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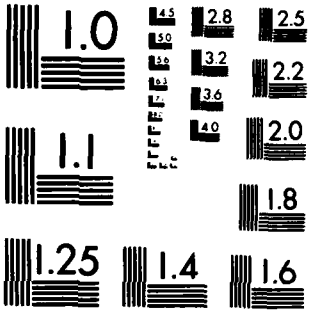

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FINAL REPORT

TO

NAVAL RESEARCH LABORATORY

Contract #N00014-82-C-2084

GYROTRON BACKWARD-WAVE OSCILLATOR

Submitted by

Dr. S. Y. Park

Omega-P, Inc.

September, 1983

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GYROTRON BACKWARD-WAVE OSCILLATOR

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backward wave oscillator

ABSTRACT

A linear theory for a gyrotron (BWO) has been developed and is presented in this report. The theory solves a reduced one-dimensional Maxwell-Vlasov equation in the form of a linear integro-differential equation using Laplace transformation. The relative amplitudes among the waveguide modes and beam modes are completely determined and enable us to determine BWO oscillation conditions.

A design is made based upon this analysis including velocity spread effects. A several tens kilowatt power level gyrotron BWO may be possible and find useful applications as rf heating sources in fusion devices and as driving sources in high power gyrotron amplifiers.

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I. INTRODUCTION

The gyrotron has demonstrated unprecedented capability of producing high power microwaves with high efficiency at high frequencies.[1] Most gyrotrons use a cyclotron maser instability (CMI) in a forward wave region requiring external feedback and thus cavity structure. The rf field in the cavity is well approximated by the cavity structure alone and thus the theory is fairly well understood.[2]

A gyrotron forward wave amplifier operates in a waveguide without external feedback. However, its gain (growth rate) is essentially determined by one growing mode which can be determined by a complex dispersion relation alone, besides some uncertainty in the initial coupling of the beam mode-to-signal. Up to this uncertainty the amplifier theory is also fairly well understood.[3]

A gyrotron backward wave oscillator (BWO) operates also in a waveguide but the electron beam itself plays the role of feedback. Due to its self-feedback mechanism, a gyrotron BWO does not require an external feedback structure and thus allows us to tune the frequency electronically. In this case, at least all three backward propagating modes (two beam modes and one waveguide mode) play equally important roles and thus their amplitude information is essential as well as the complex dispersion relation. Previously,[4] we have developed a linear theory which enables us to calculate the amplitudes from boundary conditions. This solution of a boundary value problem using Laplace transformation allows us to investigate the backward wave oscillator which we will present in this report.

Conventional BWO requires a slow wave structure (usually, helical wire) which severely limits the power handling capability. Typically, a sweep oscillator which uses a conventional BWO delivers a power level of a few

tens milliwatt at 35 GHz.

A gyrotron BWO does not require any slow wave structure and thus its power handling capability is no worse than the gyrotron itself (sub-megawatt level). Actually a gyrotron BWO operates far above cutoff while a gyrotron operates near the cutoff and therefore the wall loss of the gyrotron BWO should be even smaller than the gyrotron. Also it is tunable both by voltage and magnetic field while conventional BWO is tunable only by voltage.

A backward wave oscillator is expected to be intrinsically less efficient than a forward wave oscillator. This is due to the less favorable operating conditions with beam bunching in a strong field region and power extracting in a weak field region. This handicap is compensated by the wide band electrical tunability. Therefore, emphasis should be placed on broad band tuning capability with wide band coupler design. This study shows that a few tens killowatt level of Ku-band gyrotron tunable over almost an octave range may be possible with an existing electron gun. Such a device may find useful applications as a driving source to a gyrotron amplifier or as a source in the study of rf heating of fusion devices.

As a source of gyrotron forward wave amplifiers, one may recall that a high power amplifier usually operates at a low gain regime requiring moderately high power source to drive it. As a source of rf heating of fusion devices one may recall that heating characteristics are highly dependent on the magnetic field, temperature, density of the fusion device as well as the polarization of the rf field. The tuning capability of gyrotron BWO may be invaluable to tune the frequency according to the operating condition of the fusion device.

In this report we will present a linear theory of a gyrotron backward wave oscillator. The beam is assumed to be a hollow gyrating beam from a typical MIG gun, and the waveguide is assumed to be a rectangular waveguide. Actually, a cylindrical waveguide has also been studied and the formulation can be carried out in such a way that one-to-one comparison can be made all along the calculation. The rectangular $TE_{10}^{\square}$ mode is chosen because of its ease of mode control (no mode competetion up to an octave) and design of a wide band coupler. A test model of a coupler has been made and demonstrated wide useful band coupling capability within a few dB coupling loss over an octave besides a couple of sharp lossy spikes.

In Section 2, we reduce Maxwell equations with an electron beam as a source into one dimensional equations governing the axial growth behavior. The reduction is possible under an assumption that a single waveguide mode interacts resonantly with the beam and the result of interaction is mainly growing or decaying of the mode axially with little modification in the transverse direction. Laplace transformation is performed to convert the differential equations into algebraic equations with the boundary values included explicitly.

In Section 3, we calculate the perturbed electron beam function from the linearized Vlasov equations. With harmonic expansion, the beam modulation can be written as hysteresis integral of the form (3.9) linearly proportional to the rf field. Laplace transformation again helps to reduce this integral into a product of two Laplace transformed functions.

In Section 4, the phase space integral is carried out to obtain the induced current and thus the source term in the Maxwell equation. The integration is arranged in such a way that all the phase space integral can be done for a "cold" beam and the velocity spread effect can be introduced

afterward through "statistical average" procedure.

In Section 5, the source term is combined with Maxwell equations and the resultant algebraic equations are solved and the inverse Laplace transformation is performed to obtain the field as a function of z. The complex dispersion relation is identified as a denominator of the integrand of the inverse Laplace transformation whose zeros determine poles. The residues at the poles determine the amplitudes of each mode. For a cold beam, the denominator is a quartic function giving four poles - two waveguide modes and two beam modes.

In Section 6, the dispersion relation is analyzed in terms of scale variables which enable us to compare the relative strength of beam coupling. Eliminating the weakly coupled oppositely propagating mode , one arrives at a simple approximate cubic dispersion relation which can be easily analyzed analytically. This formulation enables us to compare directly with the conventional Pierce type approach in a conventional microwave device.

In Section 7, the velocity spread effect is examined with a "generalized Lorenzian distribution" function (7.1). Other than the regular Lorenzian distribution function, the number of modes are more than four. A more realistic distribution function with n=2, the total number of modes are six and this distribution function was used to investigate the velocity spread effect in the design.

II. MAXWELL EQUATIONS WITH A SOURCE TERM

In this section, we will derive one-dimensional equations from Maxwell equations governing RF-field growing or decaying in presence of a source (an electron beam) interacting predominantly with a single mode, particularly $TE_{no}^{\square}$ modes in a rectangular waveguide. Of course, the induced current of the beam - which will be calculated from Vlasov equation in the following two sections - is modulated due to the coupling with RF-field and, thus, Maxwell-Vlasov equation forms a self-consistent integro-differential equation which will be solved in Section V. This type of single mode analysis is valid on the basis of the following assumptions.

In general, the presence of the beam in a waveguide alter the field pattern due to its space charge and make them grow or decay due to its interaction. Expanded in terms of empty waveguide modes, since they form a complete orthonormal set, the space charge effect appears as a mode-coupling and the interaction makes them grow or decay. In some cases, this mode-coupling is important in studying the mode competetion which will be dealt with somewhere else.

However, if the beam is tenuous, only those modes interacting resonantly with the beam will participate significantly. In practice, waveguide modes are fairly well-separated and the beam parameters can be chosen to be resonant only with one interesting mode. In this case, the effect of mode coupling is negligible and the dominant effect will be the growth or decay of the resonant mode as a result of interaction, allowing us a single mode analysis.

The above observations enable us to write an ansatz for the stationary fields of $TE_{no}^{\square}$ mode in a rectangular waveguide shown in Fig. 2.1 as

