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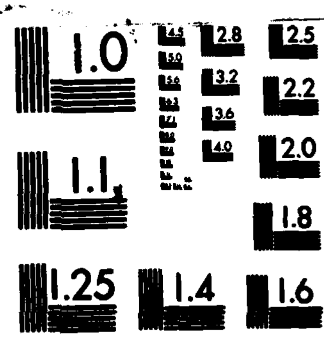


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# On the Representation of Probability Distributions as the Convolution of Symmetric and Completely Asymmetric Parts

Steven P. Ellis

## Abstract

Let F, G, and H be probability distributions on the line each having finite variance and suppose G is symmetric. F is "completely asymmetric" (c.as.) if the equation $F = G*H$ implies $G = \delta_0$, i.e. is degenerate. It's proven that F can always be written $F = G*H$ where H is c.as., but this representation may not be unique. Examples of singular and absolutely continuous (with respect to Lebesgue measure) c.as. distributions are given. Some extensions of these ideas are mentioned.

## 1. Introduction and Main Result

Let F, G, and H be distributions (= probability distributions) on $\mathbb{R}$ (= real line). Let $s(F) = \int x^2 F(dx)$. Say that F has a <u>finite variance</u> if $s(F) < \infty$. G is <u>symmetric</u> if $G(-A) = G(A)$ for every Borel set $A \subset \mathbb{R}$. ($-A = \{-a: a \in A\}$.) Say that F is <u>completely asymmetric</u> (<u>c.as.</u>) if whenever F, G, and H have finite variances, G is symmetric, and $F = G*H$, then $G = \delta_0$. ("*" = "convolution". $\delta_x$ = unit mass at $x \in \mathbb{R}$.)

The proof of the following employs Zorn's lemma and a martingale argument. It is found in the appendix along with the proof of proposition 2.1.

<u>1.1</u> <u>Proposition</u>: If F has finite variance then there exist distributions G and H s.t. (such that):

i) G and H have finite variance,

ii) G is symmetric and H is c.as.,

iii)

(1.1) $F = G*H$.

In section 2 I give examples of singular and absolutely continuous (with respect to Lebesgue measure) c.as. distributions. I also give an example in which the decomposition (1.1) isn't unique.



## 2. Examples

Some generic examples of c.as. distributions are provided by the following

<u>2.1</u> <u>Proposition</u>:

a) Let $a,b \in \mathbb{R}$ and $1/2 < \lambda \leq 1$. Then $\lambda\delta_a + (1-\lambda)\delta_b$ is c.as.

b) Let F be a distribution with finite variance and support in $[a,\infty)$, where a is finite. Suppose F has a density, f, satisfying

i) f is upper semicontinuous and finite in $(a,\infty)$,

ii) $\lim_{h \downarrow 0} f(a+h) = \infty$.

Then F is c.as.

The proof is found in the appendix. Notice that part (a) shows that the class of c.as. distributions isn't closed. E.g. as $\lambda \downarrow 1/2$ $\lambda\delta_{-1} + (1-\lambda)\delta_1 \to (1/2)(\delta_{-1}+\delta_1)$ both weakly and strongly.

An example of the non-uniqueness of the representation (1.1) follows.

<u>2.2</u> <u>Example</u>: Let $\mu = (3/5)\delta_{-1} + (2/5)\delta_1$. Then by proposition 2.1

(a), $\mu$ is c.as. Let $\nu = (1/2)\delta_{-1} + (1/2)\delta_1$, so $\nu$ is symmetric, and let $\eta$ be the symmetric signed measure

$$\eta = (1/5)\delta_{-4} - (1/10)\delta_{-2} + (4/5)\delta_0 - (1/10)\delta_2 + (1/5)\delta_4.$$

Then

$$\eta * \mu = (3/25)\delta_{-5} + (1/50)\delta_{-3} + (11/25)\delta_{-1} + (13/50)\delta_1 + (2/25)\delta_3 + (2/25)\delta_5$$

and

$$\eta * \nu = (1/10)\delta_{-5} + (1/20)\delta_{-3} + (7/20)\delta_{-1} + (7/20)\delta_1 + (1/20)\delta_3 + (1/10)\delta_5$$

