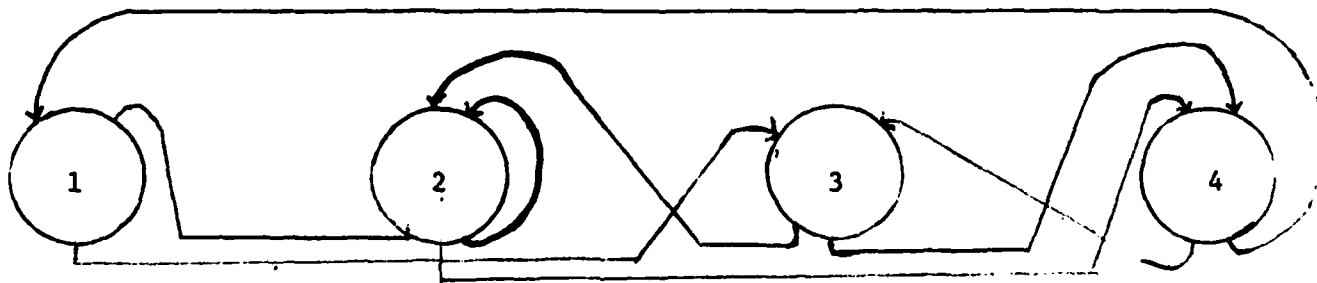
defines an autonomous switching net, but henceforth we suppress the word "autonomous." Network models and, in particular, randomly connected network models have a considerable literature. Since one may comfortably envision the flow of mass, energy or information through such networks, these models have been applied in such diverse areas as structural chemistry, sociology and information retrieval. However the widest of application has been in biology where examples include neural networks, the spread of excitation in cardiac muscles, and spread of contagious disease in a population and the spread of cancer in an organism. A few illustrative references are provided at the end of this chapter (Alben and Boutron, Bell and Dean, Blumenson, Doreian, Rashevsky, and Stubbs). But it is not our purpose here to investigate this large body of published material. We intend to focus on switching net models and as a result will draw from the above literature only that which is pertinent to the study of these models.

Formally a network is the couple (N,K) consisting of a set of N nodes or elements and K , and $N \times N$ connections matrix indicating the connections between elements. The entries in K are only "zeros" and "ones." A "one" in cell (i,j) indicates a connection from element i to element j while a "zero" indicates not. More conventional prose refers to a connection from to an element as an output and a connection to an element as an input. A "one" at any diagonal element indicates "feedback," i.e. the corresponding element draws input from its own output.

We now distinguish two terms often used interchangeably in the literature. We shall employ the word "connectance" to describe direct connection from one element to another and say that two such elements are "directly connected." In contrast we will take the work "connectively" to describe eventual connection, i.e. one element may be reached from another through a sequence of connections involving other elements. We would then say that the two elements are "connected." More precisely we will say that element i is connected to element j in p steps if the (i,j) entry in K^p (usual matrix multiplication) is "one." Obviously, then, "connectivity" includes "connectance" and "connected" includes directly connected since the latter term in each instance corresponds to the case $p = 1$.

Returning to K we define the sum of the elements in the i -th row to be the output connectance of the i -th element. Similarly the sum of the elements in the i -th column of K is called the input connectance of the i -th element. The sum of all the entries in K divided by N is the average connectance (input or output) of the network. Constant input (output) connectance k means each column (row) of K has the same sum k . A network is homogeneous if it has constant input and output connectance, i.e. each element would have as many direct connections from itself as it has to itself. A network is fully connected if $K = N$. Output connectance seems natural to consider in the study of neural nets since it would indicate the number of axons sent out by a stimulated neuron. However in switching

nets input connection is more crucial since the definition of the Boolean transformation which determines an element's responses requires specification of the number of inputs (and, in fact, a labeling of these inputs). Moreover, in the switching nets we study, we will usually specify a constant input connectance k leaving the output connectance to vary from element to element.



$$K = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 1 & 1 & 0 & 1 \\ 0 & 1 & 0 & 1 \\ 1 & 0 & 1 & 0 \end{bmatrix}$$

Figure 1: A network with $N = 4$ and associated connections matrix K

Figure 1 offers a graphic depiction of a network $N = 4$ and K is given as well. Note the constant input connectance, $k = 2$.

The study of networks entails either a deterministic selection of K which leads to the study of classical graph theory or a probabilistic selection of K which suggests an ensemble approach for

analysis. Probabilistic selection is more appropriate for studying switching net models. Moreover our applications will mandate this as they imply the existence of a complex web of interconnections among network elements but not the knowledge of specific detail.

Such randomly connected networks were first studied in detail in the late forties by Rapoport and by Shimmel for the case of unit output connectance and in the early fifties by Solomonoff and Rapoport for multiple output connectance. A survey article by Stubbs and Good notes that since that early effort there have been numerous papers discussing a variety of aspects of connectance and connectivity in random networks. The work covers material dealing with such areas as ecosystems, metabolic networks, neural nets and general theoretical development.

Structure is for our purpose the critical concept in discussing switching net models. Connectance in this sense becomes the most important aspect of a network. Rapoport observed that on the simplest level there is an isomorphism between the graph of a network and the connections matrix of a network (Figure 1 clearly illustrates this notion). We will abbreviate this idea in asserting that the network itself is isomorphic to its connections matrix. The connectance structure of a network in our models will be assigned via a random selection mechanism constrained only to a constant input connectance k . Hence the choice of k represents the only structural control we can exercise over the network.

But a switching net model is more than a network. We defined it earlier to include an associated collection of Boolean transformations, one for each element of the network. It is in the selection of particular Boolean transformations from the collection of all Boolean transformations that we achieve diverse and intricate behavior for the models. It is through constraint in the choice of Boolean transformations that more sophisticated control can be exerted in the models. It is, indeed, this embellishment which enables the useful application of these models to such complex phenomena as (1) management of large organizations, (2) dynamic behavior in genetic control systems and (3) advertising policy in consumer markets. These applications will be described shortly but first a brief perspective on the switching net literature.