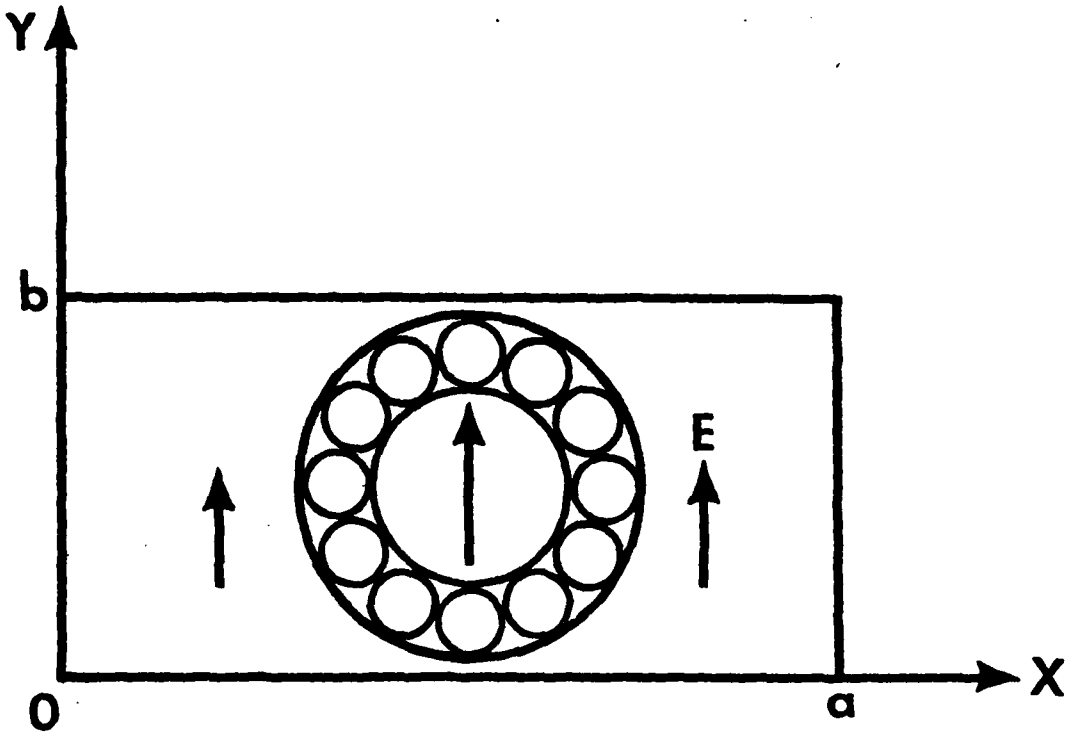

FIG. 2.1 A rectangular waveguide with a hollow electron beam.

$$
\begin{aligned}
E_y(\vec{x},t) &= -\, e^{-i\omega t}\overline{E}_y(z)e_y(x) \\
H_x(\vec{x},t) &= \quad e^{-i\omega t}\overline{H}_x(z)h_x(x) \\
H_z(\vec{x},t) &= -\, ie^{-i\omega t}\overline{H}_z(z)h_z(x) \\
&(E_x = E_z = H_y = 0)
\end{aligned}
\tag{2.1}
$$

where $e_y(x)$, $h_x(x)$, and $h_z(x)$ are $TE^{\square}_{no}$ empty waveguide mode given by

$$
\begin{aligned}
h_z(x) &= \cos k_\perp x \\
e_y(x) &= \frac{\omega/c}{k_\perp^2}\,\frac{d}{dx}\,h_z(x) = -\,\frac{\omega/c}{k_\perp}\,\sin k_\perp x \\
e_x(x) &= \frac{k_{\parallel}}{\omega/c}\,e_y(x) = -\,\frac{k_{\parallel}}{k_\perp}\,\sin k_\perp x \\
&\left(k_\perp^2 \equiv \frac{\omega^2}{c^2} - k_{\parallel}^2\right)
\end{aligned}
\tag{2.2}
$$

and $k_{\parallel}$ is a wave number of $TE^{\square}_{no}$ mode in the empty wave guide. Note that the z-dependencies of the fields in Eq. (2.1) are left undetermined functions to allow growth or decay which can be determined self-consistently with coupling to the source term.

Substituting the ansatz (2.1) into Maxwell equations, $\nabla \times \vec{E} = -\,\frac{1}{c}\frac{\partial \vec{H}}{\partial t}$ and $\nabla \times \vec{H} = \frac{1}{c}\frac{\partial \vec{E}}{\partial t} + \frac{4\pi}{c}\vec{J}$, and using (2.2), one obtains a set of one dimensional equations,

$$
\begin{aligned}
\frac{d}{dz}\,\overline{E}_y(z) &= ik_{\parallel}\overline{H}_x(z) \\
\overline{E}_y(z) &= \overline{H}_z(z)
\end{aligned}
\tag{2.3}
$$

and

$$
\left[\frac{d^2}{dz^2}\,\overline{E}_y(z) + k_{\parallel}^2\,\overline{E}_y(z)\right] e_y(x) = i\,\frac{4\pi}{c}\,\frac{\omega}{c}\,e^{i\omega t}J_y(\vec{x},t) \quad .
$$

Using $TE^{\square}_{no}$ mode projection operator, one can write the last equation as

$$\frac{d^2}{dz^2}\bar{E}_y(z) + k_{\parallel}^2\bar{E}_y(z) = i\,\frac{4\pi}{c}\,\frac{\omega}{c}\,e^{i\omega t}\bar{J}_y(z)/C_n \tag{2.4}$$

where

$$\begin{aligned}\bar{J}_y(z) &\equiv \int_0^a dx \int_0^b dy\; e_y(x) J_y(\vec{x},t)\\ C_n &\equiv \int_0^a dx \int_0^b dy\; e_y^2\,(x) = \frac{\omega^2/c^2}{k_\perp^2}\,\frac{ab}{2}\end{aligned} \tag{2.5}$$

Equations (2.3) and (2.4) form a Maxwell equation part of Maxwell-Vlasov equation, governing the growth or decay of RF field with a source term given in the form of hysteresis integral which will be calculated in the following two sections.

This type of one-dimensional integro-differential equation is best solved by Laplace transformation defined by

$$\tilde{F}(k) \equiv \int_0^\infty dz\; e^{-ikz}\; F(z) \qquad (\mathrm{Im}k \ll 0) \tag{2.6}$$

where k is a complex variable with sufficiently large negative imaginary value to guarantee the integral in (2.6) to be well-defined even when F(z) is an exponentially growing function as in the case of instability due to a resonant interaction. The Laplace transformation of Eq. (2.3) and (2.4) is

$$\begin{aligned}k_{\parallel}\tilde{H}_x(k) &= k\tilde{E}_y(k) + i\bar{E}_y(0)\\ \tilde{H}_z(k) &= \tilde{E}_y(k)\end{aligned} \tag{2.7}$$

and

$$(k^2 - k_{\parallel}^2)\tilde{E}_y(k) = \tilde{P}(k) - ik\bar{E}_y(0) - \left.\frac{d\bar{E}y}{dz}\right|_{z=0} \tag{2.8}$$

where

$$\tilde{P}(k) = -\,i\,\frac{4\pi}{c}\;\frac{\omega}{c}\,e^{i\omega t}\,\tilde{J}_y(k)/C_n \tag{2.9}$$

and $\tilde{J}_y(k)$ is the Laplace transformation of $\bar{J}_y(z)$ which is defined by (2.5).

Note that the boundary conditions $\bar{E}_y(0)$ and $(d\bar{E}_y/dz)_{z=0} = ik_{\parallel}\bar{H}_x(0)$ come in naturally in the above equations which might have been lost if we had used Fourier transformation instead of Laplace transformation. Furthermore Fourier transformation similar to (2.6), while the integration runs from $-\infty$ to $+\infty$, cannot be made well-defined in case of instability. In the following two sections, we will calculate the source term $\tilde{P}(k)$ defined by (2.9) from Vlasov equation.

III. VLASOV EQUATION - PERTURBED ELECTRON DISTRIBUTION FUNCTION

In a linear approximation, the perturbed part of electron distribution function is given by

$$f_1(\vec{x},\vec{p},t) = e\int_{t-z/v_{\parallel}}^{t} dt' \left(\vec{E}' + \frac{\vec{p}'}{mc\gamma}\times\vec{H}'\right)\cdot\frac{\partial f_0}{\partial\vec{p}'} \tag{3.1}$$

where f_0 is an unperturbed (equilibrium) electron distribution function. The integration must be done along the past history $(\vec{x}',\vec{p}',t')$ from the injection time $t_{in} = t - z/v_{\parallel}$ and the position $z_{in} = 0$ to the current time t and position z of the unperturbed characteristic. The characteristic - essentially an unperturbed "particle trajectory" - is simply a helical trajectory of an electron in a uniform magnetic field H_0 as depicted in Fig. 3.1 with

$$\begin{aligned} t' &= t + \frac{z'-z}{v_{\parallel}} \\ \phi' &= \phi + \omega_c\frac{z'-z}{v_{\parallel}} \end{aligned} \tag{3.2}$$

where $\omega_c \equiv \Omega_c/\gamma$ ($\Omega_c \equiv eH_0/mc$) is the relativistic cyclotron frequency.