Notice that both of these are probability distributions, the latter being symmetric. Let $F = (\eta * \nu) * \mu = \nu * (\eta * \mu)$. $F = (\eta * \nu) * \mu$ is obviously one representation of the type (1.1). But by proposition 1.1, $\eta * \mu$ can be written in the form $\nu' * \mu'$, where $\nu'$ is symmetric and $\mu'$ is c.as. (I systematically set out to factor $\eta * \mu$ in this way and ruled out all non-trivial factorizations. Thus, unless I made a mistake, $\eta * \mu$ is actually c.as. so $\nu' = \delta_0$.) Thus, $F = (\eta * \nu) * \mu$ and $F = (\nu * \nu') * \mu'$ are two decompositions of the type (1.1). If $\mu = \mu'$ then it would follow by considering Fourier transforms that $\eta = \nu'$. But $\nu'$ is positive while $\eta$ is not. It follows that $F$ has at least two distinct decomposition into symmetric and c.as. parts. Thus the decompositions (1.1) isn't generally unique.

## 3. Some half-baked ideas

(A detailed exposition of most of the ideas in this section can be found in Ellis (1980).)



The original motivation behind this work was to help answer the identifiability question for the following nonparametric regression problem. Let X be a random variable whose distribution has support [0,1]. Let f be a function on [0,1] and let H be the distribution on the plane of the vector (X,f(X)). Let G be a distribution on the plane satisfying some symmetry conditions and suppose we observe a random sample from the distribution $G * H$. The problem is to estimate f from the random sample.

My original idea for investigating the identifiability of f was to prove a two-dimensional analogue of proposition 1.1 (easy, I think) and then hope that the decomposition was unique so the identifiability question would be reduced to that of the complete asymmetry of H. The latter question is fairly manageable. But, of course, example 2.2 of the last section shows that the decomposition may not be unique.

To get around this problem of lack of uniqueness, I investigated the question of decompositions like (1.1) for objects other than probability distribution, namely for finite signed measures on Z and for generalized functions on R. In both cases I assumed analyticity of the Fourier transforms. For these objects the situation is much simpler than it is for probability distributions on R and thecorresponding decompositions are about as unique as one could hope for. Unfortunately, it appears to be very hard to get the same results on the plane.

Getting away from the original regression problem, here are some other, related, ideas I've explored. Since the decomposition (1.1) isn't unique, what is the structure of the set of all decompositions? To try and gain some insight into this I considered the following relation. If $F$, and $G$, $H$, and $K$ are distributions, write $F \sim G$ if $H$ and $K$ are symmetric and $F*H = G*K$. $\sim$ is an equivalence relation. (Note that one can construct random variables $U, V, X, Y$ with distributions $H, K, F, G$ resp. such that $X$ and $U$ are independent, $Y$ and $V$ are independent, $U+X = V+Y$, and $(U,X)$ and $(V,Y)$ are conditionally independent given $U+X$. This might be a useful tool for studying $\sim$.)

Let $[F] = \{G: G \sim F\}$. Then it's not hard to see that $[F]$ is convex and closed relative to a number of topologies. Furthermore, if    $= \{[F]: F$ is a distribution$\}$ then convolution of elements of    is well-defined and in fact    is a group. ($[\delta_0]$ is the identity and $[\bar{F}]$, where $\bar{F}(A) = F(-A)$, is the inverse of $[F]$.) Thus, $\sim$ has a tantalizing structure. Unfortunately, I wasn't able to get any further with these ideas.

__Appendix__: Proofs.

__A-1 Lemma__: If $H$ is a c.ss. distribution then $H*\delta_x$ is also c.ss. for every $x \in \mathbb{R}$.

__Proof__: Suppose $H$ is c.ss. but for some $x$ $H*\delta_x$ isn't. Then there are distributions $G'$ and $H'$ s.t. $G'$ is symmetric but $G \neq \delta_0$ and $H*\delta_x = G'*H'$. Then $H = G'*(H'*\delta_{-x})$, contradiction.

q.e.d.



__Proof of proposition 1.1__: By the lemma, WLOG (without loss of generality), $F$ is centered: $\int xF(dx) = 0$. Let $M$ denote the set of all distributions, $H$, on $\mathbb{R}$ s.t. $s(H) < \infty$ and s.t. there exists a symmetric distribution, $G$, on $\mathbb{R}$ s.t. $s(G) < \infty$ and $F = G*H$. $M$ isn't empty since $F = \delta_0 * F$.