The study of switching nets was initiated by McCulloch and Pitts. They defined a formal neuron and proposed its use to describe real neural interaction in the central nervous system. Subsequent researchers influenced by this work have concentrated on models of subsystems of the brain and other neural phenomena where "all or none" response of the elements is appropriate. Arbib presents a recent review of much of this work. We do not devote further attention to this area of application since it does not formalize the Boolean transformation feature of these nets. In this neural application the Boolean transformation is present but not discussed rather focusing on the equivalent (as we shall see) state diagram. Hence the aspect of functional control

available through restrictions on the transformation which we are able to usefully interpret and analyze is not at all considered. Kauffman first applied switching nets to the study of genetic systems. He describes one type of functional control which can be exerted on the models. The author in a series of papers has described the applications to management strategy and to advertising policy again from a structural point of view. Cainaello and Cull have studied these nets from a purely mathematical viewpoint. They have developed an attractive linearization of switching nets thus enabling a matrix calculus for these nets amenable to linear algebra methods. This linearization technique is particularly appealing in calculating certain behavioral characteristics of these nets and will be discussed in Section 4.

We now define the notion of a Boolean transformation. Such a function, which we shall often just call a mapping, is a rule which, for an element with k inputs, prescribes an output value for each possible vector of input values. The inputs are binary valued as is the output. Thus the mapping m is denoted by

$$m: X\{0,1\}^k \rightarrow \{0,1\} .$$

Clearly there are 2^k possible input vectors and specifications of m requires its value for each of these 2^k input vectors. Since the output is binary as well there are obviously 2^{2^k} possible Boolean transformations. Of course, one may readily conceptualize more than

binary response for the elements in a network. If we allow n responses, we refer to our systems as n -ary (instead of binary) switching nets and this number becomes n^k . The inputs to an element will be ordered and labeled as x_1, x_2, \dots, x_k . It may be convenient at times to denote a mapping m as a function of its inputs, i.e. $m(x_1, x_2, \dots, x_k)$. If all N elements in a network are governed by the same mapping, we refer to the switching net as homogeneous, otherwise heterogeneous.

A natural representation of a mapping is in tabular form with the input values arranged either in lexicographic or monotonic order. Table 1 illustrates the general representation of a mapping on three inputs with the input values in lexicographic order while Table 2 has the input values in monotonic order. Lexicographic order is more easily understood from Table 1 than it is formally defined. For k inputs the first 2^{k-1} rows will have $x_k = 0$ and the remaining 2^{k-1} rows will have $x_k = 1$. Also the first 2^{k-2} rows will have $x_{k-1} = 0$, the next 2^{k-2} rows will have $x_{k-1} = 1$, the next 2^{k-2} rows will have $x_{k-1} = 0$ and the last 2^{k-2} rows will have $x_{k-1} = 1$. The pattern is now clear and obviously all possible input vectors will be developed in this manner. Monotonic order merely arranges the input rows in terms of increasing number of "ones" from top to bottom. This order is obviously not unique, but again it is apparent that all possible input vectors will be created.

x_1	x_2	x_3	m
0	0	0	$m(0,0,0)$
1	0	0	$m(1,0,0)$
0	1	0	$m(0,1,0)$
1	1	0	$m(1,1,0)$
0	0	1	$m(0,0,1)$
1	0	1	$m(1,0,1)$
0	1	1	$m(0,1,1)$
1	1	1	$m(1,1,1)$

Table 1: General Representation of a Mapping on Three Inputs with Inputs in Lexicographic Order.

x_1	x_2	x_3	m
0	0	0	$m(0,0,0)$
1	0	0	$m(1,0,0)$
0	1	0	$m(0,1,0)$
0	0	1	$m(0,0,1)$
1	1	0	$m(1,1,0)$
1	0	1	$m(1,0,1)$
0	1	1	$m(0,1,1)$
1	1	1	$m(1,1,1)$

Table 2: General Representation of a Mapping on Three Inputs with Inputs in Monotonic Order.

According to the particular form of functional control on the Boolean transformation that we are utilizing one order may be more convenient than the other.

Returning to Figure 1 where we recall a constant connectance $k = 2$ was imposed, suppose we label the left inputs as input "1", i.e. x_1 , and the right inputs as input "2", i.e. x_2 . Then in Figure 2 we present a table specifying a choice of four Boolean transformations associated with the four elements, i.e. m_i is associated with element i , $i = 1, 2, 3, 4$.

x_1	x_2	m_1	m_2	m_3	m_4
0	0	0	1	0	1
1	0	1	1	1	0
0	1	0	1	0	1
1	1	1	0	0	0

Figure 2: Associated Boolean Transformations for the Networks in Figure 1.

Figures 1 and 2 taken together provide an example of a heterogeneous binary switching net model. With all the model components in place we next turn to a discussion of the behavior of such net models.

3. Rudimentary Net Behavior.

In describing the behavioral characteristics of a switching net, we require some further terminology. The state of the net is an ordered (left to right) N vector wherein the i -th coordinate is the current output value of the i -th net element. The net progresses

through states in a discrete manner. The state of the net at time $t+1$ is determined from its state at time t , i.e. the output values of the elements at time t become input values in determining the new output values of the elements at time $t+1$. In this manner the net moves from state to state over time in a determinant fashion. It is apparent that there are but a finite number of distinct net states, in fact 2^N of them. Because of these two remarks, it is clear that from some initial net state the net must eventually come to a state it had previously passed through. Doing so it must then repeat the sequence of intermediate states. Such a sequence of states is called a cycle. Note that every net must have at least one cycle. The sequence of states from the initial one until entering the cycle is called the run-in length; the number of net states (necessarily distinct) in the run-in is called the run-in length. The sum of the run-in length and the cycle length is called the disclosure length. Note that the cycle length may range from 1 to 2^N while the run-in length may range from 0 to 2^N .

For the switching net of Figures 1 and 2, the reader may verify that commencing for example in state 1110 the net moves through the states listed in Figure 3.

1110 → 0010 → 0101 → 1100 → 0110 → 0000 → 0101

Figure 3: A State Sequence for the Net in Figures 1 and 2.

Note that in this sequence we have both a run-in and a cycle. The run-in length is 2 and the cycle length is 4.