The unperturbed electron distribution function f_0 is an arbitrary function of invariants of the characteristic. For a hollow electron beam they can be represented by three independent invariants, $p_{\parallel}$, $p_{\perp}$ and R_G such that $f_0 \equiv f_0(p_{\parallel},p_{\perp},R_G)$ where R_G is the guiding center radius. The R_G-dependency is essential to represent an inhomogeneous plasma such as a finite electron beam. Writing that

$$\vec{p}' = \hat{z}p_{\parallel} + \hat{\phi}'p_{\perp} \quad ; \quad \hat{\phi}' = \begin{pmatrix} -\sin\phi' \\ \cos\phi' \end{pmatrix} , \tag{3.3}$$

one can cast (3.1) for $TE^{\square}_{no}$ mode into the form,

$$\begin{aligned} f_1(\vec{x},\vec{p}.t) = e\int_{t-z/v_{\parallel}}^{t} dt' \Bigg\{ &\left[E'_y\frac{\partial f_0}{\partial p_{\perp}} - H'_x\left(\frac{p_{\perp}}{mc\gamma}\frac{\partial f_0}{\partial p_{\parallel}} - \frac{p_{\parallel}}{mc\gamma}\frac{\partial f_0}{\partial p_{\perp}}\right)\right]\cos\phi' \\ &-\left[\left(E'_y + H'_x\frac{p_{\parallel}}{mc\gamma}\cos\Theta + H'_z\frac{p_{\perp}}{mc\gamma}\sin\left(\phi'-\Theta\right)\right)\right]\frac{1}{\Omega_c}\frac{\partial f_0}{\partial R_G}\Bigg\} . \end{aligned} \tag{3.4}$$

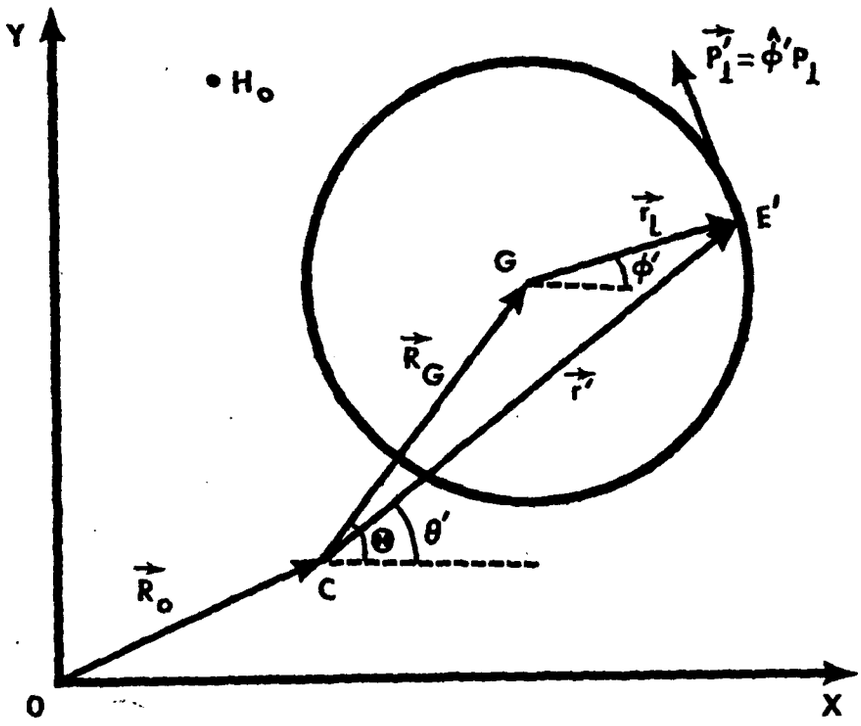

FIG. 3.1 The helical trajectory of an electron in a uniform magnetic field H_o.

Using the ansatz given by (2.1), one can write (3.3) (see Fig. 3.1)

$$f_1 = e\int_{t-z/v_\parallel}^{t} dt'\, e^{-i\omega t'}\Big[F_1'\sin\bar{X}'\cos\phi' - F_2'\sin\bar{X}'\cos\theta + F_3'\cos\bar{X}'i\sin(\phi'-\theta)\Big] \tag{3.5}$$

where

$$F_1' \equiv \bar{E}_y'\frac{\omega/c}{k_\perp}\frac{\partial f_0}{\partial p_\perp} + \bar{H}_x'\frac{k_\parallel}{k_\perp}\left(\frac{p_\perp}{mc\gamma}\frac{\partial f_0}{\partial p_\parallel} - \frac{p_\parallel}{mc\gamma}\frac{\partial f_0}{\partial p_\perp}\right)$$

$$F_2' \equiv \left(\bar{E}_y'\frac{\omega/c}{k_\perp} - \bar{H}_x'\frac{k_\parallel}{k_\perp}\frac{p_\parallel}{mc\gamma}\right)\frac{1}{\Omega_c}\frac{\partial f_0}{\partial R_G} \tag{3.6}$$

$$F_3' \equiv \bar{H}_z'\frac{p_\perp}{mc\gamma}\frac{1}{\Omega_c}\frac{\partial f_0}{\partial R_G}$$

and

$$\bar{X}' \equiv \bar{X} + a_L\cos\phi'$$

$$\bar{X} \equiv k_\perp\left(\frac{a}{2} + R_G\cos\theta\right) = \frac{n\pi}{2} + a_G\cos\theta \tag{3.7}$$

$$(a_L \equiv k_\perp r_L\ ,\ a_G \equiv k_\perp R_G)\quad .$$

Later, we will see that $\bar{E}_y'$-term of F_1' in (3.6) represents an azimuthal bunching in the phase space leading to a cyclotron maser instability and $\bar{H}_x'$-term, which vanishes if f_0 has an isotropic distribution in $p_\parallel$ and $p_\perp$, represents an axial bunching (resulting in an azimuthal bunching too) leading to Weibel instability. F_2' and F_3' vanish if f_0 is independent of R_G (homogeneous plasma) and thus represent geometrical effects of the finite beam. These alone do not lead to an instability but give substantial modification to the behavior of CMI and Weibel, particularly, at off-resonant behavior.

The trajectory integral (3.5) can be put into a more manageable form by harmonic expansion using the formula,

$$e^{ia_L\cos\phi'} = \sum_{s=-\infty}^{\infty} i^s e^{is\phi'} J_s(a_L)$$

$$e^{-ia_L\cos\phi'} = \sum_{s=-\infty}^{\infty} (-i)^s e^{is\phi'} J_s(a_L)$$

and, then,

$$\sin\bar{X}' = \sum_{s=-\infty}^{\infty} i^s e^{is\phi'} J_s(a_L) S_s$$

$$\cos\bar{X}' = \sum_{s=-\infty}^{\infty} i^s e^{is\phi'} J_s(a_L) C_s$$

$$\sin\bar{X}'\cos\phi' = -\sum_{s=-\infty}^{\infty} i^s e^{is\phi'} J_s'(a_L) C_s$$

$$\cos\bar{X}' i\sin(\phi'-\theta) = \sum_{s=-\infty}^{\infty} i^s e^{is\phi'} [\cos\theta \frac{s}{a_L} J_s(a_L) - i\sin\theta J_s'(a_L)] S_s$$

where

$$S_s \equiv \frac{1}{2i} [e^{i\bar{X}} - (-1)^s e^{-i\bar{X}}]$$
$$C_s \equiv \frac{1}{2} [e^{i\bar{X}} + (-1)^s e^{-i\bar{X}}] \quad . \tag{3.8}$$

Then, the integral (3.5) can be written as

$$f_1 = e\, e^{-i\omega t} \sum_{s=-\infty}^{\infty} i^s e^{is\phi} \int_0^z dz'\, G(z - z') F(z') \tag{3.9}$$

where

$$G(z - z') \equiv \frac{1}{v_{\parallel}} e^{i(\omega - s\omega_c)} \frac{z - z'}{v_{\parallel}}$$
$$F(z') \equiv -\bar{F}_1' C_s - (\bar{F}_2' \cos\theta + \bar{F}_3' i\sin\theta) S_s \tag{3.10}$$

and

$$\bar{F}_1' \equiv F_1' J_s'(a_L)$$
$$\bar{F}_2 \equiv (F_2' - F_3' \frac{s}{a_L}) J_s(a_L) \tag{3.11}$$
$$\bar{F}_3' \equiv F_3' J_s'(a_L) \quad .$$

The form of integral given by (3.9) is a typical hysteresis integral with $G(z - z')$ being a kernel - Green's function. The induced current corres-

ponding to (3.9) is the source term in (2.4) making it an integro-differential equation. The Laplace transformation of (3.9) is given by convolution theorem as

$$\tilde{f}_1(k) = e\, e^{-i\omega t} \sum_{s=-\infty}^{\infty} i^s e^{is\phi} \tilde{G}_s(k) \tilde{F}(k) \tag{3.12}$$

where

$$\begin{aligned} \tilde{G}(k) &= i\gamma/\Omega_s(k) \quad ; \quad \Omega_s(k) \equiv \omega\gamma - s\Omega_c - k\frac{p_{\parallel}}{m} \\ \tilde{F}(k) &= -\tilde{F}_1(k)C_s - (\tilde{F}_2(k)\cos\theta + \tilde{F}_3(k) i\sin\theta)S_s \quad . \end{aligned} \tag{3.13}$$

$\tilde{F}_1(k)$, $\tilde{F}_2(k)$, $_3(k)$ are the Laplace transformations of $\bar{F}_1'$, $\bar{F}_2'$, $\bar{F}_3'$, respectively, and, using (3.6) and(3.11),

$$\begin{aligned} \tilde{F}_1(k) &= \left[\tilde{E}_y(k)\frac{\omega/c}{k_\perp}\frac{\partial f_0}{\partial p_\perp} + \tilde{H}_x(k)\frac{k_\parallel}{k_\perp}\left(\frac{p_\perp}{mc\gamma}\frac{\partial f_0}{\partial p_\parallel} - \frac{p_\parallel}{mc\gamma}\frac{\partial f_0}{\partial p_\perp}\right)\right]\cdot J_s'(a_L) \\ \tilde{F}_2(k) &= \left[\tilde{E}_y(k)\omega\gamma - \tilde{H}_x(k)k_\parallel\frac{p_\parallel}{m} - \tilde{H}_z(k)s\Omega_c\right] J_s(a_L)\frac{1}{mc\gamma}\frac{1}{\Omega_c}\frac{\partial f_0}{\partial a_G} \\ \tilde{F}_3(k) &= \tilde{H}_z(k)\frac{p_\perp}{m} J_s'(a_L)\frac{1}{mc\gamma}\frac{1}{\Omega_c}\frac{\partial f_0}{\partial a_G} \quad . \end{aligned} \tag{3.14}$$

Notice that the interaction is greatly enhanced when the Laplace transformation of the Green's function, G(k), has poles, i.e., $\Omega_s(k) = 0$. However, exact locations of resonant interaction can be determined from the full Maxwell-Vlasov equation after we calculated the induced current corresponding to (3.12) and inserted into (2.8). In the next section we will calculate the induced current.