If $H$ is a distribution, let $m(H) = \int xH(dx)$, whenever the latter integral is defined. Let $S = \{H: H$ is a distribution and $s(H) < \infty\}$. If $H \in M$ and $F = G*H$ where $G \in S$ and $G$ is symmetric, then $m(H) = 0$ since $m(F) = 0 = m(G)$. If $H_1, H_2 \in M$, write $H_1 \le_M H_2$ if there exists a symmetric distribution, $G \in S$, s.t. $H_2 = G*H_1$. Notice that if $H_1 \le_M H_2$ and $H_2 \le_M H_1$, then there are symmetric $G_1, G_2 \in S$ s.t. $H_2 = G_1*H_1$, $H_1 = G_2*H_2$. Thus, $H_1 = G_1*G_2*H_1$. Hence, $G_1*G_2 = \delta_0$ (just consider variances) so by symmetry of $G_1$ and $G_2$, $G_1 = \delta_0 = G_2$. I.e. $H_1 = H_2$. It follows that $\le_M$ is a partial ordering.

The proposition amounts to saying $M$ contains a minimal element w.r.t. (with respect to) $\le_M$. I'll prove this using Zorn's lemma. Let $A = \{H_\alpha, \alpha \in I\}$ __SS__ $M$ can be a totally ordered w.r.t. $\le_M$. Then it suffices to show that $A$ has a lower bound in $M$. Let $v = \inf\{s(H_\alpha): \alpha \in I\}$. Choose a countable sequence $H_{\alpha(n)}$ s.t. $s(H_{\alpha(n)}) \downarrow v$ and write $H_n = H_{\alpha(n)}$, $n = 1, 2, \ldots$ . Then $m > n$ implies $s(H_m) \le s(H_n)$ and since $A$ is totally ordered it follows that $H_m \le_M H_n$.

Claim: Any lower bound for $\{H_n\}$ in $M$ (if there are any) is a lower bound for $A$. For suppose not. Then there exists $\alpha \in I$ and $H \in M$ s.t. $H_\alpha \le_M H \le_M H_n$ for all $n$ but $H_\alpha \neq H$. This implies $v \le s(H_\alpha) < s(H) \le s(H_n)$ for all $n$. But this contradicts the way $\{\alpha_n\}$ was chosen.

Thus, to find a lower bound for A, I need only find one for $\{H_n\}$. I'll do this by constructing a reversed martingale whose marginal distributions are $H_n$, $n \geq 1$ and then applying a martingale limit theorem. If $k \in N$, $H_k \leq_H H_{k-1}$ ($H_0 = F$) so there exists a symmetric distribution $K_k \in S$ s.t. $H_{k-1} = H_k * K_k$. For each n, let $X_{-n,n}$ be a r.v. (random variable) with distribution $H_n$ and let $Y_{-1,n}, \ldots, Y_{-n,n}$ be independent r.v.'s independent of $X_{-n,n}$ s.t. $Y_{-k,n}$ has distribution $K_k$. Let

$$X_{-n+k,n} = X_{-n,n} + \sum_{j=1}^{k} Y_{-n+k-1,n}, \quad k=1,\ldots,n.$$

Then $X_{-n+k,n}$ has distribution $H_{n-k}$. Note that for each k, $Y_{-n+k,n}$ is independent of $X_{-n,n},\ldots,X_{-n+k,n}$. Let $P_n$ denote the joint distribution of $(X_{0,n}, \ldots, X_{-n,n})$.

By the Kolmogorov extension theorem there exists a process $X_0$, $X_{-1}$, $X_{-2}, \ldots$ s.t. $P_n$ is the joint distribution of $X_0, \ldots, X_{-n}$. Thus, $X_k$ has distribution $H_k$, $k \geq 0$.

Let $Y_{-k} = X_{-k+1} - X_{-k}$, $k \geq 1$.
Then for $k \geq 1$,

i) $Y_{-k}$ has distribution $K_k$ (in particular $EY_{-k} = 0$),

ii) $Y_{-1}, Y_{-2}, \ldots$ are independent, and

iii) $Y_{-k}$ is independent of $X_{-k-j}$, $j \geq 0$.

It follows that $\{X_{-n}, n \geq 0\}$ is a martingale relative to its own internal history, $\underline{F}_{-n} = \sigma(X_{-m}, m \geq n)$, $n \geq 0$. Since $s(H_n) < \infty$ for



all n, $\{X^2_{-n}, n \geq 0\}$ is a submartingale relative to $\{\underline{F}_{-n}\}$. It follows from theorem 9.4.7 in Chung (1974) that $X_{-n}$ converges a.s. and in $L^2$ to a r.v. $X_{-\infty}$.