Recognizing that we have described but one state sequence suggests that we might think of the entire collection of state sequences. In referring to this collection which reflects the cyclical behavior of the net, we employ the term "cycle space." More generally, in every switching net each state either belongs to a run-in or a cycle. Analogously to Markov Chain model jargon, we may describe these states as transient or recurrent (cyclic), respectively. The cycle space of the net would partition the state space of the net into distinct cycles with each cycle having its associated transient states. Figure 4 exhibits the cycle space for our example. The lines connecting states are directed in order to graphically sequence the states.

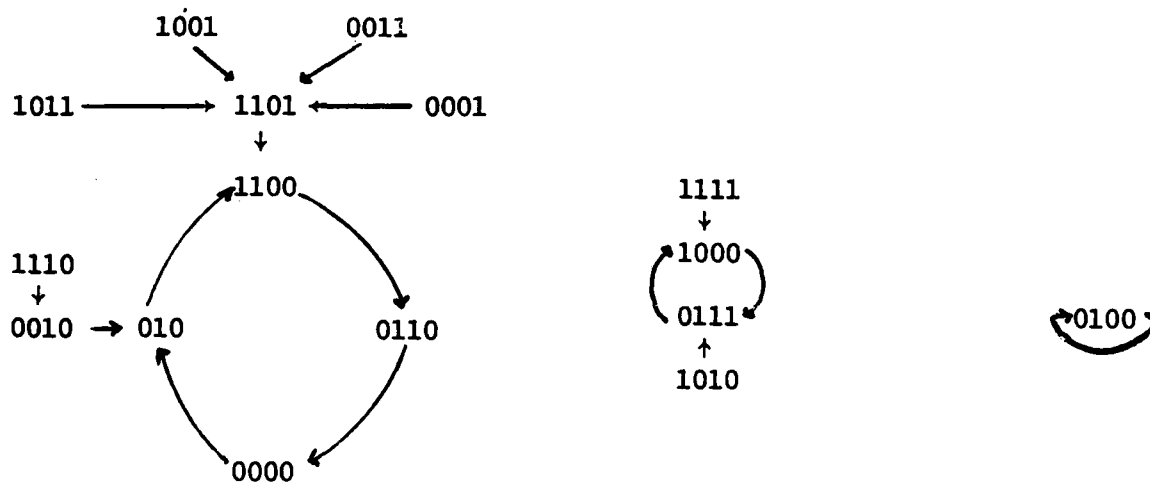


Figure 4: Cycle Space for the Net in Figures 1 and 2.

As a number of elements in the net grows large, the number of net states increases dramatically. For example, even with $N = 20$ the number of states is already $2^{20} \approx 10^6$ while for $N = 100$ the number becomes enormous -- $2^{100} \approx 10^{30}$. Hence the lengths of cycles will grow large and, moreover, it will not be feasible to explicitly detail the cycle space in these cases. Nonetheless, net models with large N are precisely the ones we would wish to use in approximating complex real world phenomena. Hence we will still want to know in an approximate sense how the number of distinct cycles and the lengths of these cycles depends on N . We would also want to know the effect on the cycle space of applying certain types of functional control to the Boolean transformations in the net.

In looking at the cycle space of a net, it becomes natural to examine the stability of the net's behavior. Suppose as the net passes through states in a cycle minor perturbation is introduced. We inquire as to whether

- (i) the net tends to return to the same cycle,
- (ii) the net tends to return to a particular subset of the cycles in the cycle space,
- (iii) the net is as likely to wind up on any cycle in its cycle space.

Case (i) represents the strongest form of stability. Case (ii) is usually referred to as "restricted local reachability." From a modeling perspective, case (iii) might render these nets less appealing in relating them to real world phenomena.

Formally we shall define a minor perturbation as changing the value of one element in a state on a cycle. After such a perturbation we would then note to which cycle the net returned. By perturbing all states on each cycle in all possible ways one element at a time, one may obtain a transition probability matrix called the flow matrix among net cycles. This is done by recording the total number of times the net returned to the perturbed cycle or ran to each other cycle. Let the rows denote the initial cycle and the columns the resultant cycle after perturbation. Then dividing the value in each cell by its row total yields the desired matrix of transition probabilities.

For our example, if we label the cycles in Figure 4 from left to right as 1, 2, and 3, then Figure 5 presents the corresponding flow matrix.

	1	2	3
1	.5	.25	.25
2	.75	.25	0
3	1	0	0

Figure 5: Flow Matrix of the Net in Figures 1 and 2.

This net would appear to have little stability and not enough distinct cycles to really discuss restricted local reachability. (The development of this flow matrix is just for illustrative purposes.)

Again, as N grows large, study of the stability of switching nets becomes difficult. We would particularly seek to examine stability in the presence of the functional control.

Lest the reader conclude that cycles are the only behavioral characteristic of a switching net which one may usefully study, we remark that the choice of system variables to be monitored usually derives from the intended application of the net model. Often they will be cycle-related variables such as we have discussed, but other possibilities exist. For example, one may monitor the proportion of ones in successive nets states in an effort to detect whether the proportion tends to be maintained or perhaps exhibits an upward or downward trend. From these observations one could compute the average behavior of the sequence and its variability. Such a collection of variables will be seen to be appropriate in our application to modeling advertising policy in consumer markets.

One may also compute distance variables between successive state vectors according to some appropriate distance measure. In this case the sequence of variables may be employed to discover whether states are becoming more similar or perhaps less similar over time. The sequence may also be utilized to attempt to detect whether a shock resulting in a change of Boolean transformation at the net elements has been applied to the system.

Of course the fact that under a static situation any set sequence must eventually fall into a cycle implies that both are the sequence of proportions and the sequence of distances will eventually be cyclical as well. In fact, because these variables

reduce state vectors to one dimensional observations, these sequences may exhibit even shorter cycles. Hence in a sense these variables are also cycle related. Does this remark deny their usefulness in modeling situations? The answer is no. With increasing net size, N , we know that cycle lengths will tend to grow quite large. Since we intend to apply frequent structural and/or functional changes to the model, in applications where these variables are appropriate it is unlikely that we will observe these sequences long enough to encounter cyclic behavior. The more crucial issue is to establish intervening control that, for example, stabilizes the sequence of proportions or decreases the sequence of distances.