IV. SOURCE TERM-INDUCED CURRENT

The induced current corresponding to the perturbed electron distribution function f_1 is given by

$$\vec{J}(\vec{x},t) = Ne \int d^3\vec{p}\, \frac{\vec{p}}{m\gamma}\, f_1(\vec{x},\vec{p},t) \tag{4.1}$$

where N is the number of electrons per unit length (the beam function f_0 is normalized to be one per unit length). The relevant component of current is $\bar{J}_y(z)$ defined by (2.5) and its Laplace transformation can be written, with the perturbed electron distribution function, (3.12) - (3.14), as

$$\begin{aligned}
\tilde{J}_y(k) &= Ne \frac{\omega/c}{k_\perp} \int dxdy\, d^3\vec{p}\, \frac{p_\perp}{m\gamma} \sin\bar{X}\cos\phi \tilde{f}_1(k) \\
&= -i \frac{Ne^2}{m} \frac{\omega/c}{k_\perp} e^{-i\omega t} \sum_{s=-\infty}^{\infty} i^s \int dp_{\parallel} dp_\perp dR_G \\
&\quad \cdot R_G \frac{p_\perp^2}{\Omega_s(k)} \int d\theta d\phi \cos\phi e^{is\phi} \sin\bar{x} \quad \cdot \\
&\quad \{\tilde{F}_1(k) C_s + [\tilde{F}_2(k)\cos\theta + \tilde{F}_3(k) i \sin\theta] S_s\}
\end{aligned} \tag{4.2}$$

where G_s and S_s are defined by (3.8) and

$$\begin{aligned}
\bar{x} &\equiv \bar{X} + a_L \cos\phi \\
(\bar{X} &= \frac{n\pi}{2} + a_G \cos\theta) \quad .
\end{aligned} \tag{4.3}$$

First, phase angle (ϕ) integration can be done with the aid of formula,

$$\frac{1}{2\pi} \int_0^{2\pi} d\phi e^{is\phi + ia_L\cos\phi} = i^s J_s(a_L)$$

$$\frac{1}{2\pi} \int_0^{2\pi} d\phi e^{-is\phi - ia_L\cos\phi} = (-i)^s J_s(a_L) \quad ,$$

and thus

$$\frac{1}{2\pi} \int d\phi \cos\phi e^{is\phi} \sin\bar{x} = -i^s J_s'(a_L) C_s \quad . \tag{4.4}$$

Using (4.4), one obtains from (4.2),

$$\tilde{J}_y(k) = 2\pi i \frac{Ne^2}{m} \frac{\omega/c}{k_\perp} e^{-i\omega t} \sum_{s=-\infty}^{\infty} (-1)^s \int dp_\parallel dp_\perp dR_G$$
$$\cdot R_G \frac{p_\perp^2}{\Omega_s(k)} J_s'(a_L) \int d\theta C_s \{\tilde{F}_1(k) C_s + [\tilde{F}_2(k)\cos\theta + \tilde{F}_3(k) i \sin\theta] \cdot S_s\} \quad . \tag{4.5}$$

Next, one can do θ-integration using the formula

$$\frac{1}{2\pi}\int_0^{2\pi} d\theta \begin{pmatrix} \cos m\theta \\ \sin m\theta \end{pmatrix} e^{ia\cos\theta} = \begin{pmatrix} i m J_m(a) \\ 0 \end{pmatrix} \quad ,$$

which can be easily proved by expanding the exponential in terms of Bessel functions,

$$\tilde{J}_y(k) = 2\pi^2 i \frac{Ne^2}{m} \frac{\omega/c}{K_\perp} e^{-i\omega t} \sum_{s=-\infty}^{\infty} (-1)^s \int dp_\parallel dp_\perp dR_G$$
$$\cdot R_G \frac{p_\perp^2}{\Omega_s(k)} J_s'(a_L) \{\tilde{F}_1(k)[(-1)^n J_0(2a_G) + (-1)^s]$$
$$+ \tilde{F}_2(k)(-1)^n J_1(2a_G)\} \quad . \tag{4.6}$$

Note that the $\tilde{F}_3(k)$-term has dropped out in (4.6). This is a unique feature of a linearly polarized mode such as TE_{no}^{α} mode where left and right helicity contribution in $\tilde{F}_3(k)$-term cancel out.

The remaining integrals in (4.6) require a more specific form of the beam function $f_0(p_\parallel, p_\perp, R_G)$. However, it is not desirable to specialize in the beam function at this stage and one useful trick to avoid this problem is to use the following identity relation. For an arbitrary function $f_0(p_\parallel, p_\perp, R_G)$ one can write as

$$f_0(p_\parallel, p_\perp, R_G) \equiv \int 2\pi R_G^0 dR_G^0 dp_\parallel^0 2\pi p_\perp^0 dp_\perp^0 \quad .$$
$$f_0(p_\parallel^0, p_\perp^0, R_G^0) \cdot \hat{f}(p_\parallel, p_\perp, R_G \; ; \; p_\parallel^0, p_\perp^0, R_G^0) \tag{4.7}$$

where

$$\hat{f}(p_\parallel, p_\perp, R_G \; ; \; p_\parallel^0, p_+^0, R_G^0)$$
$$\equiv \frac{1}{2\pi R_G^0} \delta(R_G - R_G^0) \cdot \delta(p_\parallel - p_\parallel^0) \frac{1}{2\pi p_\perp^0} \delta(p_\perp - p_\perp^0) \tag{4.8}$$

is a "cold beam" distribution function with an infinitely thin ring guiding center R_G^0. With the aid of this identity relation, one can carry out the integration over R_G, $p_{\parallel}$, $p_{\perp}$ using the "cold beam" distribution function $\hat{f}_0$ and afterward one can integrate over the actual beam distribution function $f_0(p_{\parallel}^0, p_{\perp}^0, R^0)$ - "statistical average". If the beam is actually "cold", the latter step ("statistical average") is trivial and not necessary.

This observation allows us to carry out the integrations in (4.6) for the "cold beam" function $\hat{f}_0$ and postpone the "statistical average" with the actual beam function f_0 afterward. Understanding this, Eq. (4.6) with "cold beam" function $\hat{f}_0$ becomes, after now a trivial R_G-integration with δ-function,

$$\tilde{J}_y(k) = \pi i \frac{Ne^2}{m} \frac{\omega/c}{k_\perp} e^{-i\omega t} \sum_{s=-\infty}^{\infty} \int dp_{\parallel} dp_{\perp} \frac{p_\perp^2}{\Omega_s(k)} J_s'(a_L)$$
$$\cdot \{\bar{F}_1(k)[(-1)^{n+s} J_0(2a_G^0) + 1] - 2\bar{F}_2(k)(-1)^{n+s} J_0(2a_G^0)\} \tag{4.9}$$

where

$$\bar{F}_1(k) \equiv \left[\tilde{E}_y(k) \frac{\omega/c}{k_\perp} \frac{\partial \hat{g}_0}{\partial p_\perp} + \tilde{H}_x(k) \frac{k_\parallel}{k_\perp} \left(\frac{p_\perp}{mc\gamma} \frac{\partial \hat{g}_0}{\partial p_n} - \frac{p_\parallel}{mc\gamma} \frac{\partial \hat{g}_0}{\partial p_\perp}\right)\right] J_s'(a_L)$$
$$\bar{F}_2(k) \equiv \left[\tilde{E}_y(k)\omega\gamma - \tilde{H}_x(k) k_\parallel \frac{p_\parallel}{m} - \tilde{H}_z(k) s\Omega_c\right] J_s(a_L) \frac{\hat{g}_0}{c\gamma\Omega_c} \tag{4.10}$$

and

$$\hat{g}_o \equiv \delta(p_\parallel - p_\parallel^0) \frac{1}{2\pi p_\perp^0} \delta(p_\perp - p_\perp^0) \quad . \tag{4.11}$$

The remaining momentum integrations are again trivial by integration by parts,

$$\tilde{J}_y(k) = -\frac{i}{2} \frac{Ne^2}{m} \frac{\omega/c}{k_\perp} e^{-i\omega t}$$
$$\cdot \left\{F_0(a_G^0) \frac{1}{p_\perp^0}\left[\tilde{E}_y(k) \frac{\omega/c}{k_\perp} \frac{\partial}{\partial p_\perp^0} + \tilde{H}_x(k) \frac{k_\parallel}{k_\perp}\left(\frac{p_\perp^0}{mc\gamma_0} \frac{\partial}{\partial p_\parallel^0} - \frac{p_\parallel^0}{mc\gamma_0} \frac{\partial}{\partial p_\perp^0}\right)\right]\right.$$

$$\cdot\left[\frac{p_\perp^{02}}{\Omega_s^0(k)} J_s'(a_L^0)\right] \tag{4.12}$$

$$+\left[F_0(a_G^0) - 1\right]\left[\tilde{E}_y(k)\omega\gamma_0 - \tilde{H}_x(k)k_\parallel \frac{p_\parallel^0}{m} - \tilde{H}_z(k)s\Omega_c\right]\cdot \frac{p_\perp^0}{\Omega_s^0(k)} \frac{[J_s^2(a_L^0)]'}{mc\gamma_0\Omega_c}\Bigg\}$$

where

$$F_0(a_G^0) \equiv 1 + (-1)^{n+s} J_0(2a_G^0)$$

$$\Omega_s^0(k) \equiv \omega\gamma_0 - k_\parallel \frac{p_\parallel}{m} - s\Omega_c \quad . \tag{4.13}$$