As a consequence of this, for each $n \geq 0$

$$\sum_{k=n}^{\infty} Y_{-k}$$

converges a.s. to r.v. $W_{-\infty,n}$ s.t. $W_{-\infty,n}$ is symmetric, $s(W_{-\infty,n}) < \infty$, and $X_{-\infty} + W_{-\infty,n} = X_n$. Furthermore, for each $m > n$,

$$\sum_{k=n}^{m} Y_{-k}$$

is independent of $X_{-\infty}$. Therefore, $W_{-\infty,n}$ is also independent of $X_{-\infty}$. (Just consider characteristic functions.)

But this means that if $H_\infty$ is the distribution of $X_{-\infty}$, then $H_\infty \in H$ and $H_\infty \leq_H H_0 = F$. The proposition follows.

q.e.d.

__Proof of proposition 2.1:__ a) By lemma A.1 we can assume $b = -a$. Let $1/2 < \lambda \leq 1$ and let $F = \lambda\delta_{-a} + (1-\lambda)\delta_a$. If $a = 0$ then we already know that F is e.ss. Assume that $a \neq 0$ and suppose $F = G*H$, where G is symmetric. The support of the convolution of two probability distributions is the sum of their support. It follows that either

(*) G has support on two points and H on one, or

(**) vice versa.

Suppose (*) holds, then it easily follows that $F$ is symmetric, which it isn't. Thus, (**) must hold and (a) is proved.

b) The proof is a little involved but the basic idea is as follows. Write $F = \mu * \nu$ where $\nu$ is symmetric. It will turn out that $\nu$ has bounded support. If $2b \geq 0$ is the width of that support, we'll see that $f$ must blow up at $a+2b$, contradicting (i) unless $b=0$, i.e. unless $F$ is e.as. By lemma A.1 we can assume $a=0$. By (ii) inf supp $F=a=0$. Since supp $F$ = supp $\mu$ + supp $\nu$ it follows that if $b$ = inf supp $\mu$, then $b > -\infty$ and $-b$ = inf supp $\nu$. But $\nu$ is symmetric so $b$ = sup supp $\nu \geq 0$.

In the remainder of the proof it will be convenient for $\mu$ and $\nu$ to have densities. Since $\mu$ and $\nu$ may not have densities, I change the distributions of $\mu$ and $\nu$ (and, hence, of $F$) so that they do have densities. I do this by convolving them with absolutely continuous distributions concentrated about 0.

Let $h_1$ be a symmetric unimodal probability density with support [0,1]. Suppose $h_1$ is everywhere differentiable with bounded derivative. Then $h_1(x) > 0$ if $-1 < x < 1$, $h'_1(x) \geq 0$ if $x \leq 0$, and $h'_1(x) \leq 0$ if $x \geq 0$. Let $g_1 = h_1 * h_1$. Then $g$ has support $[-2,2]$. Using the dominated convergence theorem, the mean value theorem, and the fact that $h$ has a bounded derivative, it's easy to see that the convolution of $h$ with any distribution, $\gamma$, also has a bounded derivative, $\int h'_1(x-y)\gamma(dy)$. In particular, $g_1$ has a bounded derivative. For $\varepsilon > 0$, let $h_\varepsilon(x) = \varepsilon^{-1}h_1(x/\varepsilon)$, $g_\varepsilon(x) = \varepsilon^{-1}g_1(x/\varepsilon)$. Then $h_\varepsilon$ and $g_\varepsilon$ have supports $[-\varepsilon,\varepsilon]$, $[-2\varepsilon,2\varepsilon]$, resp. and $g_\varepsilon = h_\varepsilon * h_\varepsilon$. Let $\eta_\varepsilon(dx) = h_\varepsilon(x)dx$ and $\gamma_\varepsilon(dx) = g_\varepsilon(x)dx = \eta_\varepsilon * \eta_\varepsilon(dx)$.