Increasing the response possibilities for the net elements, say to n values instead of 2, leads to the examination of what we have called n -ary switching nets. Such nets offer a wider range of variables to study, but at the expense of much less tractability both behaviorally and analytically.

4. Equivalent Representations of Switching Nets.

The cycle space or state diagram in Figure 4 which was derived from Figures 1 and 2 is, in fact, equivalent to Figures 1 and 2. That is, from Figure 4 we may retrieve the network diagram and the associated Boolean transformations. To see how this may be done, we refer to Table 3. In Table 3 at the left all sixteen possible net states are arranged by row in lexicographic order. (It will be shortly apparent that any order may be used.) At the right we have four columns corresponding to

each of the four associated mappings. For a particular row (state), to obtain the entries for m_1 , m_2 , m_3 and m_4 , we merely refer to the state diagram and find the next state and insert this state as m_1 through m_4 . For example, at the row 1101 we would find the next state to be 1100 and thus have $m_1(1,1,0,1) = 1$, $m_2(1,1,0,1) = 1$, $m_3(1,1,0,1) = 0$ and $m_4(1,1,0,1) = 0$.

Elements				Mapping			
1	2	3	4	m_1	m_2	m_3	m_4
0	0	0	0	0	1	0	1
1	0	0	0	0	1	1	1
0	1	0	0	0	1	0	0
1	1	0	0	0	1	1	0
0	0	1	0	0	1	0	1
1	0	1	0	0	1	1	1
0	1	1	0	0	0	0	0
1	1	1	0	0	0	1	0
0	0	0	1	1	1	0	1
1	0	0	1	1	1	0	1
0	1	0	1	1	1	0	0
1	1	0	1	1	1	0	0
0	0	1	1	1	1	0	1
1	0	1	1	1	1	0	1
0	1	1	1	1	0	0	0
1	1	1	1	1	0	0	0

Table 3: Reconstruction of Boolean Transformation from a Net Diagram.

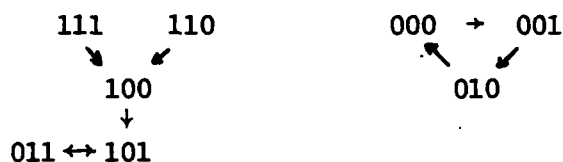
Table 3 at this point describes a fully connected net. We need to clarify the connections matrix which we may do as follows. To establish whether, say, element 2 is an input to element 3 we must ask whether $m_3(x_1, 0, x_3, x_4) = m_3(x_1, 1, x_3, x_4)$, for all x_1 , x_3 and x_4 combinations. If the answer is yes, then the value of element 2 does not affect the resultant value of element 3, i.e. element 2 does not provide input to element 3. If the answer is no, then the converse is true and element 2 does input to element 3. In this way we could establish that elements 1 and 4 input to element 3 while elements 2 and 3 do not and thus reduce the mapping m_3 in Table 3 to just two inputs (4 rows) as in Figure 2.

Without bothering to formalize the above process, it should not be clear that any binary switching net model is equivalent to its state diagram. It is still true that, for our purposes, in describing and analyzing these nets the former representation is the more convenient. This is so by virtue of the previously expressed fact that the state diagram representation of the switching net subverts the Boolean transformation notion. However, the state diagram does lead us to other net representations and, in particular, to the clever linearization of Cull and Caianello which we now discuss.

The state diagram may readily be converted to a transition matrix as follows. Identify with each state a standard basis vector in 2^N dimensional Euclidean space, i.e. state i will be isomorphic to the vector e_i which has a 1 in the i -th row and 0's in the remaining $2^N - 1$ rows.

Then clearly there is a $2^N \times 2^N$ matrix T such that $T_{ij} = 1$ if and only if state i is the successor to state j . T will be a matrix of 0's and 1's with exactly one 1 in each column. Clearly there are $(2^N)^{2^N}$ distinct T matrices corresponding to the number of mappings from a set of 2^N elements into itself, i.e. corresponding to the number of state diagrams for a switching net composed of N elements.

Thus the matrix T is equivalent to a switching net model. As a simple example with $N = 3$, consider the state diagram



Then with the states arranged in lexicographic order, T becomes

	000	100	010	110	001	101	011	111
000	0	0	1	0	0	0	0	0
100	0	0	0	1	0	0	0	1
010	0	0	0	0	1	0	0	0
110	0	0	0	0	0	0	0	0
001	1	0	0	0	0	0	0	0
101	0	1	0	0	0	0	1	0
011	0	0	0	0	0	1	0	0
111	0	0	0	0	0	0	0	0

Next we note that if $X^{(t+1)}$ and $X^{(t)}$ are N dimensional vectors representing the state of the net at time $t+1$ and t , respectively, then the switching net is equivalent to

$$(1) \quad X^{(t+1)} = F(X^{(t)})$$

where F is the "next state" function. In this notation we have F denoted by

$$F: X\{0,1\}^N \rightarrow X\{0,1\}^N .$$

In general the function F will be non-linear. But it is also a mapping from a finite set to the same finite set. This suggests that F in (1) will behave like a permutation function and since a permutation function has a matrix representation there should be a matrix representation of F .

To create this function matrix which we will denote by F , let us first recall some basic properties of the simplest finite field, the integers modulo 2. In this field there are only two numbers, 0 and 1. The two operations, addition represented by $+$ and multiplication represented by juxtaposition, are defined by the relationships

$$\begin{aligned} 0 + 0 &= 1 + 1 = 0 \\ 0 + 1 &= 1 + 0 = 1 \\ 00 &= 01 = 10 = 0 \\ 11 &= 1 . \end{aligned}$$