Noting that

$$\frac{1}{p_\perp^0} \frac{\partial}{\partial p_\perp^0} \Omega_s^0(k) = \frac{\omega/c^2}{m^2\gamma_0}$$

$$\frac{1}{p_\perp^0}\left(p_+^0 \frac{\partial}{\partial p_\parallel^0} - p_\parallel^0 \frac{\partial}{\partial p_\perp^0}\right)\Omega_s^0(k) = - k/m$$

$$\frac{1}{p_\perp^0} \frac{\partial}{\partial p_\perp^0} \left[p_+^{02} J_s'^2(a_L^0)\right] = - a_L^0\left(1 - \frac{s^2}{a_L^{02}}\right)\left[J_s^2(a_L^0)\right]' \quad ,$$

we obtain the final expression for the source term $\tilde{P}(k)$ given by (2.9) as

$$\tilde{P}(k) = - i \frac{4\pi}{c} \frac{\omega}{c} e^{i\omega t}\tilde{J}_y(k)/C_n$$

$$= \frac{2\pi\bar{\nu}}{c_n} \frac{\omega^2/c^2}{k_\perp^2} \sum_{s=-\infty}^{\infty} \Bigg\{A_2 \frac{p_\perp^{02}/m^2}{\Omega_s^{02}(k)}\left[\tilde{E}_y(k) \frac{\omega^2}{c^2} - \tilde{H}_x(k)k_\parallel k\right] \tag{4.14}$$

$$+ A_1 \frac{s\Omega_c}{\Omega_s^0(k)} \tilde{H}_z(k) + A_0 \frac{1}{\Omega_s^0(k)}\left[\tilde{E}_y(k)\omega\gamma_0 - \tilde{H}_x(k)k_\parallel \frac{p_\parallel^0}{m} - \tilde{H}_z(k)s\Omega_c\right]\Bigg\}$$

where

$$A_2 \equiv F_0 J_s'^2(a_L^0)$$

$$A_1 \equiv F_0\, a_L^0\left(1 - \frac{s^2}{a_L^{02}}\right)\left[J_s^2(a_L^0)\right]' \tag{4.15}$$

$$A_0 \equiv \left(1 - \frac{s^2}{a_L^{02}} F_0\right) a_L^0 \left[J_s^2(a_L^0)\right] \quad .$$

If the beam function is different from the "cold beam" defined by (4.8), The "statistical average" of the type (4.7);

$$\langle \tilde{P}(k) \rangle \equiv \int 2\pi R_G^0 dR_G^0 dp_{\parallel}^0 2\pi p_{\perp}^0 dp_{\perp}^0 f_0(p_{\parallel}^0, p_{\perp}^0, R_G^0) \tilde{P}(k) \tag{4.16}$$

should be done at this stage.

V. GAIN - COLD BEAM

In this section we will solve the Maxwell-Vlasov equation given by (2.7), (2.8), and (4.11) for a cold beam. Using (2.7), one can write the source term (4.14) as

$$\tilde{P}(k) = \tilde{E}_y(k)S_1(k) - i\bar{E}_y(0)S_0(k) \tag{5.1}$$

where

$$S_1(k) \equiv N_0 \sum_{s=-\infty}^{\infty} \left[\frac{\left(\frac{\omega^2}{c^2} - k^2 \; p_\perp^2/m^2\right)}{\Omega_s^{02}(k)} A_2 + \frac{s\Omega_c}{\Omega_s^0(k)} A_1 + A_0 \right]$$

$$S_0(k) \equiv N_0 \sum_{s=-\infty}^{\infty} \left[\frac{kp_\perp^2/m^2}{\Omega_s^{02}(k)} A_2 + \frac{p_\parallel^0/m}{\Omega_s^0(k)} A_0 \right] \tag{5.2}$$

$$\left(N_0 \equiv \frac{4\pi\bar{\nu}}{ab} \right) \quad .$$

Substituting (5.1) into (2.8), one obtains

$$[k^2 - k_\parallel^2 - S_1(k)]\tilde{E}_y(k) = - i\bar{E}_y(0)[k + S_0(k)] - i\bar{H}_x(0)k_\parallel$$

or

$$i\tilde{E}_y(k) = \bar{E}_y(0) \frac{N_1(k)}{D(k)} + \bar{H}(0) \frac{N_2(k)}{D(k)} \tag{5.3}$$

where

$$D(k) \equiv (k^2 - k_\parallel^2) - N_0 \sum_{s=-\infty}^{\infty} \left[\frac{\left(\frac{\omega^2}{c^2} - k^2\right) p_\perp^{02}/m^2}{\Omega_s^{02}(k)} A_2 + \frac{s\Omega_c}{\Omega_s^0(k)} A_1 + A_0 \right]$$

$$N_1(k) \equiv k + N_0 \sum_{s=-\infty}^{\infty} \left[\frac{kp_\perp^{02}/m^2}{\Omega_s^{02}(k)} A_2 + \frac{p_\parallel^0/m}{\Omega_s^0(k)} A_0 \right] \tag{5.4}$$

$$N_2(k) \equiv k_\parallel \quad .$$

Equation (5.3) gives the Laplace transformed field $\tilde{E}_y(k)$ in terms of the boundary conditions $E_y(0)$ and $\bar{H}_x(0) = \frac{1}{ik_\parallel} \frac{d\bar{E}_y}{dz}(0)$. The fields in coordinate space can be easily obtained by inverse Laplace transformation using the Bromwitz formula

$$F(z) \equiv \frac{1}{2\pi}\int_{-ic-\infty}^{-ic+\infty} dk\, e^{ikz}\tilde{F}(k) \tag{5.5}$$

where c is a sufficiently large positive so that all the poles in $\tilde{F}(k)$ must fall above the integral path. When $\tilde{F}(k)$ has only isolated poles, one can close the path by an upper semi-infinite half circle using the Jordan's lemma and the integration is reduced to a simple residue counting at the poles;

$$\begin{aligned} F(z) &= \frac{1}{2\pi i}\oint_c dk\, e^{ikz} i\tilde{F}(k) \\ &= \sum_i \mathrm{Res}[e^{ikz} i\tilde{F}(k)]|_{k=k_i;\ \text{poles}} \quad . \end{aligned} \tag{5.6}$$

In order to apply (5.6) to (5.3), recall that N_0 is small ($\sim 10^{-3}$ for a typical M1G electron gun) and only the term with $\Omega_s^0(k) \equiv \omega\gamma_0 - kp_{\parallel}/m - s\Omega_c \cong 0$ contribute substantially in (5.4). In practice, for a strong magnetic field (Ω_c; large), only one specific harmonic number s can be made resonant [$\Omega_s^0(k) \cong 0$]. In this case, (5.3) can be rationalized with quartic polynomials as

$$i\tilde{E}_y(k) = \bar{E}_y(0)\frac{n_1(k)}{d(k)} + \bar{H}_x(0)\frac{n_2(k)}{d(k)} \tag{5.7}$$

where $d(k)$, $n_1(k)$, and $n_2(k)$ are obtained from $D(k)$, $N_1(k)$, and $N_2(k)$ by multiplying $\Omega_s^{02}(k)$. Since (5.7) is now written in terms of rationalized function, the inverse Laplace transformation (5.6) is trivial

$$\bar{E}_y(z) = \bar{E}_y(0)\sum_{i=1}^{4} e^{ik_i z}\frac{n_1(k_i)}{d'(k_i)} + \bar{H}_x(0)\sum_{i=1}^{4} e^{ik_i z}\frac{n_2(k_i)}{d'(k_i)} \tag{5.8}$$

where k_i's are poles given by

$$d(k_i) = 0 \tag{5.9}$$

and

$$d'(k) \equiv \frac{d}{dk}d(k) \quad .$$

Now we have completely determined the field $\bar{E}_y(z)$ for $z > 0$ and other components of fields by (2.3) in terms of the boundary values $\bar{E}_y(0)$ and

$H_x(0) = \frac{1}{ik_{\parallel}} \frac{d\bar{E}_y}{dz}(0)$. Eq. (5.8) shows that the fields are consisted of 4-propagating modes for a cold beam and a specific harmonic number with their relative amplitudes are completely determined. This is extremely important in studying a backward wave interaction where all the 4-modes are important not like as the forward interaction where one mode eventually dominates over other modes. Note that the condition (5.9) or equivalently $D(k) = 0$ determining poles is nothing more than a usual complex dispersion relation and the residues at the poles determine the relative amplitudes among the modes. If we had used Fourier transformation - even though it is ill-defined - we would have obtained only the dispersion relation with their amplitude information lost.