Assume $F$ isn't e.as. This amounts to assuming that $b > 0$. Let $\mu_\varepsilon = \mu * \eta_\varepsilon$, $\nu_\varepsilon = \nu * \eta_\varepsilon$, $F_\varepsilon = \mu_\varepsilon * \nu_\varepsilon = f * \gamma_\varepsilon$. Then $\mu_\varepsilon$, $\nu_\varepsilon$, and $F_\varepsilon$ have differentiable densities $m_\varepsilon := \mu * h_\varepsilon$, $n_\varepsilon := \nu * h_\varepsilon$, $f_\varepsilon := f * g_\varepsilon$, resp. ("$:=$" means "is defined to be.") Notice that supp $m_\varepsilon \subseteq [b-\varepsilon,\infty)$, supp $n_\varepsilon \subseteq [-b-\varepsilon, b+\varepsilon]$.

Now,

$$m'_\varepsilon(x) = \int_{[b,\infty]} h'_\varepsilon(x-y)\mu(dy).$$

Since $h'_\varepsilon \geq 0$ on $(-\infty,0]$, it follows that $m_\varepsilon$ is increasing on $(-\infty,b]$. Similarly, $n_\varepsilon$ is increasing on $(-\infty,-b]$.

Accept for the moment the following fact which I'll prove in a moment.

(A.1) $\qquad f_\varepsilon(2b) \to \infty$ as $\varepsilon \downarrow 0$.

I'll show that this implies $f(2b)=\infty$ which contradicts (i), since by assumption $b > 0$. Part (b) then follows.

$$f_\varepsilon(2b) = \int g_\varepsilon(2b-y)f(y)dy = \int_{2b-2\varepsilon}^{2b+2\varepsilon} g_\varepsilon(2b-y)f(y)dy$$

$$\leq \sup_{2b-2\varepsilon < y < 2b+2\varepsilon} f(y).$$

so by (A.1) this supremum $\to\infty$ as $\varepsilon \downarrow 0$. On the other hand, $f$ is upper semicontinuous so

$$\sup_{2b-2\varepsilon<y<2b+2\varepsilon} f(y) \to f(2b).$$

(A.1) is a consequence of two further facts:

(A.2) $f_\epsilon(-\epsilon) \to \infty$ as $\epsilon \downarrow 0$, and

(A.3) $f_\epsilon(2b) \geq f_\epsilon(-\epsilon)$ for all $\epsilon>0$.

Proof of (A.2): Since $g_\epsilon$ is symmetric

$$f_\epsilon(-\epsilon) = \int_0^\infty g_\epsilon(\epsilon+y)f(y)dy$$

$$= \int_0^1 g_1(1+z)f(\epsilon z)dz$$

$$\geq \inf_{0<z<1} f(\epsilon z) \int_0^1 g_1(1+z)dz.$$

Now, $\int_0^1 g_1(1+z)dz > 0$ and $\inf_{0<z<1} f(\epsilon z)$

$= \inf_{0<y<\epsilon} f(y) \to \infty$ as $\epsilon \downarrow 0$ by (ii)

Proof of (A.3):

$$f_\epsilon(-\epsilon) = (n_\epsilon * m_\epsilon)(-\epsilon) = \int n_\epsilon(-\epsilon - y) m_\epsilon(y)dy$$

(A.4)

$$= \int_{b-\epsilon}^{b} n_\epsilon(-\epsilon-y)m_\epsilon(y)dy.$$

On the other hand,

$$f_\epsilon(2b) = \int n_\epsilon(2b-y)m_\epsilon(y)dy$$

$$\geq \int_{b-\epsilon}^{b} n_\epsilon(2b-y)m_\epsilon(y)dy.$$

Combining this with (A.4) yields (making the change of variables $y = 2b-z-\epsilon$),



$$f_\epsilon(2b) - f_\epsilon(-\epsilon) \geq \int_{b-\epsilon}^{b} [n_\epsilon(2b-y) - n_\epsilon(-\epsilon-y)]m_\epsilon(y)dy$$

(A.5)

$$= \int_{b-\epsilon/2}^{b} [n_\epsilon(-z-\epsilon) - n_\epsilon(z-2b)]m_\epsilon(2b-z-\epsilon)dz$$

$$+ \int_{b-\epsilon/2}^{b} [n_\epsilon(y-2b) - n_\epsilon(-\epsilon-y)]m_\epsilon(y)dy$$

$$= \int_{b-1/2\epsilon}^{b} [n_\epsilon(y-2b) - n_\epsilon(-y-\epsilon)][m_\epsilon(y) - m_\epsilon(2b-y-\epsilon)]dy.$$

Now, if $y \in (b-\epsilon/2, b)$, then $-b > y-2b > -y-\epsilon$ and $b > y > 2b-y-\epsilon$. Since $n_\epsilon$ and $m_\epsilon$ are increasing on $(-\infty,-b]$ and $(-\infty,b]$, resp., it follows that the last integral in (A.5) is nonnegative, i.e. (A.3) holds.

q.e.d.