A moment's reflection reveals that there are exactly 2^{2^N} functions of N variables over this field, i.e. each of the 2^N points in the domain can be assigned one of 2 values in the range. This is merely a restatement of the fact that there are 2^{2^N} possible Boolean transformations with N inputs, i.e. each function is equivalent to a Boolean transformation. More importantly each function is equivalent to a multinomial in N variables over this field. To clarify the notion of a multinomial, let x_1, x_2, \dots, x_N denote the N variables. There are 2^N possible subsets of the x 's which may be drawn. For the i -th subset, $i = 1, 2, \dots, 2^N$, let g_i be the function which is the product of all the x 's in this subset. (For the empty set, take $g = 1$.) Let c_i , $i = 1, 2, \dots, 2^N$, be a set of constants each of which is either 0 or 1. Then $\sum c_i g_i$ defines a multinomial in N variables over this field. For example, if $N = 3$ we have

$$g_1 = 1, \quad g_2 = x_1, \quad g_3 = x_2, \quad g_4 = x_1 x_2$$

$$g_5 = x_3, \quad g_6 = x_1 x_3, \quad g_7 = x_2 x_3, \quad g_8 = x_1 x_2 x_3$$

and if $c_3 = c_6 = c_8 = 1$ with the remaining $c_i = 0$, we obtain the multinomial

$$x_2 + x_1 x_3 + x_1 x_2 x_3.$$

Since there are 2^{2^N} distinct choices for the set of constants, there are 2^{2^N} distinct multinomials over this field. Any function of N variables over this field may be represented as a 2^N dimensional vector of 0's and 1's with the 0's and 1's being the coefficients of the multinomial representing this function. The function or multinomial may obviously be represented by an inner (dot) product on this field, i.e.

$$(2) \quad f = \underline{c}^t \underline{g}$$

where \underline{c} is the vector of c_i 's and \underline{g} is the vector of g_i 's. The function can be evaluated at a particular point (x_1, \dots, x_n) by inserting these values into \underline{g} .

It is quite apparent that given a specific multinomial, one can readily find the equivalent Boolean transformation by inserting state vectors (points) into (2) in lexicographic order and obtaining the function values. However, given a Boolean transformation, i.e. the function values at each state, it is not at all clear how to directly obtain the equivalent multinomial. That is, how do we find the corresponding \underline{c} in a systematic manner?

The answer requires a bit of insight and a bit of work. Suppose we order the states and the g_i 's in the manner suggested by our examples with $N = 3$. That is, we place them in lexicographic order. This ordering is clear for the states. For the g functions, since there are 2^N of them as well, they may be placed in correspondence with the states where ordered function g_i takes as its subset those

x's which are set at '1' in ordered state i . Now suppose we derive a $2^N \times 2^N$ matrix G whose j -th column is g_j evaluated at the j -th ordered state. Note that an element g_{ij} of G evaluated for a given state s_i can only be 1 if every x in g_j 's subset is valued at 1 in s_i . Hence by the defined order on the g 's and s 's, G_{ij} which is the i -th ordered product g_j , evaluated at the ordered state s_i must be 1 if $i = j$ and 0 if $i > j$. Therefore, G is an upper triangular matrix of 0's and 1's with all diagonal elements equal to 1. It may also be seen that G is symmetric not with respect to the main diagonal but with respect to the diagonal orthogonal to it. For our example, with $N = 3$ we have

$$G = \begin{matrix} & \begin{matrix} 000 & 100 & 101 & 110 & 001 & 101 & 011 & 111 \end{matrix} \\ \begin{matrix} 1 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{matrix} & \begin{bmatrix} 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 0 & 1 & 0 & 1 & 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 1 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix} \end{matrix}$$

We next note that G is self-inverse with respect to the field of integers mod 2, i.e.

$$(3) \quad GG = I \pmod{2} .$$

This is most easily proved by induction. Denoting by G_N the resulting $2^N \times 2^N$ matrix for a given N one may show that

$$G_{N+1} = \begin{pmatrix} G_N & G_N \\ 0 & G_N \end{pmatrix}$$

from which the induction follows directly.

Returning to (2) let V_f be the 1×2^N vector whose i -th entry is f evaluated at ordered state s_i . Then (2) may be extended to

$$(4) \quad V_f = \tilde{c}^t g$$

from which, using (3),

$$(5) \quad V_f G = \tilde{c}^t,$$

i.e. the coefficient vector c may be obtained.

Returning to our earlier example with $N = 3$, what are the multinomials f_1 , f_2 and f_3 guiding the elements 1, 2 and 3, respectively? From the state diagram we have

$$V_{f_1} = (0, 1, 0, 1, 0, 0, 1, 1)$$

$$V_{f_2} = (0, 0, 0, 0, 1, 1, 0, 0)$$

$$V_{f_3} = (1, 1, 0, 0, 0, 1, 1, 0) .$$

Thus

$$c_1^t = (0,1,0,0,0,1,1,0)$$

$$c_2^t = (0,0,0,0,1,0,1,0)$$

$$c_3^t = (1,0,1,0,1,1,0,0)$$

and

$$f_1 = x_1 + x_1x_3 + x_2x_3$$

$$f_2 = x_3 + x_2x_3$$

$$f_3 = 1 + x_2 + x_3 + x_1x_3 .$$

We are now finally ready to define the function matrix F we alluded to earlier. F is the $2^N \times 2^N$ matrix that has as its rows the coefficient vectors (c 's) of the 2^N products of the N multinomial functions computed by the N elements of the switching net. That is, analogous to what we did in developing the g 's we create 2^N products from each of the 2^N subsets of the f_1, f_2, \dots, f_N . Each such product will yield a multinomial function over our field. Corresponding to the empty set, we take the constant function 1. As before, the subset may be sequenced in the lexicographic order thus ordering the rows of F .

In taking the products of these functions, it is convenient to observe that for any x , $x^p = x$, and that for any g , $2ag = 0$ for any integer a . Thus, for example, using f_1 and f_2 above we discover

$$\begin{aligned}
f_1 f_2 &= (x_1 + x_1 x_3 + x_2 x_3)(x_3 + x_2 x_3) \\
&= x_1 x_3 + x_1 x_2 x_3 + x_1 x_3 + x_1 x_2 x_3 + x_2 x_3 + x_2 x_3 \\
&= 2x_1 x_3 + 2x_2 x_3 + 2x_1 x_2 x_3 = 0 .
\end{aligned}$$

In fact the full F matrix for $f_1 f_2$ and f_3 above becomes

	000	100	010	110	001	101	011	111
1	1	0	0	0	0	0	0	0
f_1	0	1	0	0	0	1	1	0
f_2	0	0	0	0	1	0	1	0
$f_1 f_2$	0	0	0	0	0	0	0	0
f	1	0	1	0	1	1	0	0
$f_1 f_3$	0	1	0	1	0	1	1	0
$f_2 f_3$	0	0	0	0	0	1	0	1
$f_1 f_2 f_3$	0	0	0	0	0	0	0	0

It is apparent that the matrix F is yet another equivalent representation of the net. In fact, the rows corresponding to the functions f_1 , f_2 and f_3 would suffice since we have shown that they are equivalent to the three Boolean transformations computed by the three net elements. Our reason for augmenting additional rows is to establish a relationship between F and T. That the function matrix and the transition matrix should be related does not seem surprising particularly since both are $2^N \times 2^N$.