Typically, the structure of the poles are as shown in Fig. 5.1. Other components of the field are obtained from (5.8) using (2.3) as

$$\begin{aligned} \bar{H}_z(z) &= \bar{E}_y(z) \\ \bar{H}_x(z) &= \frac{1}{ik_{\parallel}} \frac{d}{dz} \bar{E}_y(z) \\ &= \bar{E}_y(0) \sum_{i=1}^{4} \frac{k_i}{k_{\parallel}} e^{ik_i z} \frac{n_1(k_i)}{d'(k_i)} + \bar{H}_x(0) \sum_{i=1}^{4} \frac{k_i}{k_{\parallel}} e^{ik_i z} \frac{n_2(k_i)}{d'(k_i)} \quad . \end{aligned} \tag{5.10}$$

The total power flow flowing through the waveguide is given by, recalling (2.1) and (2.5)

$$\begin{aligned} P(z) &= \int_0^a dx \int_0^b dy \cdot \frac{c}{8\pi} \mathrm{Re}(- E_y H_x^*) \\ &= \mathrm{Re}[\bar{E}_y(z) \cdot \bar{H}_x^*(z)] \cdot \frac{c}{8\pi} \frac{k_{\parallel}}{\omega/c} \frac{\omega^2/c^2}{k_{\perp}^2} \frac{ab}{2} \quad . \end{aligned} \tag{5.11}$$

The quantity which is more interesting is gain defined by

$$G_F(z) \equiv P(z)/P(0) = \mathrm{Re}[\bar{E}_y(z) \cdot \bar{H}_x^*(z)]/\mathrm{Re}[\bar{E}_y(0)\bar{H}_x^*(0)] \tag{5.12}$$

in the case of foward wave interaction or

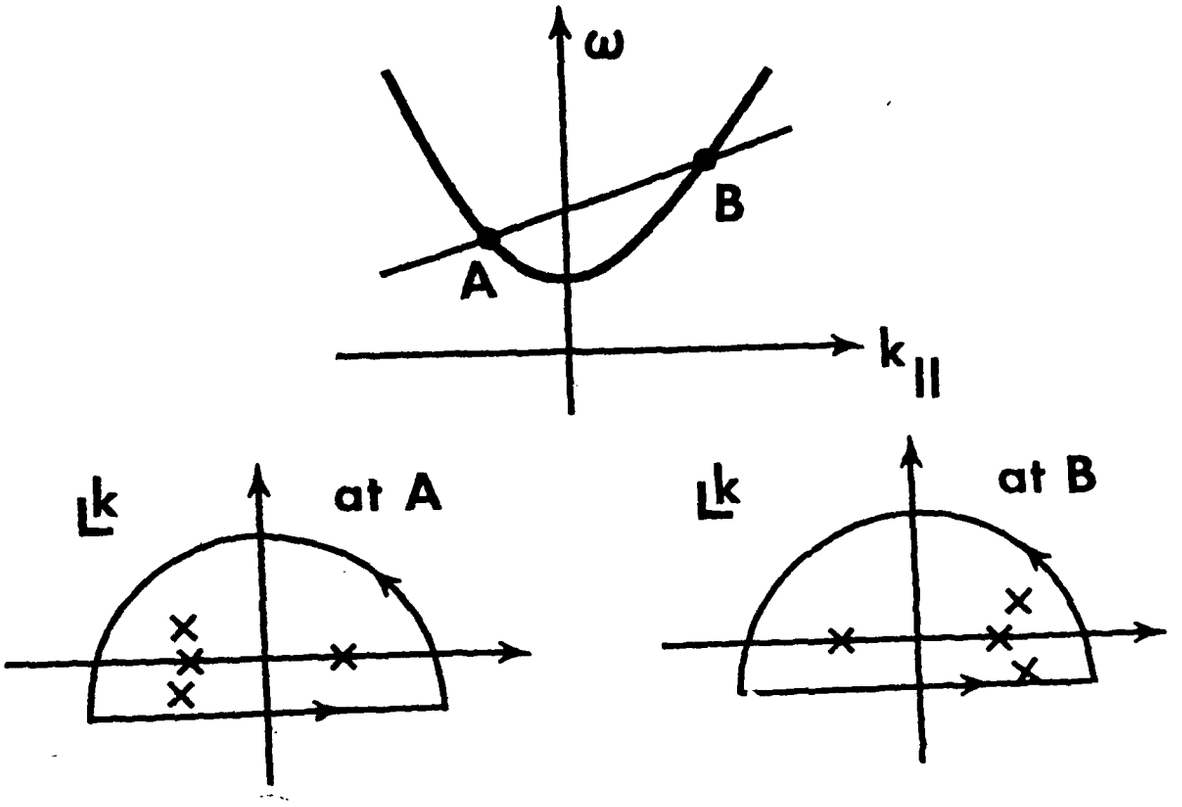

FIG. 5.1 Pole structures in a backward wave region and in a forward wave region.

$$G_B(z) \equiv P(0)/P(z) = R_e[\overline{E}_y(0)\overline{H}_x^*(0)]/Re[\overline{E}_y(z)\overline{H}_x^*(z)] \tag{5.13}$$

in case of backward wave. The gain in dB is defined by

$$G(dB) \equiv 10 \log_{10} G(z) \quad . \tag{5.14}$$

A typical gain as a function of interaction length z or a function of frequency ω/c is shown in Fig. 5.2.

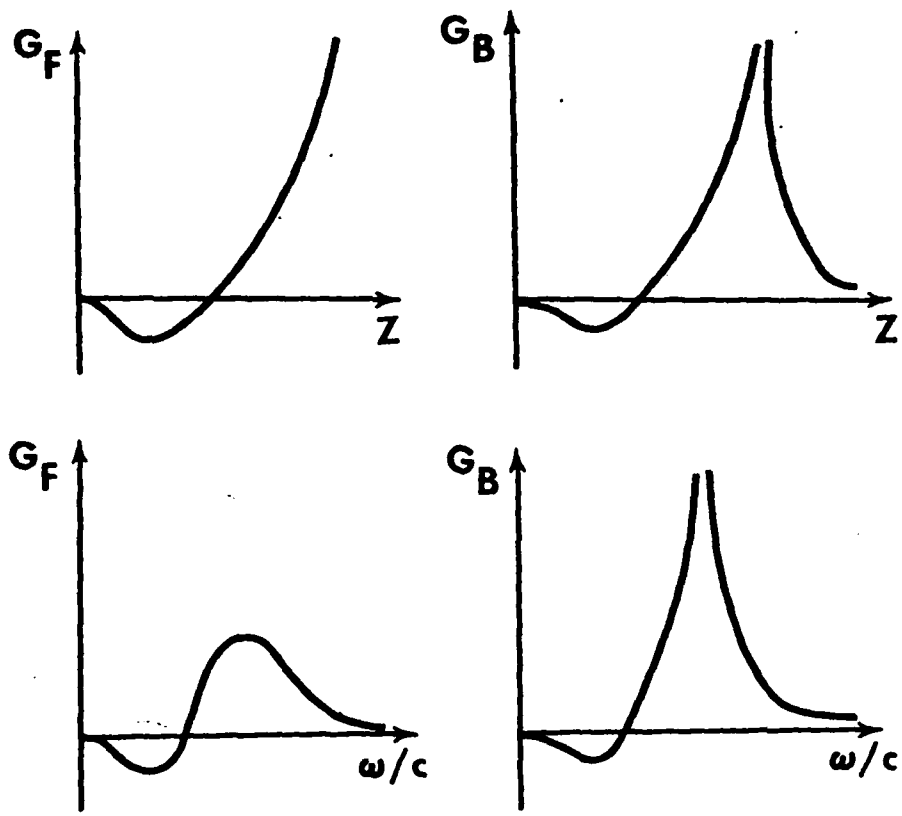

FIG. 5.2 Gain vs interaction length and frequency.

VI. THREE-WAVE ANALYSIS: PIERCE-TYPE FORMULATION

In the previous section, we have obtained the desired solution of one-dimensional boundary value problem for a TE_{no}^{a}-mode interacting with a hollow electron beam. The solution can be used to analyze a forward wave amplifier or backward wave oscillator/amplifier. The solution shows that, at least for a "cold" beam, the fields are given by four propagating waves with their relative amplitudes (including phases) completely determined by the boundary values, $\bar{E}_y(0)$ and $\bar{H}_x(0) = \frac{1}{ik_{\parallel}}\frac{d\bar{E}_y}{dz}(0)$, at the beginning of the interaction. Typically two of them are real "waveguide modes" propagating in opposite directions from each other. Two others - "Beam modes" - are complex conjugate each other, representing one growing and the other decaying and occur near the point where the beam line intersects with the waveguide dispersion line. Therefore, three waves are clustered near the intersecting point and the other one is at the point with the same ω but opposite sign of $k_{\parallel}$.

As one can see in the following, the waveguide mode propagating opposite to the beam is a nearly free propagating mode without interacting with the beam significantly. Therefore, one can eliminate this wave concentrating of three-wave analysis.