__Remark__: Notice that in the proof of proposition 2.1, no use whatsoever was made of any moment properties of the distributions involved. Thus, proposition 2.1 holds for a stronger definition of e.ss. in which no moment requirements are made.

## References

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Ellis, S.P. (1980) "Nonparametric regression with errors in both variables; decomposition of probability measures, summable sequences, and generalized functions into symmetric and completely asymmetric parts," unpublished manuscript.

Unclassified

SECURITY CLASSIFICATION OF THIS PAGE (When Data Entered)

# REPORT DOCUMENTATION PAGE

READ INSTRUCTIONS BEFORE COMPLETING FORM

1. REPORT NUMBER: 35
2. GOVT ACCESSION NO.: AD-A140 843
3. RECIPIENT'S CATALOG NUMBER:
4. TITLE (and Subtitle): On the Representation of Probability Distributions as the Convolution of Symmetric and Completely Asymmetric Parts
5. TYPE OF REPORT & PERIOD COVERED: Technical Report
6. PERFORMING ORG. REPORT NUMBER:
7. AUTHOR(s): Steven P. Ellis
8. CONTRACT OR GRANT NUMBER(s): N00014-75-C-0555
9. PERFORMING ORGANIZATION NAME AND ADDRESS: Massachusetts Institute of Technology, Statistics Center, Cambridge, MA 02139
10. PROGRAM ELEMENT, PROJECT, TASK AREA & WORK UNIT NUMBERS: (NR-609-001)
11. CONTROLLING OFFICE NAME AND ADDRESS: Office of Naval Research, Statistics and Probability Code 436, Arlington, VA 22217
12. REPORT DATE: March 1984
13. NUMBER OF PAGES: 15
14. MONITORING AGENCY NAME & ADDRESS (if different from Controlling Office):
15. SECURITY CLASS. (of this report): Unclassified

15a. DECLASSIFICATION/DOWNGRADING SCHEDULE:

16. DISTRIBUTION STATEMENT (of this Report): This document has been approved for public release and sale; its distribution is unlimited.
17. DISTRIBUTION STATEMENT (of the abstract entered in Block 20, if different from Report):
18. SUPPLEMENTARY NOTES:
19. KEY WORDS (Continue on reverse side if necessary and identify by block number): Convolution, completely assymmetric distribution, symmetric distribution
20. ABSTRACT (Continue on reverse side if necessary and identify by block number):

Let F, G, and H be probability distributions on the line each having finite variance and suppose G is symmetric. F is "completely asymmetric" (c.as.) if the equation F = G*H implies G = [delta sub 0], i.e. is degenerate. It's proven that F can always be written F = G*H where H is c.as., but this representation may not be unique. Examples of singular and absolutely continuous (with respect to Lebesgue measure) c.as. distributions are given. Some extensions of these ideas are mentioned.

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S/N 102-LF-014-6601

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## Errata in Ellis (1984) "On the representation..."

This sheet lists some misprints in the paper which appear thru my own negligence. (It's the misprints, not the paper which appears thru my negligence.) Most of the misprints have the following origin. The paper was typed on a word processor whose charactor set does not include certain mathematical symbols. I had intended to go thru and insert these symbols by hand but forgot to. So on p. 3, 4th line on the first paragraph, the symbol $\subseteq$ should b inserted between A and R. On p. 7 a $\mathcal{G}$ should be inserted before the expression "{[F]: F is a distribution}". Also in that paragraph it is $\mathcal{G}$ that is well-defined and a group. In the parenthetical statement in the same paragraph $[\check{F}]$, $\check{F}(A) = F(-A)$, is the inverse of [F].

The symbol **SS** should be replaced by $\subseteq$ wherever it is found: Third paragraph on p. 8 and in the first paragraph on p. 12.

A couple more conventional misprints are as follows. In the first paragraph beginning on p. 9, "H n" should be replaced by "$H_n$" in the 6th line. In the first line on p. 14 the limits, b-ε and b, of integration for the integral are in the wrong spot.

I found a few more misprints, but they shouldn't cause any confusion.