The relationship involves the G matrix derived earlier. This matrix has as its columns representations of the states through the

field of multinomials, i.e. the i -th column of G corresponds to the i -th ordered state. Multiplying this matrix on the left by F results for the i -th column in the state to which the i -th ordered state goes.

Now recall that T is a matrix with exactly one 1 in each column such that the particular row in which it appears indicates the successor state. Consider multiplying G on the right by T . The i -th column of this product matrix will again be the state to which the i -th ordered state goes.

In other words we have the relationship

$$(6) \quad FG = GT .$$

In our example it may be verified that

$$FG = GT = \begin{bmatrix} 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 0 & 1 & 0 & 1 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 & 0 & 1 & 1 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

Since G is self inverse (6) implies

$$(7) \quad F = GTG , \quad T = FGF$$

This means we may easily obtain the functional representation from the transitional representation and vice versa.

In the preceding development the reader may have failed to grasp the linearization of the net that has been effected through the F matrix. The next state function F given in (1) clearly needn't be linear. However employing the state representations provided by the columns of G we see from (6) that the F matrix operates linearly (matrix multiplication) to provide the next net state.

In summarizing this section we have developed 4 equivalent representations for a binary switching net. They are

- (i) a state diagram,
- (ii) a network (N,K) and an associated set of Boolean transformation,
- (iii) a transition matrix T ,
- (iv) a function matrix F .

It is obvious that representation (i) offers the most visual display of the net's behavior. However, in theoretical examination of switching net behavior representations (ii), (iii) and (iv) are most convenient.

5. Applications.

The basic structure and behavior of switching net models having been described, it is now time to turn to a discussion of several useful settings to which we may apply these models.

I. Management Strategy in Complex Organizations.

The organizational analogy we develop interprets the switching net as a control system imbedded in an organization. A given model might encompass virtually a whole organization or only some part of it; there may, however, be several control systems in any given organization. The control system is not restricted to single level activity, but rather may include heirarchical aspects as well. The net elements correspond to points in the organization where control information is used. The use may be executive, evaluative, productive and so forth according to the particular activity being modeled. Thus elements could variously be seen as governmental bureaus, mechanics, or individuals. For example, the net elements could collectively represent a group of middle managers carrying out organizational routine, staff personnel reaching a decision, foremen and their crews producing goods, or mixtures of these types. The networks structure represents the control relationships which exist among the elements, i.e. the channels through which actual control information is passed in the system. The response of a control point to the possible contingencies presented to it by its sources of information will be described by a Boolean transformation. It is recognized that dichotomizing the input information as well as the elemental response may somewhat limit the scope of activities to which these binary models may be applied. However, a considerable variety of processes are governed by yes-no, on-off, defective-nondefective, go-no go, etc. decisions. Moreover, for most of the remainder, n-ary

switching nets should provide an adequate model description. Though such nets will be a bit more complicated to examine, we are still dealing with essentially the same general systems modeling perspective.

More specifically, if the activity is essentially a production process, then the elements are seen as machines. The N machines represent N possibly different constituents each of which is either being fabricated or not. The cycle models the product, the transition states preceding the cycle model the start-up period and the entire set of cycles give the inventory of products the particular control structure and particular set of constituents is capable of producing.

While the system size N is typically conceived as quite large in switching net modeling (at least order 10^4 or 10^5 in our other applications) in the present instance N will likely be of order at most 10^2 . But the relatively smaller system size does not preclude complexity. The more important source of complexity in such real world organizations is their intricate structure. This structure is relatively fixed but unknown in detail. Even though an organizational control grid might be essentially fixed and in principle specifiable, the time required to know its details and thereby to make a prediction of system response might often exceed the time within which that prediction were needed. Therefore, it seems to us that one aspect of the practice of management requires a body of knowledge concerning what behavior may be predicted under conditions of structural uncertainty. Examining the behavior of collections of switching net models may help to increase our knowledge.

However, the Boolean transformations are a key concept in this analogy. Management strategy, i.e. the practice of management to achieve certain organizational behaviors, may be equated to restriction on the switching net model with regard to the set of available Boolean transformations. More precisely, the fact that an organization prospers or perhaps in some cases survives at all suggests that the completion of its individual processes does not frequently require excessive amounts of time. (The time frame, is of course, relative to the particular activity.) In other words, for the successful organization, repetitive activities must be completed in "reasonable" time epochs. Hence if our nets are to be effective models of real world organizations, they too must run through cycles having lengths which are empirically "reasonable." If N were, say, 100 and the net was uncontrolled then cycle lengths might be on the order of $2^{100} \approx 10^{30}$. Under any plausible assumptions as to how fast the net moves from state to state such a net could model no repetitive real world phenomenon. Therefore, we must impose constraints on the Boolean transformations governing the net elements to dramatically curtail cycle lengths.

It is clear that control systems in larger domains such as industries or economic sectors may well be examined through network models.