In order to get some feelings about the nature of the solutions, let us write the denominator of (5.7), whose zeros determine the possible modes as in the dispersion relation (5.9), in terms of dimensionless parameters. Defining the "phase shift" h and the "detuning parameter Δ as

$$\begin{aligned} h &\equiv k - k_{\parallel} \equiv k_{\parallel}\xi \\ \Delta &\equiv \omega\gamma_0 - k_{\parallel}\frac{\overset{0}{p}_{\parallel}}{m} - s\Omega_c \equiv k_{\parallel}\frac{\overset{0}{p}_{\parallel}}{m}\delta \quad , \end{aligned} \tag{6.1}$$

the "resonance factor" Ω_s^0 can be written as

$$\Omega_s^0(k) \equiv \omega\gamma_0 - k\frac{p_\parallel^0}{m} - s\Omega_c$$

$$= \Delta - h\frac{p_\parallel^0}{m} \tag{6.2}$$

$$\equiv k_\parallel \frac{p_\parallel^0}{m}\bar{\Omega}_s(\xi)$$

where

$$\bar{\Omega}_s(\xi) \equiv \delta - \xi \quad . \tag{6.3}$$

Then, factorizing the scale factor from the denominate d(k) as

$$d(k) \equiv 2k_\parallel^4 \frac{p_\parallel^{02}}{m}\bar{d}(k) \tag{6.4}$$

one obtains the dimensionless expression as

$$\bar{d}(\xi) \equiv \xi(1+\frac{\xi}{2})\bar{\Omega}_s^2(\xi) - g^3\{[1-\bar{A}_2\xi(1+\frac{\xi}{2})] + \bar{A}\,\bar{\Omega}_s(\xi) + \bar{A}_0\bar{\Omega}_s^2(\xi)\} \tag{6.5}$$

where

$$g^3 \equiv N_0 \frac{k_\perp^2}{2k_\parallel^4}\frac{p_\perp^2}{p_\parallel^{02}} A_2$$

$$\bar{A}_2 \equiv 2\frac{k_\parallel^2}{k_\perp^2}\ ,\ \bar{A}_1 \equiv \frac{s_{\Omega c}k_\parallel p_\parallel^0/m}{k_\perp^2 p_\perp^{02}/m^2}\frac{A_1}{A_2}\ ,\ \bar{A}_0 \equiv \frac{k_\parallel^2 p_\parallel^2}{k_\perp^2 p_\perp^2}\frac{A_0}{A_2} \tag{6.6}$$

and recall that N_0 is given by (5.2) and $A_i (i = 0,1,2)$ by (4.12).

For a tenuous beam, the "gain parameter" g is small in (6.5) and the solution of $\bar{d} = 0$ is roughly characterized by the solutions of

$$\xi(1+\frac{\xi}{2})\bar{\Omega}_s^2(\xi) \simeq 0$$

which gives two "waveguide modes" $\xi \simeq 0$, $\xi \simeq -2$ and two nearly degenerated "beam modes" $\bar{\Omega}_s(\xi) \simeq 0$ or $\xi \simeq \delta$. The mode with $\xi \simeq -2$ in (6.1) represents the opposite propagating mode modified by the beam only in the order of g^3 which is negligible. The other three modes are given by small $\xi \sim \delta \sim g$ in (6.5) as a solution of a simple cubic equation,

$$\xi(\delta - \xi)^2 - g^3 = 0 \quad . \tag{6.7}$$

Noting that $g > 0$, the critical value for instability (complex solution) is

$$\delta_c = \frac{3}{2^{2/3}} g \quad . \tag{6.8}$$

If the detuning factor δ is greater than δ_c, (6.7) has three real roots (stable) and, if $\delta < \delta_c$, (6.7) has one real root and two complex (conjugate) roots (unstable). Therefore, in order to get the instability, the detuning factor δ must be less than δ_c. The explicit unstable ($\delta < \delta_c$) solutions of (6.7) are given by

$$\begin{aligned} \xi_1 &= \frac{2}{3}\delta - \frac{1}{2} b_+ + i \frac{\sqrt{3}}{2} b_- \\ \xi_2 &= \frac{2}{3}\delta - \frac{1}{2} b_+ - i \frac{\sqrt{3}}{2} b_- \\ \xi_3 &= \frac{2}{3}\delta + b_+ \end{aligned} \tag{6.9}$$

when

$$\begin{aligned} b_\pm &\equiv \sqrt[3]{a_+} \pm \sqrt[3]{a_-} \\ a_\pm &\equiv \left[\frac{g^3}{2} - \left(\frac{\delta}{3}\right)^3\right] \pm \sqrt{g^3\left[\frac{g^3}{4} - \left(\frac{\delta}{3}\right)^3\right]} \quad . \end{aligned} \tag{6.10}$$

In the same approximation, the relative amplitude of each mode corresponding to (6.9) is given by

$$\frac{1}{6}\left(1 - \frac{\frac{2}{3}\delta}{\xi_i - \delta/3}\right) \quad (i = 1,2,3) \quad , \tag{6.11}$$

and the amplitude of the opposite propagating mode is $-1/2$. When the detuning factor $\delta = 0$, Eq. (6.11) shows that the three modes share equal amplitude 1/6 each. This shows that the commonly accepted equal partition rule is valid only when $\delta = 0$ (no detuning). For a forward wave amplifier, one growing mode eventually dominates and the amplitude is roughly 1/3 of the input wave at the beginning of the interaction (with small modification

depending on the detuning factor). However, for a backward wave oscillator/amplifier, all three of the relative amplitudes are important. Oscillation conditions for a BWO is given by an infinite backward wave gain defined by (5.13) and satisfy

$$\sum_{i=1}^{3} e^{ik_{\parallel}L\xi_i}\left(1 - \frac{2/3\delta}{\xi_i - \delta/3}\right) = 0 \quad . \qquad (6.12)$$

One can show that (6.12) can be satisfied only by a series of specific pairs of the frequency and the detuning factor, (ω,δ). Eq. (6.12) clearly demonstrates that the backward wave oscillation is the result of three wave interference and the correct amplitudes are essential to obtain correct oscillation conditions.

VII. VELOCITY SPREAD EFFECT

So far we have discussed the "cold beam" case defined by (4.8). If the beam is general $f_0(p_{\parallel}, p_{\perp}, R_G)$, one must do the "statistical average" on the source term as (4.13). The most sensitive velocity effect comes in through the resonance term $\Omega_s^0(k) \equiv \omega\gamma_0 - k\ \frac{p_{\parallel}}{m} - s\Omega_c$ and thus the spread in the longitudinal momentum is most important (spread in energy γ_0 is usually small).

In this section, we will discuss only this most important velocity spread effect in the longitudinal momentum. A velocity spread from a typical MIG-type electron gun may be described by a generalized Lorentian distribution of order n

$$f_0^{(n)}(p_{\parallel}) \equiv Cp\ \frac{\Delta p_{\parallel}^{2n-1}}{(p_{\parallel} - \bar{p}_{\parallel})^{2n} + \Delta p_{\parallel}^{2n}} \tag{7.1}$$

with the normalization constant defined by

$$1 = \int_{-\infty}^{\infty} dp_{\parallel} f_0^{(n)}(p_{\parallel}) \quad . \tag{7.2}$$

Defining a scale varialbe,

$$\eta \equiv \frac{p_{\parallel} - \bar{p}_{\parallel}}{\Delta p_{\parallel}} \tag{7.3}$$

one can write (7.1) and (7.2) as

$$f_0^{(n)}(p_{\parallel}) = \frac{Cp}{\Delta p_{\parallel}}\ \frac{1}{\eta^{2n} + 1} \tag{7.4}$$

and

$$1 = Cp \int_{-\infty}^{\infty} d\eta\ \frac{1}{\eta^{2n} + 1} \quad . \tag{7.5}$$

Then, a "statistical average" of any arbitrary function $F(p_{\parallel})$ is given by

$$\begin{aligned} \langle F(p_{\parallel})\rangle_n &\equiv \int_{-\infty}^{\infty} dp_{\parallel} f_0^{(n)}(p_{\parallel}) F(p_{\parallel}) \\ &= Cp \int_{-\infty}^{\infty} d\eta\ \frac{F(p_{\parallel}^0 + \eta\Delta p_{\parallel})}{\eta^{2n} + 1} \quad . \end{aligned} \tag{7.6}$$

The integral (7.6) can be done most easily by a contour integral over a closed contour with upper or lower semi-infinite circle. Then, the integral is reduced to find the poles and their residues of the integrand. The poles are given by the roots of

$$\eta^{2n} + 1 = 0 \tag{7.7}$$

and whatever poles contained in $F(p_{\parallel})$. Sometimes the poles of $F(p_{\parallel})$ occur either in the upper half plane or lower half plane depending on the parameters but not in both planes simultaneously. In this case one can close the contour to avoid these poles and the relevant poles are entirely from (7.7), which are given by

$$\eta_i = \exp\left[i\pi\left(\frac{2i-1}{2n}\right)\right], \quad (i = 1 \ldots n) \tag{7.8}$$

in the upper plane and η_i^* in the lower plane. Therefore only either η_i $(i = 1, \ldots n)$ or η_i^* are included in the contour. Therefore, the integral (7.6) is given by

$$\langle F(p_{\parallel})\rangle_n = \sum_{i=1}^{n} \eta_i F(p_{\parallel}^0 + \eta_i \Delta p_{\parallel}) / \sum_{i=1}^{n} \eta_i \tag{7.9a}$$

if F is regular in the upper plane, or

$$\langle F(p_{\parallel})\rangle = \sum_{i=1}^{n} \eta_i^* F(p_{\parallel}^0 + \eta_i^* \Delta p_{\parallel}) / \sum_{i=1}^{n} \eta_i^* \tag{7.9b}$$

if F is regular in the lower plane.