II. Genetic Control Systems.

The pioneering work in applying binary switching net theory to genetic control systems has been done by Kauffman. The net is interpreted as a cell and each element as a gene. Several idealizations

must be incorporated to develop the analogy. First of all, time must be presumed to occur in discrete clocked moments. The analytic advantages inherent in such a discrete scale offset the conceptual attractiveness of a continuous scale. Next, the pathway by which the output of a gene comes to influence another gene will be ignored. We will only acknowledge whether or not a direct connection exists. In this sense a network structure is well defined. Finally, each gene will be considered to be a binary switch capable only of being fully on or fully off. It is fair to ask whether genes tend to be able to assume finely graded levels of steady activity or whether they tend to be very active or very inactive. Kauffman offers several theoretical and experimental reasons to suggest the latter to be true. Hence, Boolean transformations become appropriate descriptions of the response of a gene to its input information. The various net states correspond to differing states of gene activity for the cell. A regular pattern of gene activity or net states would correspond to a cycle. The collection of all cycles describes the various dynamic behaviors of the cell. With regard to the connectance, gene control systems of cells are almost certainly not one input systems. In known experimental cases, we always find at least two inputs. Moreover, behaviorally there are two obvious major disadvantages to a one input specification. First is that component failure could disconnect from the system all members of the heirarchy descendent from the defective gene. Second, one input switching nets do not provide the homeostatic tendency that gene systems exhibit. So gene control systems

may be taken to have multiple inputs often including feedback. The possibility of many inputs allows the possibility of redundancy and thus of more reliable behavior. Additionally the greater the number of inputs, the more subtle and complex the cell's behavior can be. However, with increased input connectance, the net will not show the restricted patterns of activity or the homeostatic behavior found in cells. To constrain the net model to more realistic behavior will require, as in the previous application, the imposition of constraint on the Boolean transformations governing the net elements. Perhaps surprisingly the same equivalence classes of constrained Boolean transformations developed in connection with the managerial strategy application will have attractive interpretations in the biological context as well.

A second argument underscoring the need for constraint is as follows. Current estimates of the number of genes in a higher metazoan cell range from 40,000 up to 1,000,000. A metazoan with only 100,000 genes would then have $2^{100,000} \approx 10^{30,000}$ conceivable states of gene activity. At known rates of gene activity, a cell could not explore that dynamic space in billions of times the history of the universe. While it is unknown how small the subset of patterns of gene activity is, it must be quite small since one can recognize the same cell type over time and over cell divisions. This "survival" argument parallels that advanced previously on behalf of the successful organization.

That a phenomenon involving 40,000 to 1,000,000 elements is complex is obvious. It is not reasonable to expect to discover virtually all control relations among such a large number of genes. Thus switching net models provides one approach to constructing an adequate picture of the architecture of cell control systems whose full details may never be directly known. Kauffman offers a lucid statement of the philosophy behind this approach capturing its essential "small scale properties to large scale behavior" idea.

"[The] approach is to characterize any known small scale properties of the organization of cellular control systems, such as specifying the typical number of variables controlling any process and specifying the ways variations in the controlling process affect the controlled processes. Specification of such small scale local properties should be useful in two ways: (1) the local properties form the basis for hypotheses about the organization of larger control circuits; (2) the implications of the small scale properties for the large scale dynamic behavior of cellular control systems can be assessed. Systematic use of such local characteristics for both these purposes can be made by constructing a set of all the possible large control systems, each member of which is built using only these small scale properties."

Kauffman is describing an attractive modeling approach which incorporates switching nets. In his view the primary purpose in characterizing small scale properties and in constructing an ensemble of possible control

systems is to examine the implications of known small scale features for probable large scale properties.

III. Advertising Policy in Consumer Markets.

The notion that in many markets consumer purchase behavior cannot be well articulated by static models is gaining increasing acceptance. Whether the product is inexpensive such as soap or toothpaste, or costly such as a car or hi-fi equipment, the buying process is typically an exceedingly complex, many variable, time-dependent one. Hopefully some facets of the process can be usefully studied separately. One such facet is the word-of-mouth dynamics that influence prospective buyers in the choice of brand.

We can think of many markets in which it is abundantly clear that word-of-mouth is an important variable. Home buying is one example. In influencing a buyer's choice of brokerage firm, word-of-mouth has been cited (Hempel) as a factor second in importance only to specific property-oriented newspaper advertisements. The convenience and subtle influence of advice offered during the ordinary routine of social interaction ought not be underestimated. Effective marketing of a product mandates concern with these informal communication elements as well as the more familiar commercial channels. Advertising is to some extent predicated upon the assumption that this informal facet of consumer behavior can be manipulated. That is, for a particular brand, it can assist in the creation of favorable word-of-mouth for that brand in the market. Hence, successful

advertising policy in employing available commercial channels must comprehend how these channels will affect word-of-mouth produced dispositions.

The preceding preamble clearly demonstrates the worth of studying the phenomenon of word-of-mouth in certain dynamic systems. That word-of-mouth is a complex phenomenon is apparent. It will typically involve a large number of individuals whose intercommunication structure is intricate and whose response to communication is non-homogeneous. But most attractive of all for our purpose is the fact that binary switching net models are useful in studying the behavior of such systems.

The analogy for this application assumes that the net is a market area and that the net elements are households (a customary unit in marketing). For a given household, the associated influence structure is assumed to be relatively fixed. Hence we have a network. Our focus is on opinions favorable or unfavorable held by households with respect to a specific brand in the market. This dichotomy of opinion suggests that the reactions of an individual to the opinion of other households with respect to the brand may be described via a Boolean function. Function inputs are then the households influence sources and the output value at any time represents the household's opinion on the brand at that time. The heterogeneity of households leads to a collection of Boolean transformations, one for each household thus completing the switching net analogy. The network structure allows self-feedback, certainly, for many households, a component in molding opinion from influence sources.

The collective market opinion at a given time is represented by the state vector of the net at that time. One simple summary variable, which may be monitored over time, is the proportion of favorable opinions in the market. Other variables may be developed to measure opinion constancy, i.e. brand loyalty. These can be formulated as proportions as well. Note that cycles do not seem relevant in this application. In studying the behavior of these proportions an ensemble approach naturally arises. Random net models are drawn according to a fixed market size and a fixed degree of influence upon a household. These replications enable one to observe how the average and variability of these process proportion variables change with time.

Advertising efforts are presumed to be directed at how households transact their word-of-mouth opinions rather than being directed at modifying the existing transaction patterns. Again the key role of the Boolean transformation is revealed. Structural constraint in the selection of these mappings to achieve desirable behavior of the system variables may be viewed as adopting different advertising policies.

References

- Alben, R. and Boutron, P. (1975). Continuous Random Network Model for Amorphous Solid Water. Science 187, 430-432.
- Arbib, M.A. (1972). The Metaphysical Brain. J. Wiley and Sons, New York.
- Bell, A.J. and Dean, P. (1972). Structure of Vitreous Silica. Validity of Random Network Theory. Philos. Mag. 25, 1381-1398.
- Blumenson, L.E. (1970). Random Walk and the Spread of Cancer. J. Theor. Biology 27, 273-290.
- Caianiello, E.R. (1973). Some Remarks on the Temporal Linearization of General and Linearly Separable Boolean Functions. Kybernetik 12, 90-93.
- Cull, P. (1971). Linear Analysis of Switching Nets. Kybernetik 8, 31-39.
- Cull, P. (1975). Control of Switching Nets. Biological Cybernetics 19, 137-145.
- Cull, P. (1977). "A Matrix Algebra for Neural Nets," paper presented at the International Conference on Applied General Systems Research, Binghamton, New York.
- Doreian, P. (1974). On the Connectivity of Social Networks. J. Math. Sociol. 3, 245-258.
- Gelfand, A.E. and Walker, C.C. (1980). A System Theoretic Approach to the Management of Complex Organizations: Management by Consensus Level and its Interaction with other Management Strategies. Behavioral Science (to appear).
- Hempel, D. (1969). "The Role of the Real Estate Broker in the Home Buying Process," Univ. of Conn. - CREUES Real Estate Report #7.
- Kauffman, S. (1974). The Large Scale Structure and Dynamics of General Control Circuits: An Ensemble Approach. J. Theor. Biology, 44, 167-190.
- Kauffman, S. (1970). The Organization of Cellular Genetic Control Systems. Mathematics in the Life Sciences 3, 63-116.
- McCulloch, W.S. and Pitts, W.H. (1943). A Logical Calculus of the Ideas Imminent in Nervous Activity. Bull. Math. Biophys. 5, 115-133.

- Rapoport, A. (1948). Cycle Distributions in Random Nets. Bull. Math. Biophys. 10, 145-157.
- Rapoport, A. (1954). "Random Nets with Transitivity Bias," Symposium on Information Networks, Brooklyn, 187-197.
- Rashevsky, N. (1960). Mathematical Biophysics, Dover Press, New York, 230-271.
- Shimbel, A. (1948). An Analysis of Theoretical Systems of Differentiating Nervous Tissue. Bull. of Math. Biophysics 10, 131-143.
- Shimbel, A. (1951). A Note on the Problem of Cycles in Random Nets. Bull. of Math. Biophysics 13, 319-321.
- Solomonoff, R. (1952). An Exact Method for the Computation of the Connectivity of Random Nets. Bull. of Math. Biophysics 14, 153-157.
- Solomonoff, R. and Rapoport, A. (1951). Connectivity of Random Net. Bull. of Math. Biophysics 13, 131-143.
- Stubbs, D.F. (1977). Connectivity and the Brain. Kybernetics 6, 1-10.
- Stubbs, D.F. and Good, I.J. (1976). Connectivity in Random Networks. Bull. of Math. Biology 38, 295-304.
- Walker, C.C. and Gelfand, A.E. (1979). A System Theoretic Approach to the Management of Complex Organizations: Management by Exception, Priority and Input Span in a Class of Fixed Structure Models. Behavioral Science 24, 112-120.

UNCLASSIFIED

SECURITY CLASSIFICATION OF THIS PAGE (When Data Entered)

REPORT DOCUMENTATION PAGE		READ INSTRUCTIONS BEFORE COMPLETING FORM
1. REPORT NUMBER 343	2. GOVT ACCESSION NO. <i>AD-A14 1057</i>	3. RECIPIENT'S CATALOG NUMBER
4. TITLE (and Subtitle) Switching Net Models: Rudimentary Behavior, Representations and Applications		5. TYPE OF REPORT & PERIOD COVERED TECHNICAL REPORT
		6. PERFORMING ORG. REPORT NUMBER
7. AUTHOR(s) Alan E. Gelfand		8. CONTRACT OR GRANT NUMBER(s) N00014-76-C-0475
9. PERFORMING ORGANIZATION NAME AND ADDRESS Department of Statistics Stanford University Stanford, CA 94305		10. PROGRAM ELEMENT, PROJECT, TASK AREA & WORK UNIT NUMBERS NR-042-267
11. CONTROLLING OFFICE NAME AND ADDRESS Office of Naval Research Statistics and Probability Program Code 411SP		12. REPORT DATE April 4, 1984
		13. NUMBER OF PAGES 42
14. MONITORING AGENCY NAME & ADDRESS (if different from Controlling Office)		15. SECURITY CLASS. (of this report) UNCLASSIFIED
		15a. DECLASSIFICATION/DOWNGRADING SCHEDULE
16. DISTRIBUTION STATEMENT (of this Report) APPROVED FOR PUBLIC RELEASE: DISTRIBUTION UNLIMITED		
17. DISTRIBUTION STATEMENT (of the abstract entered in Block 20, if different from Report)		
18. SUPPLEMENTARY NOTES		
19. KEY WORDS (Continue on reverse side if necessary and identify by block number) Network models, Switching net models, Connectance, Boolean transformations, State diagram, Cyclic Behavior		
20. ABSTRACT (Continue on reverse side if necessary and identify by block number) PLEASE SEE REVERSE SIDE.		

UNCLASSIFIED

SECURITY CLASSIFICATION OF THIS PAGE (When Data Entered)

SWITCHING NET MODELS: RUDIMENTARY BEHAVIOR,
REPRESENTATIONS AND APPLICATIONS

Network models and in particular switching net models have shown themselves to be a useful class of abstract models to study. They exhibit considerable behavioral diversity and are comfortably amenable to both mathematical analysis and computer simulation. This report formalizes the notion of a binary switching net model. Rudimentary behavior of such nets is described. Four representations of a switching net are developed and shown to be equivalent. Finally, three attractive applications are presented. Throughout the report the need to exercise behavioral control over the nets is underscored. This provides an important direction for further research into these nets.

UNCLASSIFIED

END

FILMED

6-54

DTIC