Now one can do the statistical average integral (4.13) with the source term $\tilde{P}(k)$ given by (4.11). The $\tilde{P}(k)$ has poles in the complex logitudinal momentum plane at $\Omega_s^0 \equiv \omega\gamma_0 - k\frac{p_{\parallel}^0}{m} - s\Omega_c = 0$ and thus $p_{\parallel}^0/m = (\omega\gamma_0 - s\Omega_c)/k$. Noting that k has negative imaginary part, the poles are either in the upper plane if $\omega > s\Omega_0/\gamma_0 \equiv s\omega_c$ or in the lower plane if $\omega < s\omega_c$. Usually the operating frequency ω is higher than the cyclotron frequency $s\omega_c$ for a forward amplifier (positive doppler shift) and ω is lower than $s\omega_c$ for a backward wave amplifier (negative doppler shift). Therefore, in forward wave

region ($\omega > s\omega_c$), one can close the contour in the lower half plane to avoid the poles of $\tilde{P}(k)$ and, in the backward wave region, close in the upper half plane. Then the statistical average (5.13) picks up poles η_i^* in a forward wave region and poles η_i in a backward wave region. Using (7.9a) or (7.9b), one obtains

$$\langle\tilde{P}(k)\rangle = \sum_{i=1}^{n} \eta_i^* \tilde{P}(k, p_{\parallel}^0 + \eta_i^* \Delta p_{\parallel}) \Big/ \sum_{i=1}^{n} \eta_i^* \tag{7.10a}$$

in a forward wave region ($\omega > s\omega_c$) or

$$\langle\tilde{P}(k)\rangle = \sum_{i=1}^{n} \eta_i \tilde{P}(k, p_{\parallel}^0 + \eta_i \Delta p_{\parallel}) \Big/ \sum_{i=1}^{n} \eta_i \tag{7.10b}$$

in a backward wave region ($\omega > s\omega_c$).

With this new source term, (7.10a) or (7.10b), Eq. (5.4) is replaced by

$$D(k) = (k^2 - k_{\parallel}^2) - N_0 \sum_{s=-\infty}^{\infty} \left[\left(\frac{\omega^2}{c^2} - k^2 \right) \frac{p_{\perp}^2}{m^2} A_2 \left\langle \frac{1}{\Omega_s^{02}} \right\rangle + s\Omega_c A_1 \left\langle \frac{1}{\Omega_s^0} \right\rangle + A_0 \right]$$

$$N_1(k) = k + N_0 \sum_{s=-\infty}^{\infty} \left[k \frac{p_{\perp}^{02}}{m^2} A_2 \left\langle \frac{1}{\Omega_s^{02}} \right\rangle + \left\langle \frac{p_{\parallel}^0/m}{\Omega_s^0} \right\rangle A_0 \right] \tag{7.11}$$

$$N_2(k) = k_{\parallel}$$

where

$$\left\langle \frac{1}{\Omega_s^{02}} \right\rangle \equiv \sum_{i=1}^{n} \left(\omega\gamma_0 - s\Omega_c - k \frac{p_{\parallel}^0 + \eta_i \Delta p_{\parallel}}{m} \right)^{-2} \Big/ \sum_{i=1}^{n} \eta_i$$

$$\left\langle \frac{1}{\Omega_s^0} \right\rangle \equiv \sum_{i=1}^{n} \left(\omega\gamma_0 - s\Omega_c - k \frac{p_{\parallel}^0 + \eta_i \Delta p_{\parallel}}{m} \right)^{-1} \Big/ \sum_{i=1}^{n} \eta_i \tag{7.12}$$

$$\left\langle \frac{p_{\parallel}^0/m}{\Omega_c^0} \right\rangle \equiv \sum_{i=1}^{n} \frac{p_{\parallel}^0 + \eta_i \Delta p_{\parallel}}{m} \left(\omega\gamma_0 - s\Omega_c - k \frac{p_{\parallel}^0 + \eta_i \Delta p_{\parallel}}{m} \right)^{-1} \Big/ \sum_{i=1}^{n} \eta_i$$

in a backward wave region, or the same expression as (7.12) but η_i is

replaced by η_i^* in a forward wave region.

A major qualitative difference of (7.11) from (5.4) is that, when they are rationalized as (5.8), d(k) is no more a quartic polynomial if $n \geq 2$. This means that the number of waves are more than four. For n=1, d(k) is still a quartic polynomial and described by four-waves as in the cold beam case. However, as one can see that an ordinary Lorentian distribution (n=1) has too long a tail and may be unrealistic. The case with n=2 is seen to be much more realistic and then the number of waves are six. In general, the possible number of modes are 2 + 2n. Here we saw that the number of waves required to account for a velocity spread effect is highly dependent on the shape of velocity distribution function.

VIII. DISCUSSION

In this report we have presented a linear theory to analyze gyrotron BWO. This has been possible because our formulation allows that the axial behavior of the rf field and the electron beam is completely determined self-consistently by the boundary values. This type of solving the problem as a boundary value problem is particularily crucial for analyzing BWO because, in BWO, the electron beam itself plays the role of self-feedback.

It has turned out to be that the infinite backward gain condition in a BWO can be met by an interference among three backward propagating modes with appropriate amplitudes (including phases) and growing and decaying factors. If the conditions are right with the detuning factor and interaction length, rf fields can grow out of a noise. A typical backward wave oscillation condition as a function of the detuning frequencies and interaction lengths are shown in Fig. 8.1. There are a series of oscillation points with different detuning factors and interacting lengths. Typically the smaller the detuning factor, the longer the interacting length. On the actual device, it is interesting to see whether one can see all these modes or only the first mode (with the shortest interaction length) dominates and blocks out the remaining modes. The experimental design has a provision to test this. With a reasonable interaction length less than 15 cm, tuning over almost an octave in Ku-band seems to be possible. The velocity spread effect does not deteriorate much up to 10% or so, worse than the gyrotron oscillator but better than the forward wave amplifier. Another interesting effect of velocity spread is that, when it is above a certain critical value, it stabilizes the oscillation - acts as a backward wave amplifier rather than oscillator. This opens up a possibility to operate a gyrotron BWO in a backward wave amplifier. Certainly the required interaction length

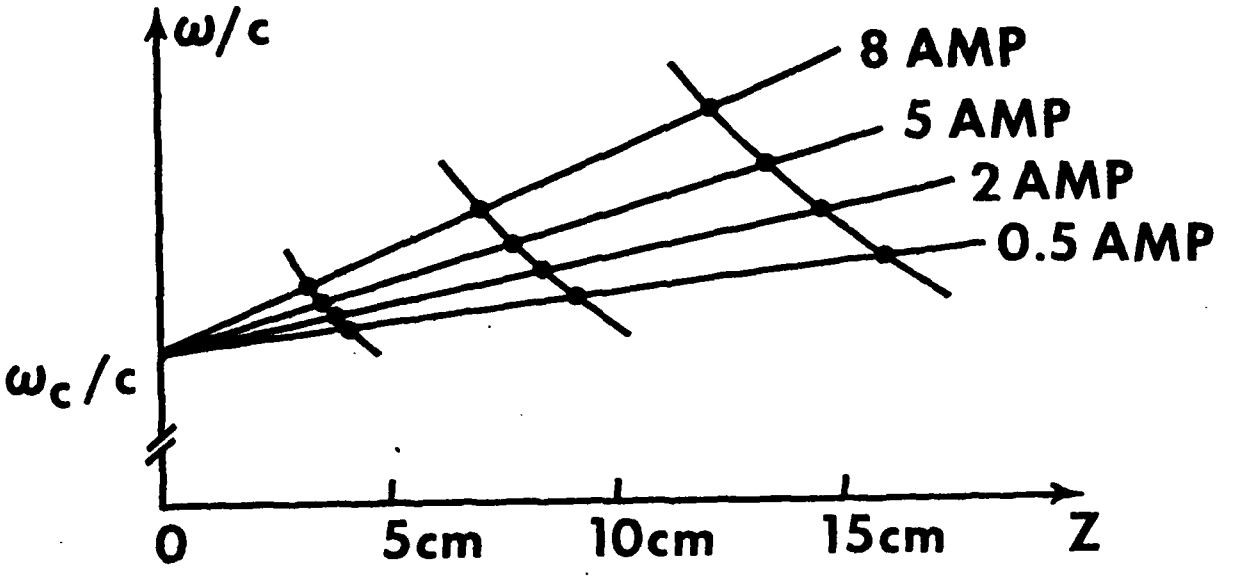

FIG. 8.1 Oscillation conditions as a function of detuning frequenc and interaction length for different current of the beam.

is large (typically 30-40 cm) and the efficiency may be low due to the large velocity spread.

In conclusion, the analysis shows that a several tens killowatt gyrotron BWO may be possible in a Ku-band tunable up to almost an octave by either magnetically, electrically, or both. The tunability may find an ideal application in studying the rf heating in fusion devices.

